CHALLENGE JOURNAL OF STRUCTURAL MECHANICS 1 (3) (2015) 124–133

Envelope analysis equations for two-span continuous girder bridges

Daud Abdoh *

Department of Civil Engineering, University of Jordan, 11942 Amman, Jordan

ABSTRACT

In this paper, envelope analysis equations for two-span continuous girder bridges were presented by deriving the analysis equation for uniformly distributed loading, concentrated loading and moving loads (single and multiple wheel loads). Most bridge engineers are using special software's to find the moment, shear and deflection envelopes for bridge girder, the complexity for this analysis increasing with the number of spans, most of cases are one-span and two-span continuous bridge, the two-span continuous bridge is more complicated which was presented in this paper, the same methodology can be applied in one-span bridge. The objective of this paper was to give all the bridge engineers direct equations for complete analysis (moment, shear and deflection) for two-span continuous bridge with more accuracy than most bridge software's by adapting continuous moving of wheel loads rather than using interval check distance to move the concentrated loads as in most of bridge software's. Pinned end boundary condition was presented here. The results were showed that shear envelope, moment envelope and maximum envelope deflection values were obtained by direct equations for two-span continuous girder bridges under single and multiple moving loads.

ARTICLE INFO

Article history:
Received 23 May 2015
Accepted 14 July 2015

Keywords:
Computational analysis
Envelope analysis
Continuous girder bridge
Two-span girder bridge
Moving load

1. Introduction

Earthquake analysis of the bridge design that won an international design competition, in which aesthetics played an important role. (Wieland and Malla, 2015). In order to reflect this phenomena analysis envelope equations (moment, shear and deflection) for two-span continuous girder bridge was derived on four stages, based on direct integration method that depends on solving linear equations to obtain the values of unknown integration constants coming from each integration process.

The chain of integration processes starting from loading to shear to moment to slope and finally to deflection, will produce one integration constant on each integration step, these unknown constants defines the analysis equations (moment, shear and deflection), the formulation of solving equations based on geometry of bridge girder, loading and boundary conditions.

The first stage was to derive the analysis equations for uniformly distributed loading, these loads mainly represent the own weight of bridge girder, super imposed dead loads and lane live loads.

The first stage was to derive the analysis equations for uniformly distributed loading, these loads mainly represent the own weight of bridge girder, super imposed dead loads and lane live loads.

The second stage was to derive the analysis equations for single concentrated load located in any specified location for any value on each span.

The third stage was to derive the envelope analysis equations for a single moving load based on the concentrated load analysis equations (second stage).

The fourth stage was to derive the envelope analysis equations for multiple moving load based on envelope analysis equations for single moving load (third stage), the motion of moving load will be continuous in third and fourth stages giving more accurate results.

2. Sign Convention

Load downward was assumed to be positive, load upward was negative, positive shear was upward, negative shear was downward, positive moment was concave

* *Corresponding author*. Tel.: +966-59-9409355 ; Fax: +962-79-6988920 ; E-mail address: daud_abdo@yahoo.com (D. Abdoh)
ISSN: 2149-8024 / DOI: http://dx.doi.org/10.20528/cjsmec.2015.07.025

down (tension at top), negative moment was concave up (compression at top), positive slope for counterclockwise rotation, negative slope for clockwise rotation, negative deflection was downward, positive deflection was upward.

3. First Stage (Uniformly Distributed Load)

The integration of uniformly load for each span gives a linear equation of shear, the integration of shear gives the second order polynomial equation for moment, the integration of moment gives slope equation as third order polynomial, and finally the integration of this slope results in deflection equation as 4th polynomial order equation, as follows:

$$V_1(X) = W_1 \times X + C_1 \,, \tag{1}$$

$$M_1(X) = \frac{W_1 \times X^2}{2} + C_1 \times X + C_2 \,, \tag{2}$$

$$\theta_1(X) = \frac{W_1 \times X^3}{6 \times EI} + \frac{C_1 \times X^2}{2 \times EI} + \frac{C_2 \times X}{EI} + \frac{C_3}{EI} \,, \tag{3}$$

$$\Delta_1(X) = \frac{W_1 \times X^4}{24 \times EI} + \frac{C_1 \times X^3}{6 \times EI} + \frac{C_2 \times X^2}{2 \times EI} + \frac{C_3 \times X}{EI} + \frac{C_4}{EI} \,, \tag{4}$$

$$V_2(X) = W_2 \times X + C_5 \,, \tag{5}$$

$$M_2(X) = \frac{W_2 \times X^2}{2} + C_5 \times X + C_6 \,, \tag{6}$$

$$\theta_2(X) = \frac{W_2 \times X^3}{6 \times EI} + \frac{C_5 \times X^2}{2 \times EI} + \frac{C_6 \times X}{EI} + \frac{C_7}{EI} \,, \tag{7}$$

$$\Delta_2(X) = \frac{W_2 \times X^4}{24 \times EI} + \frac{C_5 \times X^3}{6 \times EI} + \frac{C_6 \times X^2}{2 \times EI} + \frac{C_7 \times X}{EI} + \frac{C_8}{EI} \,. \tag{8}$$

In these equations, W_1 and W_2 are the uniformly distributed load on span 1 and 2, respectively; C_1 to C_8 is the integration constants; X is the distance from start of span 1 or 2 to required point of analysis; EI is the flexural registry of the span section, modulus of elasticity multiplied by moment of inertia around the moment axis; L_1 and L_2 are span 1 and 2 lengths, respectively,

Refer to Fig. 1 for loading, shear, moment, slope and deflection diagrams for two-span continuous girder bridges with uniformly distributed loads.

