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Least Action Principle Applied to a Non-Linear Damped Pendulum

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Abstract

The principle of least action is a variational principle that states an object will always take the path of least action as compared to any other conceivable path. This principle can be used to derive the equations of motion of many systems, and therefore provides a unifying equation that has been applied in many fields of physics and mathematics. Hamilton's formulation of the principle of least action typically only accounts for conservative forces, but can be reformulated to include non-conservative forces such as friction. However, it can be shown that with large values of damping, the object will no longer take the path of least action. Through numerical simulation, this is shown to be true for two simple systems, an object in free fall and a harmonic pendulum, both linearly and cubically damped.

MONTCLAIR STATE UNIVERSITY

Least Action Principle Applied to a Non-Linear Damped Pendulum

by

Katherine Rhodes

A Master's Thesis Submitted to the Faculty of

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College of Science and Mathematics

Department of Mathematical Sciences

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LEAST ACTION PRINCIPLE APPLIED TO A NON-LINEAR DAMPED PENDULUM

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KATHERINE RHODES

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Montclair, NJ

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Katherine Rhodes

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Chapter 1

Introduction

The least action principle (LAP) is a variational principle that can be used to derive the equations of motion for a system. This principle provides an alternative approach to deriving the equations of motion of a system versus the usual Newtonian methods and ties the equations of motion of many systems into one unifying principle. It has applications in classical mechanics, quantum mechanics, relativity, and several other branches of physics. One of the earliest examples of an optimal principle in physics is Fermat's principle of least time, which can be derived from the LAP. Light rays were shown to take the path of least time rather than any other path. This begs the question, can everything in nature be optimized? While the LAP has primarily dealt with natural phenomena related to non-living objects, recent studies have posed questions about the validity of optimality principles such as the LAP for biological phenomena. Interestingly, the LAP may also have applications in evolution and a connection to natural selection as stated by Kaila and Annila [1]. For example, Fermat's principle was used to describe the path of ants over different surfaces [2].

The principle of least action, also called the principle of stationary action, states that the actual trajectory of an object, given its initial and final configurations, is the one that has the least action as compared to all other conceivable paths with the same initial and final configurations, or

$$\delta S = 0. \tag{1.1}$$

The action S, as defined by Hamilton, is the time integral of the Lagrangian. In a simple case, with only conservative forces, the Lagrangian is defined as the kinetic energy minus the potential energy of a system, or

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}) dt = \int_{t_1}^{t_2} (K - V) dt \tag{1.2}$$

where t_1 and t_2 are the initial and final times, $\mathcal{L}$ is the Lagrangian, $x(t)$ is the trajectory, and $\dot{x}(t)$ is the velocity. K is the kinetic energy equal to $\frac{1}{2}m\dot{x}^2$, where m is the mass of the object, and V is the potential energy.

Hamilton's formulation of the action came about one hundred years after Maupertuis' first discussed the concept in 1744. The Maupertuis version states that the action is the position integral of the momentum,

$$S = \int_{x_1}^{x_2} p dx$$

where x_1 and x_2 are the initial and final positions and p is the canonical momentum and equal to $\partial\mathcal{L}/\partial\dot{x}$. The integrands in both versions of the action integral are related by the Legendre transform which states that $\mathcal{L} = p\dot{x} - H$ where $H = K + V$ is the Hamiltonian. One important result of the principle of least action is the Euler-Lagrange equation, which states

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0.$$

From this equation, the equations of motion of most classical and relativistic physics can be derived, such as Fermat's principle of least time, the equations of free fall, the harmonic pendulum, etc. This is a powerful conclusion as it means that the equations of motion can be derived solely from the energies of a system, rather than the usual Newtonian approach.

The principle of least action as defined above applies only to conservative forces. Many attempts have been made to extend this principle to include the non-conservative force of dissipation. In order to account for dissipation, a new Lagrangian must be introduced. Early attempts to formulate this Lagrangian include those by Rayleigh and Bateman [3] [4] [5].

Rayleigh [6] [7] proposed the equation

$$\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial D}{\partial \dot{x}} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) = 0$$

where the damping term D is $\frac{1}{2}\zeta\dot{x}$, a special case of Stoke's Law $f_d = -m\zeta\dot{x}$, where m is the mass of the body and ζ is the drag. However, then there is no unique Lagrangian to satisfy the least action principle (equation 1.1) and it does not relate to the dissipated energy [4].

Bateman [8] proposed a dissipative Lagrangian that was equal to the conservative Lagrangian multiplied by the factor $e^{\zeta t}$ or

$$\mathcal{L} = e^{\zeta t}(K - V). \tag{1.3}$$

However, while this Lagrangian satisfies the principle, there is nothing physically meaningful about this choice and it only applies to linear damping.

In this work, the dissipative Lagrangian proposed by Wang and Wang in [5] is used to analyze the action for both a falling object and a harmonic pendulum. Their Lagrangian is $\mathcal{L} = K - V - E_d$ where E_d is the energy lost due to dissipation. The idea is to add the force of friction to the least action principle and the Euler-Lagrange equation so that the correct equations of motion can still be derived, as they can be for conservative systems. For conservative systems, the force is given by $F = -\frac{\partial V}{\partial x}$, where F is the force, V is the potential energy, and x is the position. Similarly, let the force due to friction, f_d, be equal to the negative derivative of E_d, or $f_d = -\frac{\partial E_d}{\partial x}$. The new Euler-Lagrange equation becomes

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) = f_d \tag{1.4}$$

which is the same as

$$\partial S = \int_{t_1}^{t_2} (\partial \mathcal{L} + f_d \delta x) dt = 0. \tag{1.5}$$

Chapter 2

Least Action Principle

Calculation of the action of a system provides a real number as a result, which varies for different paths of an object. Thus we can compare the path of an object to slight perturbations of this path, using the action.

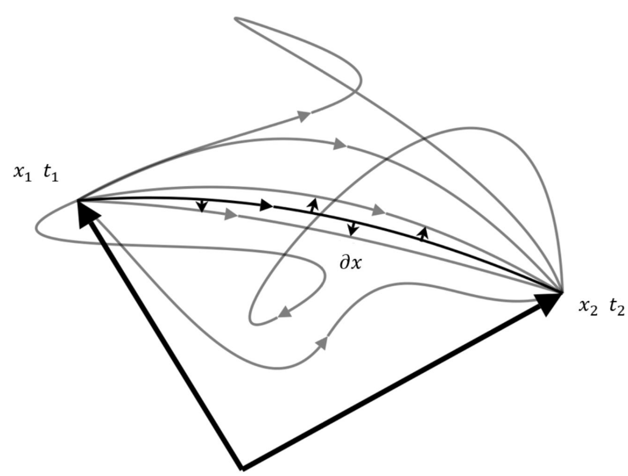

Figure 2.1: Optimal path (black) with some perturbations (gray)

The optimal path is one found using the equations of motion for the system which can be derived from the least action principle. The perturbed paths are found using a small variation of the optimal path. While it is not possible to check all conceivable alternate paths, a large number of them are generated for comparison. The principle of least action states that the action of the optimal path is a minimum compared to all other conceivable paths with the same initial and final positions. Figure 2.1 depicts one such optimal path in black with perturbed paths in gray found by moving each point of the optimal path a small variable amount ∂x while keeping the same initial and final positions x_1 and x_2 at times t_1 and t_2 respectively.

2.1 Hamilton's Principle of Least Action

Using Hamilton's principle, assume the Lagrangian depends only on the position $x(t)$, velocity $\dot{x}(t)$, and time t of the object in motion. The action, denoted S, is

$$S = \int_t \mathcal{L}(x, \dot{x}, \tau) d\tau.$$

Assume also that the true path of the object over some time period from t_1 to t_2 is $\tilde{x}(t)$, so that the action, S, achieves its minimum along this trajectory. Then take an arbitrary nearby path, say $x(t)$. This arbitrary path can be defined as the true path plus some variation, $\delta x(t)$, or

$$x(t) = \tilde{x}(t) + \delta x(t). \tag{2.1}$$

No matter which path is chosen, the particle is at the same position at both the beginning and the end of the time interval, so $\delta x(t_1) = \delta x(t_2) = 0$. Let S be stationary with small perturbations, so that the first variation is $\frac{\delta S}{\delta x} = 0$. The change in action for these two paths is defined as

$$\delta S = S(x) - S(\tilde{x}) = S(\tilde{x} + \delta x) - S(\tilde{x}).$$

Note that δS is greater than zero since $S(\tilde{x})$ is a minimum. Applying the definition of action, δS becomes

$$\delta S = \int_{t_1}^{t_2} \mathcal{L}(\tilde{x} + \delta x, \dot{\tilde{x}} + \delta \dot{x}, t) dt - \int_{t_1}^{t_2} \mathcal{L}(\tilde{x}, \dot{\tilde{x}}, t) dt.$$

The first term of δS can be expanded in a Taylor series about the path $\tilde{x}$.

$$\delta S = \int_{t_1}^{t_2} \Big[\mathcal{L}(\tilde{x}, \dot{\tilde{x}}, t) + \frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x} + \mathcal{O}(\delta x^2, \delta \tilde{x}^2)\Big] dt - \int_{t_1}^{t_2} \mathcal{L}(\tilde{x}, \dot{\tilde{x}}, t) dt$$

Then the first term of the first integral cancels with the second integral, leaving only

$$\delta S = \int_{t_1}^{t_2} \Big[\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x} + \mathcal{O}(\delta x^2, \delta x^{*2})\Big] dt$$

so that

$$\delta S = \int_{t_1}^{t_2} \Big[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)\Big] \delta x dt.$$

Since δx and dt are arbitrary,

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) = 0. \tag{2.2}$$

Equation (2.2) is the Euler-Lagrange equation. Therefore, in order for the principle of least action to be true, the Euler-Lagrange equation must be true. Two examples are used to demonstrate that the Euler-Lagrange equation can be used to derive the equations of motion.

