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AN EXPLORATION OF THE RELATIONSHIPS AMONG CATTELL-HORN-CARROLL
(CHC) THEORY-ALIGNED COGNITIVE ABILITIES AND MATH FLUENCY

A Dissertation

Submitted to the School of Education

Duquesne University

In partial fulfillment of the requirements for
the degree of Doctor of Philosophy

By

Katherine D. Piselli

December 2017

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Katherine D. Piselli

2017

**DUQUESNE UNIVERSITY
SCHOOL OF EDUCATION**

Department of Counseling, Psychology, and Special Education

Dissertation

Submitted in partial fulfillment of the requirements for the degree
Doctor of Philosophy (Ph.D.)

School Psychology Doctoral Program

Presented by:

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M.S.Ed. Child Psychology, Duquesne University, 2013
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July 20, 2017

**AN EXPLORATION OF THE RELATIONSHIPS AMONG CATTELL-HORN-CARROLL
(CHC) THEORY-ALIGNED COGNITIVE ABILITIES AND MATH FLUENCY**

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ABSTRACT

AN EXPLORATION OF THE RELATIONSHIPS AMONG CATTELL-HORN-CARROLL (CHC) THEORY-ALIGNED COGNITIVE ABILITIES AND MATH FLUENCY

By

Katherine D. Piselli

December 2017

Dissertation supervised by Ara J. Schmitt, Ph.D.

Math fluency, which refers to the ability to solve single digit arithmetic problems quickly and accurately, is a foundational mathematical skill. Recent research has examined the role of phonological processing, executive control, and number sense in explaining differences in math fluency performance in school-aged children. Identifying the links between these cognitive abilities and math fluency skills has important implications for screening, assessment, and intervention efforts in schools. As extant mathematics research in the context of Cattell-Horn-Carroll (CHC) theory has evaluated either broad mathematics performance or math calculation skills, little is known about the specific relationships between math fact fluency and broad and narrow cognitive abilities. The present study investigated the relationships among Math Fact Fluency performance and the CHC theory-aligned broad and narrow cognitive abilities using a child-age subset of the Woodcock Johnson IV standardization sample. Results of the path

analyses indicated that General Intellectual Ability (GIA) exhibited significant direct and indirect effects on Math Fact Fluency performance. With regard to broad cognitive abilities, Processing Speed had the greatest direct effect on Math Fact Fluency. Likewise, in the narrow abilities model, Perceptual Speed was most related to Math Fact Fluency, after accounting for GIA. Contrary to initial hypotheses, Working Memory, Phonetic Coding, and Attentional Control did not significantly contribute to Math Fact Fluency. Finally, the inclusion of Math Problem Solving within the cognitive abilities model resulted in a moderate direct effect on Math Fact Fluency performance. These findings are discussed in terms of directions for future research as well as implications for clinicians and educators.

DEDICATION

This dissertation is dedicated to my parents, Lea and Albert, who unequivocally supported me in any and all of my endeavors, from ballet dancer to graduate student. I am so grateful to have parents who truly encouraged my siblings and I to pursue our dreams. Mom, everything I've learned about working with children began with you. I aspire to attain your level of patience and kindheartedness. To all my family and friends, thank you for listening to me, laughing with me, and helping me grow into the person I am today.

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CHAPTER I

INTRODUCTION

Recent research has shown that early math abilities at school entry are highly predictive of future academic performance (Claessens, Duncan, & Engle, 2009; Duncan et al., 2007). Students who demonstrate knowledge of early math concepts in kindergarten are more likely to achieve math proficiency in later years. However, more than half of American students fail to demonstrate math proficiency in fourth grade, with even greater numbers of students performing below proficiency in eighth and twelfth grade (NCES, 2013). As math concepts are learned in a hierarchical sequence, it is important for students to have a strong foundational knowledge of math calculation. Fluency with basic math facts allows students to devote more attention to higher-order math calculation and problem solving skills.

Significance of the Problem

Math fluency refers to the ability to use efficient and accurate methods to solve simple calculations (NCTM, 2010). Math fact fluency, also referred to as computational fluency, is often assessed using timed tests of simple arithmetic problems. For example, the Math Facts Fluency subtest of the *Woodcock Johnson IV Tests of Achievement* (WJ IV ACH; Schrank, Mather, & McGrew, 2014a) is comprised of single-digit addition, subtraction, and multiplication problems. The student is given three minutes to complete as many problems as possible.

The National Mathematics Advisory Panel (2008) identified math fluency as a foundational skill for the development of more complex mathematical skills. Students with math fact retrieval deficits receive lower scores on mathematical measures and are more error-prone than typically achieving peers. Math fact fluency also appears to be linked to future performance in mathematics, including the undertaking of interdisciplinary tasks that involve math. Students

who are fluent in retrieving math facts are more likely to engage in math activities than students who have not yet developed fluency (Skinner, Pappas, & Davis, 2005). Additionally, fluent students are less likely to report frustration or anxiety related to math calculations (Cates & Rhymer, 2003).

Cognitive Correlates of Math Fluency

Children who exhibit a discrepancy between his or her cognitive ability and mathematical achievement are often characterized as students with a Math Learning Disability (MLD). Much of the research on math fact retrieval deficits focuses on this subset of students; however, studies have recently included children with low achievement in math fluency. Like children with MLD, children with low achievement also have at least low average intelligence and exhibit deficits in math achievement, although the achievement deficits are less severe than those seen in children with MLD. David Geary's research (e.g., 2004, 2010, 2011a) on the patterns and characteristics of math disabilities has identified a subset of children with particular difficulty in the fluent retrieval of math facts.

Geary (2011a) has proposed three mechanisms underlying math fact retrieval deficits. The first has been characterized as a semantic deficit. In this model, it is hypothesized that weakness in phonological processing tasks that measure skills including phonemic awareness and rapid automatized naming (RAN) is related to difficulty quickly and accurately retrieving math facts from memory. Given that phonological processing is a known correlate of reading fluency (e.g., Melby-Lervåg, Lyster, & Hulme, 2012), and that both tasks involve retrieval of semantic information from long-term memory, investigators have sought to understand the relationship between phonological processing abilities and math fluency. Recent research has suggested that children with phonological deficits have impaired performance in math fluency

(Chong & Siegal, 2008; Vukovic, Lesaux, & Siegel, 2010). Fuchs and colleagues (2005, 2006) have found that phonological processing measures are predictive of performance on math fluency tasks. Additionally, research has established a link between performance on RAN tasks and math fluency measures in children with MLD and low achievement (Geary, Hoard, & Bailey, 2012; Mazzocco & Grimm, 2013).

Geary's (2011a) second proposed mechanism of math fact retrieval deficits is characterized by a weakness in executive functioning, or the ability to efficiently allocate attention and cognitive resources. Specifically, Geary has hypothesized that a deficit in inhibition is related to impaired math fluency. Inhibition refers to an individual's ability to block irrelevant information from entering working memory (Geary, 2011a). In the extant literature, working memory tasks have been used to approximate this skill. Research conducted with students with math fact retrieval deficits revealed that these students have depressed performance on working memory tasks, which involve retaining and manipulating information in short-term memory (Geary et al., 2012; Geary, Hoard, & Nugent, 2012). In addition, research on general mathematical skills suggests a link between working memory and math ability in general populations (Bull, Espy, Wiebe, Sheffield, & Nelson, 2011; Kroesbergen, Van Luit, Van Lieshout, Van Loosbroek, & Van de Rijt, 2009).

Third, Geary (2011a) has proposed that weakness in numerical representation is related to deficits in math fact retrieval. Numerical representation, which may also be referred to as number sense or early numeracy, represents a domain of early math skills, including understanding of the number line, quantity representation, and determining relationship between numbers. In support of this hypothesis, Geary (2011b) was able to predict significant variance in math achievement using numerical representation tasks. Locuniak and Jordan (2008) found that math fluency in

second grade was related to performance on early numeracy tasks, even after controlling for general intelligence.

The above research illustrates the complexity of the cognitive processes underlying math fact retrieval, and that there are multiple psychoeducational abilities that may contribute to an individual's performance on these measures. It may be helpful to consider all of these individual abilities within a broader theory of cognitive ability. Simultaneously considering distinct cognitive abilities could uncover patterns of strengths and weaknesses related to math fact fluency performance. Further, it may help in identifying the most salient abilities in the prediction of math fact retrieval deficits.

CHC Theory

The Cattell-Horn-Carroll (CHC) theory of cognitive ability is the result of many years of research and collaboration between prominent intelligence theorists. Contemporary CHC theory is a reflection of Carroll's (1993) work expanding Cattell and Horn's theory of fluid (*Gf*) and crystallized (*Gc*) intelligence into three strata representing general intellectual ability, broad cognitive factors, and narrow abilities (McGrew, 2005). At present, CHC theory is the most researched and empirically validated model of intelligence (Flanagan, Ortiz, & Alfonso, 2013).

Stratum three of the CHC framework represents overall intellectual ability, or *g*. Beneath *g*, stratum two encompasses seven broad cognitive factors including the following: fluid intelligence (*Gf*), crystallized intelligence (*Gc*), long-term retrieval (*Glr*), short-term working memory (*Gsm*), visual processing (*Gv*), auditory processing (*Ga*), and processing speed (*Gs*). Stratum one is comprised of more than 70 narrow abilities subordinate to the broad abilities. These narrow abilities represent a variety of component skills, for example perceptual speed (*Gs-P*) reflects a specific skill within the general processing speed (*Gs*) ability.

Math Research using CHC-oriented Assessment Tools

The *Woodcock Johnson Tests of Cognitive Abilities, Third Edition* (WJ III COG; Woodcock, McGrew, & Mather, 2001a) is an assessment based on CHC theory. Together with the *Woodcock Johnson Tests of Achievement, Third Edition* (WJ III ACH; Woodcock, McGrew, & Mather, 2001b), this battery is used by practitioners to identify an individual's cognitive strengths and weaknesses and how this pattern relates to his or her profile of academic achievement. Research comparing subtests on the Woodcock Johnson cognitive and achievement tests has provided insight into the relationships among various cognitive factors and academic skills. Three studies have focused specifically on math achievement using the WJ III standardization sample.

Floyd, Evans, and McGrew (2003) used multiple regression analyses to investigate the link between cognitive factors and performance on math calculation and math problem solving clusters. The Math Calculation Cluster, which is comprised of an untimed math calculation subtest and a timed math fluency subtest, had a strong relationship with processing speed and crystallized intelligence in elementary and middle school students. Moderate relationships between math calculation and auditory processing and long-term retrieval were evident in early elementary age children's scores.

A second study regarding the cognitive predictors of math calculation performance was conducted by Proctor, Floyd, and Shaver (2005). This study compared the profiles of low achieving math calculation students in comparison to typically achieving peers. No significant differences were found across cognitive measures, leading the researchers to hypothesize that students who are low achievers in math calculation are likely a heterogeneous group.

Using structural equation modeling, Taub, Floyd, Keith, and McGrew (2008) examined the relationships between math achievement and cognitive factors. Crystallized intelligence, fluid reasoning, and processing speed were significantly related to overall math ability. However, as math calculation and problem solving skills were combined into one composite, it is unclear which abilities are specifically related to math calculation and/or math fluency skills.

McGrew and Wendling's (2010) analysis summarized the results of the aforementioned studies in addition to others that investigated the relationships between cognitive abilities and achievement in the context of CHC theory. Consistent with Taub et al.'s (2008) findings, crystallized knowledge, fluid reasoning, and processing speed were surmised to be most related to math calculation skills. Further analysis parsed out the narrow abilities, the discrete skills that comprise the broad factors. Both perceptual speed (*G_s-P*) and working memory (*G_{sm}-MW*) were strongly related to math calculation performance for children of all ages. For children ages 6-13, phonological processing (*G_a-PC*) had a moderate relationship with calculation skills. Of note, two of these narrow abilities (*G_{sm}-MW* and *G_a-PC*) were implicated in the relationship between cognitive abilities and math calculation, although the corresponding broad factors were not (*G_{sm}* and *G_a*).

Problem Statement

Despite research identifying the cognitive correlates for overall math achievement, no studies have examined the relationship between cognitive abilities and math fluency as an isolated skill. Recent literature suggesting that fluent math fact retrieval is a specific area of weakness for some children warrants further research investigating the cognitive correlates of these difficulties. Studies demonstrating a link between math fluency deficits and phonological processing, executive control, and number sense have revealed a complex array of factors that

may contribute to such deficits. Understanding the relationship between cognitive abilities measured by the *Woodcock Johnson, Fourth Edition* (Schrank, McGrew, & Mather, 2014) and math fact fluency has important implications for the identification and remediation of math fact retrieval deficits.

Research Questions and Hypotheses

1. Within the child age subset of the standardization sample, which broad cognitive abilities display significant effects on the Math Facts Fluency performance?
 - a. Hypothesis 1: It is hypothesized that fluid reasoning will have a direct effect on Math Facts Fluency.
 - b. Hypothesis 2: It is predicted that comprehension-knowledge will have a direct effect on Math Facts Fluency.
 - c. Hypothesis 3: Working memory is hypothesized to have a direct effect on Math Facts Fluency.
 - d. Hypothesis 4: Processing speed is predicted have a direct effect on Math Facts Fluency.
 - e. Hypothesis 5: It is hypothesized that General Intellectual Ability will have an indirect effect on Math Facts Fluency performance.
2. Which narrow abilities have significant effects on performance on the Math Facts Fluency subtest?
 - a. Hypothesis 1: It is hypothesized that perceptual speed will have a direct effect on Math Facts Fluency.
 - b. Hypothesis 2: Number facility will have a direct effect on Math Facts Fluency.
 - c. Hypothesis 3: Phonetic coding will have a direct effect on Math Facts Fluency.

- d. Hypothesis 4: It is predicted that naming facility will have a direct effect on Math Facts Fluency.
 - e. Hypothesis 5: It is hypothesized that attentional control will a direct effect on Math Facts Fluency.
3. What relationship will math problem solving abilities have with math fluency performance?
- a. Hypothesis 1: It is hypothesized that performances on the Math Problem Solving cluster will have a direct effect on performance on Math Facts Fluency.

CHAPTER II

LITERATURE REVIEW

Economists predict that the science, technology, engineering, and math industries will continue to grow at almost twice the national average, playing a vital role in the overall growth of the U.S. economy (Langdon, McKittrick, Beede, Khan, & Doms, 2011). However, the majority of U.S. students lack proficiency in the requisite math skills needed to succeed in these fields. According to the National Center for Education Statistics' 2013 Nation's Report Card, proficiency in mathematics was obtained by only 26% of 12th graders in 2013. Even in elementary and middle school, the majority of students are not performing at the Proficient level, with only 42% of fourth grade and 35% of eighth grade students obtaining at least Proficiency status on state assessments (NCES, 2013). These figures indicate that less than half of all students are able to consistently apply procedural knowledge and math reasoning skills to solve grade-level math problems. These data evidence a decreasing trend in the number of students meeting state standards in mathematics throughout the grade levels.

Duncan et al.'s (2007) seminal study on the early childhood predictors of academic achievement highlighted the importance of early math skills for later success. Specifically, Duncan and colleagues found that math skills at school entry predicted later elementary and middle school achievement better than measures of early reading, attention, behavior problems, and social skills. Similarly, Claessens, Duncan, and Engle (2009) found that kindergarten math ability predicted fifth-grade achievement in both reading and mathematics.

Early mathematical knowledge appears to be the strongest predictor of future math achievement. Recent research has focused on assessing children's "number sense." Although it has been defined in a number of different ways, number sense generally refers to the ability to

understand the meaning of numbers and the relationships of numbers with each other (Berch, 2005). Though the definition of number sense may vary by researcher, there appears to be more of a consensus on the types of skills young children should possess. Generally, math measures for young children focus on number identification, counting, quantity discrimination, and understanding of the number line.

Longitudinal research has demonstrated the relationship between number sense and later math achievement. Jordan, Kaplan, Locuniak, and Ramineri (2007) tracked children's achievement in kindergarten and first grade and found a significant (.70) correlation between fall kindergarten assessments of number sense and end of first grade math achievement. In analyzing predictors of first grade math achievement, background variables, such as reading achievement, income status, gender, and age, did not add any predictive value over number sense in kindergarten. A continuation of this study, measuring the mathematics achievement of these students in third grade, showed that kindergarten number sense skills continued to be predictive of math achievement in third grade (Jordan, Kaplan, Ramineri, & Locuniak, 2009).

In order to enhance the development of mathematical skills in young children, educators and researchers must understand children's cognitive development and capacity for mathematical reasoning. Mathematical skills are obtained in a hierarchical sequence; therefore, a foundational knowledge of numerical principles and math calculation is necessary before more complex skills can be learned (NCTM, 2000). An understanding of the cognitive correlates for discrete math skills is needed to ensure accurate assessment and remediation for math achievement deficits.

Math Development

Given that preschool children can quantitatively think and reason (Resnick, 1989), much research on the development of mathematical skills in early childhood has focused on how to

foster children's inherent mathematical understanding and encourage interest in applying mathematical concepts. The National Council of Teachers of Mathematics (NCTM) first incorporated pre-kindergarten education standards in the 2000 publication of Principles and Standards for School Mathematics (NCTM, 2000). The Principles and Standards for School Mathematics provide recommendations for high-quality mathematics education in pre-K through grade 12 classrooms. The organization emphasizes that curricula should be correctly aligned with the known progression of mathematical skills.

The Principles and Standards for School Mathematics identifies five areas of knowledge that students should develop to be proficient in mathematics. The five content areas include: (a) number and operations, (b) algebra, (c) geometry, (d) measurement, and (e) data analysis and probability. In the area of *number and operations*, young students should develop the ability to count objects, label how many objects, and answer simple addition and subtraction questions (NCTM, 2010). In the area of *algebra*, students should be given opportunities to recognize and re-create patterns of objects. In terms of *geometry*, young students should develop the ability to name shapes, use shapes to create a picture, understand simple maps, and use spatial words to describe relationships between objects. For the development of *measurement*, pre-K and early elementary age children should improve in the ability to use words to label object qualities (e.g., heavy, long) and compare objects using non-standard measuring tools, like cups or strings. Finally, student development in the area of *data analysis and probability* includes sorting objects, comparing groups, and utilizing simple graphical representations.

