

# On Graphical Calculi for Modal Logics

## *Cálculos Gráficos para lógicas modais*

### Abstract

*We present a graphical approach to classical and intuitionistic modal logics, which provides uniform formalisms for expressing, analysing and comparing their semantics. This approach uses the flexibility of graphical calculi to express directly and intuitively the semantics for modal logics. We illustrate the benefits of these ideas by applying them to some familiar cases of classical and intuitionistic multi-modal logics.*

**Keywords:** Modal logics; classical modal logics; intuitionistic modal logics; possible world semantics; graphical formulations; graphical calculi.

### Resumo

*Apresentamos uma abordagem gráfica para as lógicas modais clássica e intuicionista, capaz de fornecer formalismos uniformes para expressar, analisar e comparar suas respectivas semânticas. Tal abordagem utiliza a flexibilidade dos cálculos gráficos para expressar, direta e intuitivamente, a semântica das lógicas modais. Ilustramos os benefícios dessas ideias aplicando-as a alguns casos conhecidos de lógicas multimodais clássica e intuicionista.*

**Palavras-chave:** Lógicas modais; lógicas modais clássicas; lógicas modais intuicionistas; semântica de mundos possíveis; formulações gráficas; calculus gráficos.

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## 1. Introduction

We present a graphical approach to modal logics, which provides a flexible and uniform tool for expressing, analyzing and comparing possible-world semantics.<sup>1</sup>

This graphical approach can be regarded as a version of diagrammatic reasoning, where we can express formulas by diagrams, which can be manipulated so as to unveil properties (like consequence and satisfiability). Graphical representations and transformations, having precise syntax and semantics, give proof methods [F3V 06]. An interesting feature of this graphical approach is its two-dimensional notation providing pictorial representations that support visual manipulations [CL 95, CL 96]. These ideas have been adapted to refutational reasoning [VV 11] and applied to some versions of multi-modal logics [VVB 14].

Modal logics and graphs are closely connected. Kripke frames can be presented via directed labeled graphs for the accessibility relation of each modality [BRV 95]; it is natural to represent that  $a$  is related to  $b$  via relation  $r$  by an  $r$ -arrow from  $a$  to  $b$ . Intuitionistic modal logic is an interesting subject [FS 81, PS 86]: there seems to be little consensus on the appropriate approach to its semantics, as indicated by the diversity of Kripke-like semantics proposed (see [Ewd 86, Smp 94] and references therein).

We provide graphical calculi, having diagrams as terms and whose rules transform diagrams, capturing the semantics of the modal operators and accessibility relations. These calculi provide uniform and flexible formalisms where one can explore Kripke-like semantics for modal logics: valid formulas, consequence, etc. We illustrate these ideas by applying them to classical modal logics [BRV 95] and to versions of intuitionistic modal logics [Smp 94, Ewd 86].

We will consider a modal language  $ML$ , given by 2 sets:  $PL$ , of propositional letters, and  $RN$ , of 2-ary relation symbols [CP 08]. Its set  $\Phi$  of formulas is generated by the following grammar (where  $p \in PL$ ):

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<sup>1</sup> Previous versions of these ideas have been presented at LSFA conferences [VVB 15, VV 15].

$$\varphi ::= \perp \mid p \mid \varphi' \wedge \varphi'' \mid \varphi' \vee \varphi'' \mid \varphi' \rightarrow \varphi'' \mid \langle r \rangle \varphi \mid [r] \varphi.$$

Negation  $\neg$  is defined as usual:  $\neg \varphi$  abbreviates  $\varphi \rightarrow \perp$ .

We now introduce informally some basic ideas about our approach to modalities.<sup>2</sup> Graphical reasoning involves manipulating diagrams (for formulas).

A graphical language uses nodes to construct sets of arcs. We have two kinds of arcs: unary and binary. Unary arcs are intended to represent formula satisfaction:  $w \dashv \dashv \neg \langle r \rangle \varphi$  represents that formula  $\varphi$  holds at  $w$ . Binary arcs are used to represent accessibility:  $u \xrightarrow{r} v$  represents that  $v$  is  $r$ -accessible from  $u$ . A draft is a finite set of nodes and arcs. A page consists of a draft together with a input node (marked  $\hat{\phantom{x}}$ ).

The next example introduces a graphical approach to consequence.

**Example 1.1** (Formula consequence). We can show graphically that  $\langle r \rangle \psi$  is a consequence of  $\langle r \rangle (\psi \wedge \theta)$  as follows.

(RhS) We represent  $\langle r \rangle \psi$  by a page and manipulate it, obtaining page R:

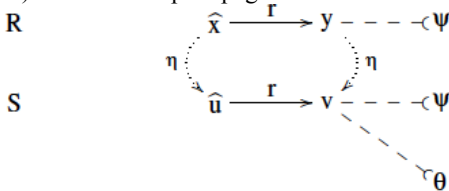
$$\hat{x} \dashv \dashv \neg \langle r \rangle \psi \quad (\langle \rangle) \quad \hat{x} \xrightarrow{r} y \dashv \dashv \neg \psi$$

(LhS) We represent  $\langle r \rangle (\psi \wedge \theta)$  by a page and manipulate it, obtaining page S as follows:

$$\hat{u} \dashv \dashv \neg \langle r \rangle (\psi \wedge \theta) \quad (\langle \rangle) \quad \hat{u} \xrightarrow{r} v \dashv \dashv \neg \psi \wedge \theta \quad (\wedge) \quad \hat{u} \xrightarrow{r} v \dashv \dashv \neg \psi$$

$\neg \theta$

( $\rightarrow$ ) We now compare pages R and S:



<sup>2</sup> These and other ideas will be formulated more precisely in Sect. 2: Graphical Reasoning.

This node translation  $\eta$  preserves arcs and input nodes.

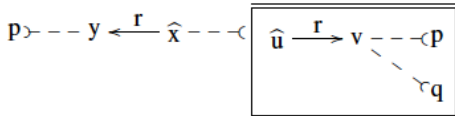
We thus have  $\langle r \rangle \psi \sim^* R$  and  $\langle r \rangle (\psi \wedge \theta) \sim^* S$  with homomorphism  $\eta: R \rightarrow S$ .

We use  $^c$  (or overbar) for complementation:  $w \not\models \varphi^c$  represents that formula  $\varphi$  does not hold at  $w$ ; note that  $\varphi^c$  is not a formula, we will call it an expression.<sup>3</sup> One can also establish consequence by refutation, as the next example illustrates.

**Example 1.2** (Satisfiable set). To show graphically that the expression set  $\{\langle r \rangle p, \langle r \rangle (p \wedge q)\}^c$  is satisfiable, we present it by the following page Q

$$\langle r \rangle p \not\models \hat{x} \not\models \neg \langle r \rangle (p \wedge q)$$

1. Page Q converts to the following page Q'



2. From this page Q' we obtain the natural structure N as follows:<sup>4</sup>

$$p \rightsquigarrow y \xleftarrow{r} x$$

3. One can see that in N, the identity assignment satisfies the arcs  $p \supset y$  and  $y \xleftarrow{r} x$  as well as the arc with complemented expression.

Thus, set  $\{\langle r \rangle p, \langle r \rangle (p \wedge q)\}^c$  is satisfied at state  $x$  of this structure N.

This example establishes that  $\langle r \rangle (p \wedge q)$  is not a consequence of  $\langle r \rangle p$ .

<sup>3</sup> See 2.1: Graphical Languages.

<sup>4</sup> We use wiggly arrows for drawing structures.

## 2. Graphical Reasoning

We now present graphical reasoning: languages (in 2.1), some concepts and results (in 2.2) and calculus rules (in 2.3).<sup>5</sup>

### 2.1. Graphical Languages

We now introduce graphical languages: syntax and semantics.

A graphical language GL is characterized by two sets of symbols:  $Sb_1$ , of unary symbols, and  $Sb_2$ , of binary symbols.<sup>6</sup> It involves an infinite set  $Nd$  of node names. We will use  $x$ ,  $y$  and  $z$  for the first 3 nodes of  $Nd$ .

The (mutually recursive) syntax (and intuitive meanings) is as follows.

(E) Expressions: unary symbols  $t \in Sb_1$ , pages, books and their complements (set of states).

(a) Arcs: unary and binary, with expressions and 2-ary symbols, respectively.

1. pair  $w \mid E$ , drawn  $w \dashv\vdash E$  ( $w$  pertains to expression  $E$ ).<sup>7</sup>

2. triple  $u \mid L \mid v$ , drawn as an L-arrow from  $u$  to  $v$  ( $v$  is L-accessible from  $u$ ).

( $\Sigma$ ) A sketch consists of a set of arcs over nodes (restrictions on states).<sup>8</sup>

(D) A draft is sketch with finite sets of nodes and arcs (restriction on states).

(P) A page consists of a draft with a node marked as input (set of states).

(B) A book is a finite set of pages (set of states).

A sketch  $\Sigma = \langle N; A \rangle$  is proper iff  $N \neq \emptyset$  (i. e.  $\Sigma$  has some node).

We also use single-node pages for expressions: the page of expression  $E$  is  $Pe(E) := E \dashv\vdash x^\wedge$ , while the page of expression-pair  $(E, F)$  is the page  $Pp(E, F) := E \dashv\vdash x^\wedge \dashv\vdash F^c$  (cf. Examples 1.1 and 1.2, in Sect. 1).

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5 For more details see, e. g. [VV 11, VVB 14] and references therein.

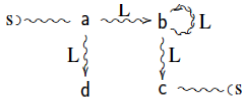
6 For the applications to modal logics, in Sets. 3 and 4,  $Sb_1$  will consist of formulas.

7 We use ‘pertains’ in an intuitive sense, see Example 2.2: Draft and page).

8 Finiteness will be important for effectiveness, but some concepts are natural for sketches.

The semantics of a graphical language  $\mathbb{G}$  is based on frames and models realizing its symbols. A  $\mathbb{G}$ -frame  $\mathfrak{F}$  consists of a non-empty set  $M$ ; together with a 2-ary relation  $L^{\mathfrak{F}}$  on  $M$ , for each binary symbol  $L \in \text{Sb}_2$ . A  $\mathbb{G}$ -structure  $\mathfrak{S}$  consists of an underlying frame  $\mathfrak{F}$ , with universe  $M \neq \emptyset$ ; together with a subset  $s^{\mathfrak{S}} \subseteq M$ , for each unary symbol  $s \in \text{Sb}_1$ . We set  $L^{\mathfrak{S}} := L^{\mathfrak{F}}$ , for  $L \in \text{Sb}_2$ .