Example 1: Object in Free Fall

For the falling object case, the kinetic energy is $K = \frac{1}{2}m\dot{x}^2$, and the potential energy is $V = mgx$, making the Lagrangian equal to

$$\mathcal{L} = K - V = \frac{1}{2}m\dot{x}^2 - mgx.$$

Then $\frac{\partial \mathcal{L}}{\partial x} = -mg$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$. Plugging into the Euler-Lagrange Equation gives

$$\frac{d}{dt}(m\dot{x}) - mg = 0$$

which simplifies to

$$m\ddot{x} = mg. \tag{2.3}$$

Equation 2.3 is the equation of motion for an object in free fall, derived from the Euler-Lagrange equation rather than the sum of the forces.

Example 2: Harmonic Pendulum

This method can also be used to derive the harmonic equation for the pendulum.

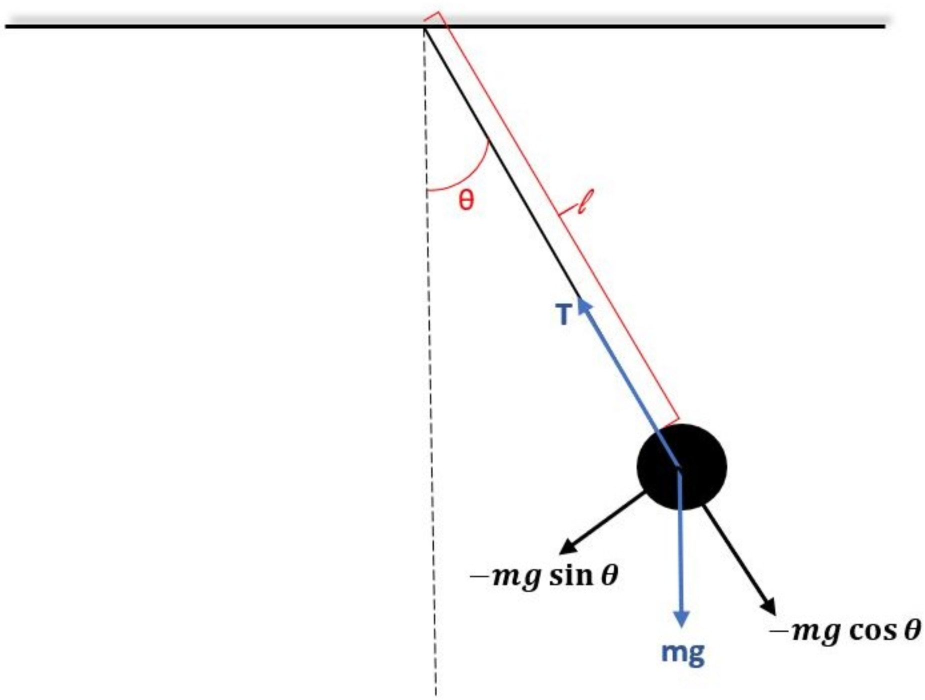

Figure 2.2: The harmonic pendulum positioned at an angle of θ from equilibrium with string length l and mass m.

In Figure 2.2, l is the length of the string and θ is the angle from equilibrium. The kinetic and potential energy are

$$K = \frac{1}{2}ml^2\dot{\theta}^2$$

and

$$V = mgl(1 - \cos\theta),$$

so the Lagrangian is

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta).$$

Then $\frac{\partial\mathcal{L}}{\partial\theta} = -mgl\sin\theta$ and $\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ml^2\dot{\theta}$. Plugging these derivatives into the Euler-Lagrange Equation gives

$$-mgl\sin\theta - \frac{d}{dt}(ml^2\dot{\theta}) = 0$$

$$-mgl\sin\theta - ml^2\ddot{\theta} = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{2.4}$$

Equation 2.4 is the equation of motion for the undamped, unforced harmonic pendulum, derived from the Euler-Lagrange equation.

Note that for both examples, the equations of motion agree with the Newtonian derivation as well.

2.2 Least Action Principle with Dissipation

The devising of the least action principle with dissipation is a persistent open question in classical mechanics. The action of a system relies on a Lagrangian or Hamiltonian. These values generally only account for conservative forces and fail to describe non-conservative interactions, such as friction. Wang and Wang describe a dissipative Lagrangian that accounts for the energy of an isolated system consisting of a damped, moving body and its environment coupled by the force of friction [5]. In this way, the energy of the total system is conserved, since energy is transfered from the body to the environment, even though the damped body is by itself a nonconservative system. The Hamiltonian, in this case, is taken to be the sum of the energy of the system. It can be written as

$$H = H_1 + H_2 + H_{int} \tag{2.5}$$

where H_1 is the Hamiltonian of the damped body and H_2 is the Hamiltonian of the environment, so that $H_1 = K_1 + V_1$ and $H_2 = K_2 + V_2 + E_d$, where $K_1 = \frac{1}{2}m\dot{x}^2$ is the kinetic energy of the body, V_1 is the potential energy of the body, K_2 is the kinetic energy of the constituents of the environment, V_2 is the potential energy of the constituents of the environment, and E_d is the energy dissipated from the body to the environment. H_{int} is the interaction energy which accounts for any changes in the interaction between the environment and the body within the neighborhood of the body.

We make the following assumptions.

1. The environment does not move so that $K_2 = 0$.

2. V_2 is equal to the potential energy at the start of motion and is constant. As a constant, it will be neglected in the calculations to follow without loss of generality.

3. There are no changes in the interface conditions. Therefore, H_{int} is also constant and will also be neglected in the following calculations.

Therefore, the effective Hamiltonian is

$$H = K_1 + V_1 + E_d. \tag{2.6}$$

The Legendre transform, which relates the Hamiltonian and Lagrangian, is $\mathcal{L} = p\dot{x} - H$, where $p = m\dot{x}$ is the canonical momentum. Then $p\dot{x} = 2K_1$, and the Lagrangian for the system is

$$\mathcal{L} = 2K_1 - H.$$

By substitution with $H = K_1 + V_1 + E_d$ we get

$$\mathcal{L} = K_1 - V_1 - E_d \tag{2.7}$$

with corresponding action,

$$S = \int_{t_i}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} (K_1 - V_1 - E_d) dt. \tag{2.8}$$

$E_d(x(t))$ is the energy dissipated from the body to the environment from the initial moment of motion to the time t due to the friction force. Hence,

$$E_d = -\int_{x_1}^{x} f_d(x(\tau), \dot{x}(\tau)) dx(\tau). \tag{2.9}$$

where f_d is a suitable function of time, position, and velocity. Since E_d depends not only on the current time, but also on the past trajectory, E_d is a nonlocal variable and causes both the Hamiltonian and the Lagrangian as defined above to be nonlocal as well.

Substituting this effective Lagrangian into the Euler-Lagrange equation gives

$$\frac{d}{dt}\Big(\frac{\partial K_1}{\partial \dot{x}}\Big) + \frac{\partial V_1}{\partial x} + \frac{\partial E_d}{\partial x} = 0 \tag{2.10}$$

The term $\partial E_d / \partial x$ is the same as

$$\frac{\partial E_d}{\partial x} = -\frac{\partial}{\partial x} \int_{x_1}^{x} f_d(x(\tau), \dot{x}(\tau) dx(\tau) = -f_d(t) \tag{2.11}$$

So equation 2.10 becomes

$$m\ddot{x} + \frac{\partial V_1}{\partial x} - f_d = 0 \tag{2.12}$$

This is the equation of motion for the system with dissipation.

2.3 Falling Object with Dissipation

For a falling object, $V_1 = mgx$ and $f_d = -\lambda \dot{x}$, so equation 2.12 becomes

$$m\ddot{x} + mg + \lambda \dot{x} = 0 \tag{2.13}$$

where m is the mass of the object, g is the gravitational constant, and λ is the damping coefficient. In order to solve the equation numerically, it can be broken up into a system of differential equations. First, let $y = \dot{x}$. Then equation 2.13 becomes

$$\dot{x} = y$$

$$\dot{y} = -g - \frac{\lambda}{m} y$$

These equations solved with Matlab's built in ODE solver, ode45.

2.4 Pendulum with Dissipation

For the damped pendulum, $V_1 = mgl(1 - \cos\theta)$ and $f_d = -\tilde{\lambda}\dot{x}^n$, where f_d is a frictional force of arbitrary power. Applying this to equation 2.12 gives

$$ml\ddot{\theta} + \frac{1}{l}(mgl\sin\theta) + \tilde{\lambda}\dot{x}^n = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{\tilde{\lambda}}{ml}\dot{x}^n = 0 \tag{2.14}$$

Letting $\omega_0 = \sqrt{\frac{g}{l}}$, $\lambda = \frac{\tilde{\lambda}}{ml}$, and adding a periodic force to the pendulum, this equation becomes

$$\frac{d^2\theta}{dt^2} + \lambda\Big(\frac{d\theta}{dt}\Big)^n + \omega_0^2 \sin\theta = \alpha\cos\omega_1 t \tag{2.15}$$

Here, λ is the damping coefficient, α is the forcing coefficient, ω_1 is the angular frequency, and n is the degree of damping. (Note that the force can easily be added to the Lagrangian using similar methods as above). To non-dimensionalize, let $\tau = \frac{t}{T_0}$ where t is the time in seconds and T_0 is the period, so that τ is non-dimensionalized time and $t = \tau T_0$.

