Reformulation of Lot-Sizing Problems with Backlogging and Outsourcing

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Abstract

Recently. we proposed a lot-sizing model with outsourcing and also developed algorithms to solve it ([8]-[10]). Since lot-sizing problems are mixed integer programming, reformulate them in compact linear programming is always a challenge. In this manuscript, we review reformulation theory developed for lot-sizing models, especially these related to outsourcing models. Although relaxed reformulation of outsourcing model is same as the one with backlogging which has been solved and published, these two models differ. We show their differences in structure of extreme optimal solutions. Finally we give a linear reformulation for discrete lot-sizing model with outsourcing for a regeneration interval.

1 Introduction

Recently, we proposed a lot-sizing model with outsourcing and also developed algorithms to solve it. Lot-sizing problems as typical mixed-integer programming, the reformulation as compact linear form is a challenge. In recent decade, various reformulations of single item lot-sizing sets have been successful achieved by some famous researchers. These researches are based on theories and technicals not only crossing wide fields, but also theories developed especially for the structure of lot-sizing problems. In this manuscript, we mainly discuss the reformulation related to the outsourcing lot-sizing problems.

For lot-sizing models, there are two types of reformulation. One is facet defining inequalities with original variables, the other one is called extended formulation with additional variables. The common point of two types is using fractional parts of coefficients of the models. Because the special structure of lot-sizing models, the number of such fractional parts is polynomial bounded by the size of the original problems.

The manuscript is organized as follows. In Section 2, the formulation related to mixing and continuous mix set is introduced ([7]). Continuous mixing polyhedron with flows is introduced in Section 3 ([1] and [2]). The relations between the problems in Sections 2-3 and lot-sizing models with outsourcing or backlogging are shown in Section 4. In sections 5, difference between outsourcing and backlogging is discussed. Final section gives linear extended reformulation for lot-sizing models with outsourcing for a regeneration interval.

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2 Mixing and continuous mixing sets

The mixing set X^{MIX} is defined as:

$$\begin{aligned} & s + y_t \geq b_t, \quad 1 \leq t \leq n \\ & s \in \mathbf{R}_+, \;\; y \in \mathbf{Z}_+^n, \end{aligned}$$

where $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq 1$. (This can be obtained by $y'_t = y_t - \lfloor b_t \rfloor$ if necessary.)

For any $I \subseteq \{1, 2, \cdots, n\}$, let $\{\pi_1, \pi_2, \cdots, \pi_{|I|}\}$ be a permutation of I such that $b_{\pi_1} \leq b_{\pi_2} \leq \cdots \leq b_{\pi_{|I|}}$. It is shown by Pochet and Wolsey ([5]) that the following mixing inequalities are enough to describe the convex hull of mixing set.

$$s \geq \sum_{i=1}^{|I|} (b_{\pi_i} - b_{\pi_{i-1}})(1 - y_{\pi_i}),$$

$$s \geq \sum_{i=1}^{|I|} (b_{\pi_i} - b_{\pi_{i-1}})(1 - y_{\pi_i}) + (1 - b_{|I|})(-y_{\pi_1}).$$

Example 2.1([6]): $X = \{(s, y) \in \mathbf{R}_+ \times \mathbf{Z} : s + y \geq 2.25\}$. The additional mixing inequality $s \geq 0.25(3 - y)$ or $s + 0.25y \geq 0.75$ cuts the polyhedron with integer values $y = 2$ and $y = 3$.

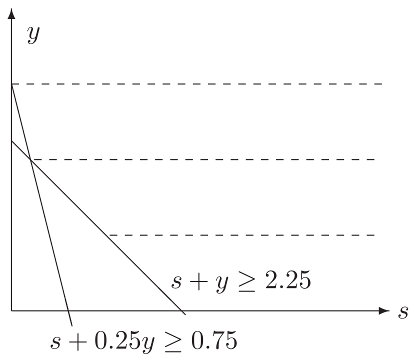

Figure 2.1. An example of mixing inequality.

The tight property of convex hull is also true when unimodular matrix is added. This property is needed when transform X^{MIX} into lot-sizing models.

The convex hull of lot-sizing model with outsourcing is related to the continuous mixing set X^{CM} given in the following, which is a generalization of X^{MIX}:

$$\begin{aligned} & s + x_t + y_t \geq b_t, \quad 1 \leq t \leq n \\ & s \in \mathbf{R}_+, \;\; x \in \mathbf{R}_+^n, \;\; y \in \mathbf{Z}_+^n. \end{aligned}$$

Although the type of the mixing inequalities is valid for X^{CM}, it is not sufficient to provide a linear-inequality description of conv(X^{CM}) ([3]). This problem was solved by Van Vyve as follows([7]).