Although NCTM defines mathematical concepts into these five areas, educational research has primarily focused on math achievement in terms of calculation and problem solving. This distinction is reflected in the current definition of a Specific Learning Disability as outlined

by the 2004 Reauthorization of Individuals with Disabilities Education Act (IDEA), which categorizes math disabilities in terms of mathematics calculation or mathematics problem solving. In terms of NCTM's content areas, mathematics calculation is most closely aligned to the domain of numbers and operations.

With respect to the area of numbers and operations, the three primary goals for pre-k through grade 12 students include: (a) understanding numbers, (b) understanding meanings of operations, and (c) computing fluently. The NCTM lists expectations for students across grade levels for pre-K through grade 2, grade 3 through 5, grade 6 through 8, and grade 9 through 12. For example, children in pre-K through grade 2 should demonstrate understanding of numbers through counting and recognizing "how many?" for a group of objects. In these grades, students should also show comprehension of words describing position or magnitude. In terms of operations, students must understand the meaning of addition and subtraction and their relationship to each other. Finally, young students should develop fluency with simple addition and subtraction problems. That is, students should be able to quickly and efficiently solve simple math problems. The developmental sequences for counting, subitizing, and calculation, three areas of growth during pre-K and elementary school years, are described below.

Counting

Within the math developmental sequence, a three year-old child can correctly count up to the number four (National Association for the Education of Young Children; NAEYC, 2010). Between ages three and four, most children are able to count up to four objects using a one-to-one correspondence. Children of this age understand that counting involves assigning only one number to one object and also begin to understand that numbers are sequenced in a fixed order. For example, the number three always comes before four. Children at age four are generally able

to understand that the last number counted for a series of objects represents the total number of objects, which is referred to as *cardinality* (Clements & Sarama, 2009). By age five, most children are able to count up to 10 objects, although rote counting ability may extend to numbers in the 20s and 30s, or beyond.

Counting is a fundamental skill, and the inability to count is linked to subsequent math disabilities. Geary and colleagues have shown that difficulties with understanding the order of numbers and the process of counting objects is related to math disabilities, controlling for the effects of IQ and reading ability (Geary, Bow-Thomas, & Yao, 1992; Geary, Hamson, & Hoard, 2000). Some differences between children with math learning disabilities' (MLD) knowledge of counting principles and typically achieving peers appear to be present. Geary et al. (1992) found that that MLD children were less likely to recognize that when counting a set of objects, one could begin at either ends of the set, and that the objects could be counted in any order.

Subitizing

Subitizing has recently been considered a core facet of children's number sense (Geary, 2010). Research has implicated subitizing in the development of counting proficiency in kindergarten (Kroesbergen et al., 2009), as well as math achievement in the elementary years (Geary, 2011a). *Subitizing* is defined as the ability to quickly recognize a quantity through visual discrimination, rather than counting each object (Kaufman, Lord, Reese, & Volkman, 1949). At age three, children can automatically answer "how many?" questions involving one to three objects. Four year old children are automatically able to recognize when four objects are present, while five year old children can recognize when five objects are present. For example, a five year old can immediately identify the number of dots when presented with a picture of five dots (Clements & Sarama, 2009). At this age, children also begin to understand that addition and

subtraction can occur between groups of subitized quantities (e.g., recognizing two groups of three objects and determining that there are six total).

Calculation

At age two or three, children demonstrate the emerging ability to count small groups of objects to determine a sum. For example, a child this age is able to correctly name how many total blocks are present after seeing a third block added to a group of two blocks (Clements & Sarama, 2012). At age four, children develop the ability to use language to solve addition problems under five digits. This child would be able to use a counting-all strategy to answer a simple addition problem; that is, the child would count each object in order to determine how many are present in all. Children at this age can use their fingers to count, understanding that each finger represents one object. At age four and five, children begin to understand small digit subtraction problems, where he or she separates objects that are taken away and counts how many objects are remaining.

School age children begin to utilize a counting-on strategy to solve basic addition problems. Using this strategy, a student would start counting after the number of the first digit, rather than starting at number one, and then count the additional numbers being added on to that digit. For example, in the equation $4 + 3$, the child would count “5, 6, 7” to arrive at the answer of 7. In addition to solving problems that require finding the sum, students may also solve missing addend problems, or “how many more” problems. Similar to counting-on, the counting-up-to strategy can be used to find how many more digits are needed to reach the total sum. Another more sophisticated strategy, decomposition, is used obtain an answer to a calculation problem by recalling answers to similar calculations (Geary, 2011a). For example, in the

calculation $5 + 6 = 11$, a student may recall $5 + 5 = 10$ and that 6 is 1 more than 5, leading to the determination that the answer is $10 + 1$, or 11.

Math Disabilities

Within the mathematics literature, students with calculation deficits are often described as students with Math Learning Disability (MLD) or low achievement. In the context of IDEA (2004), a student with a MLD must have a significant discrepancy between his or her intelligence and math achievement. In research studies, students who score at or below the 10th percentile in math are generally included within the MLD category, given an intelligence score at or above the 15th percentile is present (Geary, 2011a). Students with low achievement are characterized as having math scores within the 11th to 25th percentile and also possess at least low average intelligence (15th percentile or greater).

In terms of calculation, students with MLD continue to use less efficient strategies (Geary, 2011a). Students with typical achievement begin to exhibit more sophisticated calculation strategies in the early elementary years, while students with MLD continue to count using their fingers or by counting-all for both digits in the equation. Students with low achievement in math also show this delay relative to typically achieving peers; however, low achieving students show less significant delays than students with MLD.

Math Fluency

Math fluency refers to the ability to quickly solve simple calculations, for example single-digit addition problems. The National Mathematics Advisory Panel (2008) determined that math fluency, also referred to as computational fluency, is an important precursor for the development of higher order math skills. Math fluency is typically measured by giving a student a set of arithmetic problems to solve in a set time period. Examples include the Math Fluency

subtest on the WJ IV ACH and the Addition Fact Fluency and Subtraction Fact Fluency subtests (Fuchs, Hamlett, & Powell, 2003). Measures differ by length of time and the operations included. For example, the Math Fluency subtest on the WJ IV ACH battery includes addition, subtraction, and multiplication problems and is given a three minute time limit, while Addition Fact Fluency includes only one operation and has a one minute time limit.

Students with math fluency or math fact retrieval deficits receive lower scores on these measures and are more error-prone than typically achieving peers. These students more often make errors resulting from intrusions of counting string associates (Geary et al., 2000; Geary et al., 2012). Counting string associates refer to any number that is directly above or below one of the digits in the equation. For example, in the problem $5 + 3$, an incorrect answer of 6 would represent an intrusion of the number above 5 in the number sequence. Likewise, an incorrect answer of 4 would represent an intrusion associated with the number 3. Further, elementary students with deficits in math fact fluency tend to have growth rates similar to typically achieving peers, resulting in a maintained gap in performance (Chong & Siegel, 2008). Conversely, students with procedural deficits in math tend to have higher growth rates allowing them to “catch up” to their typical peers on procedural tasks. Given the importance of math fact retrieval skills in becoming proficient in more complex mathematics, recognizing and intervening for math fluency deficits is a fundamental goal.

Recently, researchers have sought to understand math fluency as a distinct mathematical skill. Indeed, twin studies have found that math fluency skills indeed have a unique genetic origin. Hart, Petrill, Thompson, and Plomin (2009) found that although math fluency shares genetic overlap with reading fluency, it has unique genetic influences independent of math calculation and general cognitive ability. After accounting for performance on untimed math

measures, reading comprehension, and reading fluency, approximately two thirds of the variance in math fluency remained unexplained (Petrill et al., 2012). Recent research involving the study of the cognitive correlates of mathematics achievement also reflects this distinction between math fact fluency and untimed math ability. Geary (2010) suggests that a subset of children with Math Learning Disability (MLD) and low achievement have distinct and severe deficits in the ability to efficiently retrieve basic math facts.

Characteristics of Math Fact Retrieval Deficits

David Geary's (1993) theory on the subtypes of math learning disabilities originally proposed a distinction between children with procedural, semantic memory, and visuospatial profiles of MLD. Children with math fact retrieval deficits were proposed to have the semantic memory subtype. This subtype is characterized by difficulty answering basic arithmetic problems, slow response times on math fluency measures, and frequent errors in math fact retrieval. This subtype is frequently associated with comorbid reading disabilities. Geary (2004) hypothesized that these children have deficits in phonetic and semantic representations in long-term memory.

More recently, Geary (2011a) proposed three mechanisms of retrieval deficits that may result in problems with math fact fluency for children with MLD and low achieving children. The first represents the semantic memory hypothesis, discussed in his early work. The second is related to an inhibition deficit, in which the individual fails to inhibit irrelevant number associations when attempting to retrieve a math fact from memory. The counting string intrusions, discussed previously, are one such example. Others include "table-related" intrusions, where the student recalls a number next to the correct answer on the multiplication table, or cross-operation intrusions, where the student recalls an answer to a problem using a different

operation (e.g., solving as an addition problem instead of a subtraction problem). Third, Geary proposed a deficit in number processing, or the ability to understand numerical representations of small and large quantities. Research aligned with these three subtypes is discussed below.

Phonological Processing

Phonics. Simmons and Singleton's (2008) review of research investigating the link between reading disabilities (RD) and math fact retrieval suggests that students with reading and math disabilities possessed phonological processing deficits underlying their difficulties in decoding words and retrieving math facts. Both children with MLD and RD were found to have deficits in retrieving answers to simple addition problems (Geary et al., 2000). In addition, children with RD that did not have MLD had lower scores on arithmetic measures than typical peers.

A recent study by Vukovic et al. (2010) compared the math achievement of children with RD with phonological deficits (dyslexia), children with reading comprehension difficulties absent phonological deficits, and a group of typically achieving children. Results showed that students with phonological deficits were more likely to have deficits on the WJ III ACH Math Fluency measure than the reading comprehension group or the comparison group. However, this study was limited by a small sample size, and three of the 18 students with phonological deficits showed no deficit in math fluency. A longitudinal study by Chong and Siegel (2008) also found support for deficient phonological processing skills associated with poor math fact fluency. Students with MLD and low achievement in math fluency both showed deficits in phonological processing on Word Attack, a pseudoword decoding task that requires one to use phonics skills rather than recall known sight words, as compared to typical children. Deficits in phonological

processing were evident in second, third, fourth, and fifth grades for the students with MLD and low achievement.

Further support for the link between phonological processing and math fluency has been identified in studies of typically achieving students. Fuchs et al. (2006) assessed third grade students on cognitive and math achievement measures. Using path analysis, the researchers found that performance on Word Attack was a significant predictor of performance on addition and subtract fact fluency measures. Another study by Fuchs and colleagues (2005) investigated the relationship between phonological processing and math fluency using a large sample of first grade students. Phonological processing was measured using a composite of two subtests: a sound matching subtest and a rapid digit naming subtest. Multiple regression showed that performance on this phonological processing composite was uniquely predictive of addition fact fluency beyond reading achievement and other cognitive variables.

Rapid Automated Naming. Rapid digit naming, as stated above, is often considered a measure of phonological processing. However, it is also often included as a measure of processing speed. Rapid letter, number, and color naming tasks are often referred to as measures of rapid automatized naming (RAN). These tasks assess an individual's ability to efficiently retrieve the label associated with a pictorial representation. For example, on a rapid letter naming task, an individual is asked to read a set of letters as quickly as possible within a time limit. Research on the link between RAN and reading fluency is well established (e.g., Norton & Wolf, 2012). Less work has been done on the relationship between RAN and math disabilities. As a measure of phonological processing and processing speed, RAN would appear by theory to be linked to math fluency; however, results of recent research appear mixed.

Geary et al.'s (2012) longitudinal study of MLD and low achieving students with severe and mild fact retrieval deficits examined the relationship between math fact fluency and other cognitive and achievement measures relative to typically achieving peers. Results from assessments in second, third, and fourth grade showed that RAN letter and number performance reflected significantly longer completion times for children with MLD than the typical or mild math deficits group. As expected, mean response times for the severe deficit group showed that they were slower than the typically achieving and mild deficit group. The children in the MLD group had the slowest naming speed. With respect to younger students, Georgiou, Tziraki, Manolitsis, and Fella (2013) assessed the relationships among rapid color and object naming tasks in kindergarten and math fluency in first grade. Although RAN performance was a significant predictor of reading fluency, it did not significantly predict math fluency performance in the first grade. However, research on RAN tasks within the reading literature has shown that performance on rapid naming tasks using alphanumeric symbols (i.e., letters or numbers), rather than object or colors, has proven to be most predictive of reading ability (Savage & Frederickson, 2005). This difference in task demand could relate to the lack of significant relationship between math fluency and RAN color and object tasks.

Additional research has been conducted investigating the relationship between RAN and overall math ability. Mazzocco and Grimm (2013) assessed the performance of students with MLD, low achievement, and typical achievement in kindergarten through grade eight on rapid letter, number, and color naming tasks. Results showed that children with MLD and low achievement in math were significantly slower on all tasks than typically achieving peers in kindergarten. In grade eight, children with MLD were again significantly slower when naming letters and colors compared to typically achieving peers. Low achieving students were

significantly slower than typically achieving peers on color naming only. Interestingly, children with MLD did not differ from typically achieving peers on the rapid number naming task in grade eight. Students with low achievement did not differ from typical students on letter or number naming, but did perform slower on the color naming task. A second longitudinal study investigating various math skills in kindergarten found that performance on RAN color, object, and number naming tasks did not contribute any predictive value in the identification of MLD in second and third grade after accounting for other formal (e.g., number identification) and informal (e.g., quantity discrimination) skills (Mazzocco & Thompson, 2005).

Based on these results, it appears that the relationship between phonological processing, rapid automatized naming, and math fluency may depend on the severity of math fact retrieval deficit, the age of the child, and the type of task employed. Additionally, Geary's proposed subtypes for math fact retrieval deficits suggest that children with these deficits are not a heterogeneous group.

Executive Control

Geary's second proposed mechanism underlying math fact retrieval deficits focuses specifically on a deficit in inhibiting irrelevant information intruding in working memory (Geary, 2011a). Geary et al. (2012) used two unique measures for assessing addition facts in elementary school students in second, third, and fourth grade. The first was a choice task, where the student was asked to solve simple addition problems as quickly as possible without paper and pencil. The response time was measured between presentation of the problem and the child's response. The student was also asked to describe how he or she arrived at the answer. The second measure was forced addition fact retrieval. This task was similar to the first, except that children were instructed to try to answer each problem from memory without counting or using other problem

solving strategies. Students were also assessed on the *Working Memory Test Battery for Children* (WMTB-C; Pickering & Gathercole, 2001), a battery of nine subtests in which a child is assessed using the following: three dual-task measures, whereby the child must manipulate information in working memory to produce an answer (referred to as central executive tasks); four recall tasks utilizing phonological memory; and two visuospatial memory tasks. Results showed that children with MLD had lower scores on all three working memory tasks than typically achieving children. Further, low achieving children with severe fact retrieval deficits had significantly lower scores on the central executive tasks than typical children, although the groups did not differ on phonological or visuospatial tasks. Low achieving children with mild fact retrieval deficits had lower scores on all working memory tasks, but did not differ significantly from typically achieving peers. However, all children with fact retrieval deficits showed errors on addition tasks that suggested intrusions of unrelated information. The authors suggest that this inhibition difficulty is a specific facet of working memory ability that may not be tapped by traditional working memory measures.

Geary, Hoard, and Nugent (2012) used similar measures to assess a group of children from first to fourth grade. First grade students with better performance on central executive tasks were found to perform better on addition fact retrieval tasks than children with lower scores on these working memory tasks. Although central executive measures were less predictive of fact fluency in later years, the measures did predict the development of more efficient strategies. That is, children with better performance on central executive tasks were observed to use the decomposition strategy for solving addition problems before lower performing peers.

Another function related to executive control, attention, has recently been linked to math fluency in elementary school children. The Geary et al. (2012) study used teacher-rated

inattentive behaviors. Results showed that children's attentive behavior predicted their use of more efficient calculation strategies. Similarly, Fuchs and colleagues (2006, 2008) found that ratings of attention predicted third graders' performance on basic calculation fluency.

Interestingly, Fuchs et al. (2008) found that inattentive behaviors were distinct predictors of calculation fluency deficits but not math problem solving deficits, suggesting this relationship was not simply a reflection of teacher's perception of low achieving students in the classroom.

Research has also focused on the contribution of executive functions on general math ability in young children. Using various measures assessing kindergarten and first grade age children's planning, updating, and inhibition skills, Kroesbergen et al. (2009) found that these skills contributed a significant amount of variance to these children's counting skills. Updating, which the authors define as "monitoring and coding of information relevant to the task and replacing nonrelevant information with new input," was measured using a digit span backward test, where an individual must listen to, reorder, and recite lists of increasing long digit sequences (Kroesbergen et al., 2009, p. 227). Performance on this task was determined to be the best predictor of the variance in children's early math skills. Although referred to in this study as an executive functioning skill, digit span backward is often included as a measure of working memory or central executive capacity. Regardless of how it is defined, digit span backward appears to have a significant relationship with early math skills (e.g., Geary et al., 2009; Geary, 2011a; Geary et al., 2012; Locuniak & Jordan, 2008). Finally, research on the contributions of executive functioning in preschoolers' emergent math ability suggests that these skills are unique predictors independent of the effects of crystalized (verbal) intelligence (Bull et al., 2011).