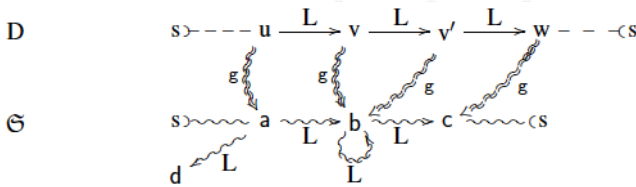
Example 2.1 (GL-structure). Consider the following diagram:



It depicts a  $\mathbb{G}$ -structure  $\mathfrak{S}$  having universe  $M = \{a, b, c, d\}$ , underlying frame  $\mathfrak{F}$ , with relation  $L^{\mathfrak{S}} = \{(a, d), (a, b), (b, b), (b, c)\}$ , and subset  $s^{\mathfrak{S}} = \{a, c\}$ .

The next example introduces some semantical ideas.

Example 2.2 (Draft and page). Given the  $\mathbb{G}$ -structure  $\mathfrak{S}$  in Example 2.1 ( $\mathbb{G}$ -structure), consider the draft  $D = s \dashrightarrow \dots u \xrightarrow{L} v \xrightarrow{L} v' \xrightarrow{L} w \dashrightarrow \dots \neg s$  and the assignment  $g$ , given by  $u \mapsto a$ ;  $v, v' \mapsto b$ ;  $w \mapsto c$ . We can visualize this situation as follows:



Assignment  $g$  satisfies every arc of  $D$ .

For instance, assignment  $g$  satisfies  $w \dashrightarrow \dots \neg s$  since  $w^g = c \in s^{\mathfrak{S}}$  and  $g$  satisfies  $u \xrightarrow{L} v$  since  $(u^g, v^g) = (a, b) \in L^{\mathfrak{S}}$ .

So,  $g$  satisfies draft  $D$ .

Now, consider page  $P = (D; u)$ , with underlying draft  $P = D$  and input node  $u$ .

State  $a = u^g$  belongs to the behaviour of page  $P$ .

We define behave our as follows.

(P): For a page  $P$ , with underlying draft  $P$  and input node  $u$ , its behaviour is the set  $[P]_s$  consisting of the values  $u^g$  for the assignments  $g$  satisfying  $P$ . (B): For a book, we set  $[B]_s := \cup \{P \in B [P]_s\}$ .

Thus, for the empty book  $\{\}$ ,  $[\{\}]_s = \emptyset$ , and for a singleton book  $\{P\}$ ,  $[\{P\}]_s = [P]_s$ .

We define the extension  $[E]_s$  of expression  $E$  as follows. For a unary symbol  $t \in \text{Sb1}$ , it is given by the structure:  $[t]_s := t_s$ . If  $E$  is a page or a book, we use its behaviour:  $[E]_s := [E]_s$ . We set  $[E^c]_s := M \setminus [E]_s$ .

**Remark 2.1** (Special pages). Consider a GL-structure  $S$  with universe  $M$ . ( $\setminus$ ): For an expression  $E$ ,  $[x^{\wedge} \text{---} \subset E^c]_s = M \setminus [x^{\wedge} \text{---} \subset E]_s$ . ( $\cap$ ): For expressions  $E$  and  $F$ ,  $[E \supset \text{---} x^{\wedge} \text{---} \subset F]_s = [x^{\wedge} \text{---} \subset E]_s \cap [x^{\wedge} \text{---} \subset F]_s$ . ( $\exists$ ): For a label  $L$  and an expression  $E$ ,  $a \in [x^{\wedge} L \rightarrow y \text{---} \subset E]_s$  iff, for some  $b$  with  $(a,b) \in LS$ ,  $b \in [y^{\wedge} \text{---} \subset E]_s$ .<sup>9</sup>

**Corollary 2.1** (Expression pages). For a GL-structure  $S$ :  $[Pe(E)]_s = [E]_s$  and  $[Pp(E,F)]_s = [E]_s \setminus [F]_s$ .

*Proof.* By Remark 2.1: Special pages.

## 2.2. Graphical Concepts and Results

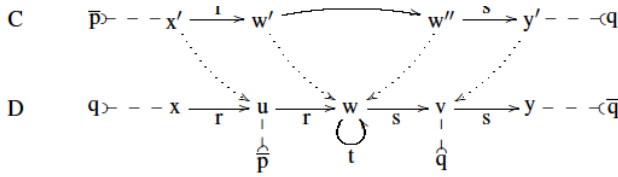
We now examine some graphical concepts and results.

We compare sketches by morphisms and pages by homomorphisms. A morphism is a node mapping that preserves arcs. A homomorphism is a morphism of underlying drafts that preserves input nodes.

**Example 2.3** (Morphism, homomorphism). A morphism  $\mu : C \rightarrow D$  is as follows:

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<sup>9</sup> Recall that  $x$ ,  $y$  and  $z$  are the first 3 nodes of  $N_d$ .



So,  $\mu$  is a homomorphism from  $(C; x')$  to  $(D; u)$ , but not from  $(C; x')$  to  $(D; x)$ .

We now define covering between pages and books. For pages:  $Q$  covers  $P$  (noted  $P \leftarrow Q$ ) iff there is a homomorphism from  $Q$  to  $P$ . Book  $H$  covers page  $P$  (noted  $P \leftarrow H$ ) iff some page  $Q \in H$  covers  $P$ . For books:  $H$  covers  $G$  (noted  $G \leftarrow H$ ) iff  $H$  covers every page  $P \in G$ . (In Example 1.1,  $S \leftarrow R$ .)

Note that the empty book  $\{\}$  is covered by any book, and  $G = \{\}$  whenever  $G \leftarrow \{\}$ .

A morphism transfers satisfying assignments by composition (see Fig. 1).

**Lemma 2.1 (Transfers).**  $(\dashv\dashv)$ : Given a morphism  $\mu : D \dashv\dashv \Sigma$ , for any assignment  $g$  satisfying  $\Sigma$  in  $S$ , the composite  $g \cdot \mu$  satisfies  $D$  in  $S$ .  $(\leftarrow)$ : If  $P \leftarrow Q$ , then  $[P]_S \subseteq [Q]_S$ ; if  $P \leftarrow H$ , then  $[P]_S \subseteq [H]_S$ ; and, if  $G \leftarrow H$ , then  $[G]_S \subseteq [H]_S$ .

**Proof.** Clear:  $(\dashv\dashv)$  is immediate and yields  $(\leftarrow)$ .

Figure 1: Satisfying assignment transfer



We will use '+' for addition of arcs (and nodes).

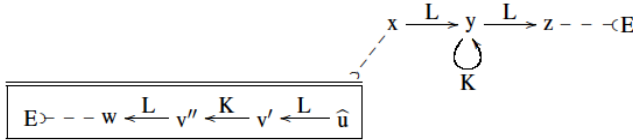
We wish to glue a page  $Q = (Q; v)$  onto a node  $w$  of a page  $P = (P; u)$ . For this purpose, we take a copy  $Q'$  of  $Q$  having no node in common with  $P$ , and identify node  $v'$  to  $w$ , obtaining page  $Q''$ . The glued page  $P_w Q$  is defined as  $(P + Q''; u)$ .<sup>10</sup>

<sup>10</sup> For instance, in Example 1.1 (Formula consequence), we have  $S = PvQ$ , with the following pages  $P := u \wedge r \rightarrow v \dashv\dashv \psi$  and  $Q := w \wedge \dashv\dashv \theta$ .



The next example introduces conflict by page witness.

**Example 2.4 (Page witness).** Consider the following draft D:



Notice that  $D = D' + x \dashrightarrow \neg \overline{Q}$ , with draft  $D' := x \xrightarrow{L} y \xrightarrow{L} z \dashrightarrow \neg E$  and page  $Q := \hat{u} \xrightarrow{L} v' \xrightarrow{K} v'' \xrightarrow{L} w \dashrightarrow \neg E$ . Now, draft  $D'$  has a copy of page  $Q$  (under morphism given by  $u \mapsto x; v', v'' \mapsto y; w \mapsto z$ ).

We consider two kinds of witness at a node  $w$  of a sketch  $\Sigma = \langle N; A \rangle$ .

(E) Expression witness: expression  $E$  with 1-ary arcs  $E \dashrightarrow w \dashrightarrow \neg E^c \in A$ .

(P) Page witness: page  $P = (P; u)$ , for which there is a morphism  $\mu : P \dashrightarrow \Sigma$  with  $w = u^\mu$  and  $w \dashrightarrow \neg P^c \in A$ .

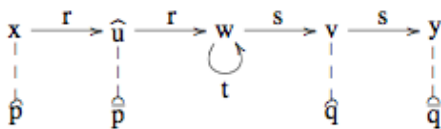
The zero objects are as follows. A sketch is zero iff it has a witness. A page is zero iff its underlying draft is zero. A book is zero iff all its pages are zero. For instance, page  $Q'$  in Example 1.2 (Satisfiable set) is not zero, and draft  $D$  in Example 2.4 (Page witness) is zero.

**Corollary 2.2 (Conflicts).** A zero sketch is unsatisfiable. A zero page or book has empty extension in every model.

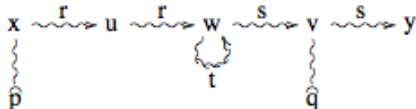
**Proof.** By Lemma 2.1: Transfers.

**Remark 2.2 (Finiteness).** ( $\mu$ ): Given drafts  $C$  and  $D$  and a function  $\mu : N_C \rightarrow N_D$ , one can effectively decide whether  $\mu$  is a morphism from  $C$  to  $D$ . ( $Z$ ): One can effectively decide whether a draft, a page or a book is zero. ( $\leftarrow$ ): One can effectively decide covering between pages or books.

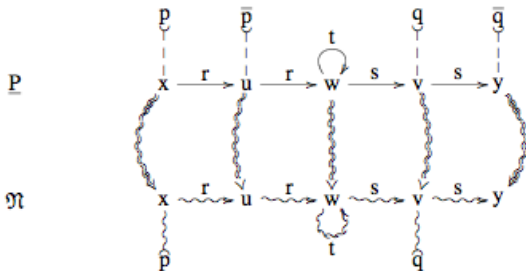
**Example 2.5 (Natural structure).** Consider the following page  $P$ :



*Its natural structure  $\mathfrak{N}$  consists of its positive information, as follows:*



The identity assignment  $1 : \underline{\mathbf{P}} \rightarrow \mathfrak{N}$  satisfies the arcs of  $\underline{\mathbf{P}}$ :



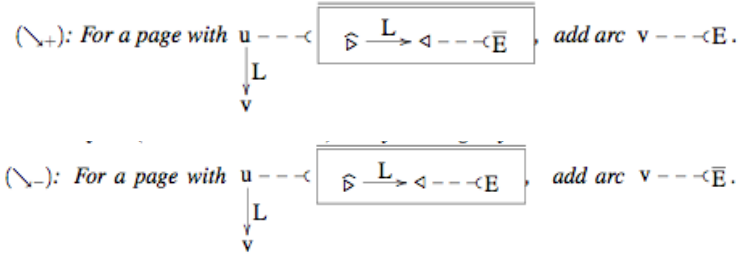
Thus,  $u$  is in the behaviour of  $P$  in  $N$ :  $u \in [P]_N$ .