Then,

$$\left(\frac{d\theta}{dt}\right)^n = \left(\frac{d\theta}{d(\tau T_0)}\right)^n = \frac{1}{T_0^n}\left(\frac{d\theta}{d\tau}\right)^n$$

$$\frac{d^2\theta}{dt^2} = \frac{d^2\theta}{d(\tau T_0)^2} = \frac{1}{T_0^2}\frac{d^2\theta}{d\tau^2}$$

Substituting into equation 2.15 gives

$$\frac{1}{T_0^2}\frac{d^2\theta}{d\tau^2} + \lambda\frac{1}{T_0^n}\left(\frac{d\theta}{d\tau}\right)^n + \omega_0^2 \sin\theta = \alpha\cos\left(\omega_1 \tau T_0\right)$$

Multiplying through by T_0^2 gives,

$$\frac{d^2\theta}{d\tau^2} + T_0^{2-n}\lambda\left(\frac{d\theta}{d\tau}\right)^n + T_0^2\omega_0^2\sin\theta = T_0^2\alpha\cos(\omega_1\tau T_0)$$

Now note that $T_0 = \frac{1}{\omega_0}$ so that,

$$\frac{d^2\theta}{d\tau^2} + \frac{\lambda}{\omega_0^{2-n}}\left(\frac{d\theta}{d\tau}\right)^n + \sin\theta = \frac{\alpha}{\omega_0^2}\cos(\frac{\omega_1}{\omega_0}\tau)$$

Lastly, let $\alpha_1 = \frac{\lambda}{\omega_0^{2-n}}$ be the coefficient of damping, let $\alpha_2 = \frac{\alpha}{\omega_0^2}$ be the forcing coefficient, and let $\alpha_3 = \frac{\omega_1}{\omega_0}$ be the angular frequency.

$$\frac{d^2\theta}{d\tau^2} + \alpha_1\left(\frac{d\theta}{d\tau}\right)^n + \sin\theta = \alpha_2\cos(\alpha_3\tau) \tag{2.16}$$

Equation 2.16 is the non-dimensionalized harmonic equation with the addition of damping and forcing. This equation can be written as a system of differential equations with $\theta = x$, $y = \dot{x}$ and $z = \alpha_3 t$. This gives:

$$\dot{x} = y$$

$$\dot{y} = -\sin x - \alpha_1 y^n + \alpha_2\cos z \tag{2.17}$$

$$\dot{z} = \alpha_3$$

These equations are solved numerically for various values of α_1, α_2, and α_3 using Matlab's built-in differential equation solver, ode45. For the remainder of this work the force is taken to be zero or $\alpha_2 = 0$.

Chapter 3

Results

Consider a linearly damped pendulum positioned at an angle of $\pi/4$ from equilibrium with an initial velocity of zero as in Figure 3.1. Let the damping coefficient be 0.1 and the total length of time be 1 with a step size of 10^{-4}. Note that for the harmonic pendulum, the equation of motion was non-dimensionalized above, so units are not used.

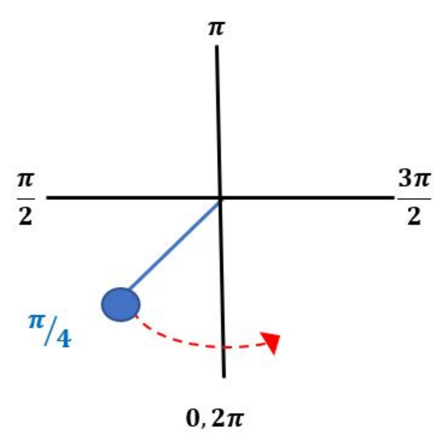

Figure 3.1: Initial position of pendulum

The position and velocity of such a pendulum are shown in Figure 3.2 as calculated with Matlab's differential equation solver. This position over time is considered the optimal path of the pendulum since it is given by the equations of motion.

For this system, the Hamiltonian is conserved. For the specific case of the linearly

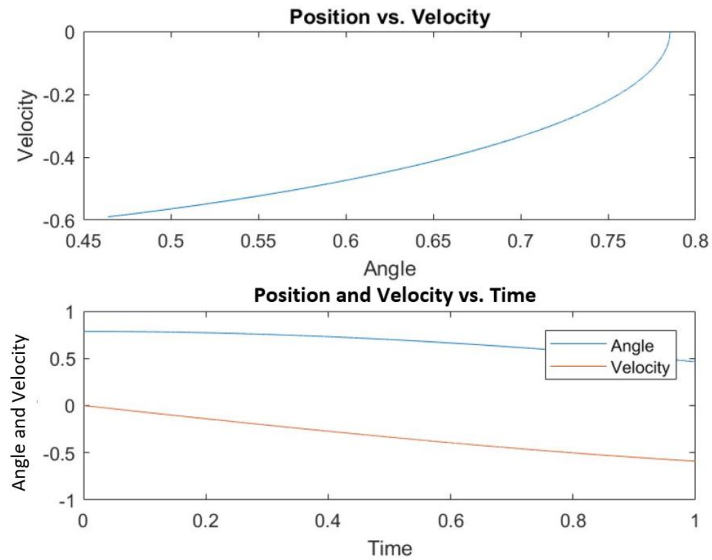

Figure 3.2: Position and velocity of a linearly damped pendulum over a time interval of length one (variables are non-dimensionalized).

damped pendulum described above, the sum of the potential, kinetic, and dissipated energy is constant (Figure 3.3). The kinetic, potential, and dissipated energy of this system is shown in Figure 3.4. The pendulum begins with only potential energy. Over time, the potential energy decreases while both the kinetic energy and dissipated energy increase.

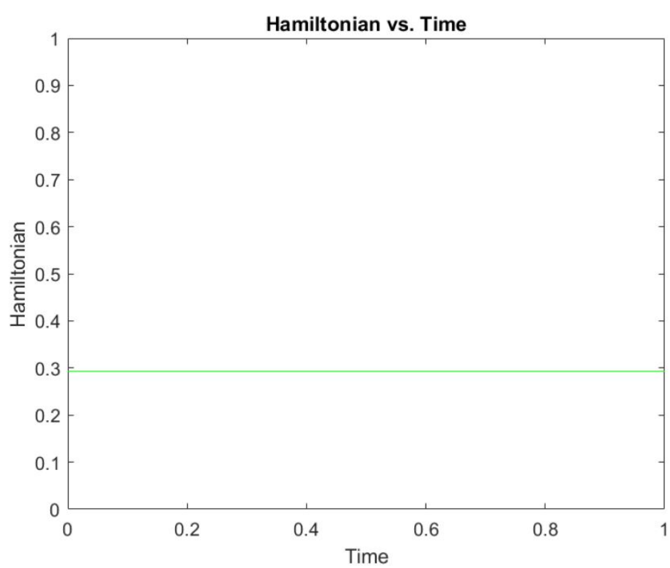

Figure 3.3: Hamiltonian of path shown in Figure 3.2.

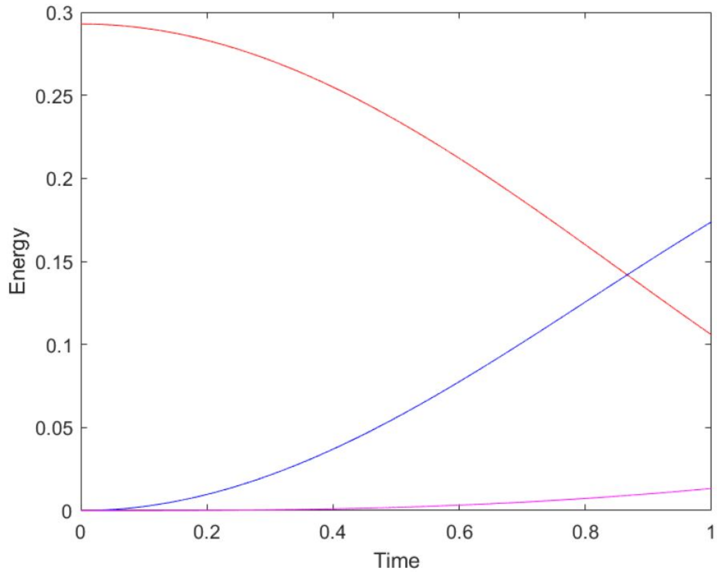

Figure 3.4: A comparison of the potential (red), kinetic (blue), and dissipative (magenta) energies of the optimal path shown in Figure 3.2.

In order to check if the variational principle of least action with dissipation holds, it is necessary to develop nearby paths. For the case of the pendulum, these paths are chosen to be polynomial approximations of a path with the same initial and final positions and one or more randomly chosen points in common, generated in Matlab (See Appendix A). Figure 3.5 shows 100 such paths (blue) that approximate the optimal path (red) for the specific pendulum described above.

