

All Error Detecting Modified Hamming Codes for Ideal Optical Channel and Application to Practical System

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Modified Hamming Codes designed to have the capability of all error detecting and one error correcting on an ideal optical channel are presented. The codes are applicable to an optical channel with only quantum noise, and are able to achieve all error excluding transmission on the optical channel in a future advanced communication system. The codes are obtained from the Modified of ordinary (7, 4) Hamming codes by adding an extra one-bit symbol to make all errors detectable. It is shown that the codes are also useful on a practical level to completely exclude almost all errors on a channel affected by not only quantum noise but also external noise.

1 Introduction

In the past four decades, high-speed and long-span optical transmission system technologies have been developed yearly [1] to realize high-performance communication networks. To achieve high-performance communication networks, signal transmission and decision problems are important for reliability. For the decision problems, the analysis of bit-error rate of product accumulate codes [2], the detection of optical turbo-product codes [3][4], and practical error free transmission [5] have been reported. Such studies are useful for the improvement of practically implemented systems. However, no approach for the realization of all error excluding optical transmission system has been considered.

It is known, however, that quantum noise remains as a final obstacle to the achievement of such an all error excluding optical transmission system, because quantum noise cannot be removed and causes unavoidable errors to optical signal decisions. Therefore, a method that can exclude all unavoidable errors would be regarded as useful for advanced future communication. For this purpose, the author investigated Polarization Shift Keying (PolSK) theoretically and experimentally [6] [7] to enable the detection of all errors caused by quantum noise and reported a method by which an all error excluding transmission system could be realized.

In order to achieve the above transmission affected by only quantum noise with no external noise, an ultra-low-noise detector is required, and single-photon detectable counting devices are important and have been reported [8]-[10]. With the addition of these detection devices, error correcting procedures based on asymmetric transition of symbols [11]-[13] on an optical transmission channel have been reported. The correction procedures are based on the asymmetrical transition property of symbols caused by quantum noise appearing specifically on an optical signal channel. However, there has been no report of a procedure including both error detection and correction, excluding all errors caused by quantum noise on an optical transmission channel.

Therefore, a procedure to enable all error detecting and one error correcting is considered in the present paper. To treat this problem, noises appearing on an optical channel must be known. Noises that must be considered are classified into two types: quantum noise and external noise. Here, the quantum noise is known as shot noise, and is represented by a Poisson distribution, and the external noise is regarded as including the concepts of spontaneous emission noise in a laser oscillator, thermal noise in free space radiation, inter-symbol interference noise in a fiber channel, electron-hole pair multiplication noise in

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an avalanche photo diode, as well as other noises. All noises except for the electron-hole pair multiplication noise are considered to be approximately represented as a Gaussian noise.

For quantum noise, the odd parity codes, such as PolSK, are useful for all error detecting. However, the codes require retransmissions for error correction. Therefore, it is better to introduce more efficient codes designed to have a capability of error correcting that does not require retransmission. As error correcting codes, the Hamming codes are known and are applicable for most types of noises. However, the Hamming codes generally do not have the capability of all error detecting, then modification of the Hamming codes to have this capability should be considered.

In Section 2, the modified Hamming codes are introduced, and application to the channel affected only by quantum noise is discussed. In Section 3, the detection and correction characteristics of errors caused by quantum and external noises are discussed. Finally, retransmission and undetectable error probabilities of both the odd parity codes and the modified Hamming codes are considered in Section 4.

2 All Error Detecting and One Error Correcting for Quantum Noise Channel

In order to analyze the error characteristics of a signal on an optical transmission channel, noises appearing on a channel have to be known as a fundamental step. Noises appearing on an optical transmission channel are basically classified into two types. One is quantum noise, which inevitably appears concurrently with the existence of an optical signal and cannot be eliminated. Another is external noise, which includes spontaneous emission noise in a laser oscillator, thermal noise, inter-symbol interference noise in fiber, and multiplication noise in avalanche photo diode.

Quantum noise has been focused since the beginning of the development of optical communication, and fundamentally affects optical communication and characterizes the quantum mechanical aspects of optical communication. Quantum noise has an essential and intuitive effect of causing errors through optical signal transmission. Since quantum noise is important as a fundamental concept, the code design for a transmission channel that is affected only by quantum noise is first considered. Next, a channel that is affected by both quantum noise and external noise is considered.

