

Fibre Spaces and Groupoids

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Abstract

In this article we research sheaves of groupoids representation of category of algebraic normal varieties, in special a category of spectra of fields. When the ground field is algebraically closed, a spectrum of a function field with the locally algebraic automorphism group dominated by a spectrum of a trivial function field is also a spectrum of a trivial function field. Next, we propose another approach for obtaining minimal models of complex projective varieties.

1 Introduction

The general theory of Grothendieck Homotopy theory and étale fundamental theory give us another description of algebraic commutative and non-commutative schemes. We shall prove that if a fibre space with the generic general fibre whose automorphic groups are locally algebraic and surjectively dominated by a product fibre space, then the fibre space is also a product. This especially includes a fibre space consisted of spectra of fields. In this paper the last statement is proved on the basis that it is sufficient to recover the fields from the absolute Galois groups acting on their algebraically separable closures, i.e., groupoids instead of regarding the groups just as abstract groups. In the next section we propose a program for obtaining minimal models for complex algebraic varieties.

2 Preliminary

2.1 The theory of the homotopy type of Grothendieck[Malt],[Cisinski], [S2], [S1]

We briefly prepare an explanation of Grothendieck homotopy theory after manuscript "Pursuing stacks" 1983 ([Malt],[Cisinski], [S2]) to apply it to small category of spectra of fields and algebraic varieties. The homotopical category *Hot* is originally the category of the CW-complexes. The homotopical category is the category whose objects are topological spaces, and whose morphisms are homotopy equivalence classes of continuous maps. Two topological spaces X and Y are isomorphic in this category if and only if they are homotopy-equivalent. This *Hot* is equivalent to the category localizing the category of all the topological spaces and continuous maps by topological weak equivalences W_∞ such that

- W_∞ is a part of the continuous
- $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ are bijections.

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- $\pi_n(f, x) : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ are isomorphisms of groups for $n \geq 1$.

That is

$$Hot \cong W_\infty^{-1}Top.$$

The category of simplexes is the category the objects of which

$$\Delta_m = \{0, 1, \dots, m\}$$

and the morphisms of which are increasing maps. The category $\hat{\Delta}$ of the presheaves over Δ is called the category of the simplicial sets. We have a unique couple of adjoint functors, i.e., topological realization and singular simplicial set functor

$$|\cdot| : \hat{\Delta} \rightarrow Top, S : Top \rightarrow \hat{\Delta}$$

A simplicial weak equivalence is a morphism of simplicial sets the topological realization of which is a homotopy equivalence. This simplicial equivalence is described W_∞ . Then the topological realization functor and the singular simplicial set functor are compatible with weak equivalences and induce categorical equivalences

$$W_\infty^{-1}\hat{\Delta} \rightarrow W_\infty^{-1}Top, W_\infty^{-1}Top \rightarrow W_\infty^{-1}\hat{\Delta}$$

which are quasi-inverses.

The homotopical category Hot can be obtained as a localization of the 2-category Cat the objects of which are the small categories and the morphisms of which are the functors. There are a couple of adjoint functors, i.e., the categorical realization functor and the nerf functor,

$$c : \hat{\Delta} \rightarrow Cat, N : Cat \rightarrow \hat{\Delta}$$

where the restriction of c to Δ is the inclusion $\Delta \subset Cat$ and the simplicial set

$$(N(C))_m = Hom_{Cat}(\Delta_m, C), m \geq 0$$

the ordered set Δ_m being as a category. A categorical weak equivalence is defined to be a functor the nerf of which is a simplicial weak equivalence. Let W_∞ be the part of $Fl(Cat)$ which are the categorical equivalences. The nerf functor is compatible with categorical weak equivalence and induces a categorical equivalence:

$$W_\infty^{-1}Cat \rightarrow W_\infty^{-1}\hat{\Delta}$$

However, the categorical realization functor is not compatible with the categorical weak equivalences. A quasi-inverse functor to N

$$Simpl : \hat{\Delta} \rightarrow Cat$$

, where it associates a simplicial set K to a category of simplexes $Simpl(K)$. The $Simpl$ functor induces a categorical equivalence