## 2.3 Graphical Calculi

A graphical calculus has rules for manipulating expressions. Some rules are special: the elimination rules hinge on the semantics of the operators, other rules involve properties of relations. The following rules, however, are general: in each one of these rules, both sides have the same extension in every structure.

The singleton rules convert a page  $P$  to its singleton book  $\{P\}$  and vice-versa (cf. 2.1). The promotion rule converts an expression  $E$  to its page  $P_e(E) = x^{\wedge} \cdots \subset E$  (cf. 2.1). The zero erase rule (Z) erases a zero page (cf. 2.2). The alternative expansion rule (!) expands a page  $P$  to the 2-page book  $\{P + w|E, P + w|E^c\}$ .

**Corollary 2.3** (Derived conversions). The following shift conversions are derived.



Proof. By rules alternative expansion ( $\downarrow$ ), zero erase ( $Z$ ) and the singleton rule.

The complement rules eliminate double complement and move complement inside (as in De Morgan laws). Table 1 gives the 3 complement rules.

Table 1: Complement rules

$(\bar{\bar{\phantom{x}}})$ Eliminate double complement	$\bar{\bar{E}} \quad \bar{\phantom{x}} \quad E$
$(\sqcup)$ Convert	$\underbrace{\{E_1, \dots, E_n\}}_{\text{book}}$ to $\underbrace{\begin{array}{c} \hat{x} \\ \swarrow \quad \searrow \\ E_1 \quad \dots \quad E_n \end{array}}_{\text{page}}$
$(\bar{\cap})$ Convert	$\underbrace{\begin{array}{c} \hat{w} \\ \swarrow \quad \searrow \\ E_1 \quad \dots \quad E_n \end{array}}_{\text{page}}$ to $\underbrace{\left\{ \begin{array}{c} \hat{w} \dashv \vdash \neg \bar{E}_1 \\ \vdots \\ \hat{w} \dashv \vdash \neg \bar{E}_n \end{array} \right\}}_{\text{book}}$

The structural rules eliminate an arc whose expression is a page (by gluing, cf. 2.2) or a book (cf. 2.1). Table 2 gives the 2 structural rules.

Table 2: Structural rules

$(\downarrow)$ Convert	$\underbrace{P + w \dashv \vdash \neg Q}_{\text{page}}$ to $\underbrace{P_w Q}_{\text{glued page}}$
$(\leftarrow)$ Convert	$\underbrace{P + w \dashv \vdash \neg H}_{\text{page}}$ to $\underbrace{\{P + w \dashv \vdash \neg Q / Q \in H\}}_{\text{book}}$

We often wish to consider restricted frames. We now introduce rules for capturing some properties of a relation:

(**Rfl**[L]) For a reflexive relation: expand node  $w$  to  $w \begin{smallmatrix} \curvearrowright \\ L \end{smallmatrix}$ .

(**Trn**[L]) For a transitive relation: add  $u \begin{smallmatrix} \curvearrowright \\ L \end{smallmatrix} w$ , when  $u \xrightarrow{L} v \xrightarrow{L} w$ .

(**Smt**[L]) For a symmetric relation: add  $u \begin{smallmatrix} \curvearrowleft \\ L \end{smallmatrix} v$ , when  $u \xrightarrow{L} v$ .

(**Asm**[L]) For an anti-symmetric relation: identify nodes  $u$  and  $v$  with  $u \begin{smallmatrix} \xrightarrow{L} \\ \xleftarrow{L} \end{smallmatrix} v$ .

We will illustrate how these rules are used in 3.2 and 4.1.<sup>11</sup>

It may be convenient to have logical relations like square  $\boxtimes$  and diversity  $\partial$ .<sup>12</sup> As the meaning of  $\boxtimes$  is  $M \times M$ , we can eliminate it by erasing  $\boxtimes$ -arcs. We do not eliminate  $\partial$ , but we can handle it by the 2 rules: erase a page with a  $\partial$ -loop and expand a page  $P$  with nodes  $u$  and  $v$  to the book  $\{P', P + u \partial v\}$ , where page  $P'$  is obtained from  $P$  by identifying nodes  $u$  and  $v$  (see also [VVB 15]).

**Remark 2.3** (Graphical calculi). Each graphical calculus is sound and complete for the corresponding structures.<sup>13</sup>

In the sequel, we will apply this graphical machinery to modal logics: classical ones (in Sect. 3) and intuitionistic ones (in Sect. 4), with language  $ML$  (cf. Sect. 1). In each case, we will formulate a graph language where we can represent formulas by expressions, to be manipulated by appropriate meaning-preserving elimination rules, so as to establish validity and consequence. We use the expression pages in 2.1: Graphical Languages. If we can convert  $Pe(\varphi^c)$  to the

<sup>11</sup> We can also handle irreflexive, serial, dense and confluent relations [VVB 15].

<sup>12</sup> We can use these relations for particular modalities (see 3.2: Special Modalities).

<sup>13</sup> For more details about graphical calculi see, e. g. [VV 11, VVB 14] and references therein.

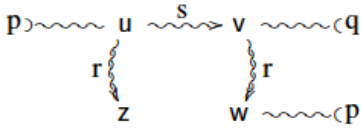
empty book  $\{ \}$ , then formula  $\phi$  is valid; otherwise, it is not valid: the natural construction (as in Example 2.5: Natural structure) will give a counter-model. To show that formula  $\theta$  is a consequence of  $\psi$ , we proceed similarly with page  $Pp(\psi, \theta)$ . One can also establish consequence directly by converting  $Pe(\psi)$  and  $Pe(\theta)$  to books  $G$  and  $H$ , respectively, with  $G \leftarrow H$  (cf. Lemma 2.1: Transfers).

### 3 Classical Modal Logics

We now consider classical modal logics: basic logic (in 3.1) and extensions (in 3.2).

The classical semantics of modal language  $\mathbf{ML}$  is based on Kripke frames and models [BRV95]. A *Kripke frame*  $\mathfrak{F}$  consists of a non-empty set  $W$ ; together with a 2-ary relation  $r^{\mathfrak{F}}$  on  $W$ , for each relation symbol  $r \in \mathbf{RN}$ . A *Kripke model*  $\mathfrak{R}$  consists of an underlying frame  $\mathfrak{R}$ , with universe  $W \neq \emptyset$ ; together with a subset  $p^{\mathfrak{R}} \subseteq W$ , for each propositional letter  $p \in \mathbf{PL}$ . We set  $r^{\mathfrak{R}} := r^{\mathfrak{R}}$ , for  $r \in \mathbf{RN}$ .

A Kripke model  $\mathfrak{R}$  with  $W = \{u, v, w, z\}$ , relations  $r^{\mathfrak{R}} = \{(u, z), (v, w)\}$  and  $s^{\mathfrak{R}} = \{(u, v)\}$ , as well as subsets  $p^{\mathfrak{R}} = \{u, w\}$  and  $q^{\mathfrak{R}} = \{v\}$  can be depicted as:



### 3.1 Basic Classical Modal Logic

We now consider basic classical modal logic: with unrestricted Kripke frames.

*Classical satisfaction* of formula  $\varphi \in \Phi$  at world  $u \in W$  of  $\mathfrak{K}$  (noted  $u \Vdash_{\mathfrak{K}} \varphi$ , or simply  $u \Vdash \varphi$ ) is recursively defined as follows.

- ( $\perp$ ) No state satisfies  $\perp$ , i. e.  $u \not\Vdash \perp$ , for every  $u \in W$ .
- (p) Satisfaction of formula  $p$ , with  $p \in \mathbf{PL}$ :  $u \Vdash p$  iff  $u \in p^{\mathfrak{K}}$ .
- ( $\wedge$ ) For conjunction:  $u \Vdash \psi \wedge \theta$  iff  $u \Vdash \psi$  and  $u \Vdash \theta$ .
- ( $\vee$ ) For disjunction:  $u \Vdash \psi \vee \theta$  iff  $u \Vdash \psi$  or  $u \Vdash \theta$ .
- ( $\rightarrow$ ) For  $\rightarrow$ :  $u \Vdash \psi \rightarrow \theta$  iff  $u \not\Vdash \psi$  or  $u \Vdash \theta$ , i. e. it is not the case that  $u \Vdash \psi$  and  $u \not\Vdash \theta$ .
- ( $\langle \rangle$ ) For  $\langle \rangle$ :  $u \Vdash \langle x \rangle \varphi$  iff for some  $v \in W$ , such that  $(u, v) \in r^{\mathfrak{K}}$ ,  $v \Vdash \varphi$ .
- ( $[]$ ) For  $[]$ :  $u \Vdash [x] \varphi$  iff for every  $v \in W$ , such that  $(u, v) \in r^{\mathfrak{K}}$ ,  $v \Vdash \varphi$ , i. e. there exists no  $v \in W$ , such that  $u r^{\mathfrak{K}} v$  and  $v \not\Vdash \varphi$ .

Now, a Kripke model  $\mathfrak{K}$  assigns to a formula  $\varphi \in \Phi$  the set of worlds satisfying it, more precisely  $\varphi^{\mathfrak{K}} := \{u \in W / u \Vdash_{\mathfrak{K}} \varphi\}$ . Thus, the above satisfaction conditions can be rewritten as follows (with  $\tilde{S} := W \setminus S$ ). ( $\perp$ ):  $\perp^{\mathfrak{K}} = \emptyset$ . ( $\wedge$ ):  $(\psi \wedge \theta)^{\mathfrak{K}} = \psi^{\mathfrak{K}} \cap \theta^{\mathfrak{K}}$ . ( $\vee$ ):  $(\psi \vee \theta)^{\mathfrak{K}} = \psi^{\mathfrak{K}} \cup \theta^{\mathfrak{K}}$ . ( $\rightarrow$ ):  $(\psi \rightarrow \theta)^{\mathfrak{K}} = \widetilde{(\psi^{\mathfrak{K}} \cap \theta^{\mathfrak{K}})}$ . ( $\langle \rangle$ ):  $u \in (\langle x \rangle \varphi)^{\mathfrak{K}}$  iff, for some  $v$  with  $u r^{\mathfrak{K}} v$ ,  $v \in \varphi^{\mathfrak{K}}$ . ( $[]$ ):  $u \in ([x] \varphi)^{\mathfrak{K}}$  iff, for no  $v$  with  $u r^{\mathfrak{K}} v$ , we have  $v \in \varphi^{\mathfrak{K}}$ .

