# Internal rays for polynomial skew products

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We consider Axiom A polynomial skew products on $\mathbb{C}^2$ of degree $d \geq 2$. We define internal rays in the stable disks of $W^s(\Lambda_{A_p})$ and show that they land on $J_2$ under some assumptions.

## 1 Introduction

In this note, we consider regular polynomial skew products on $\mathbb{C}^2$ of degree $d \geq 2$ of the form :

$$f(z, w) = (p(z), q(z, w)).$$

If we set $q_z(w) = q(z, w)$, the $n$-th iterate of $f$ is written by

$$f^n(z, w) = (p^n(z), Q_z^n(w)) := (p^n(z), q_{p^{n-1}(z)} \circ \cdots \circ q_z(w)).$$

Hence the dynamics on the $z$-plane is that of $p$. We call the $z$-plane the *base space*. The vertical planes $\{z\} \times \mathbb{C}$ are called *fibers*. Then $f$ preserves the family of fibers and this enables us to investigate the dynamics more precisely.

Let $K_p$ and $J_p$ be the *filled Julia set* and *Julia set* respectively of the polynomial $p$ and $A_p$ be the set of attracting periodic points of $p$. Let $K$ be the set of points with bounded orbits of $f$ and put $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$. The *fiber Julia set* $J_z$ is the boundary of $K_z$. The *second Julia set* $J_2$, which is a right analogue of the Julia set of a one-dimensional map, is characterized by $J_2 = \overline{\cup_{z \in J_p} \{z\} \times J_z}$. If $f$ is Axiom A, then the map $z \mapsto J_z$ is continuous in $J_p$, hence $J_2 = \cup_{z \in J_p} \{z\} \times J_z$. See Jonsson [J].

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The *stable and unstable sets* of a saddle set $\Lambda$ are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^n(y) \to \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ prehistory } \hat{y} = (y_{-n}) \to \Lambda\}. \end{aligned}$$

Let $\Lambda_{A_p}$ and $\Lambda_{J_p}$ be the saddle sets in $A_p \times \mathbb{C}$ and $J_p \times \mathbb{C}$ respectively. Since the map $f$ preserves the vertical fibers, it is easy to see that $W^u(\Lambda_{A_p}) \subset A_p \times \mathbb{C}$. Hence it follows that $W^u(\Lambda_{A_p}) \cap W^s(\Lambda_{J_p}) = \emptyset$. In Nakane [N], we got the following.

**Theorem 1.1.** *([N], Theorem 1.1) Suppose that $f$ is Axiom A. Then, the map $z \mapsto J_z$ is continuous on $K_p$ if and only if the property $W^u(\Lambda_{J_p}) \cap W^s(\Lambda_{A_p}) = \emptyset$ holds.*

Let $C_p$ and $C_z$ $(z \in \mathbb{C})$ be the set of finite critical points of $p$ and $q_z$ respectively. Jonsson [J] defined that a polynomial skew product $f$ is *connected* if $J_p$ is connected and $J_z$ is connected for all $z \in J_p$, in other words, $C_p \subset K_p$ and $C_z \subset K_z$ for all $z \in J_p$.

**Theorem 1.2.** *([N], Theorem 1.2) Suppose that $f$ is a connected Axiom A polynomial skew product on $\mathbb{C}^2$ with $C_p \subset A_p$. Then $W^u(\Lambda_{J_p}) \cap W^s(\Lambda_{A_p}) = \emptyset$.*

The *local stable manifold* $W^s_{loc}(x)$ of $x = (z_0, w_0) \in \Lambda_{A_p}$ is transversal to the fiber. That is, there exist $\epsilon > 0$ and a holomorphic function $\varphi(z, w_0)$ in $\mathbb{D}(z_0, \epsilon) := \{z \in \mathbb{C}; |z - z_0| < \epsilon\}$ such that

$$W^s_{loc}(x) = \{(z, \varphi(z, w_0)); z \in \mathbb{D}(z_0, \epsilon)\}.$$

This function $\varphi$ gives a holomorphic motion of $J_{z_0}$ over $\mathbb{D}(z_0, \epsilon)$, that is,

(1) $\varphi(z_0, \cdot) = id_{J_{z_0}}$,
(2) $\varphi(\cdot, w)$ is holomorphic in $\mathbb{D}(z_0, \epsilon)$ for each fixed $w \in J_{z_0}$,
(3) $\varphi_z = \varphi(z, \cdot)$ is injective for each fixed $z$.