$$W_\infty^{-1}\hat{\Delta} \rightarrow W_\infty^{-1}Cat$$

which is a quasi-inverse to the nerf functor N . A categorical weak equivalence is characterized by Artin-Mazur equivalence. Let $u : A \rightarrow B$ be a functor of small categories and F a locally constant over \hat{B} . A morphism of toposes $(u^*, u_*) : \hat{A} \rightarrow \hat{B}$ is Artin-Mazur equivalence if and only if

$$H^m(\hat{B}, F) \rightarrow H^m(\hat{A}, u^*F)$$

are isomorphisms for $m \geq 0$ and any locally constant sheaf F . (as sets for $m = 0$, as groups for $m = 1$, as abelian groups for $m \geq 2$.) Let A be a small category and $i_A : \hat{A} \rightarrow Cat$ ($F \rightarrow A/F$) Then

$$W_{\hat{A}} = i_A^* W_{\infty}$$

A couple of adjoint functors

$$i_A : W_{\hat{A}}^{-1} \hat{A} \rightarrow W_{\infty}^{-1} Cat$$

and

$$i_A^* : Hot = W_{\infty}^{-1} Cat \rightarrow W_{\hat{A}}^{-1} \hat{A}$$

, where

$$i_A^* : Cat \rightarrow \hat{A}C \mapsto (a \mapsto Hom_{Cat}(A/a, C))$$

are categorical equivalences, quasi-inverses each other if and only if A is said to be a weak test category. A morphism $u : A \rightarrow B$ of Cat is aspheric if $u/b : A/b \rightarrow B/b$ is a weak equivalence for any object b of B . Grothendieck says A is a weak test category if and only if for any category with a final object i_A^* is aspheric, i.e., i_A^* in W_{∞} .

A category A is a local test category if and only if for any object a A/a is a weak test category.

A category A is a test category if and only if A is a weak test category and a local test category at once.

A test category is as good as the category of simplicial sets Δ .

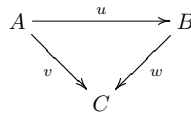
Theorem 1 (Theorem A of Quillen's) *If $u : A \rightarrow B$ is a morphism of Cat such that $u/b : A/b \rightarrow B/b$ is in W_{∞} for any object b of B , then u is in W_{∞} .*

Definition 1 (Weak Basic Localizer) *The weak basic localizer W is characterized by the following conditions.*

- (La weak saturation) *the identity is in W . If two morphisms of a commutative triangle in W , then the third of them is in W . If $i : A \subset B$, $r : B \rightarrow A$ and $r \circ i = id_A$, then $ir \in W$ implies $i \in W$.*
- (Lb final object) *If A is a small category with a final object, then the unique functor $A \rightarrow \{pt\}$ is in W .*
- (Lc Theorem A of Quillen's) *If $u : A \rightarrow B$ is a morphism of Cat such that $u/b : A/b \rightarrow B/b$ for all objects b of B , then u is in W .*

Definition 2 (Basic Localizer) *The basic localizer W is characterized by the following conditions.*

- (La weak saturation) *the identity is in W . If two morphisms of a commutative triangle in W , then the third of them is in W . If $i : A \subset B$, $r : B \rightarrow A$ and $r \circ i = id_A$, then $ir \in W$ implies $i \in W$.*
- (Lb final object) *If A is a small category with a final object, then the unique functor $A \rightarrow \{pt\}$ is in W .*
- (LC relative Theorem A of Quillen's) *Given a commutative triangle in Cat*

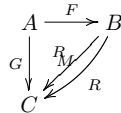


and if $u : A \rightarrow B$ is a morphism of Cat such that $u/c : A/c \rightarrow B/c$ for all objects c of C , then u is in W .

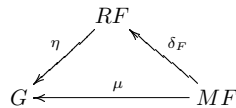
Given a weak basic localizer W , we can do an analogue to an original categorical weak equivalence W_∞ . We have analogous notions, $Hot_W = W^{-1}Cat$, W -weak test categories, W -local test categories, W -test categories.