We now consider a graphical language  $\mathbb{GL}_c$  with  $\mathbf{Sb}_1 := \Phi$  and  $\mathbf{Sb}_2 := \mathbf{RN}$ . A Kripke model  $\mathfrak{K}$  gives a structure  $\mathfrak{S}$  for  $\mathbb{GL}_c$  with  $\varphi^{\mathfrak{S}} := \{u \in W / u \Vdash_{\mathfrak{K}} \varphi\}$ .

Then, we have  $\mathbf{GL}^c$ -expressions with the appropriate extensions (cf. Remark 2.1: Special pages and Corollary 2.1: Expression pages).

( $\perp$ ) For formula  $\perp$ , we have the empty book  $\{ \}$ .

(p) For propositional letter p, we have the single-node page  $\text{Pe}(p) = \hat{x} \dashv \vdash p$ .

( $\wedge$ ) For conjunction  $\psi \wedge \theta$ , we have the single-node page  $\psi \dashv \vdash \hat{x} \dashv \vdash \theta$ .

( $\vee$ ) For disjunction  $\psi \vee \theta$ , we have the 2-page book  $\{ \psi \dashv \vdash \hat{x}, \hat{x} \dashv \vdash \theta \}$ .

( $\rightarrow$ ) For  $\psi \rightarrow \theta$ , we have the complemented single-node page  $\boxed{\psi \dashv \vdash \hat{x} \dashv \vdash \bar{\theta}}$ .

( $\langle r \rangle$ ) For formula  $\langle r \rangle \phi$ , we have the 2-node page  $\hat{x} \xrightarrow{r} y \dashv \vdash \phi$ .

( $[r]$ ) For formula  $[r]\phi$ , we have the complemented 2-node page  $\boxed{\hat{x} \xrightarrow{r} y \dashv \vdash \bar{\phi}}$ .

Thus, we can eliminate logical symbols by means of the 6 classical elimination rules given in Table 3.

Table 3: Classical elimination rules

$P + w \dashv \vdash \perp$	$(\perp)$	$\{ \}$
$P + w \dashv \vdash \psi \wedge \theta$	$(\wedge)$	$P + \psi \dashv \vdash w \dashv \vdash \theta$
$P + w \dashv \vdash \psi \vee \theta$	$(\vee)$	$\{ P + \psi \dashv \vdash w, P + w \dashv \vdash \theta \}$
$P + w \dashv \vdash \psi \rightarrow \theta$	$(\rightarrow)$	$P + w \dashv \vdash \boxed{\psi \dashv \vdash \hat{x} \dashv \vdash \bar{\theta}}$
$P + u \dashv \vdash \langle r \rangle \phi$	$(\langle r \rangle)$	$P + u \xrightarrow{r} v \dashv \vdash \phi$ <span style="float: right;">new node v</span>
$P + w \dashv \vdash [r]\phi$	$([r])$	$P + w \dashv \vdash \boxed{\hat{x} \xrightarrow{r} y \dashv \vdash \bar{\phi}}$

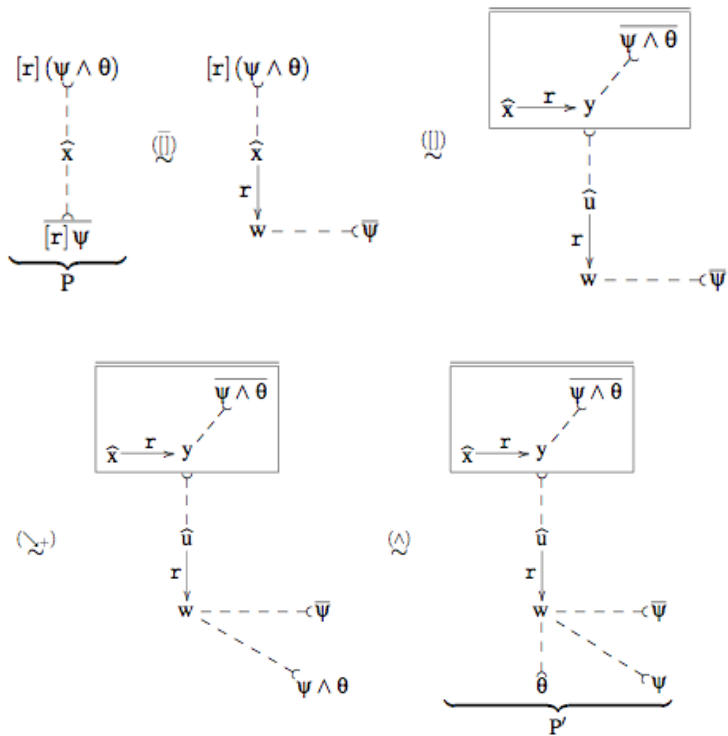
Example 1.1 (Formula consequence) shows that  $\langle r \rangle \psi$  is a classical consequence of  $\langle r \rangle (\psi \wedge \theta)$ .

Some classical derived conversions are  $P + w \dashv \vdash \neg \phi \quad (\neg) \quad P + w \dashv \vdash \bar{\phi}$   
and  $P + u \dashv \vdash \neg [r]\phi \quad ([r]) \quad P + u \xrightarrow{r} v \dashv \vdash \bar{\phi}$  (for a new node v).

Example 3.1 (Classical consequence via refutation). We can show graphically that  $[r]\psi$  is a classical consequence of  $[r](\psi \wedge \theta)$  as follows.

1. Form page  $P := \text{Pr}([r](\psi \wedge \theta), [r]\psi) = [r](\psi \wedge \theta) \rightarrow \neg \widehat{x} \rightarrow \neg \overline{[r]\psi}$ .

2. We then have the following conversions (cf. Corollary 2.3):



3. Page  $P'$  has a conflict (at  $w$ ) and is erased by rule (Z) (cf. 2.3).

### 3.2 Special Modalities

We now consider some classical modal logics with restricted frames. For these cases, we use rules such as those in 2.3: Graphical Calculi.

The next example illustrates how the transitive rule is used.



Example 3.2 (Transitive consequence). We can show graphically that  $\langle r \rangle \varphi$  is an  $r$ -transitive consequence of  $\langle r \rangle \langle r \rangle \varphi$  as follows.

(RHS) For  $\langle r \rangle \psi$ :  $\mathbb{P}\mathbb{e}(\langle r \rangle \varphi) = \widehat{x} \dashv \dashv \langle r \rangle \varphi$  converts to page  $R = \widehat{x} \xrightarrow{r} y \dashv \dashv \langle r \rangle \varphi$  (cf. Example 1.1: Formula consequence).

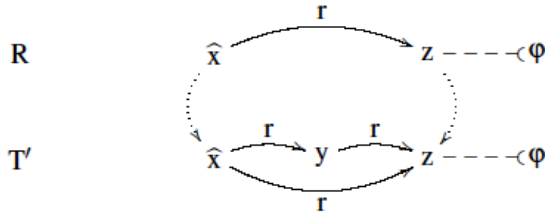
(LHS) Similarly:  $\widehat{x} \dashv \dashv \langle r \rangle \langle r \rangle \varphi \underbrace{\sim^{((\cdot))^2}}_{\mathbb{P}\mathbb{e}(\langle r \rangle \langle r \rangle \varphi)} \underbrace{\widehat{x} \xrightarrow{r} y \xrightarrow{r} z \dashv \dashv \langle r \rangle \varphi}_{T}.$

Notice that there is no homomorphism from  $R$  to  $T$ .

(LHS') We can apply transitivity ( $\mathbb{Tm}[r]$ ) (cf. 2.3) to  $T$  as follows:

$$T = \widehat{x} \xrightarrow{r} y \xrightarrow{r} z \dashv \dashv \langle r \rangle \varphi \underbrace{\sim^{(\mathbb{Tm}[r])}}_{\substack{\widehat{x} \xrightarrow{r} y \xrightarrow{r} z \\ r}} \widehat{x} \xrightarrow{r} y \xrightarrow{r} z \dashv \dashv \langle r \rangle \varphi = T'$$

( $\leftarrow$ ) We now have a homomorphism from  $R$  to  $T'$ :



Thus,  $\langle r \rangle \varphi$  is an  $r$ -transitive consequence of  $\langle r \rangle \langle r \rangle \varphi$ .

We can also show graphically that  $\langle r \rangle \varphi$  is an  $r$ -reflexive consequence of  $\varphi$  (by means of the reflexive rule  $R_{\mathbb{A}}[r]$  in 2.3).

Some interesting modalities are the universal one and difference:  $u$  satisfies  $E \varphi$  iff, for some world  $v \in W$ ,  $v$  satisfies  $\varphi$  and  $u$  satisfies  $D \varphi$  iff, for some world  $v \in W$  with  $u \neq v$ ,  $v$  satisfies  $\varphi$  [BRV 95]. We can handle them by the special relations in 2.3 (Graphical Calculi): it suffices to define  $E \varphi := \langle \boxtimes \rangle \varphi$  and  $D \varphi := \langle \hat{\cdot} \rangle \varphi$ . Then, we can show graphically that  $E \varphi$  is a consequence of  $\varphi$  and that  $D \varphi$  is a consequence of  $\neg \varphi \wedge \langle r \rangle \varphi$ .

A classical graphical calculus consists of the general graphical calculus rules (in 2.3: Graphical Calculi) together with the 6 classical elimination rules (in Table 3), extended by rules for properties of some relations. We thus have graphical calculi for classical modal logics like K, T, etc. [BRV 95]

**Theorem 3.1 (Classical graphical calculi).** Each classical graphical calculus is sound and complete for the corresponding models.

*Proof.* By Remark 2.3: Graphical calculi.

#### 4 Intuitionistic Modal Logics

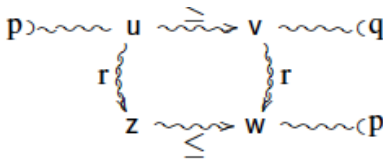
We now consider intuitionistic modal logics: flat (in 4.1) and graded (in 4.2).

##### 4.1 Flat Semantics

We first examine flat semantics for intuitionistic modal logic, as in [Smp 94].