To demonstrate the principle of least action, the action for each of the perturbed paths is found. For each path, the trapezoidal method is used to find the integral of the dissipation term and then Riemann sums are used to find the time integral of the dissipative Lagrangian. For the linearly damped pendulum previously described, the optimal Lagrangian is shown in red in Figure 3.6 alongside the Lagrangian of one of the perturbed paths shown in blue.

The actions of the perturbed paths are shown in Figure 3.7. The action of the optimal path, calculated in the same manner, is shown as a horizontal line. In this case, as the principle of least action predicts, the optimal action is shown to be a minimum compared to nearby paths.

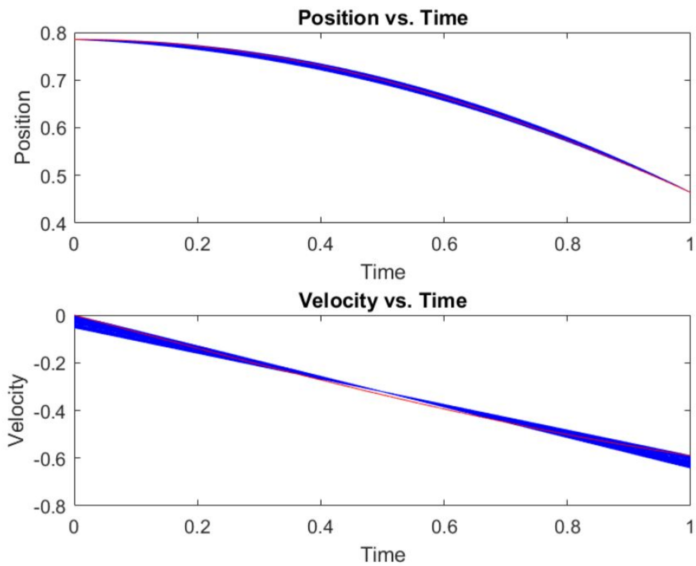

Figure 3.5: The optimal path in red compared to 100 perturbed paths in blue.

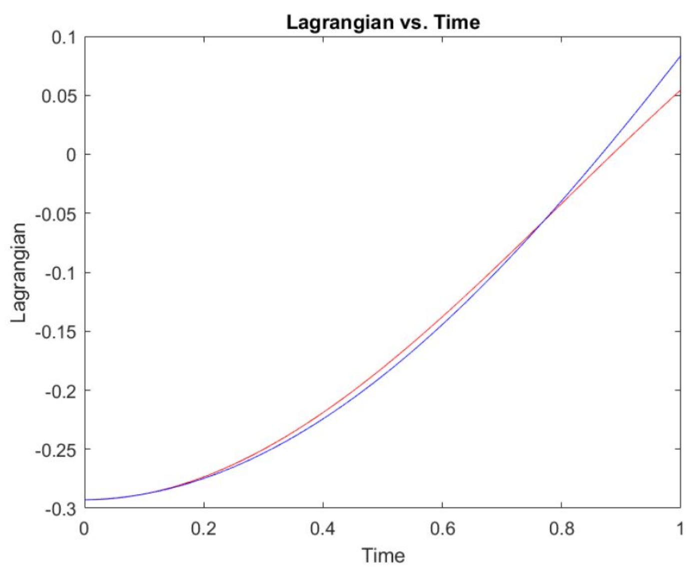

Figure 3.6: Lagrangian of the optimal path in red versus the Lagrangian for one of the perturbed paths in blue.

3.1 Falling Object

For the case of the falling object, as with the pendulum, when the coefficient of damping is 0, the action of the optimal path is always a minimum. This is also true for values of the damping coefficient below a certain threshold. Figure 3.8 shows the action of a linearly

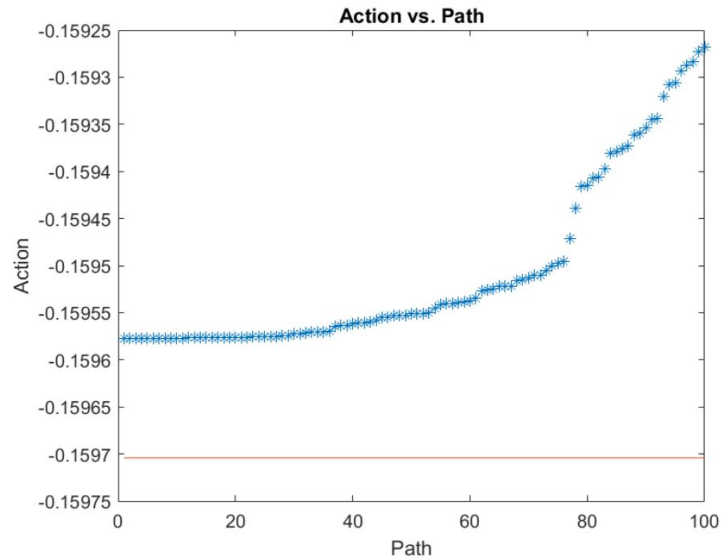

Figure 3.7: Action of a pendulum with damping coefficient 0.1 (red) compared to the action of 100 perturbed paths in blue.

damped falling object with a damping coefficient of $10^6 s^{-1}$. This object is taken to have a mass of $1.39 \times 10^{-6}\mathrm{kg}$ with initial position and velocity both zero. The gravitational constant is taken to be $10\ \mathrm{m/s^2}$. The object is considered for a time span of one second with a step size of 10^{-3}. In this case, the optimal action is again a minimum as compared to nearby paths.

However, as the coefficient of damping increases even further, the optimal action is no longer a minimum. Figure 3.9 shows the action of a falling object subject to the same conditions as the one above, except with a damping coefficient of $1.2 \times 10^6 s^{-1}$. This time, the action of some of the perturbed paths falls below the optimal action. Therefore, the optimal path is no longer the path of least action. For even larger values of damping, the optimal action is observed to be a maximum compared with nearby paths, as in Figure 3.10 where the damping coefficient is $1.5 \times 10^6 s^{-1}$.

As a check, a change in time of 10^{-4} is compared, all other conditions constant, with a change in time of 10^{-5}. Figures 3.11 and 3.12 show no discernible differences.

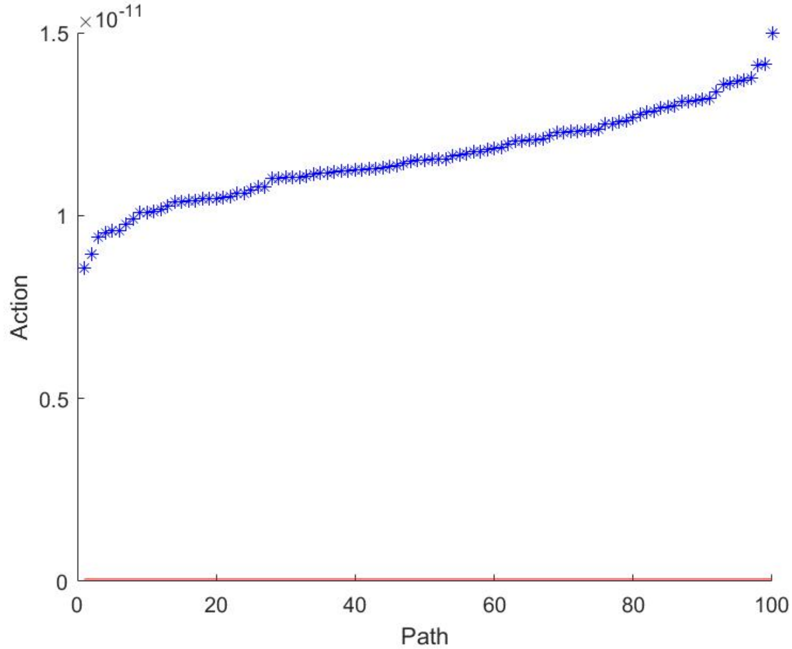

Figure 3.8: Action for a falling object with damping coefficient $10^6 s^{-1}$.

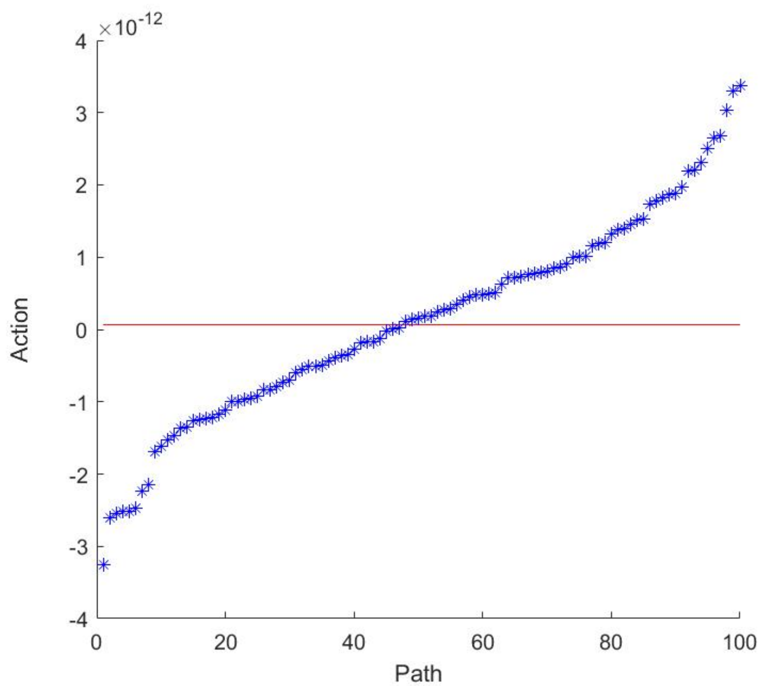

Figure 3.9: Action for a falling object with a damping coefficient of $1.2 \times 10^6 s^{-1}$.