Kan extensions are universal constructs in category theory.

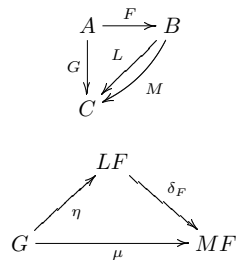
Definition 3 Let A, B, C be three categories and $F : A \rightarrow B, G : A \rightarrow C$.



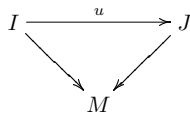
The right Kan extension of G along F consists of a functor $R : B \rightarrow C$ and a natural transformation $\eta : RF \rightarrow G$ which is co-universal so that for any functor $M : B \rightarrow C$ and a natural transformation $\mu : MF \rightarrow G$, there exists uniquely a natural transformation $\delta : M \rightarrow R$ in the following commutative diagram.



The left Kan extension of G along F is dual to the right Kan extension.



A theory of Kan homotopical extensions is the concept of Grothendieck theory of derivators. A localizer is a couple of (M, W) such that M is a category and such that W is a part of morphisms $Hot_{(M,W)}(I)$ is a localization of the category $Func(I, M)$ of functors from I to M by induced W argument-wise on natural transformations. For any functor $u : I \rightarrow J$ between small categories we have an inverse image functor $u^* : Hot_{(M,W)}(J) \rightarrow Hot_{(M,W)}(I)$



If there exists a structure of a Quillen,s closed model category on M whose weak equivalences are W and if M admits small inductive limits (resp. projective limits), then there exist the left Kan extensions (resp. the right Kan extensions). For example the localizer (Cat, W_∞) is the one. For every functor $F : I \rightarrow Cat$, we obtain a cofibred category by Grothendieck construction which is called an integration of F along I ;

$$\int_I F \rightarrow I$$

From the following diagram

$$\begin{array}{ccc} \int_I F & \xrightarrow{u_!} & \int_J F \\ \downarrow & & \downarrow \\ I & \xrightarrow{u} & J \end{array}$$

we obtain

$$u_! : \text{Hot}_{(\text{Cat}, W)}(I) \rightarrow \text{Hot}_{(\text{Cat}, W)}(J)$$

induced by localization. This is a left adjoint functor to

$$u^* : \text{Hot}_{(\text{Cat}, W)}(J) \rightarrow \text{Hot}_{(\text{Cat}, W)}(I)$$

The dual notion of integration is co-integration, denoted by $\nabla_I F = (\int_{I^o} F^o)^o$ for a contra-variant functor $F : I^o \rightarrow \text{Cat}$.

Given a basic localizer W , Grothendieck says a functor between small categories $u : A \rightarrow B$ (resp. $B' \rightarrow B$) is W -proper (resp. W -smooth) if for any cartesian square

$$\begin{array}{ccc} A' & \xrightarrow{w} & A \\ u' \downarrow & & \downarrow u \\ B' & \xrightarrow{v} & B \end{array}$$

the base change morphism

$$u'_! w^* \rightarrow v^* u_!$$

is an isomorphism, where $u_! : \text{Hot}_{(\text{Cat}, W)}(A) \rightarrow \text{Hot}_{(\text{Cat}, W)}(B)$ and $u'_! : \text{Hot}_{(\text{Cat}, W)}(A') \rightarrow \text{Hot}_{(\text{Cat}, W)}(B')$ are homotopical left Kan extensions. They are dual each other.

Let A be a small category and a functor

$$j_A : \hat{A} \rightarrow \text{Cat}/A$$

defined by $j_A X = (A/X, A/X \rightarrow A)$. Since $j_A^{-1}(W/A) = W_{\hat{A}}$, the functor j_A induces

$$j_A : H_W \hat{A} \rightarrow \text{Hot}_W \text{Cat}/A = (W/A)^{-1} \text{Cat}/A$$

which is called the category of W -types of locally homotopical constants over A and $H_W = W^{-1}$.