For a channel affected only by quantum noise, the binary odd parity codes are useful, and the codes have a detection capability of one. This means that all errors that occurred on the transmission channel are detectable because all

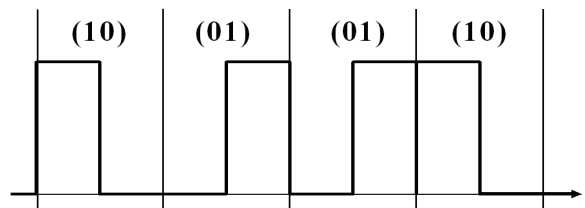
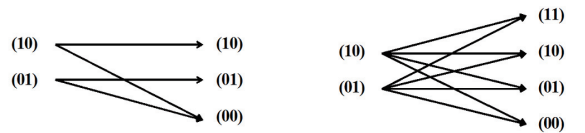


Fig.1 Binary odd parity codes



(a) Transitions by quantum noise (b) Transitions by quantum and external noises

Fig.2 Symbol transitions on an optical channel: (a) Transitions by quantum noise, (b) Transitions by quantum and external noises.

errors occur only on symbols of 1, and the number of errors in a code can only be one. The binary odd parity codes configured in time series are obtained as a direct conversion of PolSK, which was theoretically and experimentally demonstrated as all error detectable codes for the quantum noise channel. Figure 1 shows the binary odd parity codes in time series consisting of (01) and (10).

Figure 2 shows the transitions of the binary odd parity codes on the optical transmission channel. Figure 2(a) depicts the transitions on the channel affected only by quantum noise, and Figure 2(b) the transitions on the channel affected by both quantum and external noises. While quantum noise only affects errors of symbol 1, it does not have any effect on errors of symbol 0. The quantum noise appears as shot noise and is characterized statistically by the Poisson distribution, and the distribution is given by

$$p(m) = \frac{S^m}{m!} e^{-S}$$

where S and m represent the average number of photons included in an optical pulse, and the number of observed photons, respectively .

Using the above equation, the error probabilities of symbols 0 and 1 can be calculated. Let p_{e0} and p_{e1} be the error probabilities of symbols 0 and 1 decided as inverse values of 1 and 0, respectively, and the probabilities then become

$$\left. \begin{aligned} p_{e0} &= 0 \\ p_{e1} &= e^{-S} \end{aligned} \right\}$$

where S represents the average number of photons in a pulse of symbol 1. The probabilities for the transition of symbols 0 and 1 on the channel then take asymmetrical values. If the codes of both (10) and (01) are transmitted with no transition, then correct codes equal to the sent original codes are received with a probability of $1 - p_{e1}$. On the other hand, if an error occurs, the codes of both (01) and (10) always transition to (00) with the probability of p_{e1} .

Transitions of symbols on a channel affected by quantum and external noises have probabilities that differ from the values in the case of a channel affected only by quantum noise. The values for the transitions still take asymmetrical values, although the possible paths of transitions have a bidirectional pattern. The equations for the calculation of the transitions and the values of the probabilities will be given in Section 3 with detailed numerical characteristics of error probabilities caused by quantum and external noises.

To configure new all error detecting and one error correcting codes for excluding all errors caused by quantum noise on an optical signal transmission channel, a modification of a typical set (7, 4) in the Hamming codes is introduced. The Hamming codes of (7, 4) have an error

Table I Extended Hamming codes having a length of eight and an even parity

Modified Hamming codes

	Information vector	Modified Hamming Code	Hamming weight
	a b c d	$p_0 p_1 a p_2 b c d p_4$	(Number of 1)
0	0 0 0 0	0 0 0 0 0 0 0 0	0
1	0 0 0 1	1 1 0 1 0 0 1 0	4
2	0 0 1 0	0 1 0 1 0 1 0 1	4
3	0 0 1 1	1 0 0 0 0 1 1 1	4
4	0 1 0 0	1 0 0 1 1 0 0 1	4
5	0 1 0 1	0 1 0 0 1 0 1 1	4
6	0 1 1 0	1 1 0 0 1 1 0 0	4
7	0 1 1 1	0 0 0 1 1 1 1 0	4
8	1 0 0 0	1 1 1 0 0 0 0 1	4
9	1 0 0 1	0 0 1 1 0 0 1 1	4
10	1 0 1 0	1 0 1 1 0 1 0 0	4
11	1 0 1 1	0 1 1 0 0 1 1 0	4
12	1 1 0 0	0 1 1 1 1 0 0 0	4
13	1 1 0 1	1 0 1 0 1 0 1 0	4
14	1 1 1 0	0 0 1 0 1 1 0 1	4
15	1 1 1 1	1 1 1 1 1 1 1 1	8

