External rays for a regular polynomial endomorphism of $\mathbb{C}^2$ associated with Chebyshev mappings

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In this note, the dynamics of a regular polynomial map of $\mathbb{C}^2$ is investigated. Especially, landing points of the external rays are completely characterized.

1 Introduction

In this note, we consider the external rays of the map $F : \mathbb{C}^2 \to \mathbb{C}^2$ of the form :

$$F(x, y) = (x^2 - 2y, y^2 - 2x).$$

External rays were first defined for polynomial maps on $\mathbb{C}$ to investigate the combinatorial properties of the dynamics on the Julia sets. Let P be a monic centered polynomial with degree d of one variable. Let $\varphi = \varphi_P$ be its *Böttcher coordinate*, that is, a conformal map φ in a neighborhood of the point at ∞ satisfying

$$\varphi(P(z)) = \varphi(z)^d, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} = 1.$$

By this functional equation, it can be continued analytically until it meets a critical point. Especially, if $K(P)$ is connected, it extends to a conformal map $\varphi : \mathbb{C} - K(P) \to \mathbb{C} - \overline{\mathbb{D}}$. The *external ray* $R_P(\theta)$ of *external angle* θ is defined by the preimage of the ray $\{re^{2\pi i\theta}; r > 1\}$ by φ. We say it lands at a point $z \in J(P)$ if it is continued to $r > 1$ and converges to z as $r \to 1$. Recently, Bedford and Jonsson [BJ] defined external rays for regular polynomial endomorphisms of $\mathbb{C}^k$ and established a landing property with some additional assumptions. Although the map F does not satisfy the assumptions in [BJ], we can investigate the landing property from the explicit expression of its Böttcher coordinate.

The map F has dynamically distinguished properties. For example, it is *critically finite*, that is, the union of the forward orbit of the critical set forms an analytic subset of

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$\mathbb{C}^2$. This is because it is closely related to the *Chebyshev maps* of two variables. A typical example of Chebyshev maps of one variable is the quadratic polynomial $p(z) = z^2 - 2$, which is critically finite, too. A natural extention of this Chebyshev map to the two variables case is $f(z) = z^2 - 2\overline{z}$. By virtue of the distinguished properties of Chebyshev maps, Uchimura [U1] has obtained many interesting results.

Here we show why Chebyshev maps are easily analyzed. Put $z = \psi(t) = t + 1/t$. Then

$$p(\psi(t)) = (t + 1/t)^2 - 2 = t^2 + 1/t^2 = \psi(t^2).$$

Hence a branch of its inverse $\varphi = \psi^{-1}$ satisfies $\varphi(p(z)) = \varphi(z)^2$ and it gives the Böttcher coordinate of p. Then the external ray $R_p(\theta)$ is explicitly written by

$$z = \psi(re^{2\pi i\theta}) = re^{2\pi i\theta} + \frac{1}{r}e^{-2\pi i\theta}, \quad r > 1,$$

and it lands at the point

$$z_0 = e^{2\pi i\theta} + e^{-2\pi i\theta} = 2\cos 2\pi\theta.$$

Consequently, $J(p) = \{z = 2\cos 2\pi\theta; \theta \in \mathbb{T}\}$. In the sequel, we will apply this idea to the maps f and F.

2 External rays for the map f

Consider the map f studied in [U1] of the form :

$$f(z) = z^2 - 2\overline{z}.$$

The map $f : \mathbb{C} \to \mathbb{C}$ is not holomorphic but is associated with the Chebyshev maps of two variables and its dynamics is completely determined. See [U1]. Since the jacobian of f is :

$$Jac(f) = |\partial f/\partial z|^2 - |\partial f/\partial \overline{z}|^2 = 4(|z|^2 - 1),$$

its *critical set* $\mathcal{C}(f)$ is the unit circle $|z| = 1$. The *filled-in Julia set* $K(f)$, i.e., the set of points with bounded orbits, is parametrized (with some identification) as follows.

$$K(f) = \{z = z(\phi, \theta) = e^{2\pi i\phi} + e^{2\pi i\theta} + e^{-2\pi i(\phi+\theta)}; (\phi, \theta) \in \mathbb{T}^2\}. \tag{2.1}$$