The flat semantics of modal language  $\mathbf{ML}$  is based on flat frames and models. A *flat frame*  $\mathfrak{F}$  consists of a non-empty set  $W$ ; together with a 2-ary relation  $r^{\mathfrak{F}}$  on  $W$ , for each relation symbol  $r \in \mathbf{RN}$ , and a special 2-ary relation  $\leq$  on  $W$ . A *flat model*  $\mathfrak{C}$  consists of an underlying frame  $\mathfrak{C}$ , with universe  $W \neq \emptyset$ ; together with a subset  $p^{\mathfrak{C}} \subseteq W$ , for each propositional letter  $p \in \mathbf{PL}$ . We set  $r^{\mathfrak{C}} := r^{\mathfrak{F}}$ , for  $r \in \mathbf{RN}$ .

A flat model  $\mathfrak{C}$  with  $W = \{u, v, w, z\}$ , relation  $r^{\mathfrak{C}} = \{(u, z), (v, w)\}$ , special relation  $\leq = \{(u, v), (z, w)\}$ , and subsets  $p^{\mathfrak{C}} = \{u, w\}$  and  $q^{\mathfrak{C}} = \{v\}$  is as follows:



*Flat satisfaction* of formula  $\phi \in \Phi$  at world  $u \in W$  of  $\mathfrak{C}$  (noted  $u \Vdash_{\mathfrak{C}} \phi$ , or simply  $u \Vdash \phi$ ) is recursively defined as follows.

(•) For  $\perp$ ,  $p$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$ : as in classical modal logic (cf. 3.1).

( $\rightarrow$ ) For  $\rightarrow$ :  $u \Vdash \psi \rightarrow \theta$  iff for every  $v \geq u$ , if  $v \Vdash \psi$ , then  $v \Vdash \theta$ , i. e. there exists no  $v \geq u$  such that  $v \Vdash \psi$  and  $v \nVdash \theta$ .

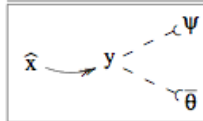
( $[ ]$ )  $u \Vdash_{\mathcal{C}} [r] \phi$  iff for all  $v, w \in W$ : if  $v \geq u$  and  $(v, w) \in r^{\mathcal{C}}$  then  $w \Vdash \phi$ , i. e. there are no  $v, w \in W$  such that  $v \geq u$ ,  $v r^{\mathcal{C}} w$  and  $w \nVdash \phi$ .

A flat model  $\mathcal{C}$  assigns to a formula  $\phi \in \Phi$  the set of worlds satisfying it, i. e.  $\phi^{\mathcal{C}} := \{u \in W / u \Vdash_{\mathcal{C}} \phi\}$ . So, we can proceed as in classical modal logic (cf. 3.1), and rewrite these satisfaction conditions as follows (with  $\tilde{S} := W \setminus S$ ).  $\perp$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$  are as before. ( $\rightarrow$ ):  $u \in (\psi \rightarrow \theta)^{\mathcal{C}}$  iff there exists no  $v \geq u$  with  $v \in \psi^{\mathcal{C}} \cap \tilde{\theta}^{\mathcal{C}}$ . ( $[ ]$ ):  $u \in ([r] \phi)^{\mathcal{C}}$  iff there are no  $v, w \in W$  with  $v r^{\mathcal{C}} w$  and  $w \in \tilde{\phi}^{\mathcal{C}}$ .

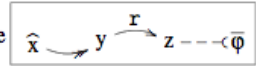
To reason graphically about flat semantics with a symbol  $wc$  for  $\leq$ , we consider a graphical language  $GL_F$  with  $^S b_1 := \Phi$  and  $^S b_2 := RN \cup \{wc\}$ .<sup>14</sup> We draw  $wc$ -arrows simply as  $\rightarrow$ . Then, we have  $GL_F$ -expressions with the appropriate extensions (cf. Remark 2.1: Special pages).

(•) For  $\perp$ ,  $\psi \wedge \theta$ ,  $\psi \vee \theta$  and  $\langle r \rangle \phi$ , we have expressions as before (cf. 3.1).

( $\rightarrow$ ) For conditional formula  $\psi \rightarrow \theta$ , we have the complemented 2-node page



( $[ ]$ ) For  $[r] \phi$ , we have the complemented 3-node page



Thus, we can eliminate logical symbols by flat elimination rules. The elimination rules for  $\perp$ ,  $p$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$  are as in Table 3 (Classical elimination rules); those for  $\rightarrow$  and  $[ ]$  appear in Table 4.

<sup>14</sup> A flat model  $\mathcal{C}$  gives a  $GL_F$ -structure  $S$  with  $\phi^S := \phi^{\mathcal{C}}$  and  $wc\ S := \leq$ .

Table 4: Flat elimination rules for  $\rightarrow$  and  $[]$

$$\begin{array}{lcl}
 P + w \dashv\vdash \neg \psi \rightarrow \theta & (\sim) & P + w \dashv\vdash \boxed{\begin{array}{c} \hat{x} \xrightarrow{\quad} y \xrightarrow{\quad} \psi \\ \phantom{\hat{x} \xrightarrow{\quad} y} \searrow \theta \end{array}} \\
 P + w \dashv\vdash \neg [r] \phi & ([]) & P + w \dashv\vdash \boxed{\begin{array}{c} \hat{x} \xrightarrow{\quad} y \xrightarrow{r} z \dashv\vdash \neg \bar{\phi} \end{array}}
 \end{array}$$

Table 5 shows some flat derived conversions.

Table 5: Flat derived conversions

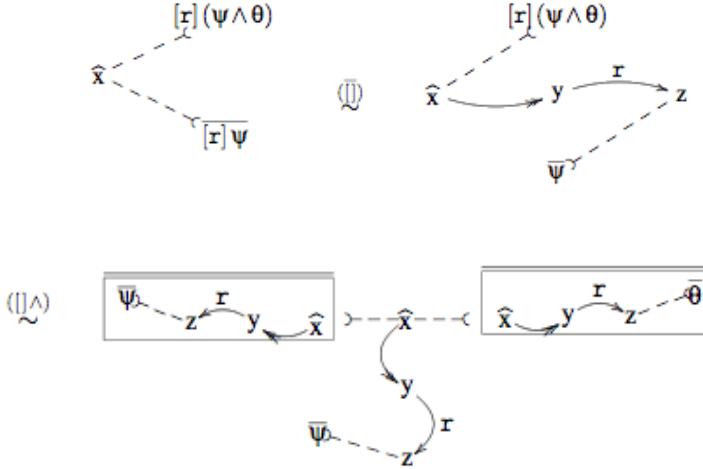
$$\begin{array}{lcl}
 P + w \dashv\vdash \neg \neg \phi & (\neg) & P + w \dashv\vdash \boxed{\hat{x} \xrightarrow{\quad} y \dashv\vdash \neg \phi} \\
 P + w \dashv\vdash \neg \neg \neg \phi & (\neg\neg) & P + w \dashv\vdash \boxed{\hat{x} \xrightarrow{\quad} y \dashv\vdash \boxed{\hat{x} \xrightarrow{\quad} y \dashv\vdash \neg \phi}} \\
 P + u \dashv\vdash \neg \neg \neg \phi & (\neg\neg) & P + u \xrightarrow{\quad} v \dashv\vdash \boxed{\hat{x} \xrightarrow{\quad} y \dashv\vdash \neg \phi} \quad \text{new node } v \\
 P + u \dashv\vdash \neg \overline{\psi \rightarrow \theta} & (\neg\rightarrow) & P + u \xrightarrow{\quad} v \dashv\vdash \neg \psi \quad \text{new node } v \\
 & & \phantom{P + u \xrightarrow{\quad} v} \searrow \theta \\
 P + w \dashv\vdash \neg \neg [r] \phi & (\neg[]) & P + w \dashv\vdash \boxed{\hat{x} \xrightarrow{\quad} y \xrightarrow{r} z \dashv\vdash \neg \phi} \\
 P + u \dashv\vdash \neg \neg [r] \phi & (\neg[]) & P + u \xrightarrow{\quad} v \xrightarrow{r} w \dashv\vdash \neg \phi \quad \text{new nodes } v, w \\
 P + u \dashv\vdash \neg [r] \phi & ([]) & P + u \xrightarrow{\quad} v \xrightarrow{r} w \dashv\vdash \neg \bar{\phi} \quad \text{new nodes } v, w \\
 P + u \dashv\vdash \neg [r] \phi & ([]) & P + u \xrightarrow{\quad} u' \xrightarrow{r} v \xrightarrow{\quad} v' \dashv\vdash \neg \bar{\phi} \quad \text{new nodes } u', v, v' \\
 \hat{x} \dashv\vdash \neg [r] (\psi \wedge \theta) & ([\wedge]) & \boxed{\psi \dashv\vdash z \xrightarrow{r} y \xrightarrow{\quad} \hat{x}} \dashv\vdash \hat{x} \dashv\vdash \boxed{\hat{x} \xrightarrow{\quad} y \xrightarrow{r} z \dashv\vdash \neg \bar{\theta}}
 \end{array}$$

Example 4.1 (Flat derivations). Formula  $\neg \langle r \rangle \perp$  is flat valid and  $[r] \psi$  is a flat consequence of  $[r] (\psi \wedge \theta)$ . We use the derived conversions in Table 5.

1. For page  $\mathbb{P}e(\neg \langle r \rangle \perp)$ , we have the flat derivation:

$$\hat{x} \dashv \vdash \neg \langle r \rangle \perp \quad (\neg \langle \rangle) \quad \hat{x} \xrightarrow{\quad} y \xrightarrow{r} z \dashv \vdash \perp \quad (\perp) \quad \{ \}$$

2. For  $\mathbb{P}p([\mathbf{r}](\psi \wedge \theta), [\mathbf{r}]\psi)$ , we have the flat derivation:



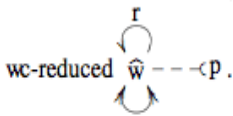
The resulting page  $Q$  is zero: page  $Q$  has as witness at node  $x$  the page  $\bar{\psi} \dashv \vdash z \xrightarrow{r} y \xrightarrow{\quad} \hat{x}$  (under morphism  $x \mapsto x, y \mapsto y, z \mapsto z$ ). So,  $Q \stackrel{(Z)}{\sim} \{ \}$ .

Thus,  $\neg \langle r \rangle \perp$  is flat valid and  $[\mathbf{r}]\psi$  is a flat consequence of  $[\mathbf{r}](\psi \wedge \theta)$ .

A PO frame is a flat frame  $R$  where special relation  $\leq$  is a partial order on  $W$ . To reason about PO frames and models, we add the 3 rules: (Rfl[wc]), (Asm[wc]) and (Trn[wc]) (cf. 2.3: Graphical Calculi).