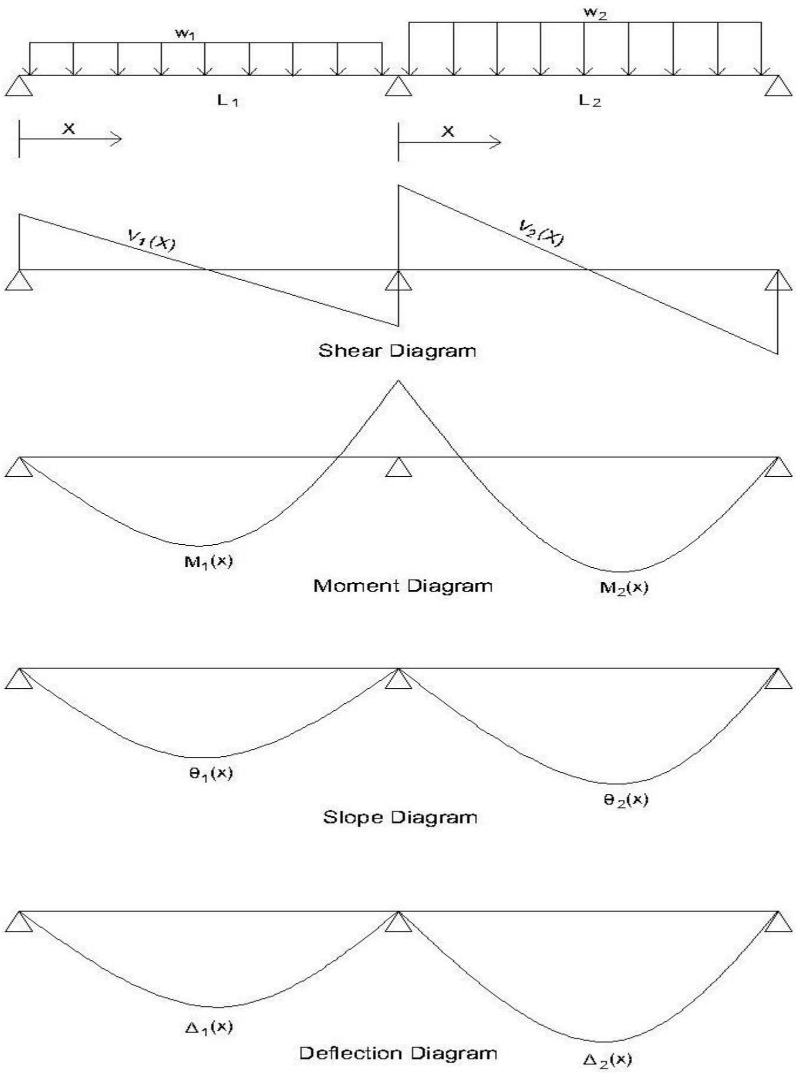

Fig. 1. Loading and analysis diagrams (shear, moment, slope and deflection) for uniformly distributed loading.

As shown, each integration process leads to one integration constant, the number of these unknown constants is eight which equals to number of spans (two) multiplied by four (number of integration constants for each step), to solve these constants, same number of equations must be formulated based on geometry, loading and boundary conditions as follows:

Moment at end span 1 is equal the moment at start span 2 (moment continuity at support).

$$\frac{W1\times L_1^2}{2}+\ C_1\times L_1+\ C_2-C_6=0\,. \tag{9}$$

Slope at end of span 1 is equal the slope at start of span 2 (rotation continuity at support).

$$\frac{W_1\times L_1^3}{6\times EI}+\frac{C_1\times L_1^2}{2\times EI}+\frac{C_2\times L_1}{EI}+\frac{C_3}{EI}-\frac{C_7}{EI}=0\,. \tag{10}$$

Deflection at end of span 1 is equal zero:

$$\frac{W_1\times L_1^4}{24\times EI}+\frac{C_1\times L_1^3}{6\times EI}+\frac{C_2\times L_1^2}{2\times EI}+\frac{C_3\times L_1}{EI}+\frac{C_4}{EI}=0\,. \tag{11}$$

Deflection at end of span 2 is equal zero (pinned):

$$\frac{W_2\times L_2^4}{24\times EI}+\frac{C_5\times L_2^3}{6\times EI}+\frac{C_6\times L_2^2}{2\times EI}+\frac{C_7\times L_2}{EI}+\frac{C_8}{EI}=0\,. \tag{12}$$

Moment at end of span 2 is equal zero (pinned):

$$\frac{W_2\times L_2^2}{2}+\ C_5\times L_2+C_6=0\,. \tag{13}$$

Moment at start of span 1 is equal zero (pinned):

$$\frac{W_1\times 0}{2}+\ C_1\times 0+C_2=0\,. \tag{14}$$

Deflection at start of span 1 is equal zero (pinned):

$$\frac{W_1\times 0}{24\times EI}+\frac{C_1\times 0}{6\times EI}+\frac{C_2\times 0}{2\times EI}+\frac{C_3\times 0}{EI}+\frac{C_4}{EI}=0\,. \tag{15}$$

Deflection at start of span 2 is equal zero:

$$\frac{W_2\times 0}{24\times EI}+\frac{C_5\times 0}{6\times EI}+\frac{C_6\times 0}{2\times EI}+\frac{C_7\times 0}{EI}+\frac{C_8}{EI}=0\,. \tag{16}$$

Solving these equations, will give the values of integration constants thus defines the analysis equations for each span as follows:

$$C_1=\frac{3\times W_1\times L_1^3-W_2\times L_2^3+\ 4\times W_1\times L_1^2\times L_2}{8\times L_1(L_2+L_1)}\,, \tag{17}$$

$$C_2=0\,, \tag{18}$$

$$C_3=\frac{W_1\times L_1^3-4\times C_1\times L_1^2}{24}\,, \tag{19}$$

$$C_4\ =0\,, \tag{20}$$

$$C_5=\frac{W_1\times L_1^2+W_2\times L_2^2-2\times C_1\times L_1}{2\times L_2}\,, \tag{21}$$

$$C_6=\frac{2\times C_1\times L_1-W_1\times L_1^2}{2}\,, \tag{22}$$

$$C_7=\frac{8\times C_1\times L_1^2-3\times W_1\times L_1^3}{24}\,, \tag{23}$$

$$C_8\ =0\,. \tag{24}$$

4. Second Stage (Concentrated Loading)

The location of concentrated load divides the span to two, a is the distance from start of span 1 or end of span 2 to the location of this concentrated load of P value, the load within all three segments will be zero, thus the shear will be constant with a different value on each segment, the moment in turns to be linear, the slope is 2nd order polynomial equation and deflection is third order polynomial equation, as follows:

$$V_1(x)=C_1\,, \tag{25}$$

$$M_1(x)=C_1\times X+C_2\,, \tag{26}$$

$$\theta_1(x)=\frac{C_1\times X^2}{2\times EI}+\frac{C_2\times X}{EI}+\frac{C_3}{EI}\,, \tag{27}$$

$$\Delta_1(x)=\frac{C_1\times X^3}{6\times EI}+\frac{C_2\times X^2}{2\times EI}+\frac{C_3\times X}{EI}+\frac{C_4}{EI}\,, \tag{28}$$

$$V_2(x)=C_5\,, \tag{29}$$

$$M_2(x)=C_5\times X+C_6\,, \tag{30}$$

$$\theta_2(x)=\frac{C_5\times X^2}{2\times EI}+\frac{C_6\times X}{EI}+\frac{C_7}{EI}\,, \tag{31}$$

$$\Delta_2(x)=\frac{C_5\times X^3}{6\times EI}+\frac{C_6\times X^2}{2\times EI}+\frac{C_7\times X}{EI}+\frac{C_8}{EI}\,, \tag{32}$$

$$V_3(x)=C_9\,, \tag{33}$$

$$M_3(x)=C_9\times X+C_{10}\,, \tag{34}$$

$$\theta_3(x)=\frac{C_9\times X^2}{2\times EI}+\frac{C_{10}\times X}{EI}+\frac{C_{11}}{EI}\,, \tag{35}$$

$$\Delta_3(x)=\frac{C_9\times X^3}{6\times EI}+\frac{C_{10}\times X^2}{2\times EI}+\frac{C_{11}\times X}{EI}+\frac{C_{12}}{EI}\,. \tag{36}$$

In these equations, $V_1(X)$, $V_2(X)$, $V_3(X)$ are the shear equations; $M_1(X)$, $M_2(X)$, $M_3(X)$ are the moment equations; $\theta_1(X)$, $\theta_2(X)$, $\theta_3(X)$ are the slope equations; $\Delta_1(X)$, $\Delta_2(X)$, $\Delta_3(X)$ are the deflection equations; C_1-C_{12} are the integration constants in parts 1, 2 and 3, respectively.

Refer to Fig. 2 for loading, shear, moment, slope and deflection diagrams for two-span continuous girder bridge with concentrated load.

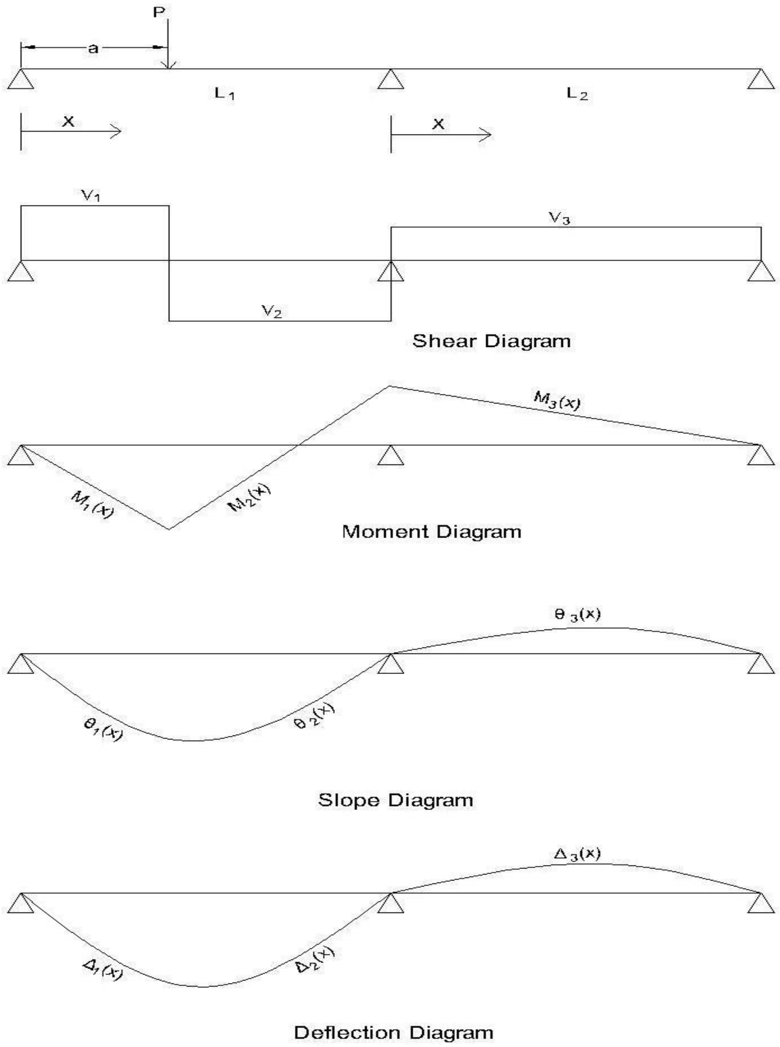

Fig. 2. Loading and analysis diagrams (shear, moment, slope and deflection) for concentrated load.