Number Sense

A third deficit in math fact retrieval proposed by Geary (2011b) involves weaknesses in numerical representations. In support of this theory, researchers employing measures tapping the broad domain of “number sense” have found that performance in this domain can predict math fluency in elementary school children. Locuniak and Jordan (2008) assessed a sample of kindergarteners on counting, number recognition, knowledge of the number line, nonverbal calculation (using manipulatives), addition and subtraction story problems, and number combinations, which were orally presented addition and subtraction problems using phrases such as “how much is x plus y ?” Calculation fluency was measured in second grade using addition and subtraction timed subtests. Results showed that children’s number sense performance in kindergarten was predictive of second grade calculation fluency even after controlling for other variables, such as intelligence and reading achievement. Digit span backward was the only cognitive measure that contributed additional variance when number sense was included in regression models.

Geary’s (2011b) longitudinal study assessing general math ability in a general sample of elementary school children found that skill on tasks assessing subitizing, or the ability to quickly recognize small quantities without counting, and quantity representation added unique variance in predicting math achievement. Using a sample of MLD and low achieving children, Geary et al. (2012) included number sets and number line measures to assess their representation of numerical quantities. The number sets measure was hypothesized to assess subitizing, the number line task was proposed to tap the ability to understand magnitude. Children in the low achieving group appeared to have deficits in these areas; however, these differences were not significant when including other measures of cognitive abilities. The authors note that these

measures may not have truly measured the concept of numerical representation. Indeed, one of the barriers in assessing number sense abilities is the lack of consensus on how number sense is defined (see Berch, 2005) and a lack of standardized instruments tapping these abilities. At present, it is unclear the degree in which deficits in math fluency can be predicted using measures of children's understanding of numerical representation.

CHC Theory

Contemporary CHC theory is the manifestation of John Carroll (1993) expanding Raymond Cattell and John Horn's theory of fluid (*Gf*) and crystallized (*Gc*) intelligence into three strata representing general intellectual ability, broad cognitive factors, and narrow abilities (McGrew, 2005). In contemporary CHC theory, general intellectual ability, or *g*, represents the third stratum. A singular construct representing overall ability has predominated over a century of intelligence research, stemming from the early work of Charles Spearman. Spearman's research was later expanded by Karl Holzinger and colleagues to include additional factors of intelligence. This groundwork eventually led to Cattell's *Gf-Cc* theory, which entails a hierarchical two-factor theory of intelligence with associated lower-order abilities (Schneider & McGrew, 2012). Cattell's collaboration with Horn over the latter half of the 20th century continued to parse out individual factors of intellectual ability utilizing factor-analytic techniques. Carroll's (1993) work reviewing existing theories and research reconciled the notion of a singular *g* ability with the multi-factored abilities discovered in Cattell and Horn's research into a three stratum hierarchy. This theory continues to be recognized as the most comprehensive and psychometrically evaluated intelligence theory, with research continuing to clarify and delineate broad and narrow cognitive abilities (Schneider & McGrew, 2012).

Broad Abilities

Schneider and McGrew (2012) conducted a review of contemporary CHC theory that included 16 broad factors in the second stratum encompassed under the umbrella of *g*. These factors were described as fluid intelligence (*Gf*), comprehension-knowledge (*Gc*; formerly referred to as crystallized intelligence), long term retrieval (*Glr*), short term memory (*Gsm*; presently referred to as working memory *Gwm*), visual processing (*Gv*), auditory processing (*Ga*), and processing speed (*Gs*), reading and writing (*Grw*), quantitative knowledge (*Gq*), domain-specific knowledge (*Gkn*), tactile abilities (*Gh*), kinesthetic abilities (*Gk*), olfactory abilities (*Go*), psychomotor abilities (*Gp*), psychomotor speed (*Gps*), and reaction and decision speed (*Gt*). The authors provide groupings according to the degree by which abilities cluster together by function, producing an acquired knowledge group (*Gc*, *Grw*, *Gq*, *Gkn*), memory group (*Gsm*, *Glr*), general speed group (*Gs*, *Gps*, *Gt*), and a motor group (*Gk*, *Gp*). Additionally, a conceptual grouping was made for sensory abilities (*Ga*, *Gv*, *Gh*, *Go*), in addition to conceptual groupings of sensory-motor domain-specific abilities (sensory and motor abilities), cognitive efficiency (memory and general speed abilities), and domain-independent general capacities (*Gf*, memory, and general speed).

Although the second stratum currently encompasses 16 broad abilities, existing intelligence batteries do not necessarily provide measures of each ability. Newton and McGrew (2010) at the time reported that intelligence measures typically include measures of fluid reasoning (*Gf*), comprehension-knowledge (*Gc*), long term retrieval (*Glr*), short term memory (*Gsm*), visual processing (*Gv*), auditory processing (*Ga*), processing speed (*Gs*), and quantitative knowledge (*Gq*). Specifically, Keith and Reynolds (2010) conducted a review of common intelligence measures that found that tests with fewer factors were the norm, with the *Kaufman*

Assessment Battery for Children (KABC-II; Kaufman & Kaufman, 2004) evidencing five factors (Gc, Gv, Gf, Glr, and Gsm), the *Stanford Binet, Fifth Edition* (SB-V; Roid, 2003) reflecting five factors (Gf, Gc, Gf-RQ, Gv, and Gsm), and the *Wechsler Intelligence Scale for Children, Fourth Edition* (WISC-IV; Wechsler, 2003) measuring four factors (Gc, Gsm, Gs, and Gf/Gv). The *Woodcock Johnson Fourth Edition Tests of Cognitive Abilities* (WJ IV COG; Schrank et al., 2014) provides measures of seven broad abilities (Gc, Gf, Gwm, Gs, Ga, Glr, Gv), which the authors state reflects the status of the most substantial research on CHC theory at the time of its development (McGrew, LaForte, & Schrank, 2014). The seven broad abilities encompassed by the WJ IV COG are described in Table 1 below.

Table 2.1

Descriptions of CHC Broad Abilities

Gc	Comprehension-Knowledge	Represents the ability to activate and access acquired, declarative knowledge
Gf	Fluid Reasoning	Measures novel problem solving ability; Requires induction, categorization, and identifying and switching rules
Gwm	Short-Term/Working Memory	Refers to the capacity to keep stimuli in immediate awareness, recode the information, and produce an output
Gs	Processing Speed	The ability to utilize attentional control to perform speeded visual perception and discrimination tasks
Ga	Auditory Processing	Represents the ability to analyze and discriminate acoustic elements; activate and retrieve semantic information
Glr	Long-Term Storage and Retrieval	Involves the construction of representations in long-term memory and the ability to retrieve this information
Gv	Visual Processing	Involves mental manipulation and rotation of visual images and retrieval of visual representations from memory

Narrow Abilities

The narrow abilities subsumed by CHC theory comprise the most fluid of the three strata, with narrow abilities continually being added, refined, or removed as the result of an evolving body of research. Over 70 narrow abilities have been proposed within stratum one (Newton & McGrew, 2010). In their 2012 chapter, Schneider and McGrew included a discussion of 81 well-supported narrow abilities falling beneath the 16 broad abilities. The narrow abilities that comprise the seven broad abilities measured by the WJ IV COG are briefly discussed below.

Comprehension-knowledge. Six narrow abilities fall within the scope of comprehension-knowledge (Schneider & McGrew, 2012). General Verbal Information (K0) refers to the store of knowledge obtained through cumulative exposure to information across various domains. Language Development (LD) refers broadly to the comprehension and application of language for expressive and receptive communication. Next, Lexical Knowledge (VL), refers specifically to vocabulary knowledge as an isolated skill. Similarly, Listening Ability (LS) refers to the ability to understand speech as a discrete ability, whereas Communication Ability (CM) is described as the ability to utilize expressive language effectively to communicate one's thoughts. Finally, Grammatical Sensitivity (MY) is reserved for the ability to understand morphological and syntactic principles and apply grammatical knowledge.

Fluid reasoning. Three narrow abilities are considered well-supported by the current empirical literature (Schneider & McGrew, 2012). The first is Induction (I). Induction refers to the ability to utilize logical reasoning for the purpose of identifying an organizing principle or rule. Second, General Sequential Reasoning (RG) refers to the ability to utilize deductive reasoning to apply known rules or principles to problem solve a through a task. Finally,

Quantitative Reasoning (RQ) describes the ability to reason using basic mathematical knowledge, including basic computation and numerical reasoning.

Short-term memory/working memory. The broad ability of *Gsm* or short-term memory has recently been re-conceptualized in light of neuropsychological and cognitive research in the domain of working memory. While Schneider and McGrew (2012) use the term short-term memory in their discussion of CHC theory, within the WJ IV manual, McGrew et al. (2014) update the broad factor by naming it short-term working memory (*Gwm*). The new conceptualization of short-term memory continues to include the narrow abilities previously identified, Memory Span (MS) and Working Memory Capacity (MW). Memory span is defined as the ability to attend to, maintain, and reproduce information from memory immediately following its presentation. Working memory capacity also includes the ability to attend to and maintain information, but reflects the capacity for mental manipulation of information before producing a response. Schneider and McGrew (2012) also note that it involves simultaneously inhibiting distracting information and performing controlled searches for additional information for memory. The shift from incorporating working memory into a broad, rather than narrow factor recognizes the higher-order function of working memory, which broadly encompasses tasks involving various levels of processing in the memory system (McGrew et al., 2014). A second update included in the WJ IV technical manual is the addition of Attentional Control (AC) as a narrow ability (McGrew et al., 2014). Attentional control refers to the ability to allocate attention efficiently to focus on a task, while ignoring irrelevant stimuli. The authors note that this ability has been referred to using various terms including focal attention, focus, control of attention, executive controlled attention, or executive attention. The addition of this ability reflects substantial research in the cognitive and neuropsychological literature.

Processing speed. Within the domain of processing speed, Schneider and McGrew (2012) outline five narrow abilities. Perceptual Speed (P) is described as the keystone ability of processing speed, involving visual scanning and discriminating for identical visual figures. In fact, the authors note that perceptual speed may qualify as an intermediate stratum ability, comprised by four lower-order abilities including pattern recognition, scanning, memory, and complex. The second narrow ability is Rate of Test-Taking (R9). This ability is described as the rate at which one can complete simple, overlearned tasks. Speed in completing learned tasks is divided into three categories. Number Facility (N) is described as the ability to rapidly perform basic arithmetic computations with accuracy. Reading Speed (RS) is defined as the ability to read text for comprehension fluently and automatically. Finally, Writing Speed (WS) refers to the rate at which one can copy or compose words or sentences.

Auditory processing. In the domain of auditory processing, Schneider and McGrew (2012) list eight narrow abilities. The first narrow ability, Phonetic Coding (PC) is perhaps the most frequently assessed by psychologists. In fact, the authors state, "...psychologists are more interested in a narrow ability (phonetic coding) than in the broad ability" (p. 132). Whereas auditory processing refers to the ability to recognize and process all auditory information (e.g. music, sound), phonetic coding refers specifically to the ability to recognize distinct phonemes. In the academic literature, this skill has been also been referred to as phonemic awareness or phonological processing. A similar but distinct skill, Speech Sound Discrimination (US), refers to the ability to the awareness of non-phonemic aspects of speech (e.g. tone, timbre, and pitch). A third ability is called Resistance to Auditory Stimulus Distortion (UR). This ability refers to one's capacity for understanding speech in the presence of background noise or other distortion. The fourth narrow ability, Memory for Sound Patterns (UM) includes a memory load and refers

to the ability to retain auditory information within short-term memory. The following three abilities are particularly pertinent to music. These abilities include Maintain and Judging Rhythm (U8), the ability to distinguish and maintain a musical beat; Musical Discrimination and Judgment (U1 U9), the ability to analyze tonal qualities of music, including harmony and complexity; and Absolute Pitch (UP), the ability identify musical pitch with perfect accuracy. The last narrow ability, Sound Localization (UL), refers to the ability to identify the location of sounds in space.

Long-term storage and retrieval. A vast array of narrow abilities fall under the domain of long-term storage and retrieval. Twelve abilities are listed as well-supported by Schneider and McGrew (2012), with limited research supporting a previously hypothesized 13th ability, Learning Abilities (Newton & McGrew, 2010). The narrow abilities have been conceptually grouped into the categories of learning efficiency and retrieval fluency, to represent abilities related to the processes of storage and retrieval, respectively. Within the category of learning efficiency, Associative Memory (MA) is defined as the ability to recall pairs of items without any meaningful relationship (e.g., wall and hat). In contrast, Meaningful Memory (MM) refers to the ability to recall information in the context of meaningful relationships (e.g., a cohesive story). A third memory ability, Free-Recall Memory (MA6) represents the ability to recall information presented in a discrete list (e.g., 12 unrelated words).

The following nine abilities involve the retrieval of learned information from memory. Schneider and McGrew (2012) organized these abilities conceptually in terms of retrieval of ideas, words, or figures. Ideational Fluency (FI) refers to the ability to generate as many verbal responses related to a word, idea, or phrase as possible. Likewise, Associational Fluency (FA) is related to the ability to generate responses to words, ideas, or phrases, but the quality of content

is evaluated, rather than simply the quantity of responses (Newton & McGrew, 2010). Expressional Fluency (FE) is the ability to express of the same information in unique ways (e.g., generate various phrases that mean you feel tired). In a more applied context, the Sensitivity to Problems/Alternative Solution Fluency (SP) ability is described as the ability to produce alternative solutions to a particular problem (e.g., name ways a person can save money on everyday expenses). Relatedly, the Originality/Creativity ability (FO) requires the ability to produce flexible and unique responses to a given situation or task. This ability has been related to the broad construct of creativity. Two word-related retrieval skills has been identified. The first is Naming Facility (NA), which refers to the ability to rapidly retrieve the name of an object, color, or letter. The authors note that this task has been referred to as Rapid Automatic Naming (RAN) within academic literature. The second word-related retrieval skill is Word Fluency (FW). Word fluency refers to the ability to generate words by phonemic, structural, or orthographic characteristics (Newton & McGrew, 2010). Finally, the two figure related retrieval abilities consist of Figural Fluency (FF), the ability to draw as many unique figural marks as possible in response to a visual stimulus, and Figural Flexibility (FX), the ability to create unique visual solutions that require adherence to specific criteria.

The WJ IV battery names an additional narrow ability, Speed of Lexical Access (LA; McGrew et al., 2014). This skill is defined as the ability to quickly retrieve information from one's lexicon, or verbal store. Although mentioned as a discrete skill from Naming Facility (NA), the definition of NA also includes the term speed of lexical access. A possible distinction is whether the task involves visual stimuli (NA) or not (LA).

Visual processing. Eleven narrow abilities are considered empirically well supported (Schneider & McGrew, 2012). The first ability, Visualization (VZ), is described as the most

dominant visual processing ability. It refers to the utilization of mental manipulation, such as rotation or transformation, in order to imagine how figures or patterns may appear. This skill is contrary to the second ability, Speeded Rotation or Spatial Relations (SR), in that it is not a measure of fluency. SR, then, refers to the ability to mentally rotate figures with speed and accuracy. Closure Speed (CS) is described as the ability to recognize visual stimuli that is incomplete in some aspect. The fourth ability, Flexibility of Closure (CF) refers to the skill whereby an individual is able to recognize a pattern of object by ignoring extraneous visual information. Similar to auditory processing, visual processing contains a narrow ability with a memory component, named Visual Memory (MV). MV is defined as the ability to store a complex visual image in memory and recall or recognize it after a short delay. The sixth ability, Spatial Scanning (SS) is the ability to visualize a route out of a maze or visual field. The seventh, Serial Perceptual Integration (PI) refers to the ability to recognize a complete object after pieces of the object are presented in rapid order. The next three abilities are relatively straightforward: Length Estimation (LE) is simply the ability to visually estimate or judge the length of an object; Perceptual Illusions (IL) is the ability to resist visual illusions; and Perceptual Alternations (PN) is the rate at which one can switch between alternating visual perspectives, rather than becoming fixed on one perspective. The final narrow ability, Imagery (IM) refers broadly to the ability to mentally visualize complex visual images and spatial location.

The Woodcock Johnson Battery

As stated previously, contemporary measures of intelligence have demonstrated alignment with multiple broad abilities (Keith & Reynolds, 2010). The *Woodcock Johnson Tests of Cognitive Abilities* is unique in that it has been developed and evaluated within the context of CHC theory since the second edition of the instrument, the *Woodcock-Johnson Psycho-*

Educational Battery-Revised (WJ-R; Woodcock, Johnson, & Mather, 1989). Much previous research has been conducted on the subsequent edition, the *Woodcock Johnson Tests of Cognitive Abilities, Third Edition* (WJ III; Woodcock et al., 2001). The fourth edition of the instrument, published in 2014, provides factor analytic validation to support its adherence to contemporary CHC theory. The subtests of the WJ IV COG and the associated broad and narrow abilities are outlined in Table 2 below. Additionally, the relevant broad and narrow abilities are provided for the *Woodcock-Johnson IV Tests of Oral Language*, a cognitive linguistic battery (WJ OL; Schrank, Mather, & McGrew, 2014b; see Table 3). In viewing the associations between cognitive subtests and CHC abilities, two important conclusions can be made. First, despite the theoretically-based nature of the battery, not all hypothesized narrow abilities are represented in this comprehensive intelligence test. Second, the overlap between distinct narrow abilities, and even broad abilities, is apparent for a variety of tasks.