Proposition 1 *Let W a basic localizer and A a W -local test category. Then*

$$j_A : H_W \hat{A} \rightarrow \text{Hot}_W \text{Cat}/A$$

is a categorical equivalence.

$$\begin{array}{ccc} \hat{A} & \xrightarrow{j_A} & \text{Cat}/A \\ p^* \downarrow & \searrow \text{dis} & \uparrow \Xi' \\ A \hat{\times} \Delta & \xleftarrow{N} & \text{Hom}(A^o, \text{Cat}) \end{array}$$

$$\Xi' : \text{Hom}(A^o, \text{Cat}) \rightarrow \text{Cat}/A \quad F \mapsto \nabla_A F$$

$$\Xi : \text{Cat}/A \rightarrow \text{Hom}(A^o, \text{Cat}) \quad a \mapsto a \setminus C$$

Representations of a presheaf of small categories over a category. Let A be a small category and A a presheaf of small categories over a category A . The category $Rep(A)$ of representations of A as follows. An object of $Rep(A)$ is a couple (X, ξ) such that X_a is a presheaf over A_a for each object a of A and for every morphism $\alpha : a \rightarrow a'$ a morphism of presheaves $\xi_\alpha : X_{a'} \rightarrow A_\alpha^* X_a$ for each object a of A and $\xi_{1_a} = 1_{X_a}$. We have a following picture;

$$X_a \quad X_{a'} \rightarrow A_\alpha^* X_a$$

$$A_a \xleftarrow{A_\alpha} A_{a'}$$

$$a \xrightarrow{\alpha} a'$$

For any composed morphisms of A

$$a \xrightarrow{\alpha} a' \xrightarrow{\alpha'} a''$$

we have the following commutative diagram

$$X_{a''} \xrightarrow{\xi_\alpha} A_{\alpha'}^* X_{a'} \xrightarrow{A_{\alpha'}^* \xi_\alpha} A_{\alpha'}^* A_\alpha^* X_a \xrightarrow{\xi'_\alpha \alpha} X_{a''}$$

To each morphism $\phi : (X, \xi) \rightarrow (Y, \eta)$ associates a morphism $\phi_a : X_a \rightarrow Y_a$ for any object a of A such that for any $\alpha : a \rightarrow a'$ the following square is commutative.

$$\begin{array}{ccc} X_{a'} & \xrightarrow{\xi_\alpha} & A_\alpha^* X_a \\ \phi_{\alpha'} \downarrow & & \downarrow A_\alpha^* \phi_\alpha \\ Y_{a'} & \xrightarrow{\eta_\alpha} & A_\alpha^* Y_a \end{array}$$

Let A be a small category, G a presheaf of small groupoids over A and $BG = \nabla G$ the classifying category associated to G .

Lemma 1 *Let A be a small category and A a presheaf of small categories over A . Then we obtain the canonical categorical equivalence.*

$$\hat{\nabla} A \cong Rep(A)$$

Theorem 2 (Grothendieck) *Let W be a basic localizer and $u : A \rightarrow B$ a W -smooth functor. Then if B is a W -local test category, A is also a W -local test category.*

We here assume an axiom of great cardinality which is said to be the principle of Vopenka. Hence every localizer over a small category is accessible.

Corollary 1 *Let W be an accessible basic localizer and A a W -local test category and A a presheaf of small categories over A . Then the category of A -representations admits a structure of closed model category with generated cofibrant the cofibrations of which are the monomorphisms and the weak equivalences of which are the W -equivalences. If moreover W is proper, then this structure is proper. There exists a categorical equivalence between*

the homotopical category $H_W \text{Rep}(A)$ and the category $\text{Hot}_W // \nabla A$ of the locally constant W -homotopy types over ∇A .

$$\text{Hot}_W // \nabla A \cong \nabla A$$

Given a presheaf of small groupoids G over a small category A , we have a classifying space BG of G with a universal G -torsors

$$EG \rightarrow BG$$

From now on suppose a basic localizer is accessible and that A is a W -local test category.