And its boundary is the hypocycloid (see Figure 1) :

$$z = 2e^{2\pi i\theta} + e^{-4\pi i\theta}, \quad \theta \in \mathbb{T}.$$

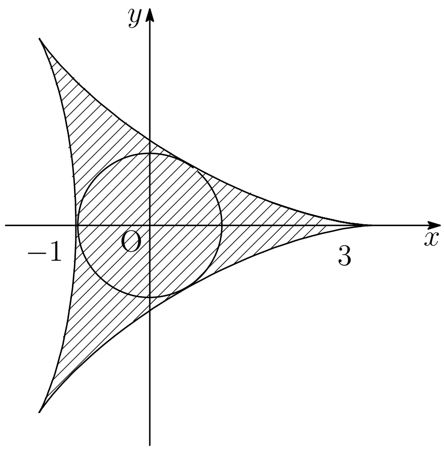

Figure 1: The dark region : $K(f)$, the circle : $\mathcal{C}(f)$

Moreover, the dynamics of f on $K(f)$ is expressed by $f(z(\phi,\theta)) = z(2\phi, 2\theta)$, which enables us to describe their dynamics by symbolic dynamics. Although this parametrization of $K(f)$ seems a bit tricky, we will give a dynamical meaning of the parameters ϕ, θ above in the next section.

First we study the Böttcher coordinate of f. Put

$$z = \psi(t) = t + \frac{1}{\bar{t}} + \frac{\bar{t}}{t}.$$

Then its jacobian $Jac(\psi)$ satisfies

$$\begin{aligned} Jac(\psi) &= |z_t|^2 - |z_{\bar{t}}|^2 \\ &= |1 - \frac{\bar{t}}{t^2}|^2 - |\frac{1}{t} - \frac{1}{\bar{t}^2}|^2 \\ &= (1 - \frac{1}{|t|^2})|1 - \frac{\bar{t}}{t^2}|^2. \end{aligned}$$

Thus ψ gives a diffeomorphism from $\mathbb{C} - \overline{\mathbb{D}}$ onto $\mathbb{C} - K(f)$ and it is easy to see

$$f(\psi(t)) = \psi(t^2), \quad \lim_{t\to\infty} \frac{\psi(t)}{t} = 1.$$

That is, the map $\varphi = \psi^{-1}$ should be the Böttcher coordinate of f and we can define the external ray as follows :

$$R_f(\theta) = \psi(\{re^{2\pi i\theta}; r > 1\}).$$

Then we have the following.

Theorem 2.1. *The external ray $R_f(\theta)$ is parametrized by*

$$z = re^{2\pi i\theta} + \frac{1}{r}e^{2\pi i\theta} + e^{-4\pi i\theta} \quad (r > 1)$$

and it lands at the point

$$z_0 = 2e^{2\pi i\theta} + e^{-4\pi i\theta} \in \partial K(f).$$

3 External rays for the map F

Now consider the following map in $\mathbb{C}^2$.

$$F(x, y) = (x^2 - 2y, y^2 - 2x),$$

which is closely related to the map f in the previous section. In fact, the map F restricted to $H = \{(x, y) \in \mathbb{C}^2; y = \overline{x}\}$ is equivalent to f. Let $\mathcal{C}(F)$ be the set of the critical points of F. By a direct calculation, it follows $\mathcal{C}(F) = \{xy = 1\}$.

Let $f(z)$ be a polynomial endomorphism of $\mathbb{C}^k$ of degree d and let $f_h(z)$ be the degree d part of $f(z)$. It is *regular* if $f_h^{-1}(0) = \{0\}$. Note that regular polynomial maps extend to analytic maps of $\mathbb{P}^k$. Let Π denote the hyperplane at ∞, which is isomorphic to $\mathbb{P}^{k-1}$. In case $k = 2$, Π is isomorphic to the Riemann sphere $\overline{\mathbb{C}}$. For a regular polynomial map f, we denote the filled-in Julia set also by $K(f)$. It is a compact subset of $\mathbb{C}^k$. And $J(f)$ denotes the smallest Julia set of f, that is, the support of $\mu = (dd^c G_f)^k$. Here G_f is the Green function of f. And we put $f_\Pi = f|_\Pi$, $J_\Pi = J(f_\Pi)$.