correcting capability of one and an error detecting capability of two. This means that error detection is not possible if errors larger than or equal to three symbols in a code of (7, 4) occur though transmission. Then, the codes are insufficient to detect all errors caused by quantum noise. Therefore, some new scheme appropriate to detect all errors caused by quantum noise must be introduced to achieve an all error excluding optical transmission system for advanced future communication.

Table 1 shows the modified Hamming codes for the above purpose. To modify the Hamming codes of (7, 4) to even parity codes, a redundant symbol is added to the end of each code. As a result of the addition of the symbol, all codes except rows 0 and 15 have the same Hamming weight of $w = 4$. For the purpose of all error detecting, codes having the same Hamming weight are desirable, because an error reduces the Hamming weight and makes all error detecting possible. It is then necessary to deselect the codes of row 0, having all 0s, and row 15, having all 1s, for the new code set.

Figure 3 shows the transmission system model using the modified Hamming codes. Before transmission, the modified Hamming codes are configured in a transmitter. The configuration is performed through the three processes. The first process is the configuration of the Hamming codes. The second process is the addition of an even parity symbol to the end of each code. The third process is the deselection of all 0 and 1 codes. Thus, the configured modified Hamming codes are transmitted to the receiver, and quantum noise on the transmission channel affects the symbols of 1 in the codes to change the value to 0.

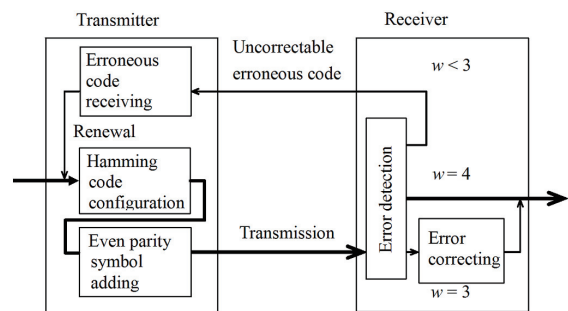


Fig.3 All error detecting and one error correcting system for a channel affected only by quantum noise.

In the receiver, the detection of errors is based on the Hamming weight. If the Hamming weight of the received code is equal to 4, then the receiver recognizes that no error has occurred and then receives as a correct code with no error. If a received code has a Hamming weight of 3, then one error has occurred in the transmission process, and the inversed symbol will be corrected. The inversed symbol is estimated by applying an error detecting and correcting procedure for the Hamming codes to a block of seven former symbols.

When the error is contained in a block of the former seven symbols and does not occur on the end symbol, the position of a inversed symbol is determined by a calculated value from the procedure for the Hamming codes, and the fact that the Hamming weight is 3 means that only one error is included. On the other hand, because the procedure for the Hamming codes provides a calculation in which no error occurs in the former seven symbols when an error occurs on the end symbol, the results having a Hamming weight of 3 and no errors among the former seven symbols leads to the assumption that errors occurs only on the end symbol.

When errors equal to or larger than 2 occur, the Hamming weight becomes equal to or smaller than 2. Since the codes for different values of the Hamming weight are not designed in the original codes configured in the transmitter, the occurrence of errors larger than 2 can still be detected. According to the above-described estimation process, one error can be corrected and the occurrence of all errors can be detected in the receiver. For the correction of received codes including errors larger than 2, a procedure for

the retransmission of the original correct codes corresponding to the received codes, including errors, can be applied. Therefore, the retransmission demand for the code including error symbols is sent back to the transmitter.