A sketch, or page, of  $GL_F$  is wc-reduced iff  $u = v$ , whenever it has arcs  $u \leftrightarrow v$ . Every  $GL_F$ -page can be contracted to a wc-reduced page.

For instance, page  $\hat{u} \xrightarrow{r} v \dashv \vdash p$  is not wc-reduced, but it contracts to the



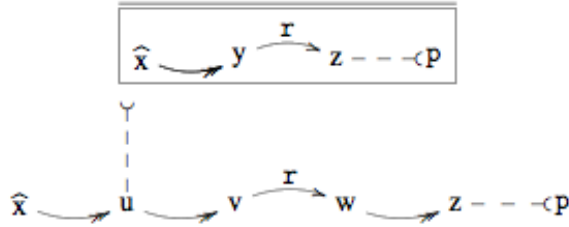
The natural construction (cf. Example 2.5: Natural structure) applied to a proper  $GL_C$ -sketch  $\Sigma$  (cf. 2.1: Graphical Languages) gives a flat model  $N[\Sigma]$ .

We can use this to show non-validity, as the next example illustrates.

**Example 4.2 (Non PO validity).** *Formula  $\neg(\langle r \rangle p \rightarrow [r] \neg p)$  is not PO valid.*

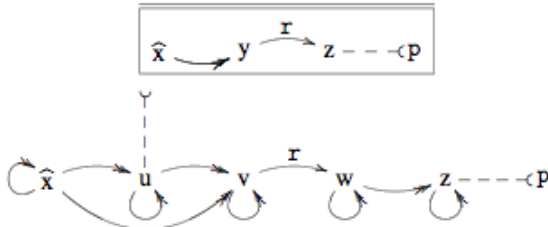
1. We start with page  $P \in (\neg(\langle r \rangle p \rightarrow [r] \neg p)) = \hat{x} \text{ --- } \neg(\langle r \rangle p \rightarrow [r] \neg p)$ .

2. We can convert this page  $P$  to the following page  $Q$



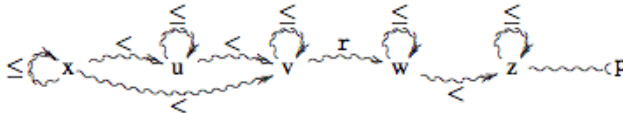
by rules  $(\Rightarrow)$ ,  $(\neg(\cdot))$  and  $(\neg(\cdot))$  (cf. Table 5).

3. Now, by the PO rules  $(Rfl[wc])$ ,  $(Ass[wc])$  and  $(Tm[wc])$ , we can convert page  $Q$  to the following page  $R$ :



Note that page  $R$  is  $wc$ -reduced and has no conflict.

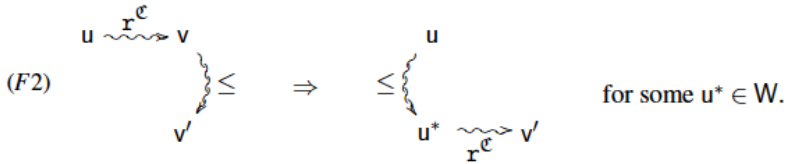
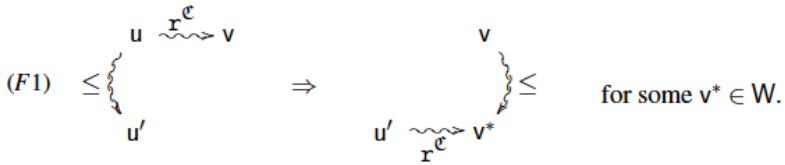
Page  $R$  gives natural model  $N[R]$  (cf. Example 2.5: Natural structure) as follows:



Notice that  $N[R]$  is a PO frame. We can also see that the identity assignment  $1 : R \rightarrow N[R]$  satisfies the arcs of  $R$ . Thus,  $N[R]$  is a PO model where, at world  $x$ , formula  $\neg\langle r \rangle p \rightarrow [r]\neg p$  does not hold.

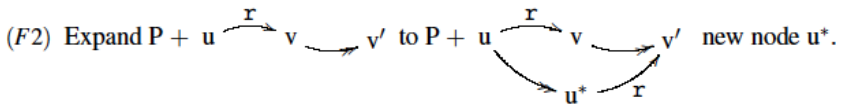
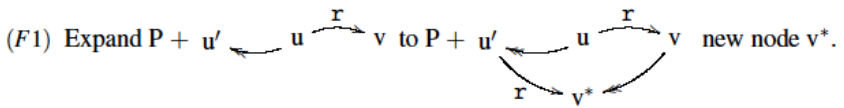
To have monotonicity of satisfaction, one restricts PO models to birelational models by imposing 3 extra requirements, for each  $p \in PL$  and  $r \in RN$  (cf. [Smp 94]).

(P) If  $u \in p^c$  and  $u \leq u'$ , then  $u' \in p^c$ .



To reason graphically about birelational models, we add the following 3 birelational rules, for each propositional letter  $p \in PL$  and relation symbol  $r \in RN$ .

(p) Contract page  $P + P \dashv \dashv u \xrightarrow{r} u' \dashv \dashv \neg\langle \bar{p} \rangle$  to the empty book  $\{ \}$ .



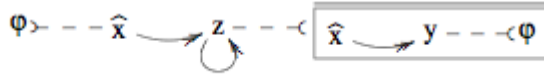
Then, we can derive the following birelational formula transfer conversion:  $P + \varphi \dashv \dashv \dashv \dashv u \rightarrow u' \sim^* P + \varphi \dashv \dashv \dashv \dashv u \rightarrow u' \dashv \dashv \dashv \dashv \varphi$ .<sup>15</sup>

15 By expansion rule (I): case  $\langle r \rangle \varphi$  follows from (F1) and case  $[r] \varphi$  follows from (Trn[wc])

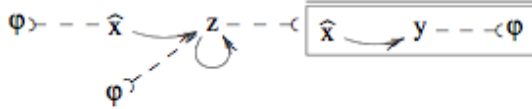
**Example 4.3 (Birelational consequence).** We can show graphically that  $\neg\neg\varphi$  is a birelational consequence of  $\varphi$  as follows.

1. We begin with page  $\text{Pp}(\varphi, \neg\neg\varphi) = \varphi \multimap \hat{x} \multimap \neg\neg\varphi$  and convert it, by  $(\neg\neg)$  (cf. Table 5), to  $\text{P} = \varphi \multimap \hat{x} \multimap z \multimap \boxed{\hat{x} \multimap y \multimap \neg\varphi}$ .

2. By reflexivity ( $\text{Rfl}[\text{wc}]$ ), page  $\text{P}$  expands to the following page  $\text{Q}$ :



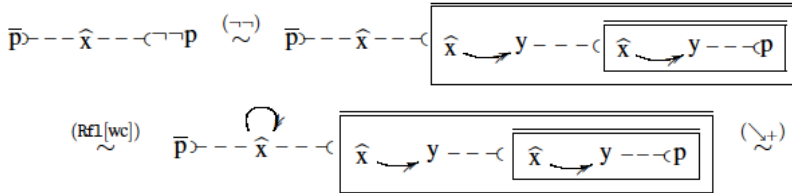
3. Now, transfer formula  $\varphi$  from  $x$  to  $z$  to obtain page  $\text{R}$  as follows:



Page  $\text{R}$  is zero: it has as witness at node  $z$  the page  $\hat{x} \multimap y \multimap \neg\varphi$  (under morphism  $x, y \mapsto z$ ). So,  $\text{R} \stackrel{(Z)}{\sim} \{ \}$ .

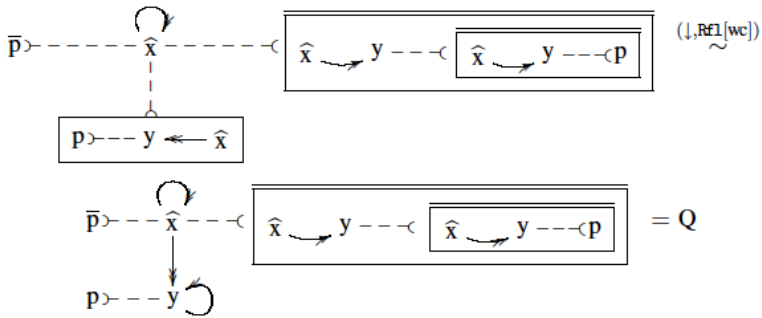
We can show the birelational validity of the following formulas (cf. [Smp 94, p. 51, 52]):  $[r](\psi \rightarrow \theta) \rightarrow ([r]\psi \rightarrow [r]\theta)$ ,  $[r](\psi \rightarrow \theta) \rightarrow (\langle r \rangle \psi \rightarrow \langle r \rangle \theta)$ ,  $\neg \langle r \rangle \perp$ ,  $\langle r \rangle (\psi \vee \theta) \rightarrow \langle r \rangle \psi \vee \langle r \rangle \theta$ ,  $\langle r \rangle \psi \rightarrow [r]\theta \rightarrow [r](\psi \rightarrow \theta)$  and  $\neg \langle r \rangle \varphi \rightarrow [r]\neg\varphi$ .<sup>16</sup>

**Example 4.4 (Non birelational consequence).** We can show that  $p$  is not a birelational consequence of  $\neg\neg p$  as follows. We have the following conversions:



<sup>16</sup> In fact, formula  $\neg \langle r \rangle \perp$  is flat valid, cf. Example 4.1: Flat derivations.





Page  $Q$  is not zero, and gives the natural model  $\mathfrak{N}[Q] = \begin{matrix} \overset{<}{\curvearrowright} & \overset{<}{\curvearrowright} \\ x & \rightsquigarrow & y \\ & \searrow & \nearrow \\ & p & \end{matrix}$ . Notice that model  $\mathfrak{N}[Q]$  is birelational. We can also see that the identity assignment  $1 : Q \rightarrow \mathfrak{N}[Q]$  satisfies the arcs of  $Q$ . Thus,  $\mathfrak{N}[Q]$  is a birelational model where, at world  $x$ ,  $\neg\neg p$  holds but  $p$  does not hold.  $\square$

The special binary relation  $\leq$  of a flat structure may be symmetric. For such cases, we use the rule (Smt[wc]) (cf. 2.3: Graphical Calculi). We can thus show graphically that  $\varphi$  is a symmetric birelational consequence of  $\neg\neg\varphi$ . We can similarly show that  $\varphi \vee \neg\varphi$  is valid in symmetric birelational structures.