Let $W^s(J_\Pi, f)$ be the stable set of J_Π :

$$W^s(J_\Pi, f) = \{z \in \mathbb{P}^k; \lim_{n\to\infty} dist(f^n(z), J_\Pi) = 0\}.$$

The inverse Böttcher coordinate Ψ is a homeomorphism $W^s_{loc}(J_\Pi, f_h) \to W^s_{loc}(J_\Pi, f)$ conjugating f_h to f. It extends to $W^s(J_\Pi, f_h)$ until it meets a critical point. Each local stable manifold $W^s_{loc}(a)$ $(a \in J_\Pi)$ is a complex disk homeomorphic to $\overline{\mathbb{C}} - \overline{\mathbb{D}_R}$ for some $R > 1$. External rays are the rays in $W^s(J_\Pi, f)$ defined by the gradient lines of the Green function G_f restricted to $W^s_{loc}(a)$. Since the Böttcher coordinate transforms the Green function into a canonical form, external rays are the images of the actual rays by the inverse Böttcher coordinate, just as for polynomials of one variable.

Bedford and Jonsson [BJ] established the continuous landing property of external rays for regular polynomial endomorphisms of $\mathbb{C}^2$.

Theorem 3.1. *([BJ], Theorem 10.2)*
Let f be a regular polynomial endomorphism of $\mathbb{C}^2$. Assume

(1) f_Π *is uniformly expanding on* J_Π*.*
(2) f *is uniformly expanding on* $J(f)$*.*
(3) The non-wandering set of f *in* $\partial K(f)$ *consists of* $J(f)$ *and a hyperbolic set* $\mathcal{S}_1$ *of unstable index* 1*.*
(4) $W^s(\mathcal{S}_1) = \cup_{\hat{x}\in\hat{\mathcal{S}}_1} W^s(\hat{x})$*.*
(5) $W^s(J_\Pi) \cap \mathcal{C}(f) = \emptyset$*.*
Then all external rays land onto $J(f)$ *and landing points vary continuously.*

As a trivial example, we consider the map $F_h(x, y) = (x^2, y^2)$. Then

$$\begin{aligned} \mathcal{C}(F_h) &= \{x = 0\} \cup \{y = 0\}, \quad K(F_h) = \{|x| \le 1, |y| \le 1\}, \\ J(F_h) &= \{|x| = |y| = 1\}, \quad F_{h,\Pi}(\zeta) = \zeta^2, \quad J_\Pi = \{|\zeta| = 1\}, \\ W^s(\zeta) &= \{y = \zeta x,\ |x| > 1\}, \quad W^s(J_\Pi, F_h) = \{|x| = |y| > 1\}. \end{aligned}$$

And all the assumptions of the above theorem are satisfied. Then external rays for F_h are labelled by two angles $(\phi, \theta) \in \mathbb{T}^2$. Here $\zeta = y/x = e^{2\pi i\phi} \in J_\Pi$ and θ is the argument of the ray in the disk $W^s(\zeta)$. Hence the external ray $R_{F_h}(\phi, \theta)$ is just the ray :

$$x = re^{2\pi i\theta}, \quad y = \zeta x = re^{2\pi i(\phi+\theta)}, \quad (r > 1),$$

which lands at $(e^{2\pi i\theta}, e^{2\pi i(\phi+\theta)}) \in J(F_h)$.

Our map F is regular but is not expanding on $J(F)$ since $J(F)$ contains critical points, as we will see later. Next lemma says that it satisfies the last condition (5). Its proof also implies that F is critically finite.

Lemma 3.1. $W^s(J_\Pi, F) \cap \mathcal{C}(F) = \emptyset$.

proof. Note that the critical set $\mathcal{C}(F)$ is parametrized as $x = t, y = 1/t$. We calculate the critical orbits and by induction, we show

$$F^n(t, t^{-1}) = (t^{2^n} + 2t^{-2^{n-1}}, t^{-2^n} + 2t^{2^{n-1}}) \quad (n \ge 2).$$

In fact, it is true for $n = 2$. Suppose it is true for $n = k$. Then the first entry of $F^{k+1}(t, t^{-1})$ is

$$\begin{aligned} (t^{2^k} + 2t^{-2^{k-1}})^2 - 2(t^{-2^k} + 2t^{2^{k-1}}) &= t^{2^{k+1}} + 4t^{2^{k-1}} + 4t^{-2^k} - 2t^{-2^k} - 4t^{2^{k-1}} \\ &= t^{2^{k+1}} + 2t^{-2^k}. \end{aligned}$$

The same holds for the second entry. Hence the case $n = k + 1$ is also true.