3.2 Pendulum with Linear Damping

Consider again the linearly damped pendulum described at the start of the chapter. Leaving all other conditions the same and only changing the damping coefficient results in a change

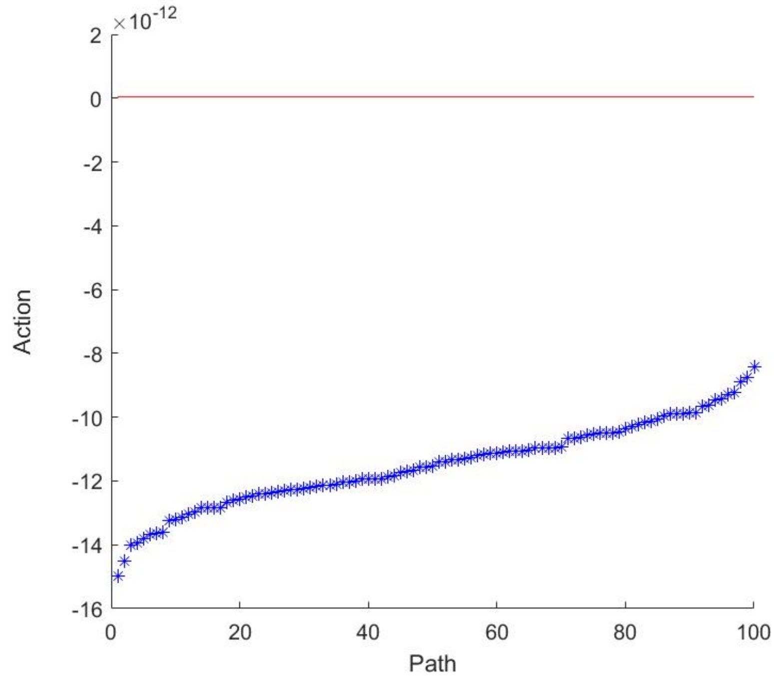

Figure 3.10: Action for a falling object with a damping coefficient of $1.5 \times 10^6 s^{-1}$.

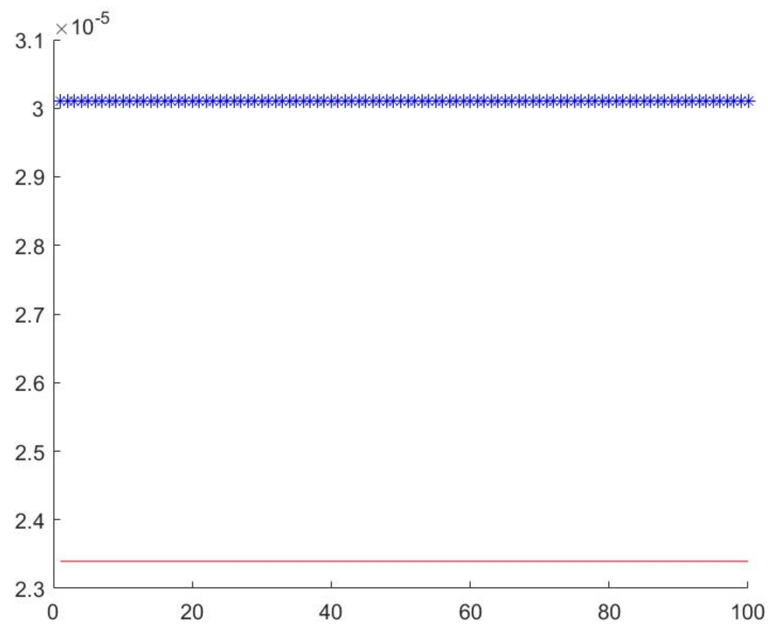

Figure 3.11: Action for a falling object with a step size of 10^{-4}.

in the optimal action. Figure 3.13 shows the optimal action with the action of the perturbed paths for a damping coefficient of 100. Here, the principle of least action holds and the

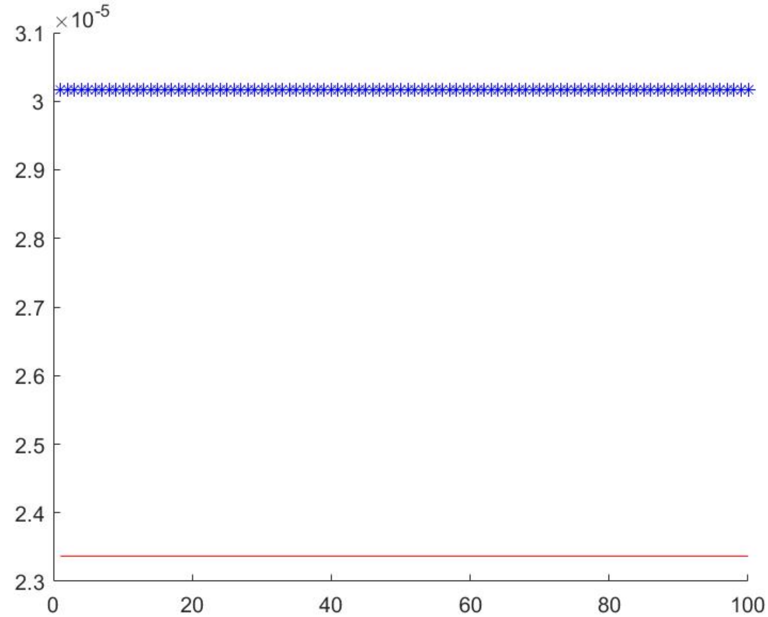

Figure 3.12: Action for a falling object with a step size of 10^{-5}.

optimal action is a minimum. However, Figure 3.14 shows the action for a pendulum with damping coefficient of 5×10^4. In this case, the optimal action is no longer the least action, as some of the perturbed paths have a smaller action and some have a larger action than the optimal. Increasing the damping coefficient to 5×10^5 results in even more of the randomized paths with action less than the optimal action, as seen in Figure 3.15.

3.3 Degree of Damping for the Pendulum

Figures 3.16 and 3.17 show the angle versus time graphs for three different degrees of damping with damping coefficients 0.1 and 10, respectively. For small values of the damping coefficient, the quadratically and cubically damped pendulums follow the same path. However, as the damping coefficient increases, the paths begin to diverge, as depicted in figure 3.17.

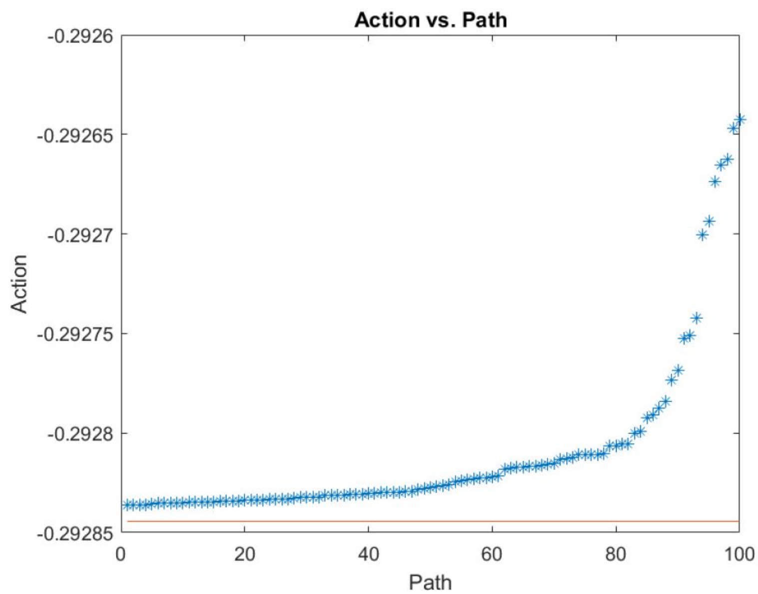

Figure 3.13: Optimal action (red) for the linearly damped pendulum with damping coefficient 100 compared with the action of perturbed paths (blue).

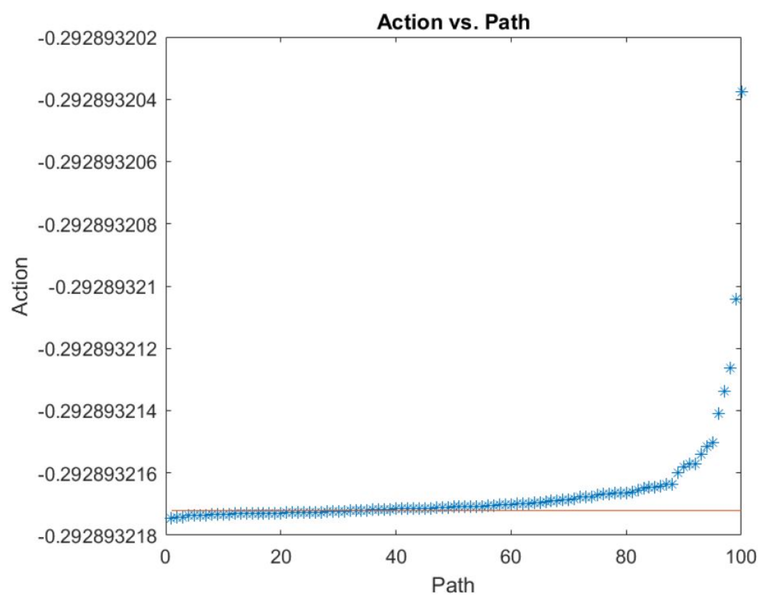

Figure 3.14: Optimal action (red) for the linearly damped pendulum with damping coefficient 5×10^4 compared with the action of perturbed paths (blue).