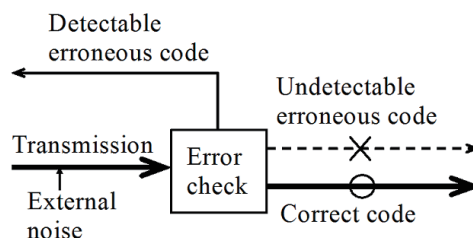
The original correct code will be retransmitted to the code demanding retransmission in renewal. Any codes including errors can be corrected by applying a combination of an error detecting procedure by the Hamming weight and an error detecting and correcting procedure for the Hamming codes, and applying the retransmission procedure to sent back codes. Finally, all errors can be excluded and a no-data-loss optical transmission system will be achieved.

Exactly other additional errors may also occur on the retransmission demand signals, in which case it is better to use echoed codes to received codes as retransmission demand. Furthermore, errors may also occur in electronic circuits in a transmitter and a receiver, and the problem should be considered in another study. While the error detecting and correcting problem in electronic circuits may be solved using a solution obtained for an electronic signal problem in a similar manner to an optical signal problem [14], this is not discussed herein because this problem is beyond the scope of the present paper.

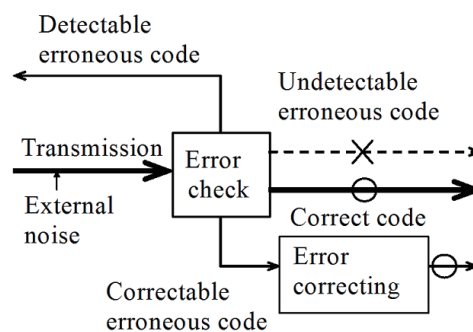
3 Application to Practical Transmission Suffering from Quantum and External Noises

In this section, the error detecting and correcting characteristics of binary odd parity codes and modified Hamming codes when both codes are transmitted on a channel affected by both quantum and external noises. Figure 4 shows the error detecting and correcting processes in a receiver, (a) for the case of binary odd parity codes, and (b) for the case of modified Hamming codes. For a channel affected by both quantum and external noises, it is not true that any errors can be detected and corrected, whereas for a channel affected only by quantum noise, any errors can be detected and corrected.

On the channel affected by both quantum and external noises, external noise causes not only a transition of a symbol 1 to a symbol 0, but also a transition of a symbol 0 to a symbol 1 while quantum noise causes only the former transition. Therefore an error probability based on the latter transition by external noise must be considered in addition to the error probability based on the former transition by quantum and external noises. In the process of an error check for the binary odd parity codes, error detection is possible in the case of a single error, in which one symbol 0 or 1 is inverted. However, it is undetectable for the case in which both symbols are inverted. According to the above-described error check process, a received code is directly accepted as a correct code, as if the



(a) Binary odd parity code



(b) Modified Hamming code

Fig.4 Error detection and correction system for a channel affected by quantum and external noises: (a) Binary odd parity codes, (b) Modified Hamming codes.

received code has no error. To the code in which one error has been detected, the retransmission demand is sent back to a transmitter. The undetectable error probability, however, cannot be eliminated because the error probability whereby both errors that occur cannot be zero. Therefore, the elimination of all errors and no-data-loss optical transmission cannot be achieved by binary odd parity codes on the channel affected by both quantum and external noises.

In the error check process of the modified Hamming codes, a received code including no error is accepted directly and a code including one error is corrected automatically. A received code that estimated the code in which errors are included sends back a retransmission demand to a transmitter, and the original correct code will be retransmitted to a receiver. However, there remains a non-zero probability of undetectable errors, and the existence of non-zero probability is an obstacle to the possibility of achieving an all error excluding optical transmission system by the modified Hamming codes on a channel affected by quantum and external noises.

Next, it is necessary to calculate error probabilities for various combinatorial cases of errors. Letting p_{e1} be an error probability of a symbol 1, and let $E_1(r)$ be the error probability for which r out of four symbols of 1 are inverted into 0, the probability is given by

$$E_1(r) = {}_4C_r p_{e1}^r (1 - p_{e1})^{4-r} \tag{1}$$

where ${}_xCy$ represents the combination number. The probabilities for the symbol 0 are also obtained in a similar manner. Letting p_{e0} be the error probability of a symbol 0, and let $E_0(r)$ be the error probability that r out of four symbols of 0 are inverted into 1, the probability is given by

$$E_0(r) = {}_4C_r p_{e0}^r (1 - p_{e0})^{4-r} \tag{2}$$

For a channel affected by quantum and external noises, the Laguerre distribution can be applied for a channel in which all external noises are assumed to be completely represented by a Gaussian distribution. The above assumption of external noises does not always hold. However, several types of noises appearing on an optical channel can be regarded as approximately Gaussian, and the distribution for photon counting is given by

$$p(m) = \frac{N^m}{(1 + N)^{m+1}} e^{-\frac{S}{1+N}} L_n^{(0)} \left(-\frac{S}{(1 + N)N} \right)$$