Table 2.2

WJ IV COG Subtests and Abilities

<u>Subtest Name</u>	<u>Broad Ability</u>	<u>Narrow Ability</u>
1. Oral Vocabulary	Gc	Lexical Knowledge (VL) Language Development (LD)
2. Number Series	Gf	Quantitative Reasoning (RQ) Induction (I)
3. Verbal Attention	Gwm	Working Memory Capacity (WM) Attentional Control (AC)
4. Letter-Pattern Matching	Gs	Perceptual Speed (P)
5. Phonological Processing	Ga	Phonetic Coding (PC) Word Fluency (Glr-FW) Speed of Lexical Access (Glr-LA)
6. Story Recall	Glr	Meaningful Memory (MM) Listening Ability (Gc-LS)
7. Visualization	Gv	Visualization (Vz)
8. General Information	Gc	General Information (K0)
9. Concept Formation	Gf	Induction (I)
10. Numbers Reversed	Gwm	Working Memory Capacity (WM) Attention Control (AC)

11. Number-Pattern Matching	G _s	Perceptual Speed (P)
12. Nonword Repetition	G _a	Phonetic Coding (PC) Memory for Sound Patterns (UM) Memory Span (G _{wm} -MS)
13. Visual-Auditory Learning	G _{lr}	Associative Memory (MA)
14. Picture Recognition	G _v	Visual Memory (MV)
15. Analysis-Synthesis	G _f	General Sequential Reasoning (RG)
16. Object-Number Sequencing	G _{wm}	Working Memory Capacity (WM)
17. Pair Cancellation	G _s	Perceptual Speed (P) Spatial Scanning (G _v -SS) Attentional Control (G _{wm} -AC)
18. Memory for Words	G _{wm}	Memory Span (MS)

Table 2.3

WJ IV OL Subtests and Abilities

<u>Subtest Name</u>	<u>Broad Ability</u>	<u>Narrow Ability</u>
1. Picture Vocabulary	G _c	Lexical Knowledge (VL) Language Development (LD)
2. Oral Comprehension	G _c	Listening Ability (LS)
3. Segmentation	G _a	Phonetic Coding (PC)
4. Rapid Picture Naming	G _{lr}	Naming Facility (NA) Speed of Lexical Access (LA)
5. Sentence Repetition	G _{wm} G _c	Memory Span (MS) Listening Ability (LS)
6. Understanding Directions	G _{wm} G _c	Working Memory Capacity (WM) Listening Ability (LS)
7. Sound Blending	G _a	Phonetic Coding (PC)
8. Retrieval Fluency	G _{lr}	Speed of Lexical Access (LA) Ideational Fluency (FI)
9. Sound Awareness	G _a	Phonetic Coding (PC)

One benefit of a theoretically-based test of cognitive abilities is that it facilitates research identifying the cognitive correlates that underlie various academic skills (Schrank, Miller, Wendling, & Woodcock, 2010). Research comparing subtests on the *Woodcock Johnson Tests of Achievement, Third Edition* (WJ III ACH; Woodcock et al., 2001) and the cognitive battery has

provided insight into the relationships between various cognitive factors and academic skills. Research focusing specifically on math achievement is discussed below.

Math Achievement Using Versions of the WJ COG and ACH

Floyd et al. (2003) investigated the relationship between cognitive variables and math achievement using the WJ III standardization sample. The analysis focused on identifying the cognitive abilities related to math calculation and math reasoning skills using multiple regression analyses. The Math Calculation composite on the WJ III consists of two subtests, Math Fluency (a timed test of simple math facts) and Calculation (an untimed test of simple and complex computations). Results showed that math calculation had a strong relationship with processing speed (*Gs*) from age 7 to 15. Moderate relationships were found with auditory processing (*Ga*) from ages 6 to 7, long term retrieval (*Glr*) from ages 6 to 8 and crystallized intelligence (*Gc*) between ages 10 and 19. These results may reflect the transition between early years where math facts are not yet automatized, but are counted out or solved by recalling newly learned strategies, to later years when children have committed math facts to memory.

A second study using the Math Calculation composite of the WJ III was conducted by Proctor et al. (2005). This study examined the differences between the cognitive profiles of 68 school-age low achievers in math (children with standard scores below 85) and typically achieving peers (standard scores above 90). Interestingly, there were no significant differences between the cognitive profiles of children in the low achieving group and children in the typical group. The researchers commented that the low achieving children represented a diverse group, with half of the children exhibiting one deficit in a cognitive domain with no clear pattern. The researchers also hypothesized that these findings may be related to non-cognitive factors

resulting in poor achievement, the exclusion of low achievers in math who were also low achievers in reading, or due to the true heterogeneity of low achieving students as a group.

More recently, Taub et al. (2008) utilized structural equation modeling to compare the relationships between cognitive abilities and math achievement. Math achievement was represented by the Quantitative Knowledge composite, a score derived from performance on the Math Calculation and Applied Problems subtests. Direct effects of fluid reasoning (*Gf*), crystallized intelligence (*Gc*), and processing speed (*Gs*) were observed across all age groups. However, as math skills were combined into one composite, unclear is the extent to which abilities are specifically related to math calculation and/or math fluency skills.

McGrew and Wendling (2010) summarized the extant research on cognitive and achievement relationships in the context of CHC theory. Based on an analysis of this research, the authors determined the following broad abilities have medium significance in the prediction of math calculation skill: comprehension knowledge (*Gc*), fluid reasoning (*Gf*) and processing speed (*Gs*). In terms of narrow abilities, phonological processing (*Ga-PC*) had a medium significance for children ages 6-13, and perceptual speed (*Gs-P*) had a high level of significance for all ages, as did working memory (*Gsm-MW*). As mentioned previously, the significance of phonological processing impacting math calculation in young children may reflect the role of counting in early computational skills. The authors note that the process of counting requires the retrieval of phonological codes. The perceptual speed finding was hypothesized to reflect a skill in subitizing or in instantly recognizing the value of numbers. Finally, working memory as a predictor of calculation ability is consistent with results of previous research both using the WJ and other measures of working memory.

One key conclusion from this summary was the importance of further research demonstrating the relationship between narrow cognitive abilities and achievement. The results of this study show that narrow abilities may sometimes be significant, even when the broad ability is not (e.g., a significant finding for *Ga-PC*, but not *Ga*). Understanding the relationships among narrow abilities and areas of achievement is also beneficial for both practitioners and researchers in determining tasks that can accurately assess for areas of particular strength and weakness.

Proposed Study

At present, there are no published studies using the Woodcock Johnson assessment battery that have investigated the relationship between cognitive abilities and math fact fluency. The theoretically and psychometrically improved WJ IV assessment battery provides the opportunity to examine this academic skill in the context of contemporary CHC theory. Given that recent research has identified math fact retrieval as a specific area of weakness for some students, it is important to understand the cognitive correlates associated with this skill to inform assessment and intervention. With the increasing popularity of academic and cognitive screening measures in schools to identify children for targeted services, it is important that educators are selecting measures that assess the fundamental cognitive predictors of later academic performance. Screening instruments that reflect known cognitive correlates of academic difficulties are necessary to improve the accuracy in which children are identified for services and to ensure that assessment practices are time and cost efficient. Furthermore, understanding the abilities related to math fluency can inform the development of evidence-based interventions by tailoring the skills taught in order to remediate specific weaknesses. With knowledge of the

relationships among cognitive abilities and achievement, interventions may also be designed to teach students alternative strategies that circumvent areas of weakness.

Studies demonstrating a link between math fluency deficits and phonics, rapid automatized naming, and executive control have revealed a complex array of factors that may contribute to such deficits. Using a child-aged subgroup of the standardization sample, the proposed research will utilize path analysis to examine the relationships between broad and narrow cognitive abilities and math fact fluency. The math fact fluency test of the WJ IV ACH assesses an individual's ability to quickly and accurately answer simple addition, subtraction, and multiplication problems within a three minute time period. This research seeks to clarify the relationships between math fluency and fluid reasoning, crystallized intelligence, working memory, and processing speed, as well its relationship with perceptual speed, number facility, phonetic coding, naming facility, and attentional control. A hypothesized relationship between math problem solving abilities and math fact fluency performance will also be evaluated.

CHAPTER III

METHODS

Sample and Participants

Data Source

A formal request to obtain the WJ IV COG, OL, and ACH standardization data was submitted to Houghton Mifflin Harcourt. The request included information regarding the name and purpose of the study and the proposed method. Following approval, the primary investigators signed a licensing agreement outlining the conditions of use. Subsequent to Duquesne University Institutional Review Board (IRB) approval, an electronic SPSS file containing the de-identified data was transmitted to the primary investigator.

The WJ IV technical manual (McGrew et al., 2014) details the normative sample, which represents a large, geographically diverse stratified sample aligned with demographics of the 2010 U.S. census. The sampling process was stratified based on the following variables: census region, sex, country of birth, race, ethnicity, community type, parent education, type of school (K-12 sample), type of college (college sample), educational attainment (adult sample), and employment status (adult sample). Data were collected from 7,416 individuals aged 24 months to 90 years. Within this group, there were 3,891 students from kindergarten through grade 12. The authors note that the school-age sample was purposefully denser than other age groups, as childhood and adolescence represent a time of growth in cognitive and academic skills.

Procedures Used to Collect the Standardization Data

Data for the standardization sample were collected over 25 months. The assessment batteries were administered by recruited examiners who demonstrated proficiency during training on the administration of the WJ IV battery. Completed protocols were reviewed by

project staff for completion and accuracy. Due to the lengthy nature of the battery, which included 18 cognitive tests, nine oral language test, 20 achievement tests, and four research tests, a planned incomplete data collection design was used. Specifically, the authors utilized a Multiple Matrix Sampling (MMS) design. The use of MMS in large-scale data collection is prevalent across educational, health, and business research and has more specifically been evaluated in terms of cognitive assessment (Rhemtulla & Little, 2012). The normative study employed a partial matrix sampling plan, which involved the administration of a core group of subtests to each participant along with an additional set of subtests administered only to certain groups of participants. McGrew et al. (2014) report that 18 tests were chosen for the core group based on previous WJ-R and WJ III research indicating strong representativeness of the broad CHC cognitive factors, and the academic areas of reading, math and writing. Complete records for each participant were generated by a statistical software program utilizing a Bayesian Multiple Imputation method. Ten complete data sets were generated, and one was randomly chosen as normative sample.

From here, weights were added for individual examinees to correct for differences between the sample and the demographic characteristics of the U.S. population. The construction of test norms utilized a bootstrap resampling procedure. Simply stated, this procedure involves the computerized generation of additional samples to the original pool in order to estimate the population distribution. The authors state that 250 resamples were generated for the norming process. Using this data, percentiles were calculated and norm curves were generated, allowing for the calculation of standard scores and percentile ranks.

Participants

For the present study, a total of 4,212 children and adolescents aged 6:0 to 19:11 were

included in this study's analyses. This age group was chosen for the purpose of comparing the current investigation to previous research utilizing the same age range (e.g., McGrew & Wendling, 2010). Additionally, this age group would allow comparison of the current results to those presented in the WJ IV technical manual (McGrew et al., 2014).

Measures

The WJ IV battery is a comprehensive assessment system used to measure the cognitive abilities, oral language, and academic achievement of individuals ages 2 to 90+ years old. The fourth edition of the Woodcock Johnson system was created to reflect updates to the most current theory of overall intelligence according to contemporary CHC theory. Improvements to the new edition of the WJ IV COG include increased cognitive complexity on subtests, higher correlations with ability scores on the WISC-IV, and a revised theoretical basis for the inclusion of working memory and memory of sound patterns as cognitive constructs (McGrew et al., 2014). The WJ IV ACH also underwent revisions to the structure of the battery, with new subtests and clusters representing an increased breadth of content.

Another major revision of the WJ IV battery is the addition of a separate battery assessing language specific abilities, the WJ IV Tests of Oral Language. The OL measure is comprised of language-based tasks formerly found on the WJ COG and ACH, as well as newly adopted tasks. The battery can be used to provide diagnostic information about specific cognitive-linguistic abilities as well as achievement in expressive language and listening comprehension. The reliability and validity of these batteries is discussed below.

Reliability

Standard error of measurement (SEM) values and reliability coefficients were reported for all subtest and cluster scores within the WJ IV battery. In the context of Classical Test

Theory, reliability coefficients (r_{11}) represent the ratio of true score variance to observed score variance (Raykov & Marcoulides, 2011). Reliability coefficients are reported as decimals between 0 and 1, with values closer to 1 representing more precise measurement. The reliability coefficient for each non-timed test is reported for each age group (for each age 2-19 individually and groups of 10 thereafter; e.g., 20-29, 30-39, etc.), in addition to the test's median reliability coefficient. Thirty-eight of the 39 non-timed tests of the WJ IV had median reliability coefficients above the desired .08 cutoff. Reliability coefficients ranged from .74 to .97 (McGrew et al., 2014), with Picture Recognition (WJ COG 14; $r_{11} = .74$) representing the only test to fall below .80. For eight of the timed tasks of the WJ IV, test-retest reliability was calculated (r_{12}). The reliability coefficients ranged from .76 to .95, with six of the eight tests above .08. Notably, the test-retest reliability of the Rapid Picture Naming subtest fell just below the .80 threshold, with a value of .79.

The WJ IV battery provides a number of cluster scores that can be obtained representing general cognitive ability (General Intellectual Ability, Brief Intellectual Ability, *Gf-Gc* Composite), broad abilities (*Gc*, *Gc*-extended, *Gf*, *Gf*-extended, *Gwm*, *Gwm*-extended, *Gs*, *Ga*, *Glr*, *Gv*), as well as several other narrow abilities, conceptual groupings, and academic clusters (e.g., perceptual speed, cognitive efficiency, reading comprehension). Reliability coefficients of the cluster scores ranged from .86 to .99, with the majority of coefficients exceeding .90.

Alternate forms reliability was reported for the speeded tests on each of the three alternate WJ ACH forms. Alternate forms reliability coefficients across all speeded tasks ranged from .76 to .96. The Math Facts Fluency subtest had reliability coefficients equal to .95 and .97, in a sample of students aged 7-11 and 14-17, respectively. A second evaluation of the speeded ACH tests was conducted to ensure that item difficulty was equivalent across the three forms.

The three forms of Math Facts Fluency subtest was administered in counterbalanced order to a group of students in grades 3 and 4 and a group of students in grades 9 through 12. The correlations among forms ranged from .94-.95 in the younger grades and .92-.94 in the older grades. An examination of the Sentence Reading Fluency subtest with these age groups revealed correlations that ranged from .85-.88. The non-speeded tests of the WJ IV ACH were systematically evaluated for equivalent item content and difficulty across three alternate forms.

Validity

The WJ IV technical manual (McGrew et al., 2014) provides evidence for the content, internal, and concurrent validity of the assessment. As mentioned above, the WJ IV battery was developed in the context of contemporary CHC theory in a manner similar to its predecessors (WJ III, WJ-R). Subtests were designed to measure a single narrow ability, and clusters were constructed by combining subtests measuring distinct narrow abilities to provide sufficient coverage of the theoretical broad ability. While the WJ COG and OL tap the seven broad abilities named previously (*Gc*, *Gf*, *Gwm*, *Gs*, *Ga*, *Glr*, *Gv*), the ACH battery measures the broad abilities of quantitative knowledge (*Gq*), reading and writing (*Grw*), and domain specific knowledge (*Gkn*), with the goal of providing broad coverage of academic skills relevant to needs of individuals conducting psychoeducational assessment. McGrew et al. (2014) state that many of the WJ IV subtests that appeared in the WJ III have been supported through independent evaluations in the context of CHC theory.

Content Validity. For the WJ IV, the authors conducted multidimensional scaling (MDS) analyses to demonstrate the content validity of the constructs measured by the battery. Using the correlations among 51 WJ IV subtests (including research tests), the authors utilized a Guttman Radex two-dimensional MDS procedure for the analysis of six different age groups (3-

5, 6-8, 9-13, 14-19, 20-39, 40-90+). A summary of these analyses revealed that the constructs tended to cluster similarly to the organization of the three batteries. The ACH battery fell into *Grw* and *Gq* domains in the upper right quadrant, while the OL battery fell into an auditory-linguistic grouping in the upper left. The COG battery spanned the lower left and right quadrants (figural-visual and cognitive efficiency speed), in addition to the upper left quadrant (auditory-linguistic). Interestingly, while speeded academic measure fell into the same conceptual grouping as the processing speed measures (the speed-fluency group), the academic fluency measures clustered separately within the upper right quadrant along with the other ACH measures.

Internal Validity. To demonstrate the internal structure of the WJ IV, the test authors generated the intercorrelations among all tests and clusters, again using the six age ranges described previously. Correlations were higher among tests that fall into related CHC domains (e.g., one *Gc* test is more highly correlated with a second *Gc* test than a *Gs* test). The same pattern was found for areas of achievement that fall in similar compared to dissimilar domains.

Second, the authors completed a three-step psychometric evaluation of the internal structure of the WJ IV utilizing exploratory and confirmatory factor analysis. The first step involved a split-sample random sample generation. Again dividing the sample into six age ranges, the groups were randomly split in half. The first group was named the model development group. This group served as the sample for the exploratory analyses. The first exploratory analysis was a cluster analysis (CA). The authors utilized Ward's hierarchical minimum variance CA, which is a technique that sorts highly correlated variables (i.e., tests), which are then merged into larger groups. Groupings aligned with the CHC framework, with all seven broad cognitive factors and two achievement factors (*Gq* and *Grw*) present. Just as in the

MDS analysis, the ACH fluency measures once again fell in the broad cognitive domain of *G_s*, however, they were organized in a distinct group from the *G_s*-COG tests.

Next, an exploratory principle components analysis (PCA) was conducted with the model development age groups. The authors state that varimax rotation was chosen due to the high degree of multicollinearity existing amongst the tests, in addition to other complicating factors of oblique solutions, including the potential removal of tests in the final solution (McGrew et al., 2014). The number of components extracted ranged from six to 10, which was informed by the results of the CA. The 8, 9, and 10 factor solutions were determined to be most interpretable. In summary of all the analyses, the broad cognitive factors of *G_c*, *G_s*, and *G_{wm}* appeared throughout. Additionally *G_f* appeared in a mixed COG-ACH factor alongside *G_q*. Finally, *G_{rw}* appeared as a standalone ACH factor. The broad cognitive ability *G_a* appeared in the nine and 10 factor solutions, while the remaining broad abilities *G_{lr}* and *G_v* appeared in varying forms across the solutions, either combined with a other broad abilities or represented by subsumed narrow abilities (e.g., *G_{lr}*-Retention abilities, *G_v*-MV/MA).