By the $\lambda$-lemma, $\varphi : \mathbb{D}(z_0, \epsilon) \times J_{z_0} \to \mathbb{C}$ is continuous.

In this note, we will define internal rays in the stable disks in $W^s(\Lambda_{A_p})$ and, under the assumptions of Theorem 1.2, we show that all internal rays land at points in $J_2$.

# 2 Internal rays

Under the situations in Theorem 1.2, we define the internal rays and investigate their landing on $J_2$.

Let $z_0 \in A_p$ be of period $m$ and $U_0$ be its immediate basin. Corollary 4.3 in Roeder [R] says that, if $f$ is connected, then the set $C_{U_0}$ of vertical critical points of $f$ over $U_0$ is contained in the Fatou set of $f$. Hence lifting by $f$, the local holomorphic motion of $J_{z_0}$ defined in the introduction extends to $U_0$. Thus, for $a \in J_{z_0}$, the connected component $W_a$ of $W^s(z_0, a)$ containing $(z_0, a)$ is the graph of a holomorphic function on $U_0$, hence is a topological disk. We call this a *stable disk*. Put $W_a^* = W_a \setminus \{(z_0, a)\}$.

From the assumption $C_p \subset A_p$, $z_0$ is superattracting and we have a Böttcher coordinate at $z_0$. That is, there exists $k \geq 2$ such that $p^m(z) = z_0 + a_k(z - z_0)^k + o(|z - z_0|^k)$, $a_k \neq 0$. The Böttcher coordinate $\phi$ at $z_0$ is a local conformal conjugacy between $p^m$ and the map $\zeta \mapsto \zeta^k$. That is, this satisfies $\phi(z_0) = 0$ and the functional equation :

$$\phi \circ p^m(z) = \phi(z)^k.$$

It is unique up to multiplication by a $(k-1)$-st root of unity. By this functional equation, it is analytically continued until it meets a critical point of $p^m$. By the assumption $C_p \subset A_p$, it never meets critical points of $p^m$, hence it can be analytically continued to the whole basin $U_0$ and gives a conjugacy $\phi : U_0 \to \mathbb{D} := \mathbb{D}(0, 1)$. Let $\psi = \phi^{-1}$ be its inverse. This satisfies $p^m \circ \psi(\zeta) = \psi(\zeta^k)$.

For $a \in J_{z_0}$, the holomorphic motion $\varphi(z, a)$ satisfies

$$f(z, \varphi(z, a)) = (p(z), \varphi(p(z), q_{z_0}(a)))$$

since $f(W^s_{loc}(z_0, a)) \subset W^s_{loc}(f(z_0, a))$. Thus it follows that

$$\begin{aligned} f^m(\psi(\zeta), \varphi(\psi(\zeta), a)) &= (p^m \circ \psi(\zeta), \varphi(p^m \circ \psi(\zeta), Q^m_{z_0}(a))) \\ &= (\psi(\zeta^k), \varphi(\psi(\zeta^k), Q^m_{z_0}(a))) \end{aligned}$$

If we define $\psi_a : \mathbb{D} \to W_a$ by $\psi_a(\zeta) = (\psi(\zeta), \varphi(\psi(\zeta), a))$, it satisfies

$$f^m \circ \psi_a(\zeta) = \psi_{Q^m_{z_0}(a)}(\zeta^k). \tag{1}$$

We now define an internal ray in $W_a$ of angle $t \in \mathbb{R}/\mathbb{Z}$ by

$$R_a(t) = \psi_a(\{re^{2\pi it} : 0 < r < 1\}).$$

By definition, $f^m(R_a(t)) = R_{Q^m_{z_0}(a)}(kt)$. We say that a ray $R_a(t)$ lands at a point $x$ if, as $r \to 1$, $\psi_a(re^{2\pi it})$ converges to $x$.