Twelve unknown constants produced to define the analysis equation for three segments in two spans, twelve equations will be formulated to solve these constants as follows:

The difference of shear at the concentrated load is equal to P load value:

$$C_1 - C_5 = P\,. \tag{37}$$

The moment at concentrated load is equal from two directions (moment continuity):

$$C_1 \times a + C_2 - C_6 = 0\,. \tag{38}$$

The slope at concentrated load is equal from two directions (rotation equality):

$$\frac{C_1 \times a^2}{2 \times EI} + \frac{C_2 \times a}{EI} + \frac{C_3}{EI} - \frac{C_7}{EI} = 0\,. \tag{39}$$

The deflection at concentrated load is equal from two directions:

$$\frac{C_1 \times a^3}{6 \times EI} + \frac{C_2 \times a^2}{2 \times EI} + \frac{C_3 \times a}{EI} + \frac{C_4}{EI} - \frac{C_8}{EI} = 0\,. \tag{40}$$

Moment is equal at interior support (moment continuity):

$$C_5 \times (L_1 - a) + C_6 - C_{10} = 0\,. \tag{41}$$

Slope is equal at interior support (rotation continuity):

$$\frac{C_5 \times (L_1 - a)^2}{2 \times EI} + \frac{C_6 \times (L_1 - a)}{EI} + \frac{C_7}{EI} - \frac{C_{11}}{EI} = 0\,. \tag{42}$$

Deflection at the end of span 1 is equal zero (at support):

$$\frac{C_5 \times (L_1 - a)^3}{6 \times EI} + \frac{C_6 \times (L_1 - a)^2}{2 \times EI} + \frac{C_7 \times (L_1 - a)}{EI} + \frac{C_8}{EI} = 0\,. \tag{43}$$

Moment is equal zero at end of span 2 (pinned):

$$C_9 \times L_2 + C_{10} = 0\,. \tag{44}$$

Deflection at the end of span 2 is equal zero (at support):

$$\frac{C_9 \times {L_2}^3}{6 \times EI} + \frac{C_{10} \times {L_2}^2}{2 \times EI} + \frac{C_{11} \times L_2}{EI} + \frac{C_{12}}{EI} = 0\,. \tag{45}$$

Moment is equal zero at start of span 1 (pinned):

$$C_1 \times 0 + C_2 = 0\,. \tag{46}$$

Deflection at the start of span 1 is equal zero (at support):

$$\frac{C_1\times 0}{6\times EI} + \frac{C_2\times 0}{2\times EI} + \frac{C_3\times 0}{EI} + \frac{C_4}{EI} = 0\,. \tag{47}$$

Deflection at the start of span 2 is equal zero (at support):

$$\frac{C_9\times 0}{6\times EI} + \frac{C_{10}\times 0}{2\times EI} + \frac{C_{11}\times 0}{EI} + \frac{C_{12}}{EI} = 0\,, \tag{48}$$

In these equations, a is the distance between start of span 1 to location of concentrated load if it is in span 1 or the distance between end of span 2 to location of concentrated load if it is in span 2 and P is the concentrated load.

Solving these equations will give the values of these constants as follows:

$$C_1 = P \times \left[\frac{a^3 - a(3{L_1}^2 + 2L_1L_2)}{2\times L_1^2\times(L_1+L_2)} + 1\right], \tag{49}$$

$$C_2 = 0\,, \tag{18}$$

$$C_4 = 0\,, \tag{20}$$

$$C_{10} = a \times C_1 + (C_1 - P) \times (L_1 - a)\,, \tag{50}$$

$$C_5 = C_1 - P\,, \tag{51}$$

$$C_6 = a \times C_1\,, \tag{52}$$

$$C_9 = \frac{-a\times C_1 - (C1-P)\times(L1-a)}{L_2}\,, \tag{53}$$

$$C_3 = \frac{-3a^2C_1 - 3\times(L_1-a)^2\times(C_1-P) - 6\times a\times C_1(L_1-a) - 2\times L_2\times a\times C_1 - 2L_2\times(C1-P)\times(L1-a)}{6}\,, \tag{54}$$

$$C_{11} = \frac{-L_2\times a\times C_1 - L_2\times(C_1-P)\times(L_1-a)}{3}\,, \tag{55}$$

$$C_{12} = 0\,, \tag{56}$$

$$C_7 = \frac{-3\times(L_1-a)^2\times(C_1-P) - 6\times a\times C_1(L_1-a) - 2\times L_2\times a\times C_1 - 2\times L_2\times(C1-P)\times(L1-a)}{6}\,, \tag{57}$$

$$C_8 = \frac{2\times(L_1-a)^3\times(C_1-P) + 3\times a\times C_1\times(L_1-a)^2 + 2\times L_2\times a\times C_1\times(L_1-a) + 2L_2\times(C1-P)\times(L1-a)^2}{6}\,. \tag{58}$$

5. Third Stage (Single Moving Load)

In this section, the analysis envelope equations were derived for single moving load for each span based on the analysis equations for a single concentrated load derived in previous section, the motion of load in span 1 starts from beginning of span 1 to end of span 1, the motion of load in span 2 starts from end of span 2 to beginning of span 2 as shown in Fig. 3.

5.1. Shear envelope equations

The maximum shear at any point in a span length when single concentrated load is moving on that span occurs when moving load acts on that point, the maximum positive and negative shear at any point is equal the positive and negative shear of the single concentrated load at this point, which means C_1 (Eq. (49)) at this point for maximum positive shear, and C_5 (Eq. (51)) at the same point for maximum negative shear, replacing a (location of concentrated load) with X as follows:

$$PVE_1(X) = P \times \left[\frac{X^3 - X\times(3\times L_1^2 + 2\times L_1\times L_2)}{2\times L_1^2\times(L_1+L_2)} + 1\right],$$

$$0 \le X \le L_1\,, \tag{59}$$

$$NVE_1(X) = P \times \left[\frac{X^3 - X\times(3\times L_1^2 + 2\times L_1\times L_2)}{2\times L_1^2\times(L_1+L_2)}\right],$$

$$0 \le X \le L_1\,. \tag{60}$$

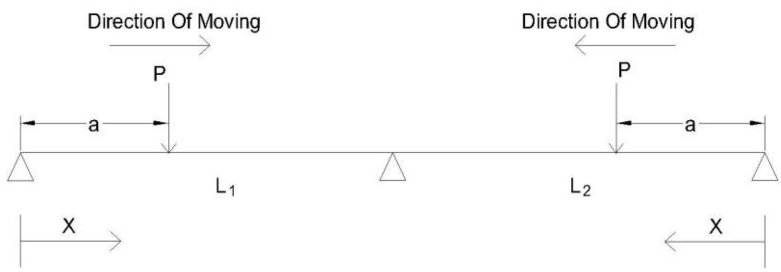

Fig. 3. Single moving load diagram.