Since EG is a G -torsor over BG and since the morphism from EG to the final object of G is a right proper W -equivalence. We can construct such a classifying space by choosing locally cofibrant representation EG that the morphism from EG to the final object of $\text{Rep}(G)$ is a trivial fibration in the meaning of the structure of the category of local closed models with respect to W . We can hence define BG as the quotient of EG under action of G . The forgetful functor $\text{Rep}G/EG \rightarrow \text{Rep}G$ is a left Quillen equivalence for the categorical structures of local or injective closed models with respect to W .

There exists a left Quillen equivalence

$$\text{Rep}(G)/EG \rightarrow \hat{A}/BG \quad (X, X \rightarrow EG \mapsto (G \setminus, G \setminus \rightarrow BG)$$

Since we have a categorical equivalence

$$H_W \text{Rep}(G)/EG \cong H_W \hat{A}/BG$$

and a canonical categorical equivalence

$$H_W \text{Rep}(G)/EG \cong H_W \text{Rep}(G)$$

we have a descent functor which is a categorical equivalence

$$\mathbf{Desc} : H_W \text{Rep}(G) \longleftarrow H_W \hat{A}/BG$$

The descent functor admits a monodromy functor as quasi-inverse

$$\mathbf{Mon} : H_W \hat{A}/BG \longleftarrow H_W \text{Rep}(G)$$

defined by

$$(X, X \rightarrow BG \xrightarrow{\text{Mon}} X \times_{BG} EG$$

The fact that functors descent and monodromy which are quasi-inverses can be considered as a generalization of the topological Galois theory giving a correspondence between the category of covering of a locally simply connected space and the category of its fundamental groupoid representations.

3 Groupoids for algebraic normal varieties

Let us consider the 2-category of categories of the sheaves of groupoids representations whose objects are the sheaf of groupoids representation categories and whose morphisms are functors compatible with sheaf of groupoid action. Apply it to the category Sch of locally noetherian connected normal schemes and take a sheaf \mathbf{G} of groupoids as

a sheaf of categories whose objects are universal schemes over opens and whose morphisms are Grothendieck etale fundamental groups acting on universal schemes.([EGA],[SGA],[GG], [TAM],[RBZL], [Shatz]) Then

$$\begin{aligned} \text{Rep}(\mathbf{G}) &= \nabla \hat{\mathbf{G}} \\ \mathcal{H}_W \text{Rep}(\mathbf{G}) &\cong \mathcal{H}_W \hat{Sch} / \nabla \mathbf{G} \\ \mathcal{H}_W \text{Rep}(\mathbf{G}) &\cong \mathbf{Hot}_W // \nabla \mathbf{G} \end{aligned}$$

Let X be a locally noetherian connected normal scheme and replace Sch by Sch/X . We obtain

$$\begin{aligned} \text{Rep}(\mathbf{G}|X) &= \nabla \hat{\mathbf{G}}|X \\ \mathcal{H}_W \text{Rep}(\mathbf{G}|X) &\cong \mathcal{H}_W \hat{Sch}/X / \nabla \mathbf{G}|X \\ \mathcal{H}_W \text{Rep}(\mathbf{G}|X) &\cong \mathbf{Hot}_W // \nabla \mathbf{G}|X \end{aligned}$$

Hence X is uniquely defined by $\mathbf{G}|X$. $\mathbf{G}|X$ is determined by the absolute Galois groups of all points on X .

Proposition 2 *Let k be a ground field. Let K be a function field over k and K^s a separable closure of K in an algebraically closed extension of K . Then to each automorphism $\sigma \in \text{Aut}_k(K)$ associates an automorphism of $\text{Gal}(K^s/K)$ such that*

$$\sigma \in \text{Aut}_k(K) \longrightarrow (g \mapsto g^\sigma) \in \text{Aut}(\text{Gal}(K^s/K))$$

which is a mono-morphism. Here extensions are given for each automorphism of $\text{Aut}_k(K)$.