In the above equation, N represents the average photon number of noise observed in an optical pulse. A Gaussian distribution, for example, can be applied to thermal noise, and approximately to spontaneous emission noise in a laser and inter-symbol interference noise. When the average signal photon number is equal to zero, the distribution becomes an asymptotic Bose-Einstein distribution and is applied to the photon counting statistics of a signal 0.

The multiplication noise in an avalanche photo diode is important as an external noise. However, the noise is difficult to treat as

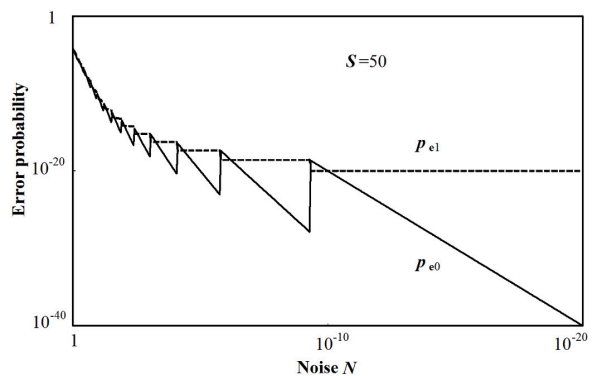


Fig.5 Error probability of a single symbol.

Gaussian distribution noise. Although the distribution of the multiplication noise is different from a Gaussian distribution, the tendency of the multiplication noise as external noise to affect signal errors is similar to other external noises. Therefore, the external noise considered herein is treated representatively as a Gaussian distribution.

Figure 5 shows the single-error characteristics of symbols 0 and 1 calculated on the basis of the Laguerre distribution. The horizontal axis indicates the average photon number of noise, and the vertical axis indicates the single error probability. Jumping positions corresponding to changing points of threshold levels according to the decrease of the average photon number of external noise appeared in the curves. The threshold level is decided as a photon number, at which the two curves for symbols 0 and 1 cross if the photon number could take a continuous real number. The photon number, however, is discrete, and so the threshold level can take a real number in the area between the lower and upper integer threshold numbers, in which the two curves cross.

In a static threshold area, according to the noise degradation, since the probability of the Bose-Einstein distribution for a photon number larger than the threshold becomes very small, the error probability of symbol 0 approaches 0. On the other hand, the error probability for the symbol 1 in the static threshold area remains approximately constant because the distribution depends mainly on the relatively large average signal photon number and does not depend on the negligibly small average noise photon number.

4 Undetectable Error Probabilities

The retransmission probability for a binary odd parity code is equal to the probability that a code is received as (00). The probability is given by the product of probability p_{e1} of a single error of symbol 1 and probability $(1-p_{e0})$ for no error of symbol 0. The undetectable probability is the probability that the received code includes two errors, which is completely changed into another code. The probability becomes equal to the product of two single-error probabilities of p_{e0} and p_{e1} .

On the other hand, the retransmission probability for the modified Hamming Code becomes the probability that two symbols are inverted because the capability of error detection of the modified Hamming code is 2 which is the same as the Hamming code, because the probability of error detection directly gives the retransmission probability. The cases of two errors are classified into two types: cases in which both errors occurred in the previous block of seven symbols, and cases in which one of two errors occurs in the former block and the remaining error occurs on the last symbol. This means that the probability is obtained independently of the positions of error that occurs because the single-error probability does not depend on the position of a symbol and is obtained as the probability for a combination of two errors of the symbols 0 and 1 in a code. The error probability is given by

$$E_1(2)E_0(0) + E_1(1)E_0(1) + E_1(0)E_0(2) \quad (3)$$

Figure 6 shows the retransmission probabilities of the binary odd parity code and the modified Hamming code, where the horizontal axis indicates the average photon number of the noise. The retransmission probability for the modified Hamming code is equivalent to the probability of correctable errors among the errors that cannot be corrected by the correcting procedure for the Hamming codes. It is shown that the retransmission probabilities for both codes decrease rapidly according to the decrease of the external noise. In particular, the above-described tendency is significant on the curves of the modified Hamming code. The reason for this tendency is thought to be that the difference in the detectable error

number for the binary odd parity code is only one, while the number for the modified Hamming code is two.