We have a hierarchy of flat graphical calculi: the flat graphical calculus consists of the general graphical calculus rules (in 2.3: Graphical Calculi) together with the flat elimination rules, which can be extended by the rules for PO, birelational and symmetry.

**Theorem 4.1 (Flat graphical calculi).** Each flat graphical calculus is sound and complete for the corresponding models.

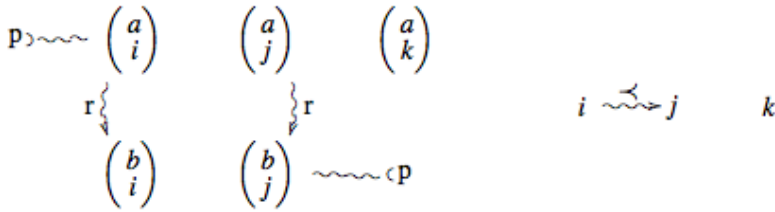
**Proof.** By Remark 2.3: Graphical calculi.

## 4.2 Graded Semantics

We now examine graded semantics for intuitionistic modal logic. The motivation comes from decoupling objects and stages, much as in [Ewd 86].

The graded semantics of modal language  $\mathbb{M}$  is based on graded frames and models. A *graded frame*  $\mathfrak{J}$  consists of a non-empty subset  $A_\times \subseteq A \times I$  (where  $A$  and  $I$  are non-empty sets); together with a 2-ary relation  $r^\mathfrak{J}$  on  $A_\times$ , for each relation symbol  $r \in \mathbb{R}\mathbb{N}$ , and a special 2-ary relation  $\preceq$  on  $I$ . A *graded model*  $\mathfrak{D}$  consists of an underlying frame  $\mathfrak{D}$ , with universe  $A_\times \neq \emptyset$ ; together with a subset  $p^\mathfrak{D} \subseteq A_\times$ , for each propositional letter  $p \in \mathbb{P}\mathbb{L}$ . We set  $r^\mathfrak{D} := r^\mathfrak{D}$ , for  $r \in \mathbb{R}\mathbb{N}$ . We will also consider some restrictions as in 4.1: Flat Semantics.

Consider the following pair of diagrams:



These diagrams represent a graded model  $\mathfrak{D}$  with sets  $A = \{a, b\}$ ,  $I = \{i, j, k\}$  and  $A_\times = \left\{ \begin{pmatrix} a \\ i \end{pmatrix}, \begin{pmatrix} b \\ i \end{pmatrix}, \begin{pmatrix} a \\ j \end{pmatrix}, \begin{pmatrix} b \\ j \end{pmatrix}, \begin{pmatrix} a \\ k \end{pmatrix} \right\}$ , special relation  $\preceq = \{(i, j)\}$ , relation  $r^\mathfrak{D}$  with  $\begin{pmatrix} a \\ i \end{pmatrix} r^\mathfrak{D} \begin{pmatrix} b \\ i \end{pmatrix}$  and  $\begin{pmatrix} a \\ j \end{pmatrix} r^\mathfrak{D} \begin{pmatrix} b \\ j \end{pmatrix}$ , and subset  $p^\mathfrak{D} = \left\{ \begin{pmatrix} a \\ i \end{pmatrix}, \begin{pmatrix} b \\ j \end{pmatrix} \right\}$ .

*Graded satisfaction* of formula  $\phi \in \Phi$  at state  $\begin{pmatrix} a \\ i \end{pmatrix} \in A_\times$  of  $\mathfrak{D}$  (which we note  $\begin{pmatrix} a \\ i \end{pmatrix} \Vdash_{\mathfrak{D}} \phi$ , or simply  $\begin{pmatrix} a \\ i \end{pmatrix} \Vdash \phi$ ) is recursively defined as follows.

(•)  $\perp$ ,  $p$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$ , as before (cf. 3.1 and 4.1), with  $u := \begin{pmatrix} a \\ i \end{pmatrix}$ .

( $\rightarrow$ )  $\begin{pmatrix} a \\ i \end{pmatrix} \Vdash \psi \rightarrow \theta$  iff for every  $j \succeq i$ , if  $\begin{pmatrix} a \\ j \end{pmatrix} \Vdash \psi$ , then  $\begin{pmatrix} a \\ j \end{pmatrix} \Vdash \theta$ , i. e. there exists no  $j \succeq i$  such that  $\begin{pmatrix} a \\ j \end{pmatrix} \Vdash \psi$  and  $\begin{pmatrix} a \\ j \end{pmatrix} \nVdash \theta$ .

$$\begin{aligned}
 ([\perp]) \quad \left(\frac{a}{i}\right) \Vdash [r]\varphi \quad & \text{iff} \quad \text{for all } j \succeq i, b \in A: \text{ if } \left(\frac{a}{j}\right) r^{\mathfrak{D}} \left(\frac{b}{j}\right) \text{ then } \left(\frac{b}{j}\right) \Vdash \varphi, \\
 \text{i. e.} \quad & \text{there are no } j \succeq i \text{ and } b \in A \text{ such that } \left(\frac{a}{j}\right) r^{\mathfrak{D}} \left(\frac{b}{j}\right) \text{ and } \left(\frac{b}{j}\right) \nVdash \varphi.
 \end{aligned}$$

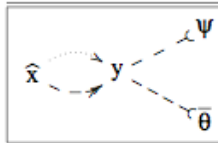
A graded model  $\mathfrak{D}$  assigns to a formula  $\varphi \in \Phi$  the set of states satisfying it:  $\varphi^{\mathfrak{D}} := \{(a, i) \in A_{\times} / (a, i) \Vdash_{\mathfrak{D}} \varphi\}$ . So, proceeding as in 3.1 and 4.1, we can rewrite these satisfaction conditions as follows (with  $\tilde{S} := A_{\times} \setminus S$ ).  $\perp$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$  are as before (now with  $u := (a, i)$ ).  $(\rightarrow)$ :  $\left(\frac{a}{i}\right) \in (\psi \rightarrow \theta)^{\mathfrak{D}}$  iff there is no  $j \succeq i$  such that  $\left(\frac{a}{j}\right) \in \psi^{\mathfrak{D}} \cap \tilde{\theta}^{\mathfrak{D}}$ .  $([\perp])$ :  $\left(\frac{a}{i}\right) \in ([r]\varphi)^{\mathfrak{D}}$  iff there are no  $j \succeq i$  and  $b \in A$  such that  $\left(\frac{a}{j}\right) r^{\mathfrak{D}} \left(\frac{b}{j}\right)$  and  $\left(\frac{b}{j}\right) \in \tilde{\varphi}^{\mathfrak{D}}$ .

We wish to express these extensions in an appropriate graphical language. For this purpose, we first introduce a binary relation  $\leq$  on  $A_{\times}$  by  $\left(\frac{a}{i}\right) \leq \left(\frac{b}{j}\right)$  iff  $i \preceq j$ . We now consider symbols  $\mathfrak{sc}$  (for  $\leq$ ) and  $\mathfrak{eo}$  (with intended meaning  $(a, i) \mathfrak{eo} (b, j)$  iff  $a = b$ ) and, then, a graphical language  $\mathbb{GL}_{\mathfrak{g}}$  with  $\mathsf{Sb}_1 := \Phi$  and  $\mathsf{Sb}_2 := \mathbb{R}\mathbb{N} \cup \{\mathfrak{wc}, \mathfrak{eo}\}$ . We draw  $\mathfrak{sc}$ -arrows as  $\dashrightarrow$  and  $\mathfrak{eo}$ -arrows as  $\dashrightarrow$ .

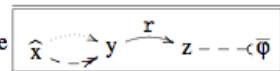
Then, we have  $\mathbb{GL}_{\mathfrak{g}}$ -expressions with the appropriate extensions (cf. Remark 2.1: Special pages).

(•) For  $\perp$ ,  $\psi \wedge \theta$ ,  $\psi \vee \theta$  and  $\langle r \rangle \varphi$ , we have the previous expressions (cf. 3.1: Basic Classical Modal Logic, and 4.1: Flat Semantics).

( $\rightarrow$ ) For conditional formula  $\psi \rightarrow \theta$ , we have the complemented 2-node page



( $[\perp]$ ) For  $[r]\varphi$ , we have the complemented 3-node page



So, we can eliminate logical symbols by graded elimination rules. The elimination rules for  $\perp$ ,  $p$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$  are as in in Table 3 (Classical elimination rules); those for  $\rightarrow$  and  $[]$  appear in Table 6.

Table 6: Graded elimination rules for  $\rightarrow$  and  $[]$

$$\begin{array}{l}
 P + w \dashv\dashv \neg \psi \rightarrow \theta \quad (\rightarrow) \quad P + w \dashv\dashv \neg \boxed{\begin{array}{c} \widehat{x} \dashv\dashv y \dashv\dashv \psi \\ \widehat{x} \dashv\dashv y \dashv\dashv \bar{\theta} \end{array}} \\
 P + w \dashv\dashv \neg [r] \varphi \quad ([]) \quad P + w \dashv\dashv \neg \boxed{\begin{array}{c} \widehat{x} \dashv\dashv y \xrightarrow{r} z \dashv\dashv \neg \bar{\varphi} \end{array}}
 \end{array}$$

So, we also have derived graded conversions as those in 4.1: Flat Semantics (cf. Table 5: Flat derived conversions), with  $\dashv\dashv$  in lieu of  $\rightarrow$ ; e. g.:

$$P + w \dashv\dashv \neg \neg \varphi \quad (\neg) \quad P + w \dashv\dashv \neg \boxed{\widehat{x} \dashv\dashv y \dashv\dashv \neg \varphi} .$$

Also, the intended meanings of  $\circ\circ$  and  $\preceq$  lead to some operational rules as follows (cf. 2.3: Graphical Calculi). For  $\circ\circ$ : ( $\text{Rfl}[\circ\circ]$ ), ( $\text{Smt}[\circ\circ]$ ) and ( $\text{Trm}[\circ\circ]$ ).

For partial order  $\preceq$ , we have ( $\text{Rfl}[\text{sc}]$ ), ( $\text{Asm}[\text{sc}]$ ), ( $\text{Trm}[\text{sc}]$ ), and the rule identifying nodes  $u$  and  $v$  such that  $u \preceq v$ . For a symmetric  $\preceq$ , we use  $\text{Smt}[\text{sc}]$ .

As in 4.1 (Flat Semantics), we also consider growing models.<sup>17</sup>

<sup>17</sup> Notice that these conditions are simpler and more intuitive than those in 4.1.

