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An Implicit Partial Pivoting Gauss Elimination Algorithm for Linear System of Equations with Fuzzy Parameters

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Abstract

This paper considers the solution of fully fuzzy linear system (FFLS) by first reducing the system to crisp linear system. The novelty of this article lies in the application of Gauss elimination procedure with implicit partial pivoting to FFLS. The method is presented in detail and we use the Matlab® software for implementing the algorithm. Numerical examples are illustrated to demonstrate the efficiency of the variant of Gauss elimination method for solving FFLS.

Keywords: fully fuzzy linear system, fuzzy number, gauss elimination, partial pivoting, implicit

1. Introduction

System of linear equations plays a crucial role in various areas such as physics, statistics, operational research, engineering and social sciences. When information is imprecise and only some vague knowledge about the actual values of the parameters is available, it is convenient to make use of fuzzy numbers (L. Zadeh 1965).

One of the major applications of fuzzy number arithmetic is in solving linear systems whose parameters are all or partially represented by fuzzy numbers. In this paper, the term fuzzy matrix is of the most importance concept, and we follow the definition proposed by D. Dubois & H. Prade (1980), that is, a matrix with fuzzy numbers as its elements. Friedman *et al.* (1998) introduced a general model for solving fuzzy linear system whose coefficient matrix is crisp and the right hand vector to be an arbitrary fuzzy vector. Friedman and his colleagues solved the fuzzy linear system by first reducing it to a crisp linear system. Review about some methods for solving these systems can be found in (M. Matinfar *et al.* 2008).

Here we consider another kind of fuzzy linear systems where all the parameters include fuzzy numbers and are named fully fuzzy linear systems (FFLS). Recently, M. Dehghan *et al.* (2006) and M. Dehghan & B. Hashemi (2006) proposed computational methods such as Cramer's rule, Gaussian elimination method, LU decomposition method and Adomian decomposition method for solving FFLS. Iterative techniques for the solution of FFLS is presented in (M. Dehghan *et al.* 2007), whereby techniques such as Richardson, Jacobi, Jacobi over relaxation, Gauss-Seidel, successive over relaxation, accelerated over relaxation, symmetric and unsymmetrical successive over relaxation and extrapolated modified Aitken are studied. For other methods to solve FFLS, one may refer to A. Kumar *et al.* (2010).

In this paper, to cater for rounding errors in the Gauss elimination process, we intend to solve the FFLS by implicit partial pivoting of the Gauss elimination method. This paper is structured as follows: In the next section, we give some preliminaries concerning fuzzy sets theory. In section 3, the new procedure based on the partial pivoting is introduced. Numerical examples are presented in section 4 to illustrate the method.

2. Preliminaries

In this section, we present some backgrounds and notions of fuzzy sets theory (D. Dubois & H. Prade 1980; M. Matinfar *et al*. 2008).

2.1 Definitions

Definition 2.1. Assume X to be a universal set, and then a fuzzy subset $\tilde{A}$ of X is defined by its membership function

$$\mu_{\tilde{A}} : X \to [0, 1],$$

where the value of

$$\mu_{\tilde{A}}(x)$$

at x shows the grade of membership of x in $\tilde{A}$. A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(x)$ and is often written as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}.$$

Definition 2.2. A fuzzy set $\tilde{A}$ in X is said to be normal if there exist $x \in X$ such that $\mu_{\tilde{A}}(x)=1$.

Definition 2.3. A fuzzy number $\tilde{A}$ is called positive (negative), denoted by $\tilde{A} > 0$ $(\tilde{A} < 0)$, if its membership function $\mu_{\tilde{A}}(x) = 0$, $\forall x < 0$ $(x > 0)$.

Definition 2.4. A triangular fuzzy number, symbolically written as $\tilde{A} = (m, \alpha, \beta)$, has the following membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m - x}{\alpha}, & m - \alpha \le x < m,\ \alpha > 0, \\ 1 - \dfrac{x - m}{\beta}, & m \le x \le m + \beta,\ \beta > 0, \\ 0, & otherwise \ . \end{cases}$$

Definition 2.5. A triangular fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is positive if and only if $m - \alpha \ge 0$.

Definition 2.6. Two triangular fuzzy numbers $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ are said to be equal if and only if $m = n,\ \alpha = \gamma,\ \beta = \delta$.

Definition 2.7. A matrix $\tilde{A}$ is called a fuzzy matrix if each of its elements is a fuzzy number. The matrix is positive if each of its elements is positive. The $n \times n$ fuzzy matrix $\tilde{A}$ may be represented as $\tilde{A} = (A, M, N)$, where $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices.

