Innovative Systems Design and Engineering www.iiste.org
ISSN 2222-1727 (Paper) ISSN 2222-2871 (Online)
Vol 2, No 4, 2011

A Fixed Point Theorem In 2-Banach Space For Non-Expansive Mapping

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Abstract:
Our object in this paper to discuss about fixed point theory in 2-Banach space also we established a fixed point theorem in 2- Banach space which generalized the result of many mathematician.

Key Words: Normed space, 2-norned space, 2- Banach space, Nonexpensive mappings.

1 Introduction

The concept of two banach space firstly introduced by (Gahler1964) This space was subsequently been studied by mathematician (Kirk1981) and (Kirk1983) in last years. (Badshah and Gupta2005) also proved some result in 2-Banach space.(Yadav et al 2007) prove the result in 2-Banach space for non contraction mapping. (Lal and Singh 1978)the analogue og Banach Contraction principle in 2-metric space for selfmap and in the present we prove a fixed point theorem in 2-Banach Spaces by taking nanexpansive mapping.

2 Preliminaries

2.1 Definition:

Let X be a real linear space and $\|.,.\|$ be a nongative real valued function defined on X satisfying the fallowing condition:

(i) $\|x, y\| = 0$ iff x and y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$ for all x,y $\in$ X.

(iii) $\|x, ay\| = |a|\|x, y\|$, a being real, for all x,y $\in$ X.

(iv) $\|x, y+z\| = \|x, y\| + \|y, z\|$ for all x,y,z $\in$ X.

then $\|.,.\|$ is called a 2-norm and the pair $(X, \|.,.\|)$ is called a linear 2-normed space.

So a 2-norm $\|x, y\|$ always satisfies $\|x, y+ax\| = \|x, y\|$ for all x,y$\in$ X and all scalars a.

2.2 Definition:

A Sequence $\{x_n\}$ in a 2-norned space $(X, \|.,.\|)$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty}\|x_m - x_n, a\| = 0$ for all a in X.

2.3 Definition:

A Sequence $\{x_n\}$ in a 2-norned space $(X, \|.,.\|)$ is said to be convergent if there is a point x in X such that $lim_{n\to\infty}\|x_n - x, y\| = 0$ for all y in X. If x_n converges to x , we write $x_n \to x$ as $n \to \infty$.

2.4 Definition:

A linear 2-normed space is said to be complete if every Cauchy sequence is convergent to an element of X. A complete 2-normed space X is called 2-Banach spaces.

2.5 Definition:

Let X be a 2- Banach space and T be a self mapping of X.T is said to continuous at x if for every sequence $\{x_n\} in X, \{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x) as\ n \to \infty.$

2.6 Definition:

A function $f : R \to R$ is said to be upper semi continuous at a point $x \in R$ if given $\in > 0$ there exist a neighourhood N of x_0 in which $f(x) < f(x_0) + \in$ for all $x \in N$.

2.7 Definition:

Let X be a 2-Banach space and C be non empty bounded closed and convex subset of X. A mapping $T : C \to X$ is said to be nonexpensive if

$$\| T(x) - T(y), a \| \le \|x - y, a\| \text{ where x,y} \in \text{C}$$

3 Main Result

3.1 Theorem

Let F and G be two non expansive mapping of a 2-Banach space X into itself . F and G satisfy the fallowing condition

(1)
FG = G = I where I is identity map.

(2)

$$\|F(x) - G(y), a\| \le \alpha\|x - F(x), a\| + \beta\|y - G(y), a\| + \gamma\|x - G(y), a\| + \delta\|y - F(x), a\| + \eta\|x - y, a\|$$

where $\alpha, \beta, \gamma, \delta, \eta, \ge 0\ \forall\ x, y \in X$ where $2\alpha + 2\beta + 3\delta + \gamma + \eta \le 2$ then F and G have common fixed point .

