On H-T Conjectures for Algebraic Cycles

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In this article we shall investigate the H-T conjectures for algebraic cycles on projective smooth varieties and give an observation, proofs and its application to some conjectures for the motivic theory.

1 Introduction

We shall investigate two H-T conjectures for algebraic cycles explained later on projective smooth varieties ([Hod], [Ta]). A counter example for the H-conjecture is well known to Kähler varieties([Zuk]). First we extend the concepts of the Ishida complex for a toric variety to that for a log smooth variety. By using Hodge theory we apply the Ishida complex to investigate the H conjecture. On the other hand we approach the H-T conjecture with the tools such as Lefschetz pencil and the relative hard Lefschetz theorem. In both case it is inevitably necessary to take into account the action of Galois group.

2 Ishida $\mathcal{O}_X$-complex

In this section we study the generalization of two compex defined by Ishida([Ish] over a toric variety to those over a log smooth variety and their applications to the H conjecture([Hod]). It seems that something resembles the filtration by the type of codimension(p.164, p.170 [Dix])(cf.a Cousin complex ([Har])) . We recall the notation and definitions for later use ([Bour]).

Definition 2.1. *Let A' be a ring and G a group operating on A'. For a prime ideal P' of A' the subgroup of the elements $\sigma \in G$ such that $\sigma P' = P'$ is said to be the decomposition group of P'. One denotes it by $G^