Proof 1 *Let $\text{Aut}_k(K^s, K) = \{g|g \in \text{Aut}_k(K^s), g(K) \subset K\}$. We have a natural homomorphism*

$$\varphi : \text{Aut}_k(K^s, K) \rightarrow \text{Aut}_k(K).$$

Then $\text{Gal}(K^s/K) = \ker(\varphi)$. Note that $\text{Gal}(K^s/K)$ is center-free. Thus for any $\sigma \neq \text{id}$ of $\text{Aut}_k(K)$ there exists an extension $\bar{\sigma}$ since K^s/K is algebraic such that $g \mapsto g^\sigma = \bar{\sigma}^{-1}g\bar{\sigma}$ is non-trivial automorphism. $g \mapsto g^\sigma = g$ for all g implies $\sigma = \text{id}$. Hence $g^\alpha = g^\beta$ for all g implies $\alpha = \beta$. Thus $\sigma \mapsto (g \in \mathbf{Gal}(K^s/K) \mapsto g^\sigma \in \mathbf{Gal}(K^s/K))$ is injective.([Breen1], [Breen2],[Gir], [Rou],[Se])

Proposition 3 *Let G denote $\text{Aut}_K(K^s) = \text{Gal}(K^s/K)$.*

1. there exists an exact sequence

$$1 \rightarrow \text{Aut}_K(K^s) \rightarrow \text{Aut}_k(K^s, K) \rightarrow \text{Aut}_k(K) \rightarrow 1$$

2. In the following diagram, the vertical homomorphism is surjective and the horizontal homomorphism is defined by g^σ for an element σ of $\text{Aut}_k(K^s, K)$ for any element g of $\text{Aut}_K(K^s)$. Here $\bar{\sigma}$ is an extension when σ is an element of $\text{Aut}_k(K)$.

$$\begin{array}{ccc} \sigma \mapsto & & (g \mapsto g^\sigma = \bar{\sigma}^{-1}g\bar{\sigma}) \\ & & \\ \text{Aut}_k(K^s, K) & \longrightarrow & \text{Aut}(\text{Aut}_K(K^s)) \\ & \downarrow & \\ & \text{Aut}_k(K) & \end{array}$$

3.

$$\text{Aut}_k(K^s) = \varprojlim_{\psi} \text{Aut}_k(K^s, \psi(K))$$

where ψ runs all $\text{Aut}_k(K^s)$.

4.

$$\text{Aut}_K(K^s) \triangleleft \text{Aut}_k(K^s, K)$$

5.

$$\text{Aut}_{\psi(K)}(K^s) \triangleleft \text{Aut}_k(K^s, \psi(K))$$

6. There exists a surjection

$$\text{Aut}(\text{Gal}(K^s/K)) \rightarrow \text{Aut}_k(K^s)$$

which is a fibre space with the same fibre $\text{Gal}(K^s/K)$.

Proof 2 1. Since K^s/K is separably algebraic, every automorphism of $\text{Aut}_k(K)$ is able to have an extension to $\text{Aut}_k(K^s)$ which keeps K into itself. Hence we have a surjective homomorphism

$$\text{Aut}_k(K^s, K) \rightarrow \text{Aut}_k(K)$$

whose kernel is $\text{Aut}_K(K^s)$. In fact K is a fixed field by action $\text{Gal}(K^s/K)$ on K^s .

2. From (1) it is obvious.

3. It follows from (1) and (2) replacing K by $\psi(K)$.

4. From (1) it follows immediately.

5. The proof is the same argument as (4).

6. Since $\text{Aut}_k(\psi(K))$ is isomorphic to $\text{Aut}_k(K)$ and

$$\text{Aut}_k(K^s) = \varprojlim_{\psi} \text{Aut}_k(K^s, \psi(K))$$

we get (6) with the aid of (1).

Definition 4 Let k be a ground field and K a function field over k . A groupoid is defined to be a category whose object is $\text{Spec}(K^s)$ and whose morphisms are $\text{Gal}(K^s/K)$.