Proof :

Taking $y = \frac{1}{2}\|(F + I)x\|, z = G(y)u = 2y - z$, then

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$\|z-x,a\| = \|G(y)-FG(x),a\|$

Now using (1) and (2) we get

$$\|z-x,a\| = \left\|G(y)-G\left(F\left(x\right)\right),a\right\| \le \alpha\|y-G(y),a\| + \beta\|F(x)-G(F(x)),a\| + \gamma\|y-G(F(x)),a\| + \delta\|F(x)-G(y),a\| + \eta\|y-F(x),a\|$$

$$\le \alpha\|y-G(y),a\| + \beta\|F(x)-x,a\| + \frac{1}{2}\gamma\|F(x)-x,a\| + \delta\|F(x)-y,a\| + \delta\|y-G(y),a\| + \frac{1}{2}\eta\|y-F(x),a\|$$

$$\le (\alpha+\delta)\mathrm{P}\, y-G(y)| + (\beta+\frac{1}{2}\gamma+\frac{1}{2}\eta+\frac{1}{2}\delta)\|F(x)-x\|$$

Now $\|u-x,a\| = \|2y-z,a\| = \|G(y)-F(x),a\|$

$$\le \alpha\|y-G(y),a\| + \beta\|x-F(x),a\| + \gamma\|y-F(x),a\| + \delta\|x-G(y),a\| + \eta\|y-x,a\|$$

$$\le \alpha\|y-G(y),a\| + \beta\|x-F(x),a\| + \frac{1}{2}\gamma\|x-F(x),a\| + \frac{1}{2}\delta\|x-F(x),a\| + \delta\|y-G(y),a\| + \frac{1}{2}\eta\|x-F(x),a\|$$

$$\le (\alpha+\delta)\mathrm{P}\, y-G(y)| + (\beta+\frac{1}{2}\gamma+\frac{1}{2}\eta+\frac{1}{2}\delta)\|F(x)-x\|$$

$$\|z-u,a\| \le \|z-x,a\| + \|x-u,a\|$$

$$\le (2\alpha+2\delta)\|y-G(y),a\| + (2\beta+\gamma+\delta+\eta)\|x-F(x),a\|$$

Now $\|z-u,a\| = \|G(y)-2y-G(y),a\| = 2\|y-G(y),a\|$

$$\le (2\alpha+2\delta)\|y-G(y),a\| + (2\beta+\gamma+\delta+\eta)\|x-F(x),a\|$$

$$\Rightarrow\ 2(1-\alpha-\delta)\|y-G(y),a\| \le (2\beta+\gamma+\delta+\eta)\|x-F(x),a\|$$

$$\Rightarrow \|y-G(y),a\| \le \frac{2\beta+\gamma+\delta+\eta}{2(1-\alpha-\delta)}\|x-F(x),a\|$$

$$\Rightarrow \|y-G(y),a\| \le S\|x-F(x),a\|$$

where, $S = \dfrac{2\beta+\gamma+\delta+\eta}{2\left(1-\alpha-\delta\right)} \le 1$

and $2\alpha+2\beta+\gamma+3\delta+\eta \le 2$

Let $T = \frac{1}{2}(F + I)$, then for any $x \in X$

$$\|T^2(x) - T(x), a\| = \|T(T(x)) - T(x), a\|$$

$$\Rightarrow \|T(y) - y, a\| = \frac{1}{2}\|y - F(y), a\|$$

$$\Rightarrow \frac{1}{2}\|FG(y) - F(y), a\| \le \frac{1}{2}\|G(y) - y, a\|$$, because F is nonexpensive function .

So, $\|T^2(x) - T(x), a\| \le \frac{S}{2}\|x - F(x), a\|$, by definition of S . We claim that $T^n(x)$ is a Cauchy sequence in X. Also by completness $T^n(x)$ converges to $T(x)$,

i.e $\lim_{n\to\infty} T^n(x) = x_0 \Rightarrow F(x_0) = x_0$ therefore x_0 is fixed point of F.

Again $\|T^2(x) - T(x), a\| \le \frac{S}{2}\|x - F(x), a\| = \frac{S}{2}\|FG(x) - F(x), a\| \le \frac{S}{2}\|x - G(x), a\|$

we can conclude that $G(x_0) = x_0$ that is x_0 is fixed point of G.

Therefore $F(x_0) = G(x_0) = x_0$, so x_0 is common fixed point of F and G.

The uniqueness part is obvious.

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