**Theorem 2.1.** *Under the assumptions in Theorem 1.2, every ray $R_a(t)$ for $a \in J_{z_0}$ and $t \in \mathbb{R}/\mathbb{Z}$ lands at a point in $J_2$. The landing points depend continuously on $a \in J_{z_0}$ and $t \in \mathbb{R}/\mathbb{Z}$.*

*proof.* The proof is an analogue of that of Theorem 10.2 in [BJ]. By the uniform expansion on $J_2$, there exist a neighborhood $E$ of $J_2$, a metric in $E$ and $\lambda > 1$ such that $f^{-m}(E) \subset E$ and the following holds for all $x \in E$ and all $v \in T_x\mathbb{C}^2$ with this metric :

$$|Df^m(x)v|^* > \lambda|v|^*. \tag{2}$$

By the proof of Theorem 1.2 in [N], $W^*_a \subset W^u(J_2)$ for any $a \in J_{z_0}$. Since the union $A_R = \cup_{a\in J_{z_0}}\psi_a(\{R^k \le |\zeta| \le R\})$ is compact in $W^u(J_2)$ for any $R < 1$, there exists $n$ such that $f^{-mn}(A_R) \subset E$. Then there exists $R < 1$ such that $\psi_a(\{R \le |\zeta| < 1\}) \subset E$ for any $a \in J_{z_0}$. Differentiating (1) and using (2), for $R \le |\zeta| < 1$,

$$|D\psi_a(\zeta)|^* \le \lambda^{-1}|D\psi_{Q^m_{z_0}(a)}(\zeta^k)|^* k|\zeta|^{k-1}. \tag{3}$$

Put $m(r) = \sup_{a\in J_{z_0}} \sup_{|\zeta|=r} |D\psi_a(\zeta)|^*$. Take $0 < \alpha < 1$ so that it satisfies $k^\alpha < \lambda$ and take $C > 0$ such that

$$m(r) \le C(1-r)^{\alpha-1}, \tag{4}$$

holds for $R \le r \le R^{1/k}$. By (3), it follows that, for $r \ge R$,

$$m(r) \le \lambda^{-1}kr^{k-1}m(r^k). \tag{5}$$

We show, by induction on $j$, that (4) holds in $R^{1/k^j} \le r < R^{1/k^{j+1}}$ for all $j \ge 0$. The case $j = 0$ is trivial. Suppose that the case $j = n-1$ is true. If

$R^{1/k^n} \leq r < R^{1/k^{n+1}}$, then $R^{1/k^{n-1}} \leq r^k < R^{1/k^n}$ and we have by (5),

$$\begin{aligned} m(r) &\leq \lambda^{-1} k r^{k-1} C(1-r^k)^{\alpha-1} \\ &= C\lambda^{-1} k r^{k-1} (1+r+\cdots+r^{k-1})^{\alpha-1}(1-r)^{\alpha-1} \\ &\leq C\lambda^{-1} k r^{k-1} (k r^{k-1})^{\alpha-1}(1-r)^{\alpha-1} \\ &\leq C\lambda^{-1} k^{\alpha} (1-r)^{\alpha-1} \\ &\leq C(1-r)^{\alpha-1}. \end{aligned}$$

Here we use the assumption $k^\alpha < \lambda$. Thus the case $j = n$ is also true. Hence we get that (4) holds for $R \leq r < 1$.

Then, for $\zeta_j = r_j e^{2\pi i t}, j = 1, 2, r_1 < r_2$, it follows that

$$\begin{aligned} d(\psi_a(r_1 e^{2\pi i t}), \psi_a(r_2 e^{2\pi i t})) &\leq \int_{\zeta_1}^{\zeta_2} |D\psi_a(\zeta)|^* |d\zeta| \\ &\leq \int_{r_1}^{r_2} m(r) dr \\ &\leq C\alpha^{-1}((1-r_1)^\alpha - (1-r_2)^\alpha) \\ &\leq C\alpha^{-1}(r_2 - r_1)^\alpha. \end{aligned}$$

In the last inequality, we use that $0 < \alpha < 1$. Thus, as $r \to 1$, $\psi_a(re^{2\pi i t})$ converges uniformly for $a \in J_{z_0}$ and $t \in \mathbb{R}/\mathbb{Z}$. This implies that every ray $R_a(t)$ lands at a point in $J_2$ and that the landing point depends continuously on $a \in J_{z_0}$ and $t \in \mathbb{R}/\mathbb{Z}$. This completes the proof. □

**Corollary 2.1.** *The map $\psi_a$ extends to a continuous surjective map : $\overline{\mathbb{D}} \to \overline{W_a}$. Hence, any point in $\partial W_a$ is the landing point of a ray. Moreover, the map $(\zeta, a) \mapsto \psi_a(\zeta)$ is continuous on $\overline{\mathbb{D}} \times J_{z_0}$, the holomorphic motion $\varphi(z, a)$ extends to a continuous function on $\overline{U_0} \times J_{z_0}$ and $\varphi_z : J_{z_0} \to J_z$ is surjective for $z \in \overline{U_0}$.*