Proposition 4 Let k be an algebraically closed field. Let E be a groupoid extension of a profinite group P by a groupoid G . Assume

- E has a group section over P . Here it is necessary to assume k is algebraically closed.
- P is the absolute Galois group $\Gamma_K = \text{Gal}(\hat{K}/K)$.
- G is a groupoid whose objects is the separably algebraic closure of a field in an algebraically closed field, i.e., $\text{Spec}(K^s)$ and whose morphisms are the absolute Galois groups acting the object.
- the automorphism group $\text{Aut}_{\Gamma_L}(G_L)$ of G_L is locally algebraic group for every algebraic extension field L of K . ([Mat], [MO])

Then P is trivial extension after base-change by a finite extension K' of K .

Proof 3

$$\begin{array}{ccccc}
 H^1(P, \text{Aut}(G)) & \longrightarrow & H^1(P, (G \rightarrow \text{Aut}(G))) & \longrightarrow & H^2(P, G) \\
 & \searrow & \downarrow & & \\
 & & H^1(P, \text{Out}(G)) & &
 \end{array}$$

A given extension class is in $H^1(P, (G \rightarrow \text{Aut}(G)))$. If E has a group section, $P \rightarrow E$, $E \rightarrow \text{Inn}(E)$ and $\text{Inn}(E) \rightarrow \text{Aut}(G)$ gives an element $H^1(P, \text{Aut}(G)) \cong \text{Hom}(P, \text{Aut}(G))$. By assumption this image is a finite group. After base-change E attains a trivial extension.

We will next show a special case, which is a prototype of the precedent. Let p be a prime number. Let K be a subfield of a finitely generated field extension of \mathbb{Q}_p , which is called a sub- p -adic field. Let $\text{Gal}(/K)$ be the absolute Galois group $\text{Gal}(\hat{K}/K)$, where \hat{K} is the algebraic closure of K in an algebraically closed field containing K . In the following commutative diagram

$$\begin{array}{ccc}
 \text{Spec}K(X) & \xrightarrow{\quad} & \text{Spec}K(Y) \\
 & \searrow \quad \swarrow & \\
 & \text{Spec}K(S) & \\
 & \downarrow & \\
 & \text{Spec}K &
 \end{array}$$

where $K(X)$, $K(Y)$ and $K(S)$ are function fields which are regular extensions over K , i.e., X, Y, S are varieties geometrically irreducible, reduced over K , if X is trivial, i.e., $X = X_0 \times_K S$ for some variety X_0 over K , when does Y decompose $Y = Y_0 \times_{K'} S$ for some variety Y_0 over a finite extension K' of K ? The answer is yes when $\text{Aut}_{\hat{K}}(\text{Spec}\hat{K}(Y))$ is a group scheme which is locally of finite type, i.e., locally algebraic group.

Remark 1 1. The object of our groupoid is a universal covering of a scheme to act by the absolute Galois group, where the universal covering must also a universal covering of such a scheme that we should prove in the conclusion. ([Zuo]) For example it is suitable for the aim to take $\text{Spec}(K)$ as a scheme since its universal covering $\text{Spec}(K^s)$ is also a universal covering of any type sub-covering.

2. Let K, L be function fields over a ground field k . Assume L is an extension of K . Let $\text{Gal}(K^s/K), \text{Gal}(L^s/L)$ be the absolute Galois groups of K, L , respectively. Then we have a homomorphism $\text{Gal}(L^s/L) \rightarrow \text{Gal}(K^s/K)$, which canonically factors in the following way:

$$\text{Gal}(L^s/L) \rightarrow \text{Gal}(K^s/(L \cap K^s))$$

is a surjection.

$$\text{Gal}(K^s/(L \cap K^s)) \rightarrow \text{Gal}(K^s/K)$$

is an injection.

3. Non commutative analogue above is available in the case we can define the algebraically closed fields ([Cohn]).

4 Program for minimal models

Definition 5 ([MP], [Mori], [I], [Kaw], [Mats]) Let X be a normal variety.

1. X is said to be Cohen-Macaulay if each local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay for every point $x \in X$.
2. Let ω_X be the canonical sheaf over X . The ω_X is defined to be $i_*\Omega_{X_{reg}}^d$ where $i : X_{reg} \subset X$ is an inclusion and $\Omega_{X_{reg}}^d$ is a sheaf of regular d -forms with $d = \dim X$. It is a reflexive sheaf and a dualizing sheaf.
3. A canonical divisor K_X is a Weil divisor associated to ω_X . Further we denote

$$\omega_X^{[m]} = i_*(\Omega_{X_{reg}}^d)^{\otimes m}$$

to which is associated a Weil divisor mK_X .

