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Two basic results on translations between logics[1]

Dois resultados básicos sobre traduções entre lógicas

Abstract

The aim of the present paper is to show two basic results concerning translation between logics:

[1] The first result establishes that given two logics S1 and S2 with languages L1 and L2, and a translation F of L1 into L2 that interprets S1 into S2, then, given any intermediate logic S3 between S1 and S2, the same translation F interprets S1 into S3.

[2] The second result establishes that the translation F cannot interpret S3 into S2.

Keywords: Translations; Intermediate Logics.

Resumo

O objetivo do presente trabalho é mostrar dois resultados básicos relativos à tradução entre lógicas:

O primeiro resultado estabelece que, dadas duas lógicas S1 e S2, com linguagens L1 e L2, e uma tradução F de L1 em L2 que interpreta S1 em S2, então, dada qualquer lógica intermediária S3 entre S1 e S2, a mesma tradução F Interpreta S1 em S3.

[2] O segundo resultado estabelece que a tradução F não pode interpretar S3 em S2.

Palavras-chave: Traduções; Lógicas intermediárias.

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1. Introduction

In the late twenties and early thirties of last century several results were obtained connecting different logics and theories. These results assumed the form of translations/interpretations of one logic/theory into another logic/theory. In 1925 Kolmogorov defined a translation from classical logic into minimal logic aiming to show that "classical mathematics" could be translated into "intuitionistic mathematics" ([1] and [2]). In 1929 Glivenko [3] proved two fundamental results connecting provability in classical propositional logic to provability in intutionistic propositional logic. In 1933, Gödel ([4] and [6]) and Gentzen [7] independently defined an interpretation of Classical/Peano Arithmetic (PA) into Heyting´s arithmetic (HA). In 1933 Gödel [5] also defined an interpretation of intuitionistic propositional logic into classical modal logic S4. All these results were obtained in a foundational environment: the main motivation was to reduce foundational problems in a classical setting, like the consistency problem, to the same problems in an intuitionistic/constructive setting, aiming to show that if the problems could be solved in the intuitionistic setting, they could also be solved in the original classical setting.

The first general approach to translations was proposed by Prawitz and Malmnäs in [8]. According to Prawitz and Malmnäs, a translation from a deductive system/Logic S1 into a deductive system/Logic S2 is simply a function F from the language L1 of S1 into the language L2 of S2. S1 is said to be interpretable in S2 if the translation F satisfies the following condition:

$$\vdash_{S1} A \quad \text{iff} \vdash_{S2} F[A]$$

If for each set $\Gamma \cup \{A\}$ of formulas of L1 we have

$$\Gamma \vdash_{S1} A \quad \text{iff } F[\Gamma] \vdash_{S2} F[A]$$

Where $F[\Gamma] = \{F[B]: B \in \Gamma\}$

We say that S1 is interpretable in S2 with respect to derivability.

The aim of the present paper is to show two basic results concerning translation between logics:

[1] The first result establishes that given two logics S1 and S2 with languages L1 and L2, and a translation F of L1 into L2 that interprets S1 into S2, then, given any intermediate logic S3 between S1 and S2, the same translation F interprets S1 into S3.

[2] The second result establishes that the translation F cannot interpret S3 into S2.

2. Results

Let S2 and S1 be two logics such that every deductive relation in S2 is also a deductive relation in S1, but not every deductive relation in S1 is a deductive relation in S2. A well-know example is obtained if we take S2 = Intuitionistic Logic and S1 = Classical Logic. A logic S3 is said to be an intermediate logic between S1 and S2 if and only if:

(i) Every deductive relation in S2 is also a deductive relation in S3;
(ii) Not every deductive relation in S3 is a deductive relation in S2.
(iii) Every deductive relation in S3 is also a deductive relation in S1;
(iv) Not every deductive relation in S1 is a deductive relation in S3.

Theorem 1: Let S1 and S2 be two logics formulated in the languages L1 and L2 respectively, and let F be a translation from L1 into L2 that interprets S1 into S2. Let S3 be an intermediate logic between S1 and L2. Then, if F also satisfies the property that A -||- F(A) in S1, then F is a translation from L1 into the language L3 of S3 that interprets S1 into S3.

Proof: By hypothesis, $\Gamma \vdash_{S1} A \Leftrightarrow F(\Gamma) \vdash_{S2} F(A)$

Given that S3 is intermediate between S1 and S2,
$\Gamma \vdash_{S1} A \Rightarrow F(\Gamma) \vdash_{S2} F(A)$ and $F(\Gamma) \vdash_{S3} F(A)$.

The other direction follows directly from the hypothesis that

A -||- F[A] in S1

Remark: From this basic result we can conclude that double-negation translations (Gentzen/Gödel) can also interpret Classical logic into the logic of Constant Domains (CD) and into Dummett's logic. We can also give a general explanation why we can use the same translation to interpret classical logic into intuitionistic logic and minimal logic.

Theorem 2: The translation F of theorem 1 cannot be a translation from S3 into S2.

Proof: Assume F is a translation from S3 into S2. Then, $\Gamma \vdash_{S3} A \Leftrightarrow F(\Gamma) \vdash_{S2} F(A)$. Take any Γ and A such that $\Gamma \vdash_{S1} A$. Given that F is a translation from S1 into S2, we have that $F(\Gamma) \vdash_{S2} F(A)$ and hence that (because F is also a translation from S3 into S2) $\Gamma \vdash_{S3} A$. So, every deductive relation in S1 is also a deductive relation in S3 and S3 would not be an intermediate logic, as we had assumed.

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