The same equations are applicable in span 2 but replacing L_1 with L_2, and vice versa, X start from zero (end of span 2) to L_2 (start of span 2) as follows:

$$PVE_2(X) = P \times \left[\frac{X^3 - X \times (3 \times L_2^2 + 2 \times L_1 \times L_2)}{2 \times L_2^2 \times (L_1 + L_2)} + 1\right],$$
$$0 \le X \le L_2\ , \tag{61}$$

$$NVE_2(X) = P \times \left[\frac{X^3 - X \times (3 \times L_2^2 + 2 \times L_1 \times L_2)}{2 \times L_2^2 \times (L_1 + L_2)}\right],$$
$$0 \le X \le L_2\ . \tag{62}$$

In these equations, $PVE_1(X)$ is the positive shear envelope equation for span 1; $NVE_1(X)$ is the negative shear envelope equation for span 1; $PVE_2(X)$ is the positive shear envelope equation for span 2; $NVE_2(X)$ is the negative shear envelope equation for span 2.

Refer to Fig. 4 for Positive and negative shear envelope diagrams for both spans with single moving load.

5.2. Moment envelope equations

The maximum negative moment (downward) in each span at any point occurs when moving load acts on that point, replacing a with X in the moment equation for concentrated load at any point will give the moment envelope equation as 4th degree polynomial equation for each span as follows:

$$NME_1(X) = P \times X \times \left[\frac{X^3 - X(3{L_1}^2 + 2L_1L_2)}{2 \times L_1^2 \times (L_1 + L_2)} + 1\right],$$
$$0 \le X \le L_1\ , \tag{63}$$

$$NME_2(X) = P \times X \times \left[\frac{X^3 - X(3{L_2}^2 + 2L_1L_2)}{2 \times L_2^2 \times (L_1 + L_2)} + 1\right],$$
$$0 \le X \le L_2\ . \tag{64}$$

In these equations, $NME_1(X)$ is the negative moment envelope equation in for span 1 and $NME_2(X)$ is the negative moment envelope equation in for span 2.

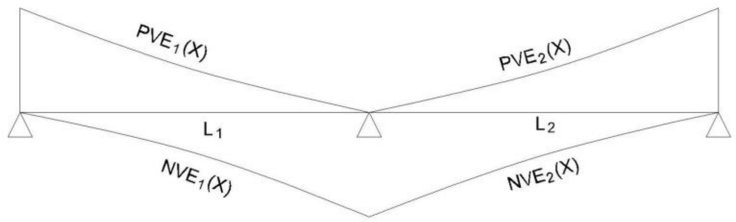

Fig. 4. Shear envelope diagram for single moving load.

The maximum positive moment will occur generally when the concentrated load acts on the other span. In case of equal span lengths (L_1=L_2),the positive moment envelope equation will be line extending from the maximum positive moment at interior support (*MPMIS*) due to moving load to the end of span (zero moment at pinned) based on the shape of positive moment equation for the concentrated load on the another span, by Replacing a with X in the moment equation for concentrated load at the location of interior support, refer to Eq. (34), and derive this equation according to X , equal this equation by zero to find the X which gives the location of moving load that producing the maximum positive moment at interior support, substituting this value in previous equation (Eq. (34)) will give the maximum positive moment at interior support. Thus defines the positive moment envelope equation as follows:

If $L_1 = L_2 = L$,

$$a = \frac{L}{\sqrt{3}}, \tag{65}$$

$$MPMIS = \frac{L \times P}{6 \times \sqrt{3}}\ , \tag{66}$$

where, *MPMIS* is the maximum positive moment at interior support.

Refer to Fig. 5 for positive and negative moment envelope diagram for both spans with single moving load for equal span case.

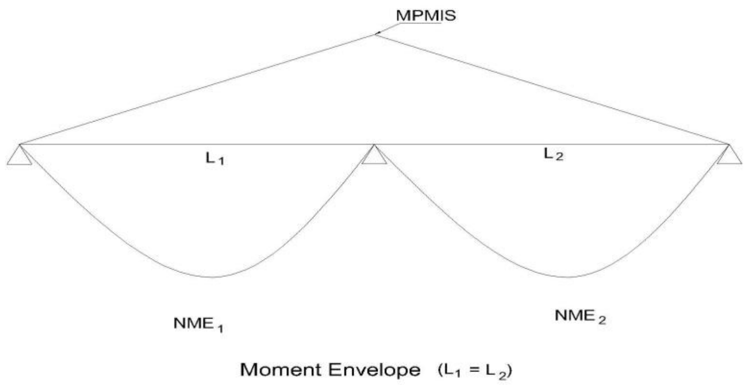

Fig. 5. Moment envelope diagram for single moving load when L_1=L_2.

In case of unequal spans, the positive moment envelope equation is divided into two lines on the longer span and one line for shorter span, due to that maximum positive moment at interior support on longer span will come from moving load on the same span not on the other one, the derivation of maximum positive moment at interior support (*MPMIS*) is derived as discussed above when moving load acts on the longer span only, the derivation of the maximum positive moment near the interior support on the longer span (*MPMNIS*) is the same as in equal span case when moving load acts on shorter span only, by deriving these values, the location of intersection of two positive moment line equations (*POI*) can be found as follows:

If $L_1 > L_2$,

$$a = \frac{L_1}{\sqrt{3}}, \tag{67}$$

$$MPMIS = \frac{L_1 \times P}{6\times\sqrt{3}} , \tag{68}$$

$$MPMNIS = \frac{L_1^2 \times P}{3\times\sqrt{3}\times(L_1+L_2)}, \tag{69}$$

$$POI = \frac{\frac{L_1^4 - L_2^4}{3\times\sqrt{3}\times\left(L_1^3 + L_2\times L_1^2\right)}}{\frac{3\times(3\times L_1+2\times L_2)-L_1}{6\times\sqrt{3}\times(L_1+L_2)} - \frac{2\times L_2^2}{6\times\sqrt{3}\times L_1\times(L_2+2\times L_1)}}, \tag{70}$$

where, *MPMNIS* is the maximum positive moment near interior support; *POI* is the point of intersection.