A *growing graded* model is a PO graded model  $\mathfrak{D}$  with growing universes, predicates and relations. For  $i \preceq j \in I$ : if  $(a, i) \in A_\times$  then  $(a, j) \in A_\times$ ; if  $(a, i) \in p^\mathfrak{D}$  then  $(a, j) \in p^\mathfrak{D}$  and if  $(a, i) r^\mathfrak{D} (b, i)$  then  $(a, j) r^\mathfrak{D} (b, j)$ . For the growing graded conditions, we have the following 3 *growing graded rules*.

For domain: given  $u \dashrightarrow v$ , add  $u \dashrightarrow u^* \dashrightarrow v$  (with new node  $u^*$ ).

For  $p \in \text{PL}$ : erase any page with  $p \dashrightarrow u \dashrightarrow v \dashrightarrow \neg p$ .

For  $r \in \text{RN}$ : given  $u' \dashrightarrow u \xrightarrow{r} v \dashrightarrow v'$ , add  $u' \xrightarrow{r} v'$ .

Then, we can derive the growing formula transfer (cf. 4.1: Flat Semantics):

$$P + \varphi \dashrightarrow u \dashrightarrow v \stackrel{(\varphi)}{\sim} P + \varphi \dashrightarrow u \dashrightarrow v \dashrightarrow \neg \varphi.$$

We can establish validity and consequence as in 4.1 (Flat Semantics), with



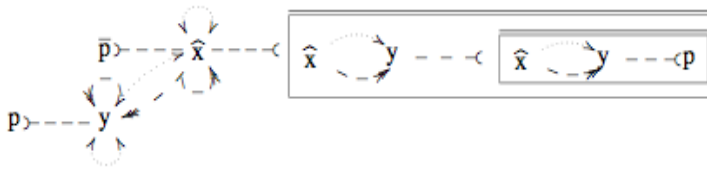
$\dashrightarrow$  in lieu of  $\rightarrow$ . We can show that  $[r] \psi$  is a graded consequence of  $[r] (\psi \wedge \theta)$  as in Example 4.1 (Flat derivations), that  $\neg \neg \varphi$  is a growing graded consequence of  $\varphi$  as in Example 4.3 (Birelational consequence), and also that  $\varphi$  is a symmetric growing graded consequence of  $\neg \neg \varphi$ .<sup>18</sup>

We can also establish non-consequence much as in 4.1 (Flat Semantics), even though the natural construction is now more involved, as it involves quotients. The next example illustrates the main ideas.<sup>19</sup>

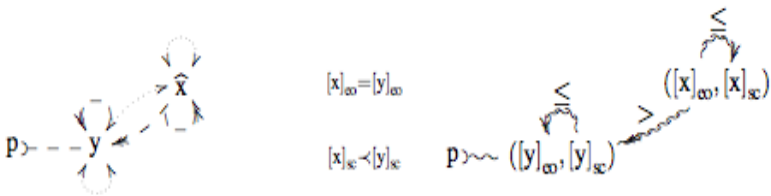
To see that  $p$  is not a growing graded consequence of  $\neg \neg p$ , we proceed as in Example 4.4 (Non birelational consequence), with  $\dashrightarrow$  in lieu of  $\rightarrow$ , as well as  $(\text{Rfl}[\text{sc}])$ ,  $(\text{Rfl}[\text{eo}])$  and  $(\text{Smt}[\text{eo}])$ . We obtain the following final page  $Q'$ :

<sup>18</sup> Symmetric growing graded structures have constant universes, predicates and relations.

<sup>19</sup> For details, see [VV 15].

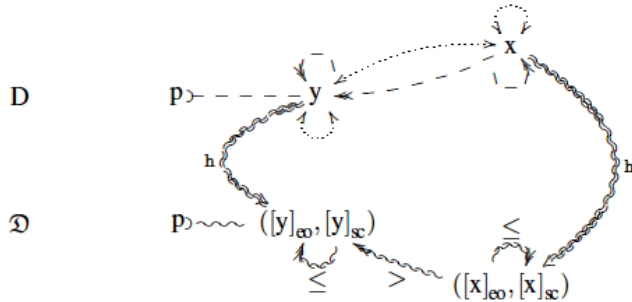


The positive part  $D$  of  $Q'$  and corresponding natural model  $\mathcal{D}$  are:



Notice that model  $\mathcal{D}$  is growing graded.

We see that the natural assignment  $h$  satisfies draft  $D$  in model  $\mathcal{D}$  as follows:



We can also see that the natural assignment  $h$  satisfies draft  $Q'$  in model  $\mathcal{D}$ .

We have a hierarchy of graded graphical calculi: the graded graphical calculus consists of the general graphical calculus rules (in 2.3: Graphical Calculi) together with the graded elimination rules and operational rules for eo, which can be extended by the PO, growing and symmetry rules.

**Theorem 4.2 (Graded graphical calculi).** Each graded graphical calculus is sound and complete for the corresponding models.

**Proof.** By Remark 2.3: Graphical calculi.

## 5 Comparison of Modal Logics

We now consider classical, flat and graded modal logics: the first two (in 5.1) and the other two (in 5.2); they have similar semantics for  $\perp$ ,  $p$ ,  $\wedge$ ,  $\vee$  and  $\langle \rangle$ . We will compare these modal logics graphically by means of the graph languages GLc (in 3.1: Basic Classical Modal Logic), GLf (in 4.1: Flat Semantics) and GLg (in 4.2: Graded Semantics).

### 5.1 Classical and Flat Modal Logics

We now compare graphically classical and flat modal logics. The result is not unexpected, but the development will serve to introduce the method to be used in 5.2.

Table 7 shows some formula representations in classical and flat semantics; other formulas have the same representations (cf. 3.1 and 4.1).

**Table 7: Classical and flat representations for  $\rightarrow$  and  $\langle \rangle$**

Formula	Classical	Flat
$\psi \rightarrow \theta$		
$\langle r \rangle \varphi$	$\hat{x} \xrightarrow{r} y \text{ --- } \neg \varphi$	

Consider the transformation FtCl: identify nodes  $u$  and  $v$  that are  $wc$  connected, i. e. such that  $u \rightarrow v$ . It transforms formula representations from flat to classical semantics (cf. Table 7).

**Remark 5.1** (Flat and classical rules). The transformation FtCl maps flat rules to classical rules.

$GL_c$ -expression  $F$  is associated to  $GL_f$ -expression  $E$  (noted  $E \Rightarrow F$ ) iff  $F$  is the result of applying FtCl to  $E$ . For derivations:  $F_1, \dots, F_n$ , in  $GL_c$ , is associated to  $E_1, \dots, E_n$ , in  $GL_f$ , iff  $E_i \Rightarrow F_i$ , for  $i = 1, \dots, n$ .

**Lemma 5.1** (Flat and classical derivations). Every flat  $GL_f$ -derivation  $\Pi_f$  has an associated classical  $GL_c$ -derivation  $\Pi_c$ .

Proof. By Remark 5.1: Flat and classical rules.

**Proposition 5.1** (Flat and classical formulas). If a modal formula  $\phi$  is flat (PO or birelational) derivable, then  $\phi$  is classically derivable.

Proof. By Lemma 5.1: Flat and classical derivations.

We now apply completeness of the classical and flat graphical calculi.

**Theorem 5.1** (Flat and classical semantics). If a modal formula  $\phi$  is flat (PO or birelational) valid, then  $\phi$  is classically valid.

Proof. By Proposition 5.1 (Flat and classical formulas) and Theorems 3.1 (Classical graphical calculi) and 4.1 (Flat graphical calculi).

## 5.2 Flat and Graded Modal Logics

We now compare graphically flat and graded modal logics.

Table 8 shows some formula representations in flat and graded semantics; other formulas have the same representations (cf. 4.1 and 4.2).



Table 8: Flat and graded representations for  $\rightarrow$  and  $[]$ 

Formula	Flat	Graded
$\psi \rightarrow \theta$		
$\langle r \rangle \varphi$		

Consider the replacement FtGr of wc by sc and eo. It transforms formula representations from flat to graded semantics (cf. Table 8). A  $GL_g$ -expression is neat iff sc and eo occur only in parallel arcs. A  $GL_g$ -derivation is neat iff it consists of neat expressions.

**Remark 5.2** (Graded and flat rules). The expressions in graded rules are neat. The replacement FtGr transforms flat rules to graded rules and vice-versa.

$GL_g$ -expression  $F$  is similar to  $GL_f$ -expression  $E$  (noted  $E \approx F$ ) iff  $F$  is the result of applying FtGr to  $E$ . For derivations:  $F_1, \dots, F_n$ , in  $GL_g$ , is similar to  $E_1, \dots, E_n$ , in  $GL_f$ , iff  $E_i \approx F_i$ , for  $i = 1, \dots, n$ .

**Lemma 5.2** (Flat and graded derivations). ( $\rightarrow$ ) Every flat  $GL_f$ -derivation  $\Pi_f$  has a neat graded  $GL_g$ -derivation  $\Pi_g$  similar to it, so that  $\Pi_g$  is PO or growing if  $\Pi_f$  is PO or birelational. ( $\leftarrow$ ) Every neat graded  $GL_g$ -derivation  $\Pi_g$  is similar to a flat  $GL_f$ -derivation  $\Pi_f$  (which is PO or birelational if  $\Pi_g$  is PO or growing).

*Proof.* By Remark 5.2: Graded and flat rules.

**Proposition 5.2** (Flat and graded formulas). A modal formula  $\varphi$  is flat (PO or birelational) derivable iff  $\varphi$  is graded (PO or growing) derivable.

*Proof.* By Lemma 5.2: Flat and graded derivations.

**Theorem 5.2** (Flat and graded semantics). The same modal formulas hold in flat (PO or birelational) and graded (PO or growing) structures.

*Proof.* By Proposition 5.2 (Flat and graded formulas) and Theorems 4.1 (Flat graphical calculi) and 4.2 (Graded graphical calculi).

## 6 Concluding Remarks

We have presented a flexible and uniform graphical approach to pictorial formalisms for multi-modal logics where one can express, analyse and compare possible-world semantics, validities and consequences.

Our approach explores the flexibility of graph calculi [VV 11, VVB 14] to express directly and graphically Kripke-like semantics of modal logics. This approach is uniform: once we have expressed the semantics (including properties of relations), we employ the corresponding (sound and complete) graph-calculus.

We have illustrated these ideas by applying them to some classical and intuitionistic modal logics (in Sects. 3: Classical Modal Logics and 4: Intuitionistic Modal Logics), which we have compared (in Sect. 5: Comparison of Modal Logics). We can also consider some variants as in [Smp94] (see [VV 15]) and some operations on relations (see [VVB 14, VVB 15]).

We thus have a flexible and uniform approach for constructing rigorous and intuitive formalism for analysis and visual exploration of multi-modal logics.

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