A third exploratory analysis was conducted on the model development groups using MDS. The results of this analysis are similar to those described above in regards to content validity. Using the results of these three exploratory analyses, three models to be used in confirmatory analysis (CFA) were generated. The first model was a single factor model representing the latent factor *g*. The second model was a top-down model including *g* as a higher order factor and the broad cognitive and achievement abilities *G_c*, *G_{rw}*, *G_f*, *G_s*, *G_q*, *G_v*, *G_{lr}*, *G_{wm}*, and *G_a*. The third model was a bottom-up model including three tiers: narrow abilities, broad abilities, and *g*. Narrow abilities that were not supported were removed in favor of the related broad ability. Four narrow abilities were included in the final model: Verbal Language-

Based Reasoning (*Gf-Vbl*), Quantitative Reasoning (*Gf-RQ*), Associative Memory (*Glr-MA*), and Speed of Lexical Access (*Glr-LA*). Results indicated that model one did not fit the data as well as model two or three. All parameters in models two and three were deemed positive, significant, and meaningful (McGrew et al., 2014).

The third stage involved conducting a CFA using the model cross-validation samples to compare to the model development samples. Both samples demonstrated very similar fit statistics. Model fit indices were provided for models two and three across all six age groups using both samples. The authors concluded that model two and three demonstrated comparable fit for all ages 6 and above (McGrew et al., 2014). In terms of fit, the models resulted in RMSEA (Root Mean Squared Error of Approximation) values ranging from 0.115 to 0.123 across age groups, which is higher than desired (.05 or below). Additionally, the TLI (Tucker-Lewis non-normed fit index) values ranged from 0.607 to 0.684, which is lower than desired (.90 or above). However, both of these fit indices are reliant on maximum-likelihood estimation, which is sensitive to violations of assumptions, including multivariate normality and multicollinearity. As the data violates these assumptions, scale-free least squares (SFLS) fit indices were calculated. The Adjusted Goodness of Fit Index (AGFI) calculated using SFLS resulted in acceptable fit values (.956-.984).

Concurrent Validity. An investigation of the concurrent validity of the WJ IV battery was conducted using various cognitive, oral language, and achievement measures. In an evaluation of 174 students, correlations between the WJ IV COG and the WISC-IV demonstrated a strong (.86) relationship between the overall intelligence as measured by the two assessments. In addition, high correlations were found between general intellectual ability on the WJ COG and the equivalent metric on the KABC-II (.77) and the SB-V (.80). Validation of the WJ OL

was performed with two commonly used language measures, the *Clinical Evaluation of Language Fundamentals, Fourth Edition* (CELF-4; Semel, Wiig, & Secord, 2003), and the *Peabody Picture Vocabulary Test, Fourth Edition* (PPVT-4; Dunn & Dunn, 2007). In general, the clusters available on the WJ OL demonstrated moderate to high correlations (.60 to .70) with the CELF-4 composites. However, exclusions were noted for the working memory composite, which had low correlations with all measures in the older children group (10-18), and the speed of lexical access domain. The authors note that the CELF-4 does not have comparable measure on the latter ability. The correlations between the OL and the PPVT-4 resulted in a similar pattern of generally moderate to high correlations. In terms of achievement, correlations among the WJ ACH and the *Wechsler Individual Achievement Test, Third Edition* (WIAT-III; Wechsler, 2009) show moderate to strong relationships between related academic clusters (.60 or above). The math fluency composite of the WIAT-III was highly correlated (.85) with the math calculation cluster on the WJ ACH.

Research Questions and Hypotheses

Research Question 1

Within the child age subset of the standardization sample, which broad cognitive abilities display significant effects on Math Facts Fluency performance?

Hypothesis 1: It is hypothesized that fluid reasoning will have a direct effect on Math Facts Fluency.

Hypothesis 2: It is predicted that comprehension-knowledge will have a direct effect on Math Facts Fluency.

Hypothesis 3: Working memory is hypothesized to have a direct effect on Math Facts Fluency.

Hypothesis 4: Processing speed is predicted to have a direct effect on Math Facts Fluency.

Hypothesis 5: It is hypothesized that General Intellectual Ability will have an indirect effect on Math Facts Fluency performance.

Research Question 2

Which narrow abilities have significant effects on performance on the Math Facts Fluency subtest?

Hypothesis 1: It is hypothesized that perceptual speed will have a direct effect on Math Facts Fluency.

Hypothesis 2: Number facility will have a direct effect on Math Facts Fluency.

Hypothesis 3: Phonetic coding will have a direct effect on Math Facts Fluency.

Hypothesis 4: It is predicted that naming facility will have a direct effect on Math Facts Fluency.

Hypothesis 5: It is hypothesized that attentional control will have a direct effect on Math Facts Fluency.

Research Question 3

What relationship will math problem solving abilities have with math fluency performance?

Hypothesis 1: It is hypothesized that performances on the Math Problem Solving cluster will have a direct effect on Math Facts Fluency performance.

Data Analysis

This study used path analysis, a statistical technique that falls in the Structural Equation Modeling (SEM) family. Path analysis is unique to other SEM techniques in that it may be used

evaluate only manifest or observed variables (i.e., variables that have been directly measured), rather than observed and latent variables (Kline, 2011). However, path analysis has the primary benefit of allowing the investigation of direct and indirect effects, which is lacking in other manifest variable techniques, such as multiple regression. A second benefit of SEM techniques, including path analysis, is the ability to revise and reevaluate the hypothesized model to ensure the closest fit to the data.

Two models were evaluated using path analysis. The first model included the broad cognitive factors hypothesized to have effects on math fact fluency performance, as well as General Intellectual Ability representing a high-order ability. This model is depicted in Figure 1. The subtests comprising the broad cognitive factors of the WJ IV COG are displayed in Table 4. Data included all broad ability composite scores, which were utilized in the analysis.

Table 3.1

Tests Included in the Broad Abilities

<u>Broad Ability</u>	<u>Test Number</u>	<u>Associated Test Name</u>
Fluid Intelligence (GF)	COG 2	Number Series
	COG 9	Concept Formation
Comprehension-Knowledge (GC)	COG 1	Oral Vocabulary
	COG 8	General Information
Short-Term Working Memory (GWM)	COG 3	Verbal Attention
	COG 10	Numbers Reversed
Cognitive Processing Speed (GS)	COG 4	Letter-Pattern Matching
	COG 17	Pair Cancellation
Auditory Processing (GA)	COG 5	Phonological Processing
	COG 12	Nonword Repetition
Long-Term Retrieval (GLR)	COG 6	Story Recall
	COG13	Visual-Auditory Learning

The second path analysis model evaluated included the narrow abilities thought to be related to math fact fluency. This second model is shown in Figure 2. The composition of the narrow abilities is provided in Table 5. Composite scores for Perceptual Speed (PERSPD),

Number Facility (NUMFAC), and Phonetic Coding (PHNCOD) were all provided within the dataset. The hypothesized narrow ability called Rapid Picture Naming (RPCNAM) was measured by just one subtest; therefore this subtest was used to represent this narrow factor. No composite for Attention Control (ATTCTRL) is currently available. However, according to the WJ IV manual, three subtests: Verbal Attention, Numbers Reversed, and Pair Cancellation, measure this narrow ability. For the current study, an Attention Control composite score was created using the subtests Verbal Attention and Pair Cancellation. The Numbers Reversed subtest was excluded, as it appears in the Number Facility (NUMFAC) factor. The Attentional Control composite is therefore a mean of the two aforementioned subtests. Although this process resulted in decreased variance as compared to the other composite scores, it allowed all variables to exist on a common standard score scale. In order to answer the final research question, the Math Problem Solving cluster, which is comprised of the applied problem and number matrices subtests on the WJ ACH, was added to the best fitting model.

Table 3.2

Tests Included in the Narrow Abilities

<u>Narrow Ability</u>	<u>Test Number</u>	<u>Associated Test Name</u>
Perceptual Speed (PERSPD)	COG 4	Letter-Pattern Matching
	COG 11	Number-Pattern Matching
Number Facility (NUMFAC)	COG 10	Numbers Reversed
	COG 11	Number-Pattern Matching
Phonetic Coding (PHNCOD)	OL 3	Segmentation
	OL 7	Sound Blending
Rapid Picture Naming (RPCNAM)	OL 4	Rapid Picture Naming
Attention Control (ATTCTRL)	COG 3	Verbal Attention
	COG 17	Pair Cancellation

Data were first imported into SPSS Version 24 for preliminary analyses. The dataset was first analyzed for missing data. Cases with missing data on any of the variables within the

models were excluded from analysis. All variables within the dataset are provided as standard scores, with a mean of 100 and a standard deviation of 15. As described below, data were examined for univariate and multivariate normality, as well as univariate outliers. Additionally, data were inspected for collinearity and homoscedasticity. Next, the dataset was imported into the R Studio program, running R version 3.1.2 along with the lavaan 0.5-17 package (Yves Rosseel, 2012).

Estimation Method

Within structural equation modeling techniques, the estimation method refers to the algorithm used to generate parameter estimates. The Maximum Likelihood (ML) method is suitable for most studies and is the default method in R (Kline, 2011). However, ML has strict requirements. First, it is a full-information method, meaning that all parameter estimates are calculated simultaneously; therefore, ML requires a complete dataset, absent of missing data. The ML method also assumes that all variables are continuous variables that are normally distributed. According to Kline (2011), ML may result in inaccurate estimates if variables are standardized. Therefore, an alternative estimation method was chosen for this study. Fully Weighted Least Squares (WLS) estimation is a family of methods that provide alternatives to ML. Within the lavaan package, alternative methods include Generalized Least Squares (GLS), Weighted Least Squares (WLS), Diagonally Weighted Least Squares (DWLS), and Unweighted Least Squares (ULS). Additionally, robust variations can be used that correct for non-normal standard errors. Given the standardized scores within the dataset, the WLSMVS (Weighted Least Squares with robust standard errors and a Mean and Variance adjusted test statistic). This method uses the Satterthwaite approach, which does not assume equal variances (Satterthwaite, 1946). In R, fit statistics are provided using a Diagonally Weighted Least Squares (DWLS)

partial-information method as well as the robust formula that accounts for mean and variance adjustments. The latter corrects for bias in the fit statistics inherent in utilizing a partial-information method (Li, 2016).

Alternative Models

After accepting a final model, one must consider the possibility that an alternative model may explain the data equally well. Models that produce equivalent fit indices are referred to as equivalent models (Kline, 2011). It is incumbent on the researcher to identify equivalent models and provide an argument for the theoretical model. However, near-equivalent models, those that produce a similar, but different, covariance matrix may just as easily prove a threat to the proposed model (Kline, 2011). The first alternative model considered aligns with the Cattell and Horn theory of intelligence, which posits that Fluid Reasoning (GF) and Comprehension-Knowledge (GC) are at the crux of general ability. Although the Cattell-Horn-Carroll aligns other broad abilities, such as Working Memory and Processing Speed, on the same stratum of ability, some remnants of this theory remain. For example, the most recent iteration of the WISC-V continues to put the greatest weight on Verbal Comprehension and Fluid Reasoning when calculating the Full Scale IQ (Wechsler, 2014). Therefore, the first alternative model identified Comprehension Knowledge and Fluid Reasoning as higher-level abilities between General Ability and Working Memory and Processing Speed. Paths remained between Math Facts Fluency and all other variables (see Figure 3).

A second alternative model was created by changing the directionality between GIA and the broad factors. This model was generated to test the theory that GIA is the most important predictor of math fluency performance. Within this model, the broad variables were positioned as higher-order variables that contribute to general ability (see Figure 4).

CHAPTER IV

RESULTS

Preliminary Statistics

Missing Data and Outliers

As stated previously, the purpose of this study was to investigate relationships among cognitive abilities and math fact fluency performance in a child sample. The sample provided contained 4,212 cases, which is the number of children aged 6 through 19 within the normative sample. Of this sample, 23 cases contained missing data on at least one of the variables within the study and were removed from the dataset. Next, the data were examined for univariate and multivariate outliers in SPSS. Multivariate outliers were identified in a regression analysis by calculating the Mahalanobis Distance statistic for each case. Because of the large sample size ($n > 500$), a chi-square distribution was used to determine the cut-off value of 32.910 ($p = .001$). Forty-nine cases were determined to be significant multivariate outliers and were excluded from the sample. Data were also analyzed for univariate outliers. Each variable contained extreme values (absolute value of 3 SD greater than the mean), representative of a standard score less than 55 or above 145. However, when these cases were removed, a visual analysis of the q-q plots indicated that the variables deviated from normal at the extreme ends of the distribution. Therefore, these cases were included in the final sample. For all analyses, the sample size was 4,140. The mean age of final sample was 12.32 years ($SD = 3.98$).

Normality and Homoscedasticity

All variables were examined for univariate normality. Significant skew was characterized by any variable that exceeded 3.0 on the ratio of the skewness statistic to the standard error statistic (Kline, 2011). General Intellectual Ability (GIASTD) showed a significant negative

skew (-5.00), as did Perceptual Speed (PERSPD; -3.32). Additionally, the Perceptual Speed variable somewhat was somewhat leptokurtic (kurtosis ratio = 8.38); however, this values falls below 10.0, which has been described as a conservative metric (Kline, 2011). All other variables had acceptable skewness and kurtosis values. Because the estimation method used in this study is robust to violations of normality, variables evidencing significant skew were not transformed. Additionally, the retention of the current distribution allows for comparison between variables, which all contain standard scores. Means and standard deviations for all variables in the study are presented in Table 6.

Table 4.1

Descriptive Statistics for All Variables

	Mean	Std. Error	Std. Deviation	Variance
General Intellectual Ability (GIASD)	99.922	.239	15.393	236.946
Comprehension-Knowledge (GC)	100.290	.240	15.437	238.302
Fluid Reasoning (GF)	99.881	.240	15.435	238.247
Short-Term Working Memory (GWM)	100.787	.239	15.374	236.355
Cognitive Processing Speed (GS)	99.796	.236	15.167	230.040
Number Facility (NUMFAC)	100.310	.242	15.566	242.314
Perceptual Speed (PERSPD)	99.834	.240	15.427	238.005
Attentional Control (ATTCTRL)	100.295	.190	12.194	148.691
Phonetic Coding (PHNCOD)	100.334	.243	15.628	244.229
Rapid Picture Naming (RPCNAM)	100.189	.238	15.329	234.975
Math Problem Solving	100.591	.238	15.335	235.161

(MTHPRB)				
Math Facts Fluency	99.994	.246	15.798	249.562
(MTHFLU)				

Finally, to examine homoscedasticity, two linear regressions were conducted using the variables associated with the broad and narrow models. Within these analyses, normality probability plots of the standardized residuals both showed a linear pattern of distribution. Additionally, frequency charts of the standardized residuals both followed a normal distribution.

Correlation Analysis

Bivariate correlations were calculated for all variables included within the study (see Table 7). All correlations were significant at the $p = .001$ level. As expected, Number Facility (NUMFAC) and Perceptual Speed (PERSPD), which both contained the Number-Pattern Matching subtest, were highly correlated (.851). Within the narrow factors model, this collinearity was accounted for by a covariance path between the two variables. Additionally, Processing Speed (GS) and Perceptual Speed were also highly correlated (.843). Both variables also share a common subtest (Letter Pattern Matching). These variables were not within analyzed within the same model, as one represents a broad factor and the other represents a narrow factor. Finally, Working Memory (GWM) also exhibited strong correlations with Number Facility (.738) and Attentional Control (.727). Again, both narrow factors contain a shared subtest with the broad factor (Numbers Reversed and Verbal Attention, respectively), although collinearity was avoided by separating broad and narrow factors into separate models.

General Intellectual Ability (GIASD) had strong correlations with Fluid Reasoning (GF), Working Memory, Number Facility, Attentional Control, and Math Problem Solving (MTHPRB), as well as moderate correlations with all other variables. This is consistent with CHC theory, which posits that General Intellectual Ability is a higher-level ability that

encompasses all broad and narrow abilities. Notably, although Math Problem Solving was strongly correlated with General Intellectual Ability and Fluid Reasoning, Math Fluency was only moderately correlated with these variables. In fact, Math Facts Fluency was moderately correlated with all variables included in the study, with the exception of Phonetic Coding (PHNCOD; .201) and Rapid Picture Naming (RPCNAM; .314).

Table 4.2

Bivariate Correlations for All Study Variables

	1	2	3	4	5	6	7	8	9	10	11	12
1. GIASTD	1	.678	.807	.735	.622	.703	.682	.719	.559	.374	.775	.629
2. GC		1	.486	.470	.327	.366	.314	.434	.404	.288	.577	.436
3. GF			1	.536	.406	.557	.474	.474	.462	.299	.768	.513
4. GWM				1	.419	.738	.469	.727	.414	.337	.552	.445
5. GS					1	.656	.843	.754	.291	.384	.430	.550
6. NUMFAC						1	.851	.643	.373	.352	.529	.584
7. PERSPD							1	.619	.304	.352	.436	.615
8. ATTCRL								1	.373	.437	.537	.496
9. PHNCOD									1	.218	.446	.201
10. RPCNAM										1	.250	.314
11. MTHPRB											1	.593
12. MTHFLU												1

Note: GIASTD = General Intellectual Ability - Standard; GC = Comprehension-Knowledge; GF = Fluid Reasoning; GWM = Working Memory; GS = Processing Speed; NUMFAC = Number Facility; PERSPD = Perceptual Speed; ATTCRL = Attention Control; PHNCOD = Phonetic Coding; RPCNAM = Rapid Picture Naming; MTHPRB = Math Problem Solving; MTHFLU = Math Fact Fluency. All correlations are significant at the $p < .001$ level.