3.4 Action and Cubic Damping

Consider a pendulum with the same conditions as the pendulum above except for damping coefficient and degree of damping. Specifically, consider a cubically damped pendulum.

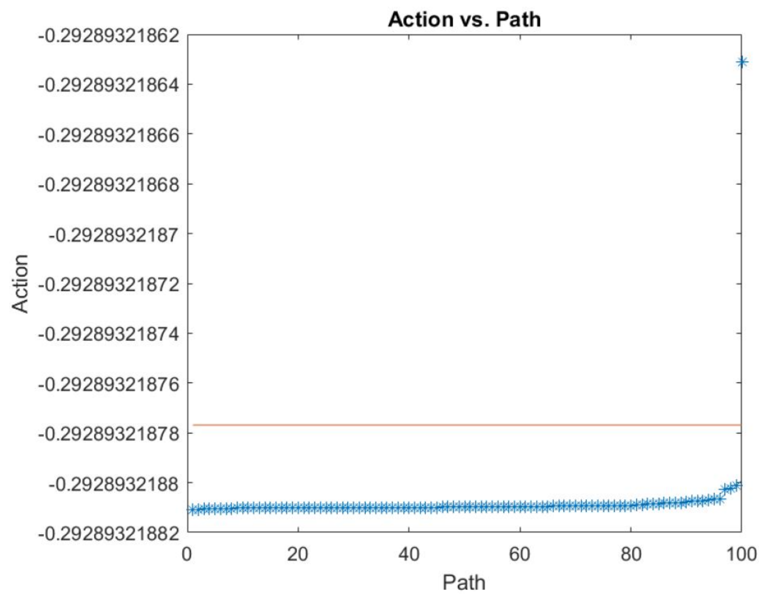

Figure 3.15: Optimal action (red) for the linearly damped pendulum with damping coefficient 5×10^5 compared with the action of perturbed paths (blue).

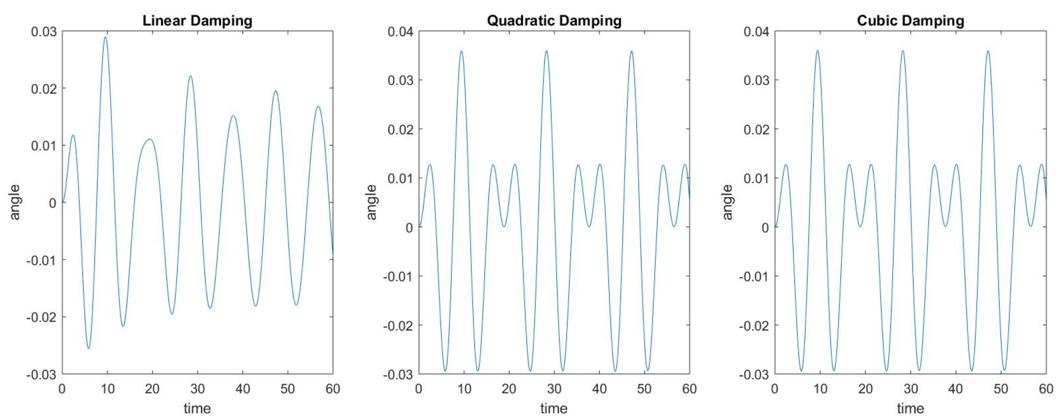

Figure 3.16: Angle vs. time graph for first, second, and third degree damping with damping coefficient 0.1.

For the damping coefficient 0.1, the optimal action is a minimum (Figure 3.18). However, if the damping coefficient is increased to 1, the optimal action is no longer a minimum (Figure 3.19). For even larger values of the damping coefficient the optimal action remains above the action for perturbed paths as seen in Figure 3.20 for a damping coefficient of 500. For the cubically damped pendulum, the optimal action moves out of minimum regime at smaller values of the damping coefficient then for the linearly damped pendulum.

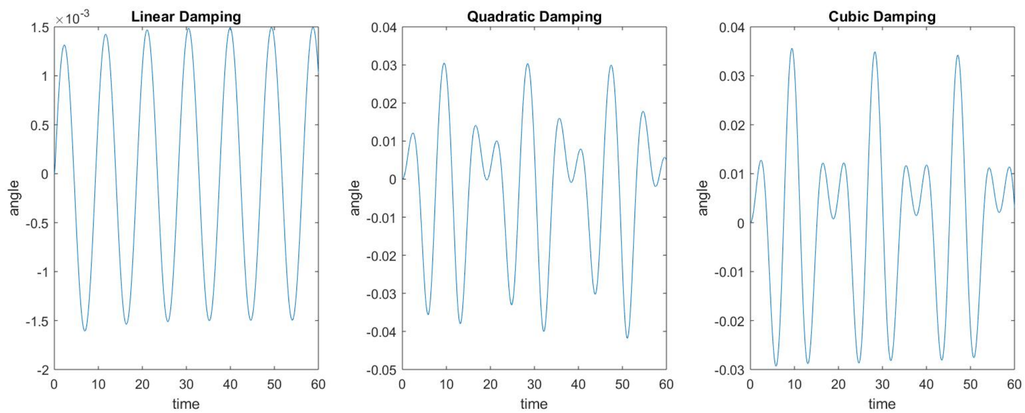

Figure 3.17: Angle vs. time graph for first, second, and third degree damping with damping coefficient 10.

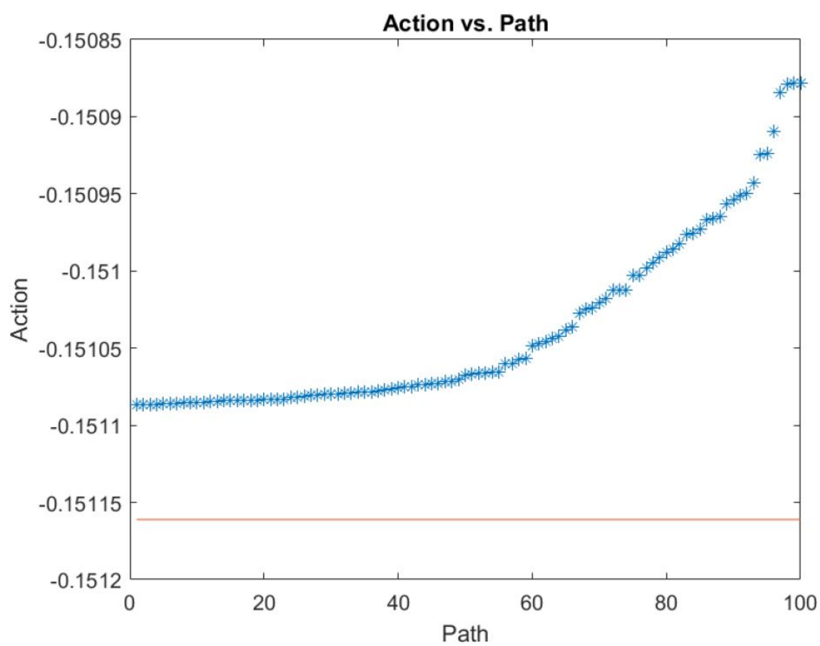

Figure 3.18: Optimal action for the cubically damped pendulum with damping coefficient 0.1 compared with the action of perturbed paths.

3.5 Results for a Longer Time

For the cubically damped pendulum, consider a longer period of time, specifically $T = 10$. For this case, take the initial position to be $\pi/4$ and the initial velocity to be zero. The step size is 10^{-4}. In this case, let the damping coefficient be equal to 1. The optimal position

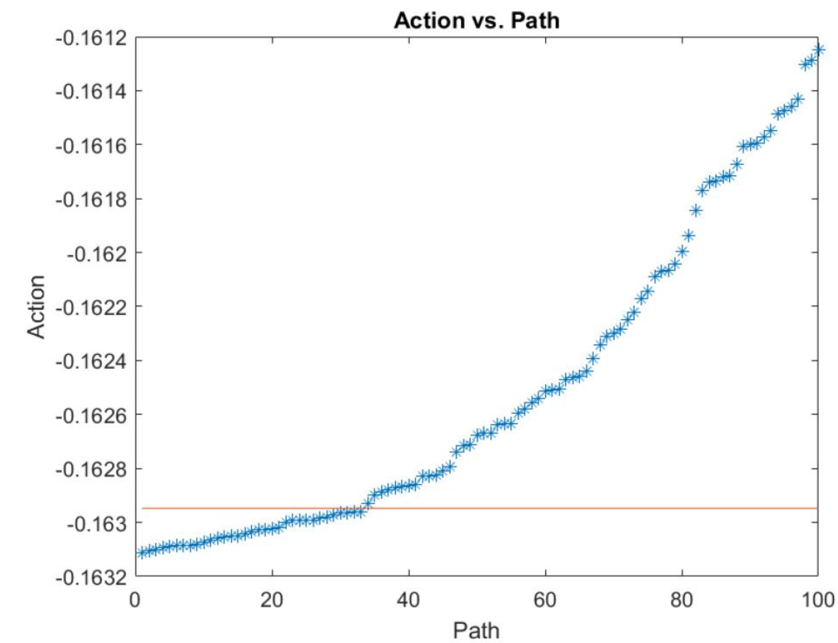

Figure 3.19: Optimal action for the cubically damped pendulum with damping coefficient 1 compared with the action of perturbed paths.