Let $G = (V, A)$ be a directed graph with loops, where $V = 1, \cdots, n$. To each arc $(j, k) \in A$, a linear express $\phi_{jk}(s, r, z)$ is defined:

$$\phi_{jk}(s,r,z) = \begin{cases} s + r_j + f_j^k z_j - f_k & \text{for forward arcs } (j,k) \\ r_j + f_j^k z_j & \text{for backward arcs } (j,k) \\ s + r_j + z_j - f_j & \text{for loops } (j,j), \end{cases}$$

where,

$$f_j^i = \begin{cases} f_j - f_i & \text{if } f_j \geq f_i \\ f_j - f_i + 1 & \text{if } (f_j < f_i). \end{cases}$$

By summing the expression associated to the arcs belonging to a cycle of the network, some types of inequalities contain the mix inequalities as special cases, and some new types inequalities are introduced. These valid inequalities are enough to express conv(X^{CM}).

3 Continuous mixing polyhedron with flows

The general model of lot-sizing with outsourcing is related to the flow version of the continuous mixing set X^{CMF} defined as follows ([2]):

$$\begin{aligned} & s + x_t + z_t \geq b_t, \quad 1 \leq t \leq n \\ & x_t \leq y_t, \quad 1 \leq t \leq n, \\ & s \in \mathbf{R}_+, \ x \in \mathbf{R}_+^n, \ z \in \mathbf{R}_+^n, \ y \in \mathbf{Z}_+^n. \end{aligned}$$

Note that replace $x_t \leq y_t, (1 \leq t \leq n)$ in the first set inequalities in X^{CMF}, we get X^{CM}.

Extended formulation of conv(X^{CMF}) is solved via other systems and a key lemma. We outline these in the following.

First, set Z is defined as follows ([2]):

$$\begin{aligned} & s + r_t + y_t \geq b_t, \quad 1 \leq t \leq n \\ & s + r_k + x_k + r_t + y_t \geq b_t, \quad 1 \leq k < t \leq n \\ & s + r_t + x_t \geq b_t, \quad 1 \leq t \leq n \\ & s \in \mathbf{R}_+, \ x \in \mathbf{R}^n, \ z \in \mathbf{R}_+^n, \ y \in \mathbf{Z}_+^n. \end{aligned}$$

The relation between Z and X^{CMF} is shown in the following when they are defined on the same vector b.

$$X^{CMF} = Z \cap \{(s, r, x, y) : 0 \leq x \leq y\}. \qquad (1)$$

The above relation is not difficulty. To obtain conv(X^{CMF}) = conv$(Z) \cap \{(s, r, x, y) : 0 \leq x \leq y\}$ based on (1), the following lemma is needed ([1]).

For a polyhedra P in $\mathbf{R}^n$ and a vector $a \in \mathbf{R}^n$, let $\mu_P(a)$ be the value of min $\{ax, x \in P\}$ and $M_P(a)$ be the face $\{x \in P : ax = \mu_P(a)\}$, where $M_P(a) = \emptyset$ whenever $\mu_P(a) = -\infty$.

Lemma 3.1 *Let $P \subseteq Q$ be two pointed polyhedra in $\mathbf{R}^n$, with the property that every vertex of Q belongs to P. Let $Cx \geq d$ be a system of inequalities that are valid for P such that for every inequality $cx \geq \delta$ of the system, $P \not\subset \{x \in \mathbf{R}^n : cx = \delta\}$. If for every $a \in \mathbf{R}^n$ such that $\mu_P(a)$ is finite*

but $\mu_Q(a) = -\infty$, $Cx \geq d$ ***contains an inequality*** $cx \geq \delta$ ***such that*** $M_P(a) \subseteq \{x \in \mathbf{R^n} : cx = \delta\}$, ***then*** $P = Q \cap \{x \in \mathbf{R}^n : Cx \geq d\}$.

The proof of first part of the lemma is direct. Note in the proving of second part, $P \not\subseteq \{x \in \mathbf{R}^n : cx = \delta\}$ can be omitted when *P* is full dimension. For the last part of the lemma, find all extreme rays of *Q\P* that lead to minus infinity of value of μ. If each of these directions can been bounded by equality in $Cx \geq d$, we are done.