Refer to Fig. 6 for positive and negative moment envelope diagram for both spans with single moving load when $L_1 > L_2$.

If $L_1 < L_2$,

$$a = \frac{L_2}{\sqrt{3}}, \tag{71}$$

$$MPMIS = \frac{L_2 \times P}{6\times\sqrt{3}} , \tag{72}$$

$$MPMNIS = \frac{L_2^2 \times P}{3\times\sqrt{3}\times(L_1+L_2)}, \tag{73}$$

$$POI = \frac{\frac{L_2^4 - L_1^4}{3\times\sqrt{3}\times\left(L_2^3 + L_1\times L_2^2\right)}}{\frac{3\times(3\times L_2+2\times L_1)-L_2}{6\times\sqrt{3}\times(L_1+L_2)} - \frac{2\times L_1^2}{6\times\sqrt{3}\times L_2\times(L_1+2\times L_2)}}. \tag{74}$$

Refer to Fig. 7 for Positive and negative moment envelope diagram for both spans with single moving load when $L_1 < L_2$.

5.3. Maximum envelope deflection

During load moving, the maximum deflection at each point occurs when slope (θ) is equal zero, there is only one point on each span where maximum deflection occurs at same location of moving load, this point will be called point of maximum envelope deflection (*PMED*), this point divides the span into two parts, the first one, where the maximum deflection occurs after any required point in that part, the second part, where the maximum deflection occurs before any required point in that part, the maximum envelope deflection at each span occurs at the point of maximum envelope deflection (*PMED*).

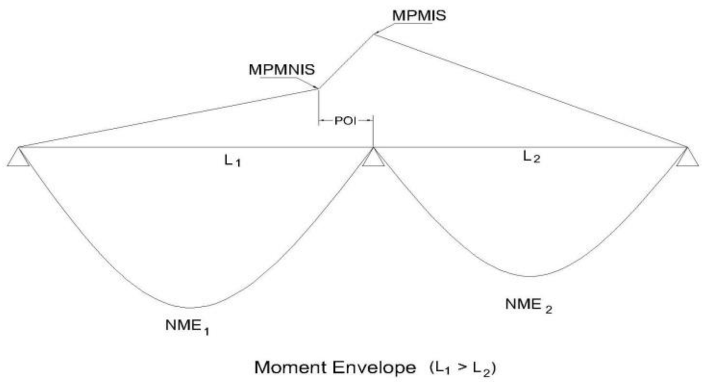

Fig. 6. Moment envelope diagram for single moving load when $L_1 > L_2$.

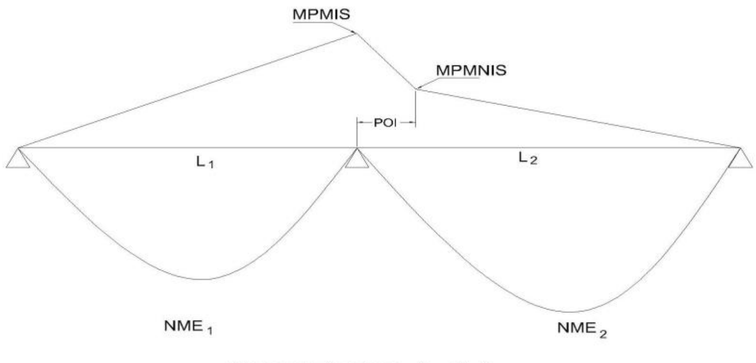

Fig. 7. Moment Envelope diagram for single moving load when $L_1 < L_2$.

The *PMED* occurs at the location of moving load where the slope value is equal zero under this load, this can be derived by equaling the slope equation at moving load location a by zero, the following demonstration is for span 1:

$$\theta_1 = \frac{C_1 \times a^2}{2 \times EI} + \frac{C_2 \times a}{EI} + \frac{C_3}{EI} = 0\,, \tag{75}$$

$$\frac{3 \times a \times P \times L_1^3 + 3 \times a^3 \times P \times L_1 + 4 \times L_1^2 \times L_2 \times P \times a + 2 \times a^3 \times P \times L_2 - 6 \times a^2 \times L_1^2 \times P - 6 \times a^2 \times P \times L_2 \times L_1}{12 \times L_1^2 + 12 \times L_1 \times L_2} - \frac{a^2 \times C_1}{2} = 0\,, \tag{76}$$

$$\frac{a \times [12 \times a^2 \times P \times L_1^2 + 8 \times a^2 \times P \times L_1 \times L_2 - 12 \times L_1^3 \times P \times a - 12 \times a \times P \times L_2 \times L_1^2 + 3 \times P \times L_1^4 + 4 \times L_1^3 \times L_2 \times P - 3 \times a^4 \times P]}{12 \times L_1^3 + 12 \times L_1^2 \times L_2} = 0\,, \tag{77}$$

Solving this equation to obtain the location of *PMED* directly is hard, instead of that, the variable a will be replaced with $R_1 \times L_1$ and L_2 with $R_2 \times L_1$.

$$R_1 = \frac{a}{L_1}\,, \tag{78}$$

$$R_2 = \frac{L_2}{L_1}\,, \tag{79}$$

where, R_1 is the ratio between the location of moving load (a) and span 1 length (L_1); R_2 is the ratio of span 2 length (L_2) with span 1 length (L_1).

Converting the above equation into following form:

$$Y = L_1^4 \times [12 \times R_1^2 + 8 \times R_1^2 \times R_2 - 12 \times R_1 - 12 \times R_1 \times R_2 + 3 + 4 \times R_2 - 3 \times R_1^4]\,. \tag{80}$$

Substituting R_2 with a range of values from 0.1 to 10, and finding the intersection of Y curve with X-axis for each value of R_2 to find the value of R_1 that makes the above equation is zero, arranging R_1 values with R_2 values, gives a curve that defines the relationship between span 1 length to span 2 length ratio (R_2) and the location of *PMED* as a ratio of span length where deflection is to be calculated (R_1), substituting this value in a parameter in deflection equation (Eq. (28) or Eq. (32))for concentrated load to obtain the value of *PMED*.