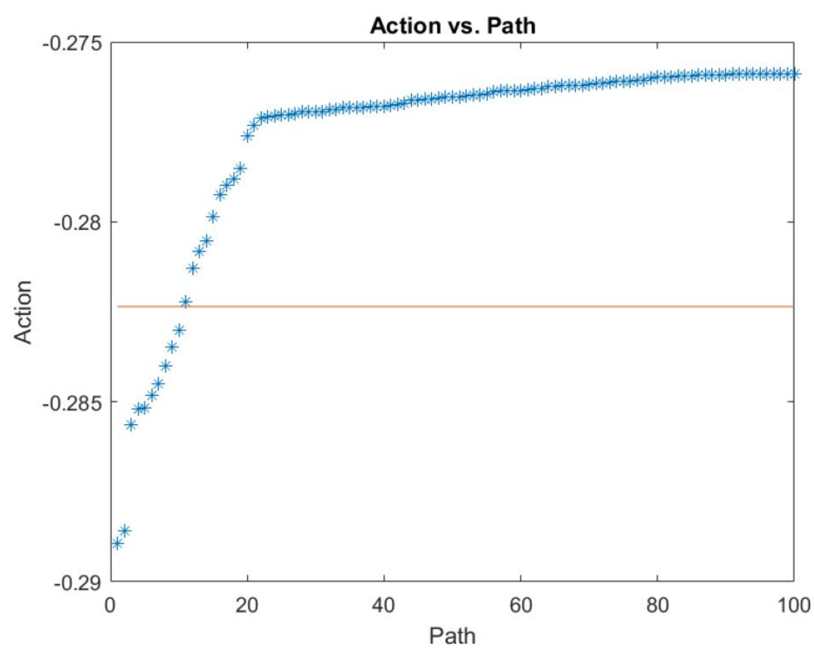

Figure 3.20: Optimal action for the cubically damped pendulum with damping coefficient 500 compared with the action of perturbed paths.

and velocity of such a pendulum is depicted in Figure 3.21.

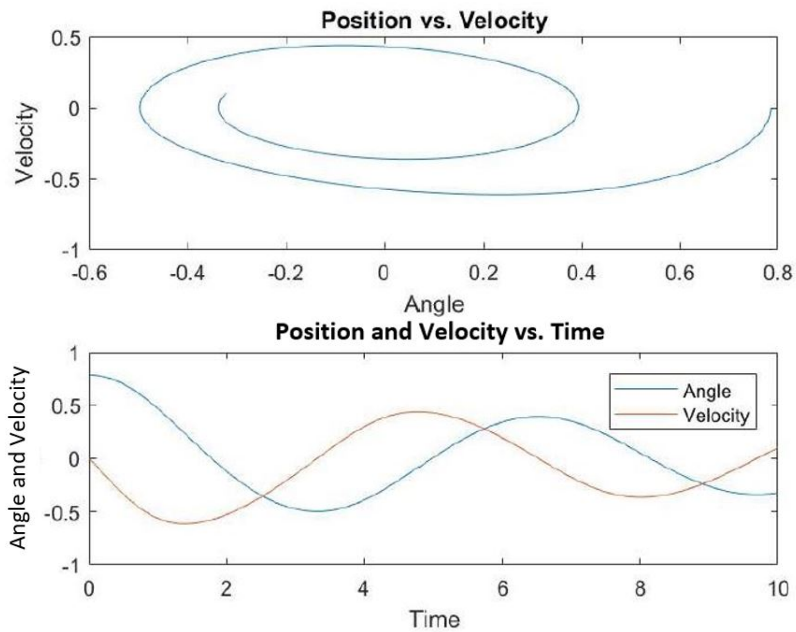

Figure 3.21: Position and Velocity of a cubically damped pendulum with damping coefficient 1.

For a pendulum under these conditions, the kinetic, potential, and dissipated energy are shown in Figure 3.22. As the amplitude of the kinetic and potential energies decreases, the dissipated energy increases. However, the sum of the energies remains constant (Figure 3.23).

One hundred perturbed polynomial paths are generated that have the same initial and final positions with three other randomly chosen points in common with the optimal path. Figure 3.24 shows the optimal path in red with 100 randomly generated perturbed paths in blue.

The Lagrangian for the optimal path and for each perturbed path is calculated. Figure 3.25 shows the optimal Lagrangian in red with the Lagrangian of just one of the randomly generated paths.

Figure 3.26 shows the optimal action with the action of each of the perturbed paths. Here, the action is clearly not a minimum, even though the damping coefficient is only

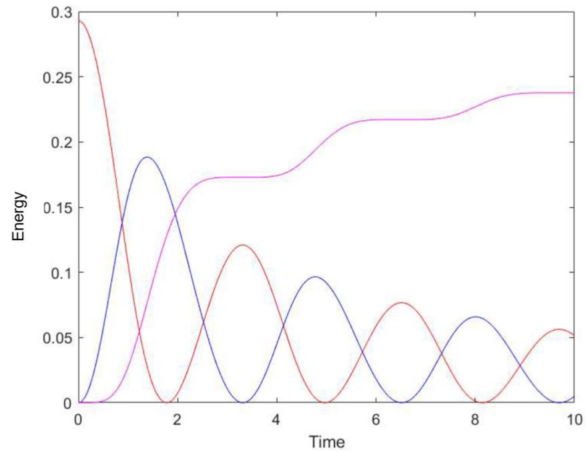

Figure 3.22: Potential (red), kinetic (blue), and dissipated (magenta) energy for the cubically damped pendulum over a time interval of length 10.

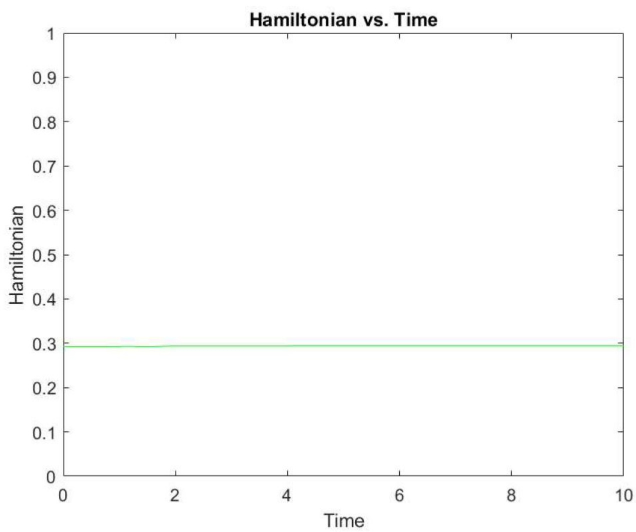

Figure 3.23: The Hamiltonian for the pendulum described by Figure 3.21.

1, much less than damping coefficient of 500 needed to see a case where the action is not minimum as in the previous section. Since all other variables are held constant, this suggests that for longer time intervals, the regime where the action is no longer a minimum occurs for smaller values of the damping parameter.

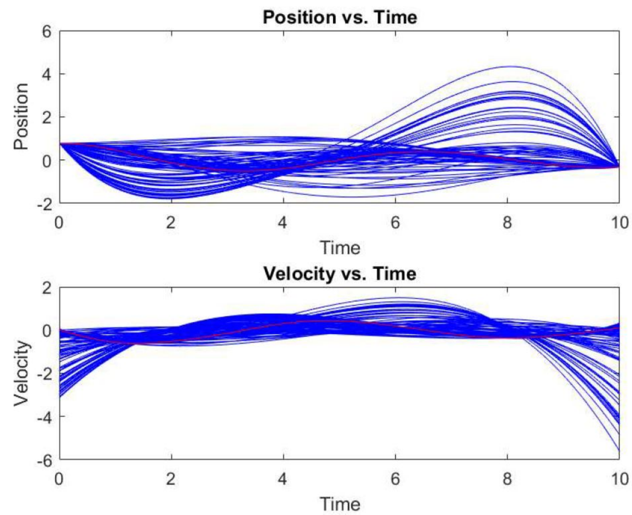

Figure 3.24: Optimal path shown in red with polynomial variations in blue.

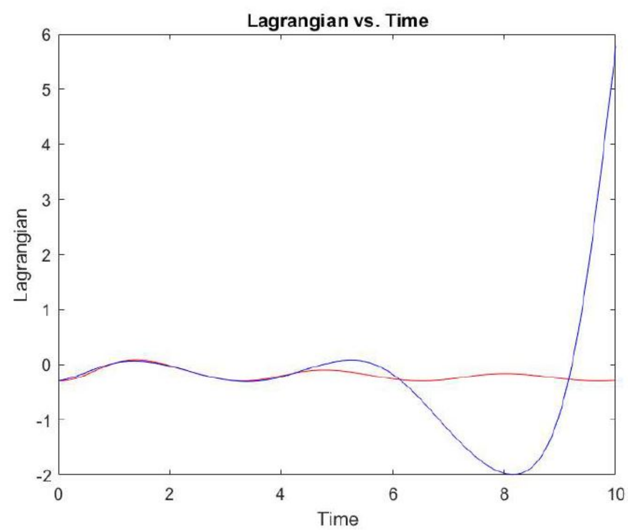

Figure 3.25: The Optimal Lagrangian (red) compared to the Lagrangian of a perturbed path (blue).