*proof.* We put $\psi_a(e^{2\pi i t}) := \lim_{r\to 1} \psi_a(re^{2\pi i t})$. Since this convergence is uniform for $(t, a) \in (\mathbb{R}/\mathbb{Z}) \times J_{z_0}$, the map $(t, a) \mapsto \psi_a(e^{2\pi i t})$ is continuous on $(\mathbb{R}/\mathbb{Z}) \times J_{z_0}$. For any $\epsilon > 0$, there exists $\delta > 0$ such that, if $|a' - a|, |t - s| < \delta$ and $1 - \delta < r < 1$,

$$d(\psi_{a'}(re^{2\pi i t}), \psi_a(e^{2\pi i s})) \leq d(\psi_{a'}(re^{2\pi i t}), \psi_{a'}(e^{2\pi i t})) + d(\psi_{a'}(e^{2\pi i t}), \psi_a(e^{2\pi i s})) < 2\epsilon.$$

This implies that the map $(\zeta, a) \mapsto \psi_a(\zeta)$ is continuous also on $\partial\mathbb{D} \times J_{z_0}$. Since $\psi_a(\overline{\mathbb{D}})$ is a compact set containing $\psi_a(\mathbb{D}) = W_a$, it easily follows that $\psi_a(\overline{\mathbb{D}}) = \overline{W_a}$.

As for $\varphi$, note that $\partial U_0$ is a Jordan curve. Then the map $\psi$ extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{U_0}$, hence the map $\varphi(z, a) = \pi_2 \circ \psi_a \circ \psi^{-1}(z)$ extends to a continuous map on $\overline{U_0} \times J_{z_0}$. If we take $z' \in \partial U_0$, there exists a sequence $z_n \in U_0$ tending to $z'$. By the continuity of the maps $z \mapsto J_z$ and $z \mapsto \varphi_z$ on $\overline{U_0}$, we have

$$J_{z'} = \lim_{n\to\infty} J_{z_n} = \lim_{n\to\infty} \varphi_{z_n}(J_{z_0}) = \varphi_{z'}(J_{z_0}).$$

Thus the equality $\varphi_z(J_{z_0}) = J_z$ holds also for $z \in \overline{U_0}$. □

In general, the map $\varphi_z : J_{z_0} \to J_z$ is not injective for $z \in \partial U_0$. In fact, for the following Example 2.1 with $c = -1$, pinching occurs as $z \to \partial U_0$.

**Example 2.1.** $f(z, w) = (z^2, w^2 + cz)$.

If we set $g_c(w) = w^2 + c$, then $f^n(z, w) = (z^{2^n}, z^{2^{n-1}} g_c^n(\frac{w}{\sqrt{z}}))$. We have $C_p = \{0\} = A_p$. It easily follows that $f$ is Axiom A (resp. connected) if and only if $g_c$ is hyperbolic (resp. $J_{g_c}$ is connected). Thus, $f$ satisfies the assumptions of Theorem 1.2 if $c$ lies in a hyperbolic component of the Mandelbrot set.

Let $\phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_{g_c}$ be the inverse Böttcher coordinate of $g_c$. Then it follows that

$$\varphi_z(w) = \phi_z(w) = \sqrt{z}\phi_c(\frac{w}{\sqrt{z}}), \quad z \in \mathbb{D},$$

which depends holomorphically on $z$ because $\phi_c$ is an odd function. The internal ray $R_a(t)$ for $a \in J_0 = \partial\mathbb{D}$ is written as

$$R_a(t) = \{(re^{2\pi it}, \sqrt{r}e^{\pi it}\phi_c(\frac{a}{\sqrt{r}e^{\pi it}})); r < 1\}.$$

It lands at the point $(e^{2\pi it}, e^{\pi it}\phi_c(ae^{-\pi it})) \in J_2$. The fiber Julia set $J_z = \phi_z(\partial\mathbb{D})$ is a Jordan curve if $z \in \mathbb{D}$, while it is a rotation of the Julia set $J_c$ if $z \in \partial\mathbb{D}$. Thus, pinching occurs as $z$ approaches $\partial\mathbb{D}$.

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