Refer to Fig. 8 for relationship of R_2 (X-axis) with R_1 (Y-axis) for location of point of maximum deflection (*PMED*).

The same equations and curve used to derive the point of maximum envelope deflection *PMED* in span1 was used in span 2 by replacing L_1 with L_2 and vice versa as follows:

$$R_1 = \frac{a}{L_2}\,, \tag{81}$$

$$R_2 = \frac{L_1}{L_2}\,. \tag{82}$$

In case of equal span length ($L_1=L_2=L$), R_1 is equal 0.46875, Substituting a=0.46875×L in X and a in Eq. (28) or Eq. (32) (and in corresponding integration constants equations) gives the same value of *PMED* for both spans 1 and 2.

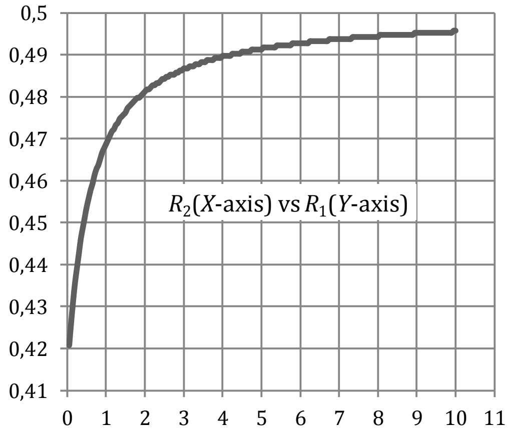

Fig. 8. Relationship between R_2 and R_1 for single moving load.

6. Fourth Stage (Multiple Moving Load)

In this stage, Analysis envelope equations for multiple loads will be derived based on the single load derivations, refer to Fig. 9 for three moving loads diagram.

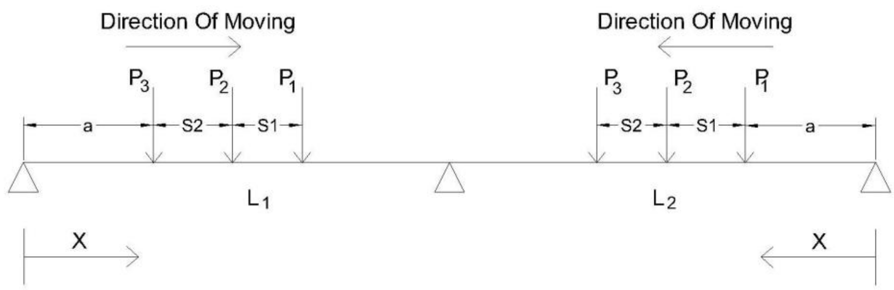

Fig. 9. Three moving loads diagram.

6.1. Shear envelope equations

The maximum positive shear at any point in span 1 due to moving of multiple loads occurs when the last load (P_3) acts on it as follows:

$$P(X) = \sum_{N=1}^{M} P_N \times \left[\frac{\left(X+\sum_{N}^{M-1} S_N\right)^3 - \left(X+\sum_{N}^{M-1} S_N\right)\times\left(3\times L_1^2+2\times L_1\times L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \ , \ 0 \le X \le L_1 \,. \tag{83}$$

The maximum negative shear at any point in span 1 due to moving of multiple loads occurs when the first load acts (P_1) on it as follows:

$$NVE_1(X) = \sum_{N=1}^{M} P_N \times \left[\frac{\left(X-\sum_{1}^{N-1} S_N\right)^3 - \left(X-\sum_{1}^{N-1} S_N\right)\times\left(3\times L_1^2+2\times L_1\times L_2\right)}{2\times L_1^2\times(L_1+L_2)}\right] \ , \ 0 \le X \le L_1 \,. \tag{84}$$

The maximum positive shear at any point in span 2 due to moving of multiple loads occurs when the first load (P_1) acts on it as follows:

$$PVE_2(X) = \sum_{N=1}^{M} P_N \times \left[\frac{\left(X+\sum_{1}^{N-1} S_N\right)^3 - \left(X+\sum_{1}^{N-1} S_N\right)\times\left(3\times L_2^2+2\times L_1\times L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1\right] \ , \ L_2 \ge X \ge 0 \,. \tag{85}$$

The maximum negative shear at any point in span 2 due to moving of multiple loads occurs when the last load (P_3) acts on it as follows:

$$NVE_2(X) = \sum_{N=1}^{M} P_N \times \left[\frac{\left(X-\sum_{N}^{M-1} S_N\right)^3 - \left(X-\sum_{N}^{M-1} S_N\right)\times\left(3\times L_2^2+2\times L_1\times L_2\right)}{2\times L_2^2\times(L_1+L_2)}\right] \ , \ L_2 \ge X \ge 0 \,. \tag{86}$$

In these equations, $PVE_1(X)$ is the positive shear envelope equation for span 1; $NVE_1(X)$ is the negative shear envelope equation for span 1; $PVE_2(X)$ is the positive shear envelope equation for span 2; $NVE_2(X)$ is the negative shear envelope equation for span 2; S_N is the distance between N wheel axis loads; P_N is the wheel axial load number N; M is the total number of wheel loads.

6.2. Moment envelope equations

The maximum negative moment at each point on both spans, occur when one of any multiple moving load acts on it due to moment diagram shape for multiple concentrated load (vertex of moment diagram), for demonstration, three moving loads (most general wheel load, truck load) were taken, and accordingly the negative moment envelope equations were developed.