Note that the map F has two super-attracting fixed points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ in Π and $W^s(J_\Pi, F)$ is contained in the common boundary of their basins. The above calculation implies that the parts $|t| > 1$ and $|t| < 1$ are contained in the basins of the points $[1:0:0]$ and $[0:1:0]$ respectively and the part $|t| = 1$ is contained in $K(F)$. Thus $\mathcal{C}$ never intersects $W^s(J_\Pi, F)$. This completes the proof. $\square$

Now we consider the external rays for F. Fortunately, we have an explicit expression of an inverse Böttcher coordinate of F and we can define them directly. Put

$$(x, y) = \Psi(u, v) = (u + \frac{1}{v} + \frac{v}{u}, v + \frac{1}{u} + \frac{u}{v}).$$

Then it satisfies the functional equation

$$F \circ \Psi(u, v) = \Psi(u^2, v^2) = \Psi \circ F_h(u, v).$$

The jacobian $Jac(\Psi)$ is written by

$$Jac(\Psi)(u, v) = (1 - \frac{1}{uv})(1 - \frac{u}{v^2})(1 - \frac{v}{u^2}).$$

Hence it is invertible on $W^s(J_\Pi, F_h)$. The inverse $\Phi = \Psi^{-1}$ is a Böttcher coordinate of F.

Then each stable manifold $W^s_F(\zeta)$ of $\zeta \in J_\Pi$ for F is the image of $W^s_{F_h}(\zeta) = \{(t, \zeta t); |t| > 1\} \cong \mathbb{C} - \overline{\mathbb{D}}$ by Ψ. This coordinate gives the Böttcher coordinate of the restriction of F on $W^s_F(\zeta)$. Hence the external ray $R_F(\phi, \theta)$ is the image of $R_{F_h}(\phi, \theta)$ by Ψ.

Theorem 3.2. *The external ray $R_F(\phi, \theta)$ is expressed by*

$$\begin{aligned} x &= re^{2\pi i\theta} + \frac{1}{r}e^{-2\pi i(\phi+\theta)} + e^{2\pi i\phi} \\ y &= re^{2\pi i(\phi+\theta)} + \frac{1}{r}e^{-2\pi i\theta} + e^{-2\pi i\phi} \quad (r > 1). \end{aligned}$$

Its landing point depends continuously on $(\phi, \theta) \in \mathbb{T}^2$:

$$\begin{aligned} x_0 &= e^{2\pi i\theta} + e^{-2\pi i(\phi+\theta)} + e^{2\pi i\phi} \\ y_0 &= e^{2\pi i(\phi+\theta)} + e^{-2\pi i\theta} + e^{-2\pi i\phi} = \overline{x_0}. \end{aligned}$$

Thus $(x_0, y_0) \in H$. Recall that this parametrization of x_0 coincides with that of $K(f)$ described in (2.1) in the previous section.

Lemma 3.2. $K(F) = \{(x, \overline{x}) \in H; x \in K(f)\}$.

proof. Note that the numbers $u, \frac{1}{v}, \frac{v}{u}$ (resp. $v, \frac{1}{u}, \frac{u}{v}$) in the definition of Ψ are the roots of the cubic equation $t^3 - xt^2 + yt - 1 = 0$, (resp. $t^3 - yt^2 + xt - 1 = 0$.) Thus the map $\Psi : (\mathbb{C} - \{0\})^2 \to \mathbb{C}^2$ is surjective. Hence, for any $(x, y) \in \mathbb{C}^2$, there exists a point

$(u, v) \in (\mathbb{C}-\{0\})^2$ such that $(x, y) = \Psi(u, v)$. Then we have $F^n(x, y) = \Psi \circ F_h^n(u, v)$ and it easily follows that $F^n(x, y) \to \infty$ if and only if $F_h^n(u, v) \to \infty$. Since $\Psi(\frac{1}{v}, \frac{1}{u}) = \Psi(u, v)$, it is easy to see that $(x, y) \in K(F)$ if and only if $|u| = |v| = 1$. This completes the proof. □

Lemma 3.3. $J(F) = K(F)$.

proof. Note that $J(F) \subset K(F)$. Since the critical value set of Ψ intersects $K(F)$ only at the boundary of $K(f)$, Ψ is locally invertible in the interior of $K(F)$ in H. Let Φ_j, $0 \le j \le 5$ be the branches of Ψ^{-1} there. Then, we have

$$(dd^cG)^2 = \frac{1}{3}\sum_{j=0}^{5} \Phi_j^*(dd^cG_h)^2.$$

Hence, $J(F) = supp(dd^cG)^2$ contains the image of $J(F_h) = \{|u| = |v| = 1\}$ by Ψ. Thus $K(F) \subset J(F)$. This completes the proof. □

Now the parameters ϕ and θ turn out to be the external angles for F. Note that $\mathcal{C}(F) \cap H = \{(x, \overline{x}); |x| = 1\}$ coincides with $\mathcal{C}(f)$ and is contained in $J(F)$. See Figure 1. Thus F is not expanding on $J(F)$.