3.6 Future Work

In order to confirm these findings, more tests should be run for various values of damping parameters, degrees of damping, and other variations, such as the addition of a forcing term. These results show the switch from minimum damping to non-minimum happening at extremely high values of damping over a time interval of one second. However, the results

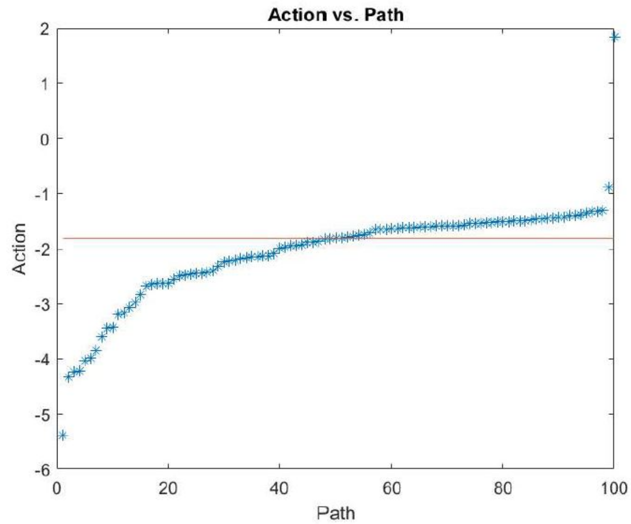

Figure 3.26: Optimal action seen with the action of 100 perturbed paths.

for the cubically damped pendulum show this switch happening at much lower damping coefficient.

Another topic to be explored is even powers of damping which require a path which is not sinusoidal. This can be done for small angles where $\theta << 1$.

Lastly, the least action principle with dissipation could be tested with other non-linear systems to see if they provide similar results.

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Appendix A

Matlab Code for Damped Harmonic Pendulum

```
clc; clear; close all

% INITIAL CONDITIONS

ti = 10;               % length of time
d = 3;                 % degree of damping
y = 1;                 % damping parameter
x = pi/4;              % initial Position
v = 0;                 % initial velocity
i=[x;v];               % initial conditions
dt = .0001;            % change in time
tspan=0:dt:ti;         % time span
le=length(tspan);      % length of vectors
w = 2/3;               % angular frequency
a = 0;                 % force coefficient
g = 1;                 % gravity
m = 1;                 % mass
l = 1;                 % string length

% SHM: Includes damping and forcing

f = @(t,x) [ x(2) ; -sin(x(1))-y*x(2).^d+a*cos(w*t)];
[t, xa] = ode45(f,tspan,i);

figure
subplot(2,1,1); plot(xa(:,1),xa(:,2));
xlabel Angle; ylabel Velocity; title('Position vs. Velocity')
subplot(2,1,2); plot(t,xa(:,1));
xlabel Time ; ylabel Angle; title('Position vs. Time')
```

```
hold on
subplot(2,1,2); plot(t,xa(:,2))
legend('Angle','Velocity');

% ENERGY

figure
K = .5*m*l^2*xa(:,2).^2;
V = m*g*l*(1-cos(xa(:,1)));
plot(t,V,'r'), xlabel Time, ylabel Potential
hold on
plot(t,K,'b'), xlabel Time, ylabel Potential


% LAGRANGIAN, HAMILTONIAN, AND ACTION

W = zeros(1,le);
for i = 2:le
x1 = xa(1:i,1);
x2 = xa(1:i,2);
W(i) = trapz(x1,y*x2.^d);
end

W = W';

hold on
plot(t,W,'m')
legend('Potential Energy','Kinetic Energy', 'Dissipation')

L = .5*m*l^2*xa(:,2).^2 - m*g*l*(1-cos(xa(:,1))) - W;
H = .5*m*l^2*xa(:,2).^2 + m*g*l*(1-cos(xa(:,1))) + W;
figure
plot(t,H,'g'), xlabel 'Time', ylabel Hamiltonian, axis([0 ti 0 1])

S = 0;
for i = 2:le
S = S + L(i)*dt;                              % Right Reimann Sum
end

disp(S)

% COMPARE TO NEARBY PATHS

N = 100;                  % Number of paths
SV = zeros(1,N);          % Initialize action vector
```

```
figure

for k = 1:N
I = randi([1,length(tspan)],1);
J = randi([1,length(tspan)],1);
H = randi([1,length(tspan)],1);

T0 = [tspan(1), tspan(I), tspan(J), tspan(H), tspan(length(tspan))];
T0 = sort(T0);

Ix = find(tspan==T0(2));
Jx = find(tspan==T0(3));
Hx = find(tspan==T0(4));

X0 = [xa(1,1), xa(Ix,1), xa(Jx,1), xa(Hx,1), xa(length(tspan),1)];

P = polyfit(T0,X0,4);

Y = zeros(le,1);

X = P(5) + P(4)*tspan + P(3)*tspan.^2 + P(2)*tspan.^3 + P(1)*tspan.^4;
X = X';

for i = 2:(le-1)
Y(i) = (X(i+1)-X(i))/dt;
end

Y(1)=Y(2);
Y(le)=Y(le-1);

dW = zeros(1,le);
for i = 2:le
X1 = xa(1:i,1);
X2 = xa(1:i,2);
dW(i) = trapz(X1,y*X2.^d);
end

dW = dW';


dL = .5*Y.^2 - g*(1-cos(X)) - dW;  % Calculate the Lagrangian
dH = .5*m*l^2*Y.^2 + m*g*l*(1-cos(X)) + dW;
for i = 2:le
SV(k) = SV(k) + dL(i)*dt;                    % Left Reimann Sum
```

```
end


subplot(2,1,1); plot(tspan,X,'b')
hold on
subplot(2,1,2); plot(tspan,Y,'b')
hold on
end

subplot(2,1,1); plot(t,xa(:,1),'r')
xlabel Time, ylabel Position, title 'Position vs. Time'
subplot(2,1,2); plot(t,xa(:,2),'r')
xlabel Time, ylabel Velocity, title 'Velocity vs. Time'


figure
plot(t,L,'r')
xlabel Time, ylabel Lagrangian, title 'Lagrangian vs. Time'
hold on
plot(t,dL,'b')
legend('Optimal Lagrangian','Approximation')

figure
SV = sort(SV);
plot(SV,'*'); ylabel Action ; xlabel Path; title 'Action vs. Path'
hold on
plot(1:N,S*ones(1,N))
legend('Optimal Action','Action of Perturbed Path')
```

Appendix B

Matlab Code for Damped Falling Object

```
clc; clear; close all

% INITIAL CONDITIONS

m  = 1.39*10^(-6);  % Mass
g  = 10;            % Gravitational Constant
T  = 1;             % Total Time
dt = 10^(-3);       % Change in Time
tspan = 0:dt:T;     % Timespan
i = [0, 0];         % Initial Position and Velocity
y = 1;              % Damping Parameter
d = 1;              % Degree of Damping
le=length(tspan);   % Length of Vectors

% FALLING OBJECT WITH DAMPING

f = @(t,x) [ x(2) ; -g-y*x(2)];
[t, xa] = ode45(f,tspan,i);

figure
K = .5*m*xa(:,2).^2;
V = m*g*xa(:,1);
plot(t,V,'r'), xlabel Time, ylabel Energy
hold on
plot(t,K,'b'), xlabel Time, ylabel Engery

% LAGRANGIAN, HAMILTONIAN, AND ACTION

W = zeros(1,le);
for i = 2:le
x1 = xa(1:i,1);
```

```
x2 = xa(1:i,2);
W(i) = m*trapz(x1,y*x2.^d);
end

W = W';

hold on
plot(t,W,'m')

L = .5*m*xa(:,2).^2 - m*g*xa(:,1) - W ;   % Calculate the Lagrangian
H = .5*m*xa(:,2).^2 + m*g*xa(:,1) + W ;

figure
plot(t,H), xlabel Time, ylabel Hamiltonian, axis([0, T, 0, 1])

figure
plot(t,L), xlabel Time, ylabel Lagrangian

S = 0;                                  % Initialize action
for i = 2:le
S = S + L(i)*dt;                        % Right Reimann Sum
end

% COMPARE TO NEARBY PATHS

N = 100;                        % Number of paths
SV = zeros(1,N);                % Initialize action vector

for k = 1:N
X=xa(:,1);                          % X is the angle vector
Y=xa(:,2);                          % Y is the velocity vector
for i = 2:(le-1)                    % Find another path
X(i) = X(i) + .0001*rand;        % Amplitude of variation is .1 mm
end

for i = 2:(le-1)
Y(i) = -(X(i)-X(i+1))/dt;
end

Y(1)=Y(2);
Y(le)=Y(le-1);

dW = zeros(1,le);
for i = 2:le
x1 = X(1:i);
```

```
x2 = Y(1:i);
dW(i) = m*trapz(x1,y*x2.^d);
end

dL = .5*m*Y.^2 - m*g*X - dW;  % Calculate the Lagrangian
for i = 2:le
SV(k) = SV(k) + dL(i)*dt; % Left Reimann Sum
end
end

figure
SV = sort(SV);
hold on
plot(1:N,S*ones(1,N),'r')
plot(SV,'b*')
disp(S)
```