4. If X is Cohen-Macaulay and there exists a number m (resp. $m=1$) such that mK_X is Cartier, X is said to be \mathbf{Q} -Gorenstein (resp. Gorenstein).

Definition 6 1. Let X be a normal \mathbf{Q} -variety. X has only terminal (resp. canonical) singularities if $\mu_i > 0$ (resp. $\mu_i \geq 0$) for every i where

$$K_{X'} = \pi^*K_X + \sigma_i\mu_iE_i$$

for a desingularization $\pi : X' \rightarrow X$ and for components E_i of exceptional divisors of π . μ_i is called discrepancy at E_i .

2. Let X be a normal variety and $D = \sigma_i d_i D_i$ an effective \mathbf{Q} -Weil divisor such that $K_X + D$ is \mathbf{Q} -Cartier. Then (X, D) is called a log pair. A desingularization $\pi : X' \rightarrow X$ is said to be a log resolution if the exceptional locus of π and $f^{-1}(D_{red})$ have only normal crossings. We there have

$$K_{X'} = \pi^*(K_X + D) + \sigma a_i E_i$$

where a_i are rational numbers and E_i are irreducible.

3. (X, D) is log-terminal if $d_i < 1$ and if $a_i > -1$ for all i .
4. (X, D) is Kawamata log terminal, i.e., klt if $d_i \leq 1$, $a_i > -1$ and if there exists a π -ample divisor whose support is the exceptional locus.
5. Let X be a normal (\mathbf{Q} -factorial) projective variety with at most terminal singularities. X is said to be a minimal variety if K_X is nef.

Conjecture 1 Every function field has a minimal model variety if a projective model is not uni-ruled.

Let k be a complex number field. We shall propose a program in the following. Let X be a projective variety. Let $k(X)$ be a function field. There exists a function field of transcendental degree $\dim X - 1$. Hence we have a fibre space between projective normal varieties

$$f : X \rightarrow S$$

such that there exist effective \mathbf{Q} -Cartier divisors D_X and D_S over X and S , respectively such that

$$f^*(m(K_S + D_S)) \rightarrow m(K_X + D_X)$$

where $(K_S + D_S)$ and $(K_X + D_X)$ are \mathbf{Q} -Cartier and some $m > 0$. ([Mum]) To prove it we make use of Iitaka-Vieweg conjecture ([Vieh], [Vieh2], [Vieh3], [Vieh4]) and its log version. In our case infinitesimal Torrelli method is useful since relative dimension is one for a fibre space X/S . Assume $\kappa(K_X + D_X) \geq 0$ over X . Then we can take D_S over S such that $(K_X + D_X) - f^*(K_S + D_S)$ has no components pull-back of effective \mathbf{Q} -divisors over S . Then $\kappa(K_S + D_S) \geq 0$ since $\kappa(K_X + D_X) \geq 0$. By induction argument, we may take a model S such that $K_S + D_S$ is nef (resp. abundant). It suffices to show

$$f_*\mathcal{O}_X(m(K_X + D_X) - f^*(m(K_S + D_S)))$$

is nef (resp. abundant). We construct a fibre space $f : X \rightarrow S$ in the following way. Take a product $S \times P^1$ and an integral closure of $S \times P^1$ in the function field $k(X)$. We rename this integral closure by X . Thus X is normal and projective. The structure map

$$\tau : X \rightarrow S \times P^1$$

is finite. There is a trace map

$$\tau_*\omega_X \rightarrow \omega_{S \times P^1}$$

If they are true, $K_X + D_X$ is nef (resp. abundant) since the tensor between $f_*\mathcal{O}_X(m(K_X + D_X) - f^*(m(K_S + D_S)))$ and $f^*(m(K_S + D_S))$ is nef (resp. abundant) for some $m > 0$. Then MMC shall be a special case.

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