Now Lemma 3.1 says $W^s(J_\Pi) \cap \mathcal{C}(F) = \emptyset$. Then it follows from Theorem 7.4 in [BJ] that Ψ extends to a homeomorphism from $W^s(J_\Pi, F_h)$ onto $W^s(J_\Pi, F)$ conjugating F_h to F. In our case, this is trivial and we have a global parametrization of $W^s(J_\Pi, F)$ as the union of the stable manifolds $W_F^s(\zeta)$ with $\zeta = e^{2\pi i\phi}$.

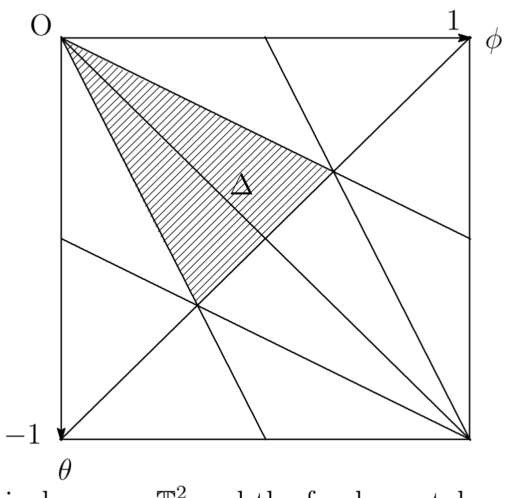

Figure 2: Equivalence on $\mathbb{T}^2$ and the fundamental region Δ

Note that (ϕ, θ) and the parameters

$$\begin{aligned}\rho_1(\phi,\theta) &= (1+\theta, \phi-1)\\ \rho_2(\phi,\theta) &= (\phi, -\phi-\theta)\\ \rho_3(\phi,\theta) &= (-\phi-\theta, \theta)\end{aligned}$$

give a same landing point (x_0, y_0). That is, several rays land at a same point. We will investigate this in details. We remark that ρ_1, ρ_2 and ρ_3 are the reflections with respect to the lines $\phi = \theta + 1$, $\phi = -2\theta$ and $\theta = -2\phi$ respectively. These reflections give an equivalence relation in $\mathbb{T}^2$. The fundamental region is the closed triangular region Δ surrounded by the three lines :

$$\phi = \theta + 1, \quad \phi = -2\theta, \quad \theta = -2\phi.$$

Figure 2 shows the torus $\mathbb{T}^2$, where the dark region indicates the fundamental region Δ. Each triangle is equivalent to one of the two halves of Δ. Now the next lemma is easy to see.

Lemma 3.4. *The equivalence class of a point in the interior of* Δ *consists of* 6 *points, while that of a point on one of the three edges of* $\partial\Delta$ *consists of* 3 *points and that of a vertex of* $\partial J(F)$ *consists of a single point itself.*

Since Δ and $\partial\Delta$ correspond respectively to $J(F)$ and $\partial J(F)$, we have the following.

Theorem 3.3. *Each point* $z = (x, y)$ *in* $J(F)$ *is the landing point of exactly one,* 3 *or* 6 *external rays if* z *is a cusp point on* $\partial J(F)$*,* z *is a non-cusp point on* $\partial J(F)$ *or* $z \in int\, J(F)$*, respectively.*

Finally note that the restriction of Ψ to H is

$$\Psi(t,\bar{t}) = (t + \frac{1}{\bar{t}} + \frac{\bar{t}}{t}, \bar{t} + \frac{1}{t} + \frac{t}{\bar{t}}) = (\psi(t), \overline{\psi(t)}),$$

where ψ is the Böttcher coordinate of f. Thus the external rays for the map f studied in the previous section are just the restriction of the rays for the map F to H.

References

[BJ] E. Bedford and M. Jonsson: Dynamics of regular polynomial endomorphisms of $\mathbb{C}^k$. Amer. J. Math. 122 (2000), pp. 153–212.

[U1] K. Uchimura: The dynamical systems associated with Chebyshev polynomials in two variables. Int. J. Bifurcation and Chaos 6 (1996), pp. 2611–2618.

[U2] K. Uchimura: Dynamics of symmetric polynomial self maps of $\mathbb{C}^2$. Preprint.