Results for Research Question 1

The aim of the first model was to examine which broad cognitive abilities exhibit significant effects on Math Fluency performance. The data set (N = 4,140) was imported into R version 3.1.2 running the lavaan 0.5-17 package. Given the large sample size, all analyses have sufficient power (.80) to detect a close-fit (RMSEA <.05; MacCallum, Brown, & Sugawara, 1996). For Model 1, which is depicted in Figure 1, the robust chi-square value, $\chi^2 = 98.755$, $df =$

3.967, was significant ($p = .000$), indicating that the model is statistically discrepant from the covariance matrix (Kline, 2011). Additionally, the results using the DWLS method also resulted in a significant chi-square, $X^2 = 19.941$, $df = 6$, $p = .003$. For the broad factor model, the Root Mean Square Error of Approximation (RMSEA) exceeded the .05 threshold for the close-fit hypothesis, with an RMSEA = .076. The robust Comparative Fit Index (CFI) of .960 indicates acceptable fit (Kline, 2011), but is below the optimal value for rejecting a misspecified model, given the large sample size (Sivo, Fan, Witta, & Willse, 2006). The Standardized Root Mean Square Residual (SRMR) value of .017 indicates that the correlated residuals suggest an adequate fit (Hu & Bentler, 1999), with values closer to 0 denoting better fit. Among the correlated residuals, General Intellectual Ability (GIASD) and Processing Speed (GS) had the greatest disturbance (.056). However, all values fell below the .100 threshold, which is representative of a significance discrepancy between the model and the sample correlation (Kline, 2011).

In order to improve the model fit, the residuals were examined in order to determine where covariance paths could be added. As noted above, the largest correlated residual values occurred between General Intellectual Ability and the broad ability factors, which were already specified with a direct path from GIA to the individual factors. An examination of the remaining variables led to an addition of covariance paths between Processing Speed and Comprehension Knowledge (GC; -.037) and Processing Speed and Fluid Reasoning (GF; -.024). The resulting model is discussed below.

The adjusted broad model resulted in a decrease in the robust chi-Square value, $X^2 = 47.510$, $df = 2.951$). Again, the robust chi-square p value was significant ($p = .000$), although the value computed using the DWLS method was not ($X^2 = 6.381$, $df = 4$, $p = .172$). Within this

model, the RMSEA = .060. The CFI value improved to an acceptable level (.981). Additionally, the SRMR value decreased slightly to .010. An examination of the correlated residuals again indicated all values fell well below .100. The greatest magnitude was now between General Intellectual Ability and Fluid Reasoning (.038).

Path estimates for the final broad factors model are displayed in Table 8. With regard to indirect effects, General Intellectual Ability had significant positive effects on all broad abilities. Strong relationships with GIA were exhibited for Fluid Reasoning (B = .760) and Working Memory (B = .700). General Intellectual Ability evidenced moderate effects on Comprehension-Knowledge (B = .648) and Processing Speed (B = .587). Finally, GIA exhibited a weak, but significant positive direct effect on Math Facts Fluency performance (B = .281). In terms of significant direct effects from the broad abilities to Math Facts Fluency, Processing Speed had a moderate effect (B = .310), Fluid Reasoning had a weak effect (B = .134), and Comprehension-Knowledge had a significant, but weak effect (B = .090). The path from Working Memory to Math Fluency was not significant (B = .007).

Table 4.3

Path Estimates for the Final Broad Factor Model

Path	B	SE	β
GIASD - GC	.648*	.012	.655
GIASD - GF	.760*	.010	.769
GIASD - GWM	.700*	.011	.711
GIASD - GS	.587*	.013	.605
GC - MTHFLU	.090*	.015	.088
GF - MTHFLU	.134*	.017	.131
GWM - MTHFLU	.007	.017	.007
GS - MTHFLU	.310*	.015	.297
GIASD - MTHFLU	.281*	.022	.278

* $p < .01$

Results for Research Question 2

The purpose of the second model was to examine which narrow cognitive abilities exhibit significant effects on Math Fluency performance. Using the dataset from the previous model, the theoretical model shown in Figure 2 was analyzed. For this model, the robust chi-square value, $X^2 = 284.873$, $df = 5.296$, was significant ($p = .000$) as was the DWLS value, $X^2 = 90.566$, $df = 9$, $p = .000$. Fit indices indicated poorer fit than the broad model, with a robust RMSEA value of .113 and a CFI of .883. However, the SRMR was still relatively low (.030) and all correlated residual values fell below .100.

As with the first model, modifications were made to the narrow model by examining the correlated residuals for areas where covariance paths could be added to improve overall model fit. Covariance paths were added incrementally, until all residual correlations values fell below .020. The resulting model included covariance paths between Perceptual Speed and all three remaining narrow variables, Attentional Control, Phonetic Coding, and Rapid Picture Naming. Note that a covariance path between Perceptual Speed and Number Facility was specified in the original narrow model. Additionally, covariance paths were added between Number Facility and Attentional Control, Number Facility and Rapid Picture Naming, and Attentional Control and Rapid Picture Naming.

For the adjusted narrow model, the robust chi-Square value dropped significantly, $X^2 = 7.647$, $df = 2.365$ and was no longer significant at the $p < .01$ level ($p = .031$). Additionally, the DWLS solution resulted in a non-significant p value ($X^2 = 2.094$, $df = 3$, $p = .553$). The model evidenced good fit according to the robust RMSEA of .023 and the robust CFI of .998. Finally, the SRMR also decreased to .004.

Path estimates for the final narrow factors model are displayed in Table 9. Consistent

with the broad factors model, General Intellectual Ability had significant positive effects on all narrow abilities within the model. The indirect effects of GIA were moderate ranging from $B = .699$ (Number Facility) to $B = .376$ (Rapid Picture Naming). Within this model, GIA also had a moderate direct effect on Math Fluency ($B = .481$). The narrow factors all exhibited weak or negligible effects on Math Fluency. Significant positive effects were present for Perceptual Speed ($B = .266$), Number Facility ($B = .091$), and Rapid Picture Naming ($B = .054$). In contrast, Phonetic Coding had a significant negative effect on Math Fluency ($B = -.179$). Finally, the relationship between Attentional Control and Math Fluency was not significant ($B = -.011$). Given the positive correlations identified previously, it is likely that effects of these variables are suppressed. The latter relationship is unsurprising, given the results of the first research question and the similarity between the Working Memory and Attentional Control clusters.

Table 4.4

Path Estimates for the Final Narrow Factor Model

Path	B	SE	β
GIASD - NUMFAC	.699*	.013	.695
GIASD - PERSPD	.677*	.012	.679
GIASD - ATTCRL	.557*	.010	.707
GIASD - PHNCOD	.552*	.015	.546
GIASD - RPCNAM	.376*	.015	.379
NUMFAC - MTHFLU	.091*	.026	.090
PERSPD - MTHFLU	.266*	.024	.260
PHNCOD - MTHFLU	-.179*	.014	-.177
RPCNAM - MTHFLU	.054*	.014	.053
ATTCRL - MTHFLU	-.011	.025	-.009
GIASD - MTHFLU	.481*	.022	.471

* $p < .01$

Results for Research Question 3

After the effects of broad and narrow factors on math fact fluency were explored, the final research question was proposed to examine the relationship between Math Problem Solving

(MTHPRB) and Math Facts Fluency. For this investigation, the best fitting model was chosen. Although the broad model was most parsimonious, the adjusted narrow model exhibited the best fit according to the RMSEA and CFI. Additionally, it was the only model that passed the robust chi-square significance test. Therefore, the Math Problem Solving variable was added to this model for analysis. This model is depicted in Figure 7.

The resulting model failed the robust chi-square test ($X^2 = 70.203$, $df = 4.849$, $p = .000$), although the DWLS value was not significant ($X^2 = 18.664$, $df = 8$, $p = .017$). Compared to the narrow factors model, the new model fit was relatively poorer according to both the RMSEA (.057) and the CFI (.975). The SRMR increased slightly to .012, but was still well within the acceptable range. As with the previous models, the correlated residuals were examined to determine areas where covariance paths could be added. Based on this information, it was determined that a covariance path could be added between Math Problem Solving and Perceptual Speed (-.040).

The following fit statistics describes the math problem solving model given this adjustment. The robust chi-square value decreased, but remained significant ($X^2 = 29.380$, $df = 4.849$, $p = .000$). Again, the DWLS value was not significant ($X^2 = 7.968$, $df = 7$, $p = .335$). The RMSEA of .037 suggested good fit, as did the CFI (.990) and the SRMR (.007).

Path estimates for the final model are displayed in Table 9. As expected, GIA had a large positive effect on Math Problem Solving ($B = .759$). When examining the direct effects, Perceptual Speed ($B = .374$) and Math Problem Solving ($B = .361$) both exhibited moderate positive effects on Math Facts Fluency. In contrast to the preceding model, GIA now showed a weak positive effect on Math Facts Fluency ($B = .187$). Additionally, the Number Facility path was no longer significant ($B = .016$). The paths from Phonetic Coding, Rapid Picture Naming,

and Attention Control maintained the same directionality and approximately the same magnitude.

Table 4.5

Path Estimates for the Final Math Problem Solving Model

Path	B	SE	β
GIASD - NUMFAC	.693*	.013	.690
GIASD - PERSPD	.674*	.012	.677
GIASD - ATTCRL	.551*	.010	.701
GIASD - PHNCOD	.561*	.015	.557
GIASD - RPCNAM	.356*	.015	.360
GIASD - MTHPRB	.795*	.011	.767
NUMFAC - MTHFLU	.016	.025	.016
PERSPD - MTHFLU	.374*	.024	.366
PHNCOD - MTHFLU	-.176*	.013	-.174
RPCNAM - MTHFLU	.059*	.013	.057
ATTCRL - MTHFLU	-.017	.023	-.013
GIASD - MTHFLU	.187*	.025	.184
MTHPRB - MTHFLU	.361*	.018	.350

* $p < .01$

Alternative Models

Two alternative models were generated to compare to the final broad model. The broad model was chosen for comparison, as it was the most parsimonious and has the greatest theoretical support. As stated previously, the first alternative model posited that GF and GC mediated the relationship between GIA and the remaining broad abilities, GWM and GS (see Figure 3). Fit indices indicated the model was misspecified. The model resulted in large, statistically significant chi-square values (Robust $\chi^2 = 1036.416$, $df = 2.638$, $p = .000$; DWLS $\chi^2 = 13176.529$, $df = 15$, $p = .000$). The RMSEA value of .308 indicates extremely poor fit, as does the CFI value of .563; the SRMR was still within the acceptable range (.051). An examination of the parameter estimates indicated that model generated a moderate negative effect of Fluid Reasoning on Math Facts Fluency, which further supports the conclusion that the model is

misspecified.

A second alternative model was generated to test the theory that General Intellectual Ability was the sole contributor to Math Facts Fluency performance. Within this model, it was proposed that GIA mediated the relationship between Fluid Reasoning, Comprehension Knowledge, Working Memory, and Processing Speed (see Figure 4). The robust chi-square value, $\chi^2 = 260.680$, $df = 3.124$, was statistically significant ($p = .000$), as was the DWLS chi-square value, $\chi^2 = 44.664$, $df = 4$, $p = .000$. The robust RMSEA of .141 indicated a poor fit. Similarly, the CFI value of .890 fell below the desired level. Again, the SRMR was acceptable (.024). An examination of the parameter estimates identified that all paths were significant and positive as expected. However, given the poor fit statistics, the model was rejected in favor of the broad factor model outlined in the first research question.

CHAPTER V

DISCUSSION

Math fact fluency, which refers to the accuracy and speed with which one can perform simple arithmetic, is a foundational skill for the development of more complex, high-order mathematic calculations. Research has linked math fact retrieval deficits with lower scores on mathematic assessments, a reluctance to engage in mathematical activities, as well as increased frustration and anxiety related to performing mathematical calculations. Given that mathematical skills build in a hierarchical sequence, it is important that deficits are accommodated or remediated early. When schools use cognitive and academic screening measures to identify children for targeted services, it is also important that these measures assess the fundamental cognitive predictors of later academic performance. As such, the present study was the first of its kind to utilize the *Woodcock Johnson IV* assessment battery to investigate the relationship between cognitive abilities and math fact fluency.

Because no existing studies have evaluated math fact fluency performance in the context of the Catell-Horn-Carroll theory of intelligence, the present study relied on CHC research identifying the cognitive predictors of general mathematic achievement as well as individual studies assessing math fact fluency performance with narrow-band cognitive measures. The following discussion highlights the findings of the present study in the context of this research and generates general conclusions and implications for practice. Additionally, the study's limitations are outlined, in addition to avenues of future research.

Findings Regarding Broad Abilities

The first aim of this study was to identify the relationships among General Intellectual Ability, broad cognitive abilities, and Math Facts Fluency performance. Previous research has

revealed that Crystallized Intelligence (Comprehension-Knowledge), Fluid Reasoning, Short-Term Memory, and Processing Speed have the strongest relationships with overall mathematics achievement. A preliminary correlational analysis showed that GIA and these four broad abilities all were moderately correlated with Math Facts Fluency. When the abilities were analyzed together, GIA, Comprehension-Knowledge, Fluid Reasoning, and Processing Speed all exhibited positive direct effects on Math Facts Fluency performance, consistent with the hypotheses. As expected, GIA had moderate to strong effects on all broad abilities. The largest direct effect on Math Facts Fluency was evident for Processing Speed, which had a moderate positive effect. In comparison, GIA, Fluid Reasoning, and Comprehension Knowledge all had weak direct effects. Contrary to the hypothesis, Working Memory did not display a significant direct effect on Math Facts Fluency.

Overall, results of the broad abilities analysis are consistent with the findings of Taub et al. (2008) that utilized the WJ III battery and found that Crystallized Intelligence, Fluid Reasoning, and Processing Speed were most related to mathematics achievement. In addition, McGrew and Wendling's (2010) meta-analysis concluded that these three broad abilities were most related to math calculation skills. The present findings are in contrast to McGrew and Wendling's conclusions that Working Memory (which was categorized as a narrow ability in the WJ III battery) was related to math calculation skills at all ages. Hypotheses for this discrepancy are discussed below.

When comparing the results of this study to previous research, one must be cognizant of the fact that the Working Memory domain is new to the WJ IV battery. Previously, this factor was named Short-Term Memory and was derived from a working memory task and a relatively simple Short-Term Memory task (Memory for Words). The current cluster is comprised of two

subtest measuring Working Memory, which is characterized by the *manipulation of information* within immediate awareness. Although as noted previously, Taub et al. (2008) did not find significant effects of Short-Term Memory on mathematics performance, it was hypothesized that Working Memory ability would have a significant effect, in line with the results of McGrew and Wendling's (2010) meta-analysis. Although Working Memory has a moderate bivariate correlation with math fact fluency, it has no significant direct effect on Math Facts Fluency when GIA and the other broad abilities are taken into account. This would suggest that the correlation between Working Memory and Math Facts Fluency is actually representative of a third, mediating variable. This mediating variable is likely GIA, given its strong correlation with Working Memory.

The absence of a direct effect from Working Memory in the broad abilities model of Math Facts Fluency is also theoretically plausible, given the assertion that math fact fluency skills represent rote retrieval within this study. Whereas Working Memory is understandably important for more complex, multi-step calculations, which require one to sequence the order of operations and process information simultaneously, it appears less so for this relatively simple task. To the extent that executive functioning skills, such as inhibiting and shifting, are potential predictors of Math Facts Fluency is unknown in this model, as there are no such executive function measures in the WJ battery. However, it can be tentatively concluded that Working Memory ability is less important than initially hypothesized when considering the Math Facts Fluency performance of children and adolescents.

Unsurprising is that GIA contributed indirectly and directly to Math Facts Fluency performance, given the wealth of research that has shown general intelligence to be predictive of academic achievement. As GIA is composite of a broad array of cognitive skills, it is

hypothesized that the direct relationship to math fact fluency is an artifact of the presence of other abilities not measured within the model. Given the power of intelligence to predict achievement, it is also fitting that Fluid Reasoning, which is the broad ability most highly related to GIA, is also a significant predictor of Math Facts Fluency. Additionally, Fluid Reasoning has repeatedly demonstrated a strong relationship with math achievement (e.g., Floyd et al., 2003; Proctor et al., 2005; Taub et al., 2008). Within the correlational analysis employed in this study, the Math Problem Solving cluster had a strong relationship with Fluid Reasoning, similar in magnitude to its relationship with GIA. In this study, it was hypothesized that Math Problem Solving would serve as a representation of early numeracy in young children, given that the content of these problems involves comparing quantities and determining the relationships between numbers. If Math Facts Fluency performance is indeed related to early numeracy and number sense, then it is logical that children with stronger Fluid Reasoning skills, and thereby, stronger math reasoning skills, would perform better on math fluency measures.

As stated previously, Processing Speed had the greatest direct effect on Math Facts Fluency performance. The importance of Processing Speed on Math Facts Fluency performance is largely absent in the framework of math fact retrieval deficits proposed by David Geary, with the exception of Rapid Automatized Naming (RAN), which falls under the Phonological Processing subtype (Geary, 2011a). However, important to consider are the effects of age in comparing the two theories. Geary and others investigating the relationships between cognitive abilities and academic performance (e.g., Fuchs, et al., 2006; Geary et al., 2012; Jordan et al., 2007) have largely focused on the early elementary age, where skills are emerging and intervention is most fruitful. However, this study utilized a 6-19 population, with a mean age of just over 12 years. As such, the majority of individuals in this study are in late elementary or

beyond, when skills such as Math Facts Fluency have moved from the acquisition phase to the fluency and generalization phase. Therefore, it is prudent to conclude that Processing Speed is moderately related to Math Facts Fluency performance in individuals who have automatized the skill.