2.2 Arithmetic operations on fuzzy numbers

In this section, we present arithmetic operations of triangular fuzzy numbers. Let $\tilde{A} = (m, \alpha, \beta)$ *and* $\tilde{B} = (n, \gamma, \delta)$ be two triangular fuzzy numbers, then the following rules are valid:

1) $\tilde{A} \oplus \tilde{B} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta)$.
2) $-\tilde{A} = -(m, \alpha, \beta) = (-m, \beta, \alpha)$.
3) If $\tilde{A} > 0$ and $\tilde{B} > 0$ *then*

$$\tilde{A} \otimes \tilde{B} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) = (mn, n\alpha + m\gamma, n\beta + m\delta).$$

4) If λ is any scalar then $\lambda \otimes \tilde{A}$ is defined as

$$\lambda \otimes \tilde{A} = \begin{cases} (\lambda m,\ \lambda\alpha,\ \lambda\beta),\ \lambda \geq 0, \\ (\lambda m,\ -\lambda\beta,\ -\lambda\alpha),\ \lambda < 0. \end{cases}$$

3. A New Method for Solving FFLS

Consider the $n \times n$ fully fuzzy linear system of the form (M. Dehghan *et al.* 2006; M. Matinfar *et al.* 2008)

$$\tilde{A} \otimes \tilde{x} = \tilde{b}, \tag{1}$$

with $\tilde{A} = (\tilde{a}_{ij}),\ 1 \leq i,\ j \geq n$ and $\tilde{x}_j, \tilde{b}_j \in F(R)$, where $F(R)$ is the set of all fuzzy numbers. In extended form (1) can be expressed as

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \ldots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{cases}$$

Here we are trying to solve for a positive solution of the FFLS (1). Let $\tilde{A} = (A,\ M,\ N) > \tilde{0}$, $\tilde{b} = (b,\ h,\ g) > \tilde{0}$ and $\tilde{x} = (x,\ y,\ z) > \tilde{0}$. Thus we have

$$(A,\ M,\ N) \otimes (x,\ y,\ z) = (b,\ h,\ g).$$

Therefore using the multiplication rule 3 of Section 2.2 followed by Definition 2.6, we obtain the following crisp system

$$\begin{cases} Ax = b, \\ Ay + Mx = h, \\ Az + Nx = g. \end{cases} \tag{2}$$

From (2), we observe that once a solution to x, say x_{sol}, is obtained, solution to y and z may be derived by solving the crisp linear systems

$$\begin{cases} Ay = h - Mx_{sol}, \\ Az = g - Nx_{sol}. \end{cases} \tag{3}$$

We further note that the coefficient matrix for the linear system remains the same for all x, y and z. Next section we present a method for the solution of the crisp linear system of equations.

3.1 Gauss Elimination Method with Partial Pivoting

Consider the crisp linear system of the form

$$Ax = b,\ A = (a_{i\,j}), \tag{4}$$

where the matrix A is of order n, and $x = (x_1, x_2, \ldots, x_n)^T$ and $b = (b_1, b_2, \ldots, b_n)^T$ are column vectors of length n. One of the methods for solving (1) is the Gauss elimination procedure which can be summarized by the following statement.

Theorem 3.1 (A. Gourdin & M. Boumahrat 2003): *If A is an arbitrary non-singular matrix of order n then there exists an invertible matrix S such that SA=U, where U is an upper triangular matrix.*

The triangularization process of Theorem 3.1 can be obtained by pre-multiplying elementary matrices (S.

Lipschutz 2005) (whereby there may be row exchange operations) with the augmented matrix $[A \mid b]$. At each step in the triangularization process, an assumption is made that the term a_{kk} is non zero. This term is called the pivot which is used to eliminate x_k from the rows $(k + 1)$ to n. In terms of floating point arithmetic, dividing by small pivots should be avoided to minimize rounding errors. The partial pivoting is a well-known strategy to cater for that drawback.

Next we present the Gauss elimination with partial pivoting algorithm where p_k is the k^{th} pivot found in the row l_k for $k = 1, 2, \ldots, n$. We note the algorithm is an implicit approach as there is no exchange of the rows or columns of the augmented matrix.