The maximum negative moment at any point on span length can be caused by one of the three moving loads acting on it directly, three moment envelope equations were developed by taking each load acting on required point, the other loads will add moment to this point according to their value and relative location to demonstrate moving load (Distance between moving loads, axis distance), each distance interval on each span, one of the three negative moment envelope equation will be maximum, thus the resultant negative moment envelope equation is the maximum of the three developed negative moment envelope equation at any point as follows ($0 \le X \le L_1$):

$$NME_{1P1}(X) = X \times \left\{\begin{matrix} P_1 \times \left[\frac{X^3 - X\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] + P_2 \times \left[\frac{(X-S_1)^3 - (X-S_1)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \\ +P_3 \times \left[\frac{(X-S_1-S_2)^3 - (X-S_1-S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \end{matrix}\right\}, \tag{87}$$

$$NME_{1P2}(X) = X \times \left\{\begin{matrix} P_2 \times \left[\frac{X^3 - X\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] + P_1 \times \left[\frac{(X+S_1)^3 - (X+S_1)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \\ +P_3 \times \left[\frac{(X-S_2)^3 - (X-S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \end{matrix}\right\}, \tag{88}$$

$$NME_{1P3}(X) = X \times \left\{\begin{matrix} P_3 \times \left[\frac{X^3 - X\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] + P_2 \times \left[\frac{(X+S_2)^3 - (X+S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \\ +P_1 \times \left[\frac{(X+S_1+S_2)^3 - (X+S_1+S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_1^2\times(L_1+L_2)} + 1\right] \end{matrix}\right\}. \tag{89}$$

In these equations, $NME_{1P1}(X)$ is the negative moment envelope equation for span 1 as load P_1 is the main load; $NME_{1P2}(X)$ is the Negative moment envelope equation for span 1 as load P_2 is the main load; $NME_{1P3}(X)$ is the Negative moment envelope equation for span 1 as load P_3 is the main load; P_1, P_2, P_3 are the moving loads (represent vehicle axis loads); S_1, S_2 is the distance between P_1 and P_2, the distance between P_2 and P_3 respectively.

The same concept applied in span 2, by replacing L_1 with L_2 and vice versa, as follows ($0 \le X \le L_2$):

$$NME_{2P1}(X) = X \times \left\{ \begin{array}{c} P_1 \times \left[\frac{X^3 - X\left(3{L_1}^2 + 2L_1L_2\right)}{2\times L_2^2 \times (L_1+L_2)} + 1 \right] + P_2 \times \left[\frac{(X+S_1)^3 - (X+S_1)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1 \right] \\ +P_3 \times \left[\frac{(X+S_1+S_2)^3 - (X+S_1+S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1 \right] \end{array} \right\}, \tag{90}$$

$$NME_{2P2}(X) = X \times \left\{ \begin{array}{c} P_2 \times \left[\frac{X^3 - X\left(3{L_1}^2 + 2L_1L_2\right)}{2\times L_2^2 \times (L_1+L_2)} + 1 \right] + P_1 \times \left[\frac{(X-S_1)^3 - (X-S_1)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1 \right] \\ +P_3 \times \left[\frac{(X+S_2)^3 - (X+S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1 \right] \end{array} \right\}, \tag{91}$$

$$NME_{2P3}(X) = X \times \left\{ \begin{array}{c} P_3 \times \left[\frac{X^3 - X\left(3{L_1}^2 + 2L_1L_2\right)}{2\times L_2^2 \times (L_1+L_2)} + 1 \right] + P_2 \times \left[\frac{(X-S_2)^3 - (X-S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1 \right] \\ +P_1 \times \left[\frac{(X-S_1-S_2)^3 - (X-S_1-S_2)\times\left(3{L_1}^2+2L_1L_2\right)}{2\times L_2^2\times(L_1+L_2)} + 1 \right] \end{array} \right\}. \tag{92}$$

For the positive moment envelope equation for multiple moving loads, another approach was followed which is to consider the location of multiple loads as the same of the location of centroid of these loads, to calculate the maximum positive moment at interior support (*MPMIS*), centroid of multiple loads put on a distance as given by Eqs. (65), (67) or (71) depending which span is longer, MPMIS is equal the sum of moments caused by each load on their specific location as described in moment equation of concentrated load case, to calculate the maximum positive moment near interior support (*MPMNIS*), the centroid of multiple load put on *POI* as defined in Eqs. (70) or (74) according to span length case. *MPMNIS* is equal the sum of moments caused by each load on their specific location according to moment equation for concentrated load case.

6.3. Maximum envelope deflection

To calculate the maximum envelope deflection for multiple moving load, the centroid of these loads are put on the point of maximum envelope deflection (*PMED*) as shown in section 5.3 according to span length case, then the deflection was calculated by summing the deflection caused by each load on their specified location at *PMED* point as shown in deflection equations for concentrated load.

7. Conclusions

Bridge engineers can follow the equations presented in this paper to conduct complete analysis for two-span continuous girder bridge in uniformly distributed loading and wheel loading for any number of axis loads without using a software with more accuracy, making the bridge girder analysis more practical and accurate.

The future challenge in this field is to find direct analysis envelope equations for girder bridges with more than two spans with wider boundary conditions such as support settlement or deformation in the bearing pad.

References

Barker R, Puckett J (2013). Design of Highway Bridges: An LRFD Approach (2nd edition). John Wiley & Sons Inc., New York.

Fu CC, Wang S (2014). Computational Analysis and Design of Bridge Structures. CRC Press, Boca Raton.

Gambhir ML (2011). Fundamentals of Structural Mechanics and Analysis. Prentice-Hall Inc., Englewood Cliffs, NJ.

Hibbeler RC (2005). Mechanics of Materials (6th edition). Prentice-Hall Inc., Englewood Cliffs, NJ.

Liew JYR, Shanmugam NE (2005). Structural Analysis. *In: Chen WF, Lui EM, editors. Handbook of Structural Engineering* (*2nd edition*). CRC Press, Boca Raton.

Rattan SS (2008). Strength of Materials. McGraw-Hill, New York.

Wieland M, Malla S (2015). Earthquake design of a viaduct with full seismic isolation of bridge deck. *Challenge Journal of Structural Mechanics*, 1(1), 27–31.