The inequality $s + r_k + x_k + r_t + y_t \geq b_t$ of *Z* includes two indexes ***k***, ***t***, so *Z* is equivalent to a difference set X^{DIF} ([2]):

$$\begin{aligned} &\sigma_k + r_t + y_t \geq b_t - b_k, \quad 0 \leq k < t \leq n \\ &\sigma \in \mathbf{R}_+^{n+1}, \ \ r \in \mathbf{R}_+^n, \ \ y \in \mathbf{Z}_+^n. \end{aligned}$$

A compact extended formulation of conv(*Z*) was solved via conv(X^{DIF}) with an affine transformation([2]).

For a number $a \in \mathbf{R}$, denote the *fraction part* of ***a*** by *f*(***a***), i.e., *f*(***a***) = ***a*** − $\lfloor a \rfloor$. By extended formulation of a polyhedron, we mean introducing some additional variables.

In X^{DIF}, both σ_k and r_t are replaced by integer and fractional parts. The integer part is simple an integer variable, and fraction part is convex combination of fractional numbers related to $b_t - b_k$ $(1 \leq k < t \leq n)$, i.e.,

$$\mathrm{conv}(\{f(b_t - b_k) : \ 1 \leq k < t \leq n\}).$$

From $b_{tk} = b_t - b_k$ $(1 \leq k < t \leq n)$ we know that the number of fractional parts are polynomial bounded by ***n***, or $O(n^2)$. An example of b_{tk} $(1 \leq k < t \leq n)$ is given in [2] where distinct fractional parts is exponential in ***n***.

If the result inequalities system is total unimodular matrix(TU), the corresponding extended formulation of polyhedron is integral. In the case here, the constraint matrix is a dual network matrix, so it is TU.

Finally, project the polyhedron to original variable space, we obtain conv(X^{DIF}).

4 Lot-sizing with outsourcing and backlogging model

The single-item constant capacity lot-sizing problem with outsourcing $X^{LS-CC-O}$ over ***n*** periods, which can be formulated as:

$$\begin{aligned} &s_{k-1} + \sum_{u=k}^{t} w_u + \sum_{u=k}^{t} v_u = \sum_{u=k}^{t} d_u + s_t, \ 1 \leq k \leq t \leq n \qquad (2) \\ &w_u \leq C z_u, 1 \leq u \leq n; \ s \in \mathbf{R}_+^{n+1}, \ w \in \mathbf{R}_+^n, \ v \in \mathbf{R}_+^n, \ z \in \{0,1\}^n. \end{aligned}$$

Here d_u is the demand in period ***u***, s_u is the stock at the end of period ***u***, w_u and v_u are production and outsourcing in period ***u***, z_u takes value 1 if there is a set-up in period ***u*** allowing production to take place. Without loss of generality, we set production capacity *C*=1 (or divide by *C* on both sides and reset variables needed). Fix ***k***, and set $s = s_{k-1}$, $x_t = \sum_{u=k}^{t} w_u$, $y_t = \sum_{u=k}^{t} z_u$, $r_t = \sum_{u=k}^{t} v_u$ and $b_t =$

$\sum_{u=k}^{t} d_u$. Then the relaxation of above problem becomes:

$$s + x_t + r_t \geq b_t,\ k \leq t \leq n \quad (3)$$
$$0 \leq x_u - x_{u-1} \leq y_u - y_{u-1} \leq 1,\ k \leq u \leq n \quad (4)$$
$$s \in \mathbf{R}_+,\ x \in \mathbf{R}_+^{n-k+1},\ r \in \mathbf{R}_+^{n-k+1},\ y \in \mathbf{Z}^{n-k+1}.$$

Note inequality (3) is obtained from $s_t \geq 0$. Summing (4) over $k \leq u \leq t$ and dropping the upper bound on y_t, one obtains X^{CMF}.

In the case of backlogging model $X^{LS-CC-B}$, equation (2) becomes ([1])

$$s_{k-1} + \sum_{u=k}^{t} w_u + r_t = \sum_{u=k}^{t} d_u + s_t + r_{k-1},\ 1 \leq k \leq t \leq n.$$

The difference between backlogging and outsourcing in relaxation process is, $s_t + r_{k-1} \geq 0$ is dropped, not $s_t \geq 0$, and also $r_t = r_t$, not $r_t = \sum_{u=k}^{t} v_u$. Both have the same relaxation inequality (3).

5 Structures of optimal solutions

Although, in above section, we obtain the same relaxation inequalities system both for lot-sizing models with outsourcing and backlogging, but there are obvious difference in structure of optimal solutions.