Algorithm 3.1: Input – Non-singular matrix A and vector b

Output – Vector x

For $k = 1{:}n - 1$,

1. For $i = 1{:}n$ with $i \neq l_1, l_2, \ldots, l_{k-1}$ select the pivot element p_k as $p_k = \max\{a_{ik}\}$.
2. For $i \neq l_1, l_2, \ldots, l_k$ and $j = k, k + 1, \ldots, n + 1$, triangularize the augmented matrix by using the formula

$$a_{ij} = a_{ij} - (a_{ik} a_{l_k j}) / p_k .$$

3. Solve for x by using the formulae

$$x_n = (a_{l_n n+1}) / p_n ,$$

$$x_i = (a_{l_i n+1} - \sum_{j=i+1}^{n} a_{l_i j} x_j) / p_i , \; i = (n-1), \ldots, 1.$$

4. Numerical examples

In this section, we apply Algorithm 3.1 for solving fully fuzzy linear system. We implement the algorithm in the Matlab® software and for the first example we illustrate as well the resulting matrix when eliminating the x_i variable from the remaining equations of the system (3).

4.1 Example 1

Consider the following FFLS (M. Dehghan *et al.* 2006):

$$\begin{bmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{bmatrix} \tilde{x} = \begin{pmatrix} (58, 30, 60) \\ (142, 139, 257) \\ (316, 297, 514) \end{pmatrix}.$$

The augmented matrix for the system (4) of order 3 is

$$\begin{bmatrix} 6 & 5 & 3 & | & 58 \\ 12 & 14 & 8 & | & 142 \\ 24 & 32 & 20 & | & 316 \end{bmatrix}.$$

Using Algorithm 3.1, we found that when $k = 1$, $p_1 = 24$, $l_1 = 3$ and the resulting matrix when eliminating the x_1 variable from the remaining equations is given as

$$\begin{bmatrix} 0 & -3 & -2 & | & -21 \\ 0 & -2 & -2 & | & -16 \\ 24 & 32 & 20 & | & 316 \end{bmatrix}.$$

When k = 2, p_2 = -2, l_2 = 2 and the resulting matrix when eliminating the x_2 variable from the remaining equation is

$$\begin{bmatrix} 0 & 0 & 1 & | & 3 \\ 0 & -2 & -2 & | & -16 \\ 24 & 32 & 20 & | & 316 \end{bmatrix}.$$

In a similar procedure, we obtain the third pivot element p_3=1 and l_3 = 1. So with the implicit row exchange operation we have

$$x = (x_1, x_2, x_3)^T = (4, 5, 3)^T.$$

Thus using (3) and Algorithm 3.1, we get

$$y = (y_1, y_2, y_3)^T = (1, 0.5, 0.5)^T,$$

and

$$z = (z_1, z_2, z_3)^T = (3, 2, 1)^T.$$

Therefore the fuzzy solution of the problem is

$$\tilde{x} = \begin{pmatrix} (4, 1, 3) \\ (5, 0.5, 2) \\ (3, 0.5, 1) \end{pmatrix},$$

the same solution with LU decomposition method as given in (M. Dehghan *et al.* 2006)

4.2 Example 2

Consider the following FFLS (M. Dehghan & B. Hashemi 2006):

$$\begin{bmatrix} (19,1,1) & (12,1.5,1.5) & (6,0.5,0.2) \\ (2,0.1,0.1) & (4,0.1,0.4) & (1.5,0.2,0.2) \\ (2,0.1,0.2) & (2,0.1,0.3) & (4.5,0.1,0.1) \end{bmatrix} \tilde{x} = \begin{pmatrix} (1897, 427.7, 536.2) \\ (434.5, 76.2, 109.3) \\ (535.5, 88.3, 131.9) \end{pmatrix}.$$

The augmented matrix for the system (4) of order 3 is

$$\begin{bmatrix} 19 & 12 & 6 & | & 1897 \\ 2 & 0.1 & 0.1 & | & 434.5 \\ 2 & 2 & 4.5 & | & 535.5 \end{bmatrix}.$$

So using Algorithm 3.1, we have $x = (37, 62, 75)^T$, $y = (7, 5.5, 10.2)^T$ and $x = (13.3016, 4.5794, 13.9196)^T$. Thus the fuzzy solution is

$$\tilde{x} = \begin{pmatrix} (37, 7, 13.3016) \\ (62, 5.5, 4.5794) \\ (75, 10.2, 13.9196) \end{pmatrix}.$$

5. Conclusion

In this paper, a new method is applied to compute the solution of fully fuzzy linear system. Here, an implicit partial pivoting Gauss elimination procedure is used as solver and the validity of the proposed algorithm is examined with two numerical examples which were used in (M. Dehghan *et al.* 2006) and (M. Dehghan & B. Hashemi 2006).

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