Given the previous conclusion that the majority of individuals within this study have automatized math facts, then it follows that the ability measuring crystalized knowledge would be related to Math Facts Fluency. Although crystalized intelligence has been renamed and restructured in the WJ IV battery as Comprehension-Knowledge, the results of the present study suggest that this area continues to have a significant effect on math performance. A potential hypothesis for this relationship is that Math Facts Fluency requires the fluent retrieval of information stored within long-term memory, which is also required for the vocabulary and general knowledge tasks within the Comprehension-Knowledge domain. This proposed relationship would lend credence to the semantic deficit subtype of math fact fluency weakness, which posits that some children have a specific weakness in the fluent retrieval of information from memory. This hypothesis is further discussed when considering the results of the narrow factors model.

Before exploring the second model, it is worth mentioning that the broad abilities model discussed previously was mathematically superior to both alternative models generated. The first model, which identified Fluid Reasoning and Comprehension-Knowledge as mediating variables between GIA and the other broad abilities, Working Memory and Processing Speed, exhibited extremely poor fit. Thus, the current research supports contemporary CHC theory, which organizes Fluid Reasoning, Comprehension-Knowledge, Working Memory, and Processing Speed as broad abilities falling under the umbrella of general ability. The second model reversed

the relationships between GIA and the narrow abilities; in this alternative model, the narrow abilities exhibited direct effects on GIA, which in turn had a direct effect on Math Facts Fluency. Although better than the first alternative model, the second alternative model also showed poor fit. Therefore, the theoretical broad factor model was chosen as the representation of the current data.

Findings Regarding Narrow Abilities

The second goal of this study was to identify the relationships among General Intellectual Ability, narrow cognitive abilities, and Math Facts Fluency performance. McGrew and Wendling's (2010) meta-analysis suggested that Perceptual Speed, Working Memory, and Phonological Processing were most related to math calculation performance. Support for the importance of Phonological Processing, including Rapid Automatized Naming (RAN), and Working Memory has also been found elsewhere within the literature (e.g., Bull et al., 2011; Chong & Siegal, 2008; Kroesbergen et al., 2009; Mazzocco & Grimm, 2013; Vukovic et al., 2010). Therefore, Perceptual Speed, Phonetic Coding, and Rapid Picture Naming were included in the narrow model. Within CHC theory, narrow abilities are numerous and still evolving. Within the updated Working Memory composite, WJ IV authors proposed that Attentional Control was a contributing narrow ability (McGrew et al., 2014). However, as no narrow-band composite is provided within the battery, one was created for this study. It was anticipated that this narrow ability would approximate executive control. Another new addition to the WJ IV is the Number Facility cluster, which is a narrow ability that measures skills with numbers across the domains of Working Memory and Processing Speed.

In the correlational analysis, moderate positive correlations with Math Facts Fluency were present for Number Facility, Perceptual Speed, and Attentional Control. In contrast,

Phonetic Coding and Rapid Picture Naming displayed weak positive correlations with Math Facts Fluency. When analyzed within the model, GIA had moderate positive effects on all narrow abilities. Additionally, GIA had a moderate direct effect on Math Facts Fluency, which suggests that GIA accounted for a sizeable amount of variance not explained by the narrow factors within the model. Indeed, all of the positive direct effects of the narrow abilities were weak. Of these narrow abilities, the greatest relationship was found between Perceptual Speed and Math Facts Fluency, followed by Number Facility, then Rapid Picture Naming. Phonetic Coding had a significant negative impact on Math Facts Fluency. Finally, Attentional Control had a negative, though non-significant, effect on Math Facts Fluency.

The results of the narrow factors analysis partially supported extant research. The positive direct relationship from Perceptual Speed to Math Facts Fluency supports findings from the meta-analysis regarding the WJ III (McGrew & Wendling, 2010). Additionally, the positive direct relationship from Number Facility to Math Facts Fluency supports the assertion that this narrow ability is responsible for “the speed at which basic arithmetic operations are performed accurately” (McGrew et al., 2014, p. 246). Finally, the small, but significant effect of Rapid Picture Naming on Math Facts Fluency supports research linking RAN and math fact fluency (Geary et al., 2012). However, the absence of a positive direct effect of Phonological Coding on Math Facts Fluency is at odds with Geary and others’ hypothesis that phonological processing weaknesses underlie a math fact retrieval deficit. Lastly, although the lack of relationship between Attentional Control and Math Facts Fluency is expected given the results of the broad factor model, it stands in contrast with research suggesting an executive functioning deficit responsible for a weakness in math fact fluency performance (Geary et al., 2012).

When considering the results of the broad factor model, it is fitting that Perceptual Speed, a narrow ability underlying Processing Speed, had the strongest relationship with Math Facts Fluency of all the narrow abilities. In terms of bivariate correlations, Perceptual Speed had the second strongest relationship with Math Facts Fluency, preceded only by GIA. Both Perceptual Speed and Processing Speed share a subtest (Letter Pattern Matching), but they differ on the demands of the second subtest. Whereas Perceptual Speed contains a second alphanumeric matching task (Number Pattern Matching), Processing Speed's second task requires the identification of a pair of pictures in an array (Pair Cancellation). Apparently, the speed at which one can identify alphanumeric symbols (i.e., letter and numbers) is of primary importance in Math Facts Fluency performance. Therefore, important to keep in mind when examining the weak relationship between Rapid Picture Naming and Math Facts Fluency is that a stronger relationship may have been present had the rapid naming task involved letters or numbers. This hypothesis is in line with research regarding rapid naming and reading fluency (Savage & Frederickson, 2005).

When considering the significant, yet smaller effect of Number Facility on Math Facts Fluency, likely is that this cluster also somewhat represents the positive effect of Number Pattern Matching. Although the Numbers Reversed subtest also involves the use of numbers, the fundamental skill employed in this task is Working Memory. Indeed, this subtest is a component of the broad Working Memory factor, which had a non-significant relationship with Math Facts Fluency in the broad model. The narrow factors model replicated the broad factors model in the relative unimportance of Working Memory tasks on Math Facts Fluency. Of note, the Attentional Control composite in this study included both a Working Memory task (Verbal Attention) and a Processing Speed task (Pair Cancellation); however, the seemingly positive

weight of the Processing Speed task was not enough to generate a significant relationship with Math Facts Fluency. One hypothesis for this finding is that Pair Cancellation is a cognitively complex task that requires inhibition and interference control. In contrast, other speeded processing tasks lack this executive control component. As stated previously, the findings of this research is in direct contrast to the executive functioning type of math fact fluency deficit (Geary, 2011a). However, it is worth repeating that the present study utilized a sample of participants across childhood and adolescence and that the importance of particular cognitive abilities in acquiring math fact fluency may not continue through the fluency and generalization stage. Consistent with this hypothesis, Geary et al. (2012) acknowledged that central executive measures were less predictive of fact fluency in fourth grade.

Perhaps the most surprising finding resulting from the second model was the significant negative relationship between Phonetic Coding and Math Facts Fluency. This finding stands in opposition to previous research using the WJ III battery as well as independent studies identifying phonological processing measures as predictive of math fact fluency (e.g., Fuchs et al., 2005; Fuchs et al., 2006). Although suppression from other narrow abilities is likely the cause of the negative relationship within the model, the bivariate correlation between Phonetic Coding and Math Facts Fluency indicates that the two are only weakly positively correlated. Again, it is important to consider that the effect of Phonetic Coding in acquiring Math Facts Fluency may be relatively diminished by late elementary and thus would be masked within the current sample. This hypothesis is supported by a similar trend in reading fluency performance; although phonemic awareness is strongly related to reading acquisition in young children, its importance declines across time, when reading fluency is better established (Phillips & Torgesen, 2006). Second, it is important to note that the Phonetic Coding cluster provided in the WJ IV is

distinct from the WJ III Phonological Processing cluster, given the restructuring of the battery and the addition of the Tests of Oral Language, and may measure slightly different abilities. The phonological processing tasks in the Phonetic Coding cluster, Segmentation and Sound Blending, are both measures of one's ability to manipulate phonemes. However, other tasks measuring alphabetic principle (sound-symbol associations) could also fall under the umbrella of phonological processing. Thus, there may be important distinctions between tasks given the same general classification.

Findings Regarding Math Problem Solving

A third aim of this study was to investigate the relationship between an achievement cluster, Math Problem Solving and Math Facts Fluency performance. Math Problem Solving was hypothesized to have a significant direct effect on Math Facts Fluency, given the evidence supporting the hypothesis that a number sense weakness is present in children with math fact fluency deficits (Geary, 2011b; Locuniak & Jordan, 2008). Given the previous finding that Math Problem Solving is strongly correlated with Fluid Reasoning, the narrow factors model was used to explore the relationship between Math Facts Fluency and Math Problem Solving.

As expected, the Math Problem Solving cluster had a significant positive effect on Math Facts Fluency performance. When this variable was included within the model, other relationships shifted slightly. The direct path from GIA to Math Facts Fluency remained significant, although decreased to a weak positive effect. Similarly, the effect of Number Facility on Math Facts Fluency decreased and was no longer significant. In contrast, the direct effect of Perceptual Speed increased in magnitude and exhibited the strongest direct effect on Math Facts Fluency. The effects of Phonetic Coding and Rapid Picture Naming were relatively unchanged.

In line with the previous statement that Fluid Reasoning and Math Problem Solving are highly correlated, Math Problem Solving exhibited a positive direct effect on Math Facts Fluency. However, within the narrow factors model, Math Problem Solving took on relatively stronger importance than Fluid Reasoning did in its respective model. Well established from longitudinal research is that early math achievement is predictive of future achievement (Duncan et al., 2007). The current findings suggest that performance on an applied mathematics achievement measure may be more predictive of Math Facts Fluency than other measures of cognitive abilities in children and adolescents. The exception, of course, is Processing Speed/Perceptual Speed, which have proven to be the most important individual cognitive ability within this sample of students. Nonetheless, to the degree that Math Problem Solving measures an underlying number sense, the current research lends support to the theory that number sense is predictive of math fact fluency performance.

Study Limitations

One of the foremost limitations evident when discussing the results of this study is the large age range of students within the sample. Although the purpose of the study was to make broad conclusions regarding the relationships between cognitive abilities and math fact fluency in school-age children and adolescents, these results may not accurately reflect the dynamic importance of particular cognitive abilities when considering cross-sections of students. Previous research (e.g., Floyd et al., 2003) has demonstrated the differential impact of cognitive abilities across the development academic skills. Thus, it is important to be cognizant of this fact if these results are to be applied to inform assessment or screening practices in young children.

Second, although this study provided evidence of a moderate relationship between math problem solving and math fact fluency, it was beyond the scope of the study to determine the

predictive value of other measures of academic achievement. Given that all data within this study was collected at a single point in time, no generalizations can be made to the use of math problem solving to predict future math fact fluency performance. Additionally, quite possible is that other inter-achievement relationships may exist within the battery, for example, between reading fluency and math fluency. Further research is needed to address these questions.

A final, practical limitation concerns the nature of the data used within this study. Many of the subtests that create the narrow and broad composites are cognitively complex tasks that span various abilities. For example, the Pair Cancellation task is thought to involve three narrow abilities: Perceptual Speed, Spatial Scanning, and Attentional Control (McGrew et al., 2014). Therefore, when used as a predictor, it is difficult to determine which narrow ability is most salient in determining the outcome variable. Although it is worth exploring the narrow abilities in order to isolate specific areas of weakness, it is to be expected that a high degree of similarity may be present across these tasks and their corresponding broad abilities. For example, the broad ability Processing Speed and the narrow ability Perceptual Speed both share a common subtest; similarly, the narrow ability Number Facility shares a common subtest with the Processing Speed and Working Memory broad ability clusters. Given the degree of overlap between subtests comprising the broad and narrow abilities, a model exploring the relationships among broad and narrow factor concurrently would contain significant multicollinearity. Therefore, broad and narrow abilities were analyzed separately for this study. The difficulty with parsing out theoretical narrow abilities in an assessment instrument is not inherent to the WJ IV. When considering the input, processing, and output demands of any assessment task, it is likely one will be able to identify multiple underlying abilities. Therefore, it is imperative that the findings be viewed in the context of the task demands produced within the assessment battery. One

should take caution in generalizing these findings to other assessment batteries with distinct measures.

Future Directions of Study

The aforementioned limitations of the current research provide logical areas of future research. It is recommended that future research examine the applicability of the current model of Math Facts Fluency performance in age-defined subsamples. Future research may find that the abilities contributing to math fact fluency should be conceptualized differentially in the acquisition stage of the learning process (ages 6-10) than in the in fluency stage (ages 11+). In contrast, research comparing students in the acquisition phase versus the fluency, regardless of age, may be the best suited for uncovering true differences in the importance of cognitive abilities throughout the learning process. As mentioned previously, the relationship between math fluency and reading and writing fluency was not explored within this study. It is likely that future research regarding both the shared and distinct mechanisms for identifying fluency deficits using the WJ IV battery would prove useful for clinicians.

As the WJ IV battery is still a relatively new instrument, potential areas of future research with this instrument are abundant. Replication studies examining the relationships between the narrow and broad cognitive abilities and general mathematics achievement would be valuable in determining the potential effects of the restructured cognitive tasks. This information would provide an important context for analyzing the results of the current study. Further, as previous WJ research has typically examined Math Calculation Skills, which combines the assessment of Math Calculation and Math Facts Fluency, it remains to be seen which relationships with Math Calculation are most important when Math Facts Fluency is removed from the equation.

Implications for Practice

This is the first known study to use the Woodcock Johnson battery to examine the cognitive abilities related to math fact fluency as an isolated skill. As such, the results of the present study provide a comprehensive analysis of math fluency within the framework of the most empirically validated theory of intelligence. From here, further research is needed to clarify the ability of cognitive measures to predict math fact fluency performance across the learning process. Nonetheless, some implications for practice are provided.

First, the current research supports the assertion that assessment of one's general cognitive ability can provide information about his or her expected achievement, even with a relatively straightforward task such as completing single-digit computations. When assessing broad domains of functioning, it appears that measuring processing speed performance would be most important in identifying students with potential difficulty in the area of math fact fluency, regardless of age. If a student exhibits a deficit in processing speed, clinicians are urged to consider an accommodation that would bypass this area of weakness.

In determining the most appropriate accommodation for a student, clinicians must consider the student's pattern of performance on a math fluency task. Students who are slow but accurate in their computation skills may benefit from extended time. In terms of accommodations, extended time is widely used and easy to implement in the classroom (Bolt & Thurlow, 2004). Students who are inaccurate in their computation skills would likely benefit from the use of a calculator. This accommodation is appropriate in instances where math fact fluency is not the primary skill being measured by the assessment. For example, on a comprehensive mathematics exam, the use of the calculator may allow the examiner to measure the student's performance on higher-level math skills involved in solving algebraic equations.

In addition to considering appropriate accommodations for students with math fact fluency deficits, clinician and educators should also be familiar with empirically supported interventions that can be used to remediate these deficits. Cover-Copy-Compare (CCC) is a drill-and-practice intervention that can be administered individually or in a group, which has demonstrated sustained increases in math fact fluency relative to control conditions or a constructivist-oriented intervention (Poncy, McCallum, & Schmitt, 2010; Skinner, Turco, Beatty, & Rasavage, 1989). Additionally, the Taped Problems intervention is a second drill-and-practice method for increasing fact fluency in the classroom setting (McCallum, Skinner, Turner, & Saecker, 2006). A comparison of the two interventions revealed that both were effective in increasing the math fact fluency of an elementary student with impaired cognitive functioning (Poncy, Skinner, & Jaspers, 2007). The authors noted that Taped Problems was more efficient in terms of times spent implementing the intervention; however, both interventions were implemented in fewer than 10 minutes per day. Thus, the research supports the use of targeted interventions to increase the math fact fluency of students, which have proven to be efficacious and relatively low-burden for educators.

Second, the results of this research have implications for universal screening practices. Previous research has shown that early academic skills, such as early numeracy or number sense, can predict later math achievement. In accordance with this finding, the present research showed that math problem solving skills were moderately predictive of math fact fluency performance. Although comprehensive assessment is certainly warranted in some cases, educators may be able to simplify their universal screening process by using a mathematical reasoning measure to identify children at-risk for mathematical difficulties. Whereas calculation and math fact fluency are learned throughout elementary school, research suggests that young children possess

mathematical reasoning abilities upon entering formal schooling. Relatedly, processing speed tasks are relatively simple and can be administered to young children before math fluency skills are solidified. Given the results of this study, educators may be able to use these measures to identify children who may go on to develop math fact fluency deficits. With this information, these children can be targeted with early intervention to support their future academic achievement.

Summary

The purpose of this research was to explore the relationships among the Cattell-Horn-Carroll (CHC) Theory-Aligned Cognitive Abilities and Math Fact Fluency performance using the Woodcock Johnson IV standardization sample. Using path analysis, the broad and narrow abilities thought to influence Math Fact Fluency performance were modeled. The broad ability model revealed that General Intellectual Ability exhibited significant direct and indirect effects on Math Fact Fluency. With regard to the broad factors, Processing Speed had a moderate direct effect on Math Fact Fluency, followed by weak direct effects from Fluid Reasoning and Comprehension Knowledge. Contrary to the initial hypothesis, Working Memory did not have a significant direct effect on Math Fact Fluency. Within the narrow factors model, Perceptual Speed, Number Facility, and Rapid Picture Naming all exhibited weak positive direct effects. In contrast, Phonetic Coding and Attentional Control were not positively related to Math Fact Fluency. Inclusion of the Math Problem Solving composite revealed that Math Problem Solving had a moderate direct effect on Math Fact Fluency, which was similar in magnitude to Perceptual Speed. Overall, the current research points to general ability, speed of processing, and math reasoning abilities as the most important contributors to Math Fact Fluency performance. As previous research with the WJ battery has studied general mathematics achievement, the current

findings shed light on the unique relationships among cognitive abilities and Math Fact Fluency, with implications for clinicians and educators.