Finally, undetectable probabilities for the codes have to be considered. The undetectable error probabilities for the binary odd parity code is derived from the previous discussion in this section and is given by

$$P_{e0}P_{e1} \tag{4}$$

The undetectable error probability for the modified Hamming code is calculated as the probability that three or more symbols are inverted. The probability is obtained from the sum of the probabilities for combinations of errors greater than or equal to three symbols. Although the method is not complex, the resulting equation is long and redundant, then only the first approximation that is important for calculation is given here. The approximation then becomes

$$\begin{aligned} &Pr\{\text{Number of errors is larger than or equal to three}\} \\ &\approx E_1(3)E_0(0) + E_1(2)E_0(1) + E_1(1)E_0(2) + E_1(0)E_0(3) \end{aligned} \tag{5}$$

In Fig. 7, it is shown that the characteristics of the undetectable error probability Eq. (4) for the binary odd parity code and those for modified Hamming code are computed from Eqs. (1), (2), and (5). This shows that both curves have a similar tendency, whereby the probabilities decrease rapidly with the decreasing of noise, and the tendency is more remarkable than that of the retransmission probabilities discussed above. As shown in the figure, the probabilities of undetectable errors cannot be zero unless the noise becomes be 0. The probabilities, however, become very small, and are less than a level of 10^{-40} , for example. When the probabilities take values smaller than such the level, a signal transmission may practically be regarded as a transmission on a channel with only quantum noise. The transmission approximately realizes the ideal all error excluding optical transmission system, which is permitted only for a channel with quantum noise.

5 Conclusion

The modified Hamming codes designed to offer a capability of all error detecting and one error correcting are presented for an all error excluding optical transmission system. The codes are obtained by adding one bit parity symbol to the ordinary Hamming codes to make the detection of all errors possible. It

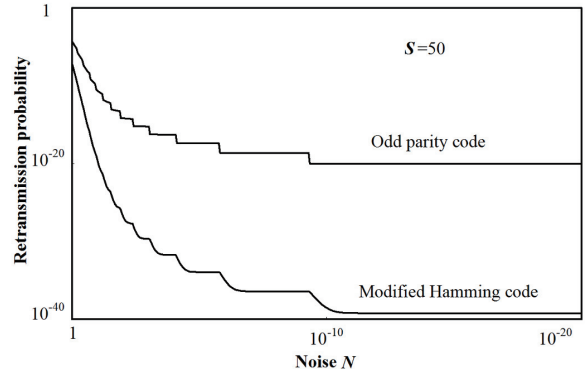


Fig.6 Retransmission probability of a binary odd parity code and a modified Hamming code.

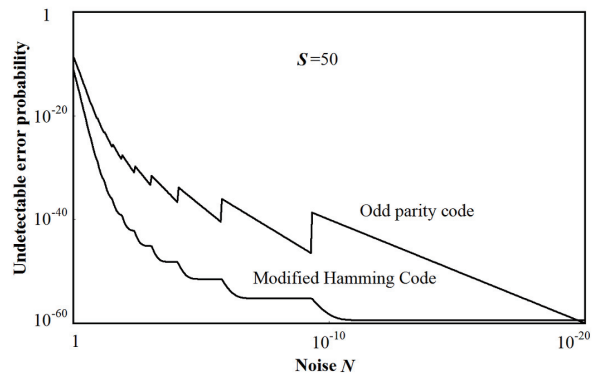


Fig.7 Undetectable probability of a binary odd parity codes and a modified Hamming codes

is shown that all error excluding transmission can be achieved if the modified Hamming codes are applied to the channel with quantum noise, despite the occurrence of decision errors being unavoidable because of errors caused by physically irremovable quantum noise. Furthermore, the codes are of a practically sufficient level to exclude most errors on the channel affected by quantum noise and low-level external noise.

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