Lot-sizing problems in general can been seen as minimum cost network flow problems. For a minimum cost network flow problem, a well-known and fundamental property of minimum cost network flow problem tells:

Observation 5.1: *For a basic feasible solution of a minimum cost network flow problem, the arcs corresponding to variables with flows strictly between their lower and upper bounds form an acyclic graph.*

Before giving the structure of optimal solutions, we need a concept defined as follows.

Definition 5.1: Planning time period n are partitioned into intervals $[t_1, t_2 - 1]$, $[t_2, t_3 - 1]$, $\cdots$, $[t_r - 1, t_r]$, where no stock entering or leaving each interval, are called *regeneration intervals*.

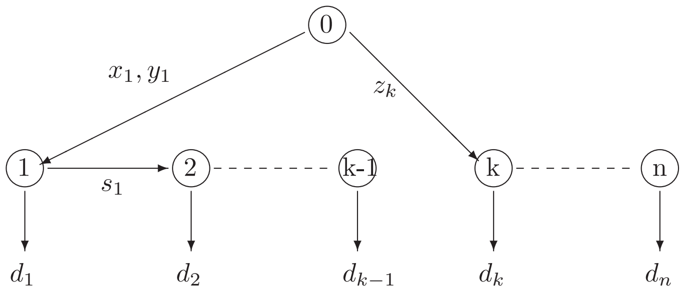

Figure 5.1. Structure of an optimal solution of X^{LS-U-O}

Now we first consider the simple case when the production capacity is un- bounded. In order to have a more comprehensive difference, we also give a dynamic programming (DP) here. Note, by the flow balance equalities in constraint, stock variables, or production variables can be canceled (or omitted).

Let $G(t)$ be the minimum cost of solving the problem over the first t periods, and let $\phi(k,t)$ be the minimum cost of solving the problem over the first t periods subject to the additional condition that the last production or outsourcing periods is k for some $k \leq t$. From the definition we have

$$G(t) = \min_{k:k\leq t} \phi(k,t).$$

By extreme optimal solution structure also *Principle of Optimality* for DP, $\phi(k,t)$ can be calculated as

$$\phi(k,t) = G(k-1) + \min[q_k + p_k d_{kt}, g_k d_{kt}].$$

Where, q_k is set up cost, p_k and g_k are production cost and outsourcing cost.
Note, stock cost is omitted.

A (forward) dynamic programming recursion for X^{LS-U-O}

$$\begin{aligned} G(0) &= 0 \\ G(t) &= \min_{k:k\leq t}[G(k-1) + \min[q_k + p_k d_{kt}, g_k d_{kt}] \qquad \text{for } t = 1,\cdots,n. \end{aligned}$$

Now, let us see the unbounded lot-sizing model with backlogging X^{LS-U-B}.

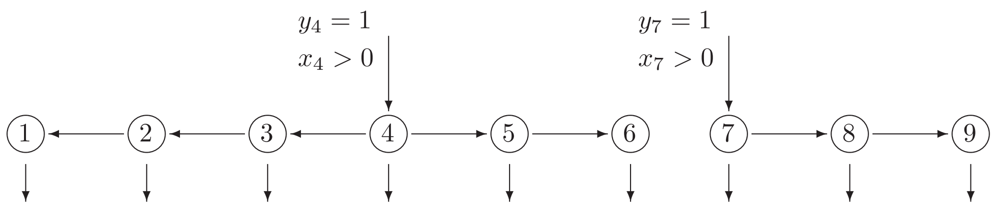

Figure 5.2. Structure of an extreme point solution of X^{LS-U-B}

For periods $u, v \in \{1,\cdots,n\}$, let $\phi(u,v)$ denote the minimum cost of satisfying demands for periods $v, \cdots, n$ in which the demand for period v is satisfied by production in u if $u \geq v$, and let $G(v)$ denote the minimum cost solution of production X^{LS-U-B} defined over the horizon $v, \cdots, n$.
Here production cost is omitted.

A Backward dynamic programming algorithm for X^{LS-U-B} ([6])

$$G(v) = \min_{u\geq v} \phi(u,v)$$

$$\phi(u.v) = (\sum_{t=u}^{v-1} h_t) d_v + \min[\phi(u,v+1), G(v+1)] \qquad \text{for } u < v$$

$$\phi(u.v) = (\sum_{t=v}^{u-1} b_t)d_v + \phi(u, v+1) \qquad \text{for } u > v$$

$$\phi(u.u) = q_u + \min[\phi(u, u+1), G(u+1)] \qquad \text{for } u = v$$

Where h_t and b_t are stock cost and backlogging cost.

For constant capacity, we only give an example of structure of optimal solution for $X^{LS-CC-O}$ in Figure 5.3. For X^{LS-CC}, z_1, C should be replaced by x_1. For $X^{LS-CC-B}$, the directions of flows on stocks arcs are not known. While the DP for X^{LS-CC} can been founded in [6], and DP for $X^{LS-CC-O}$ is supposed recently by author in [8], which is much complex that the one of X^{LS-CC}. To our knowledge, the DP for $X^{LS-CC-B}$ is not known, if it is possible, it will be a hard job.

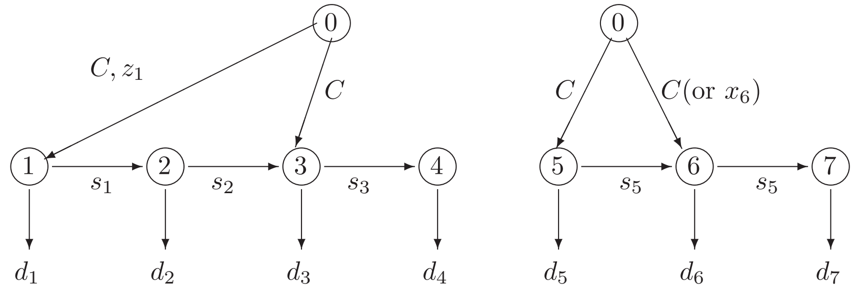

Figure 5.3. Structure of extreme optimal solution for constant capacity

6 Extended reformulation for lot-sizing problems with outsourcing for a regeneration interval

To keep the linear property, we have to revise the outsourcing as in [4], but with zero outsourcing set-up cost.

For a particular regeneration interval $[k, l]$, let $\delta_i = 1$ if $z_i > 0$ and $\delta_i = 0$ if $z_i = 0$. Also let n_i be the number of full capacity C of z_i, let $\chi_i = 1$ if $x_i = C$ and $\chi_i = 0$ if $x_i = 0$. Now we have the relaxed linear program (LP_1^{kl}):

$$\begin{aligned}
\min \quad & \sum_{i=k}^{l}(p_i C + f_i)\chi_i + \sum_{i=k}^{l} g_i(\rho_{kl}\delta_i + Cn_i) \\
s.t. \quad & \sum_{i=k}^{\tau}\chi_i + \sum_{i=k}^{\tau} n_i + \sum_{i=k}^{\tau}\delta_i \geq \lceil \frac{d_{k\tau}}{C} \rceil \ \ \tau = k, \cdots, l-1, \text{ with } \rho_{k\tau} \leq \rho_{kl}, \\
& \sum_{i=k}^{\tau}\chi_i + \sum_{i=k}^{\tau} n_i \geq \lceil \frac{d_{k\tau}}{C} \rceil \qquad \tau = k, \cdots, l-1 \text{ with } \rho_{k\tau} > \rho_{kl}. \\
& \sum_{i=k}^{l}\chi_i + \sum_{i=k}^{l} n_i = \lceil \frac{d_{kl} - \rho_{kl}}{C} \rceil, \\
& \sum_{i=k}^{l}\delta_i = 1, \\
& \chi_i, \delta_i \leq 1, \ \ \chi_i, \ n_i, \ \delta_i \geq 0.
\end{aligned}$$

Comparing the fractional batch production indicator in [4], the value of δ_i differs if $\rho_{kl} = 0$, i.e., outsourcing may occur even if $\rho_{kl} = 0$. Inequalities of constrains are same before the period where fractional batch production or out-sourcing occurs.

Note that $\sum_{i=k}^{l} \delta_i = 1$ should be $\sum_{i=k}^{l} \delta_i \leq 1$, we exclude $\sum_{i=k}^{l} \delta_i = 0$ because it means that no outsourcing occurs, therefore we can keep the inequalities (include equalities) in one direction and coefficients with plus only.

Hence, when outsourcing occurs, the formulation of (LP_1^{kl}) is valid. The constraints constitute a $0-1$ matrix with the property that 1's are consecutive in each column if arranging appropriately, i.e., it is a interval matrix. Therefore it is total unimodular.

Summarize above arguments, we have:

Proposition 6.1 *The linear programing* LP_1^{kl} *s feasible with regeneration interval[k, l] when outsourcing occurs.*

With the strict inequality condition $s_t > 0$ within regeneration interval, inequalities shall be strict in above system. These do not affect the coefficients of variables of inequalities, hence the claim. We remain the formulation as the treatments in other papers.

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