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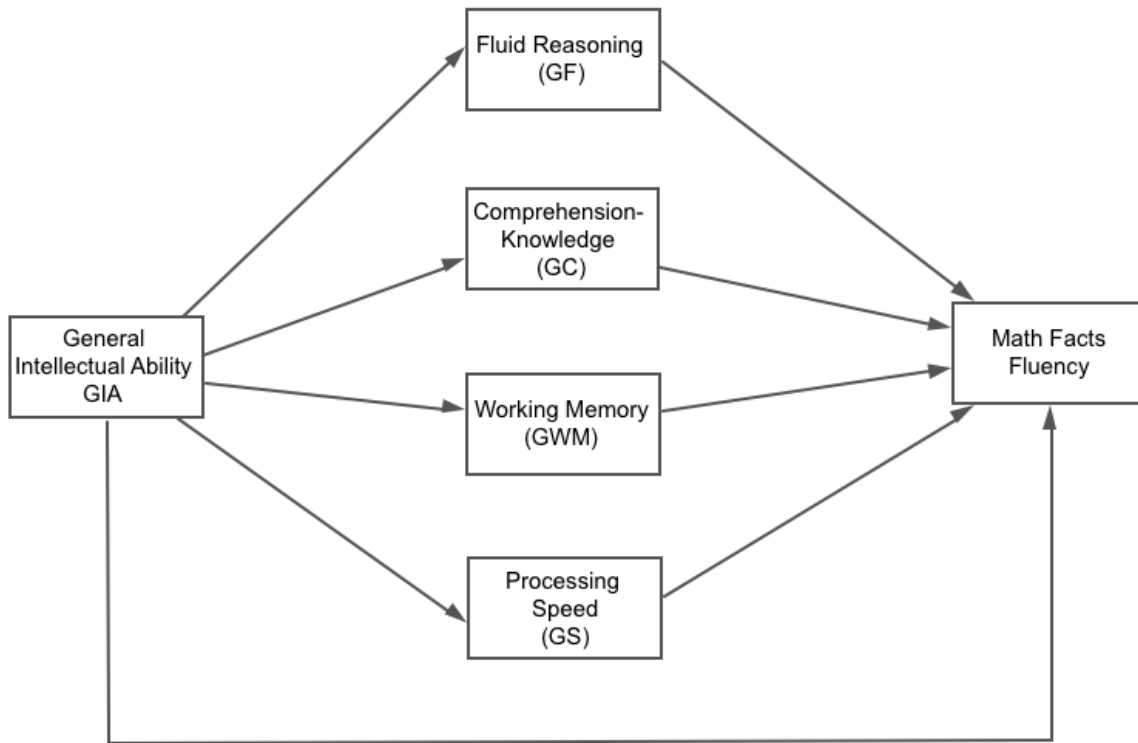


Figure 1. Theoretical broad factor model.

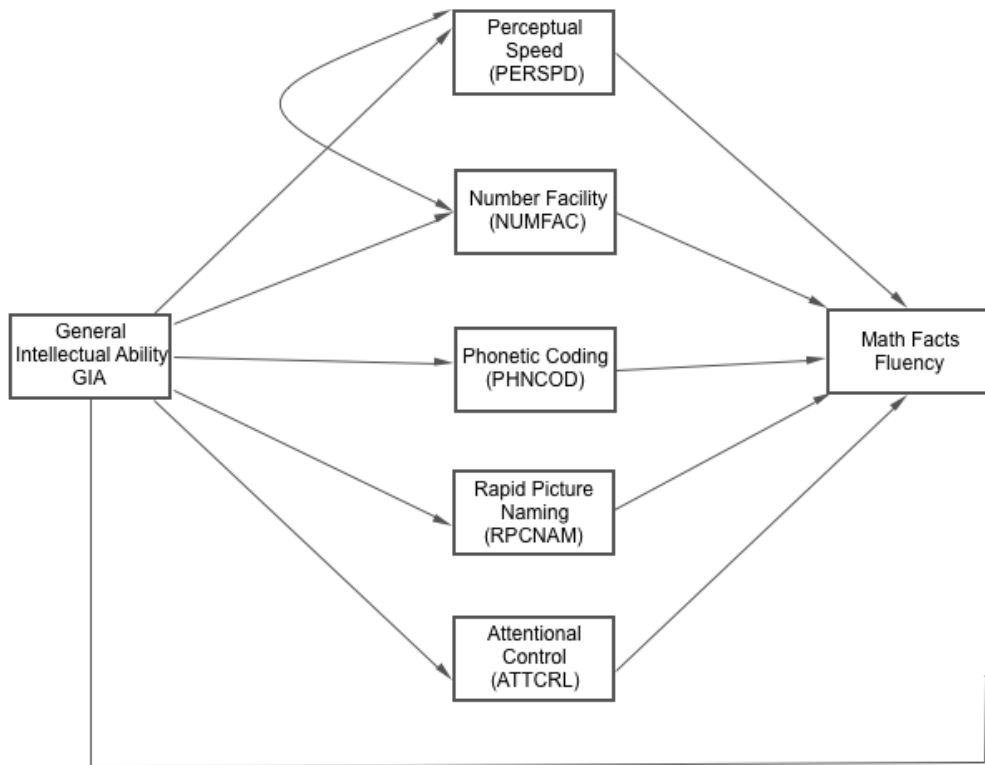


Figure 2. Theoretical narrow factor model.

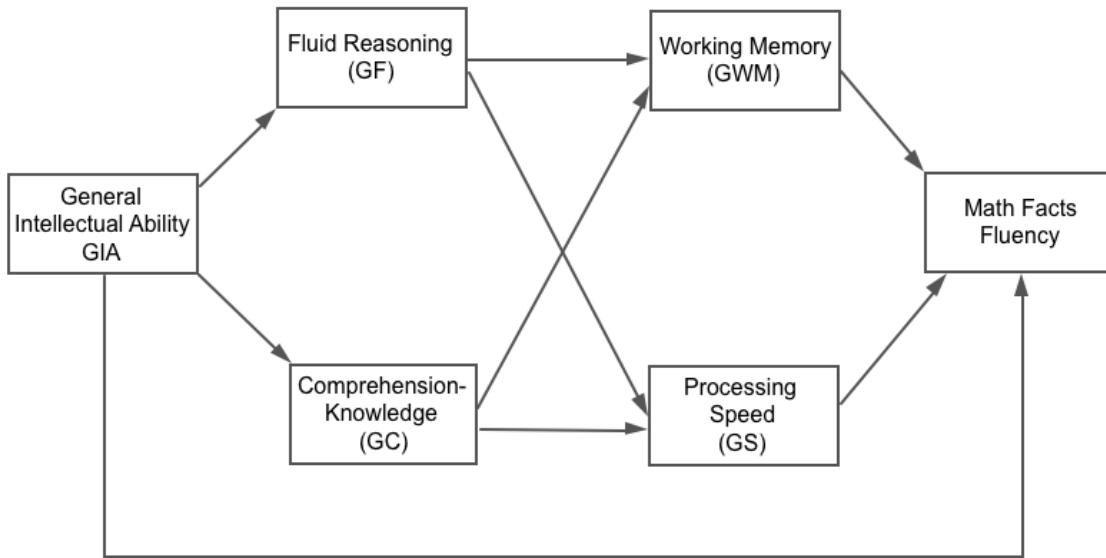


Figure 3. First alternative broad factor model.

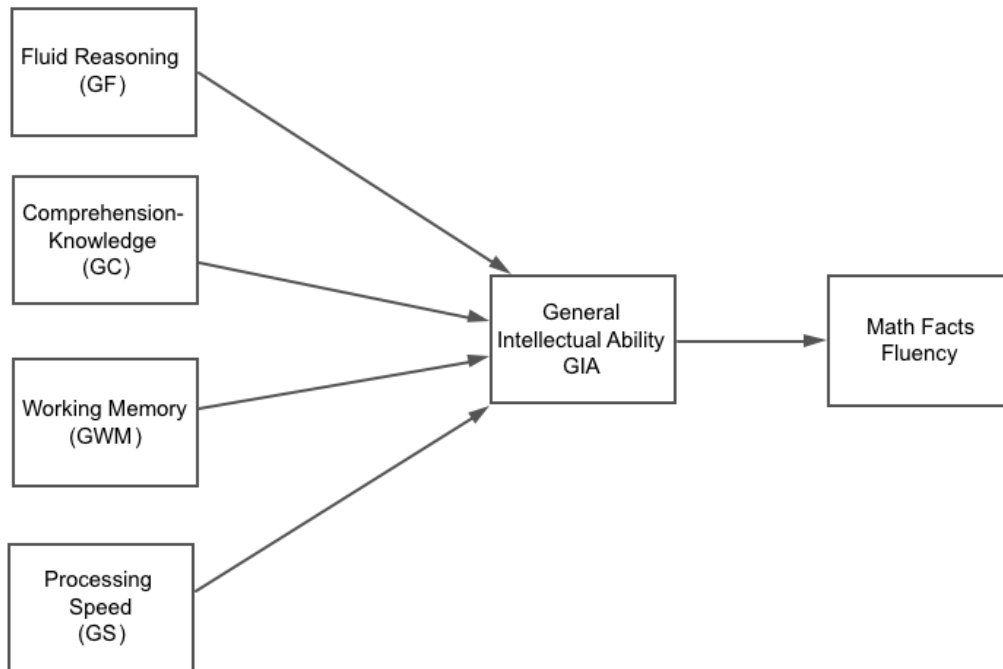


Figure 4. Second alternative broad factor model.

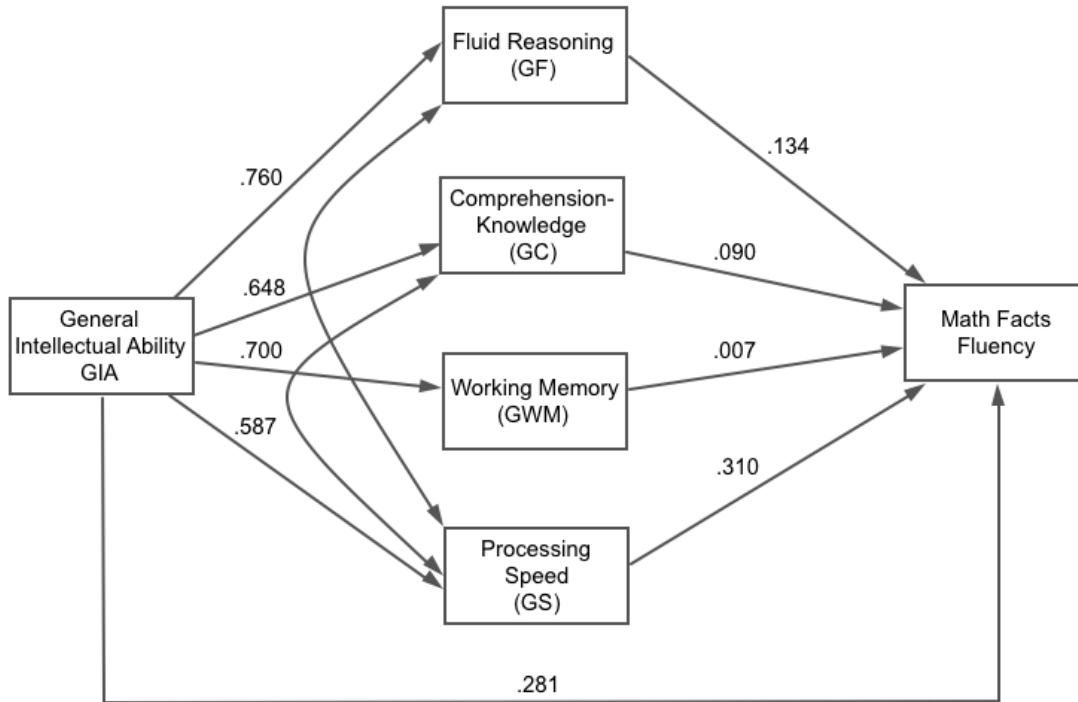


Figure 5. Final broad factor model with parameter estimates.

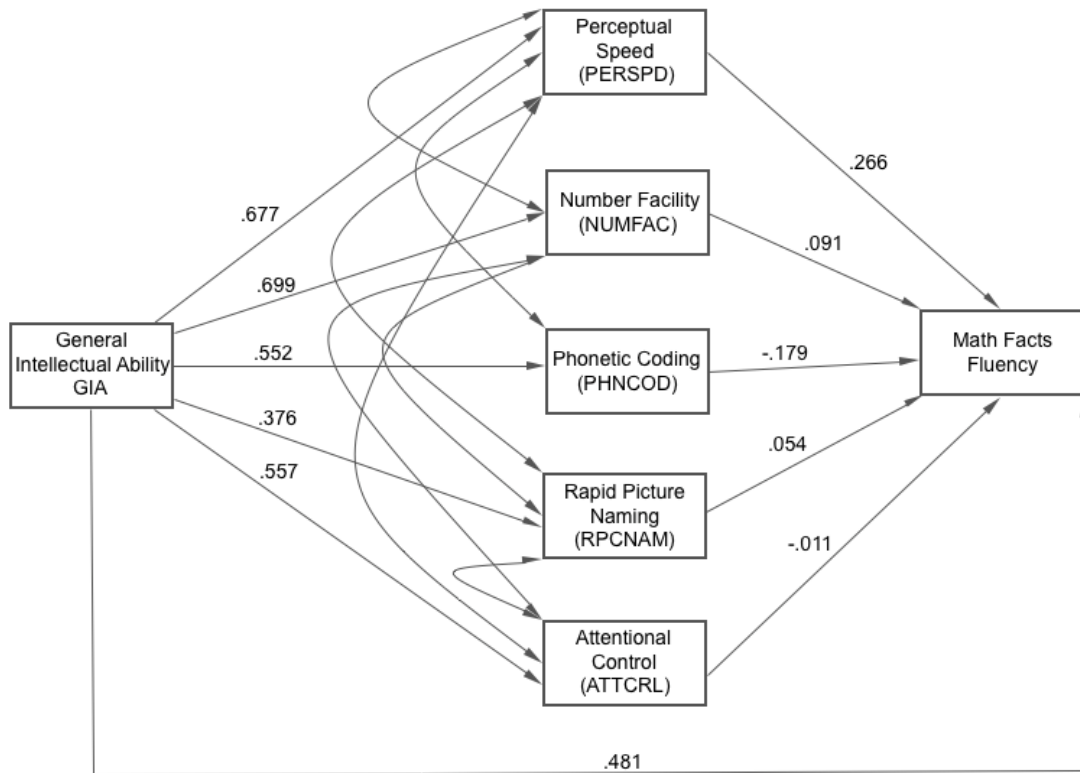


Figure 6. Final narrow factors model with parameter estimates.

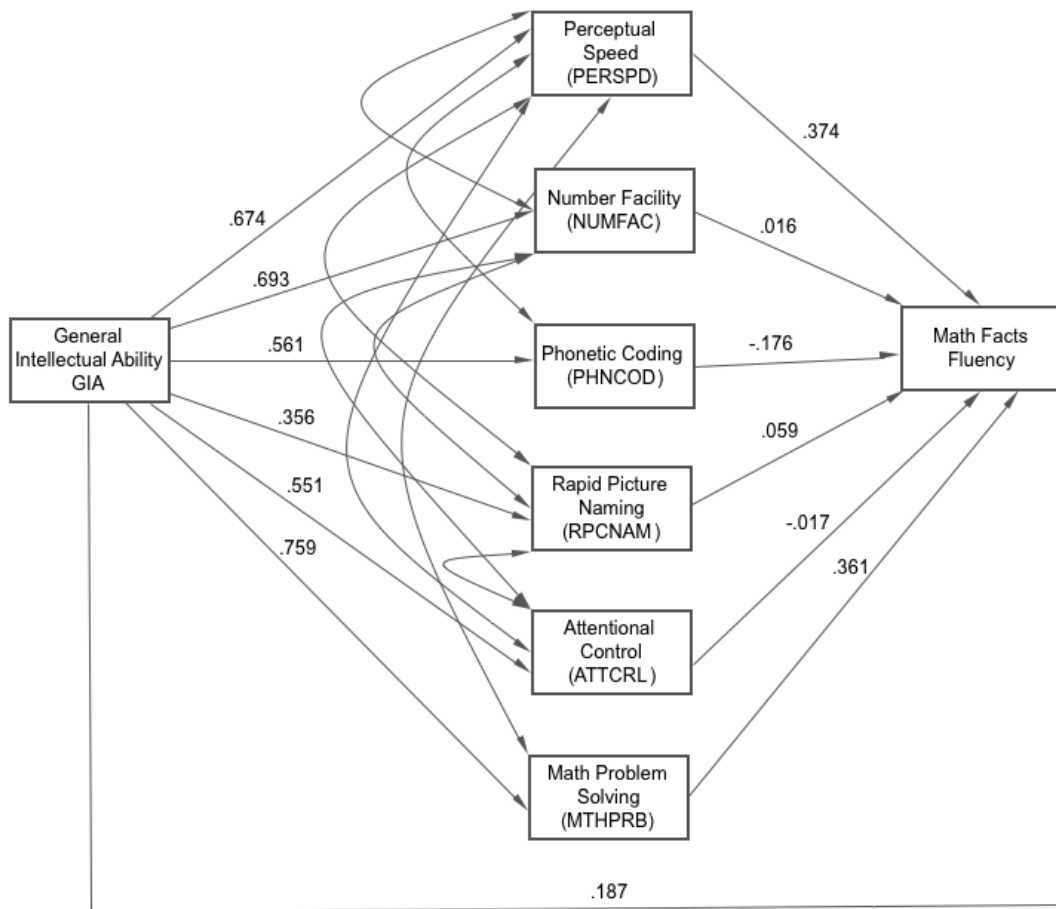


Figure 7. Final math problem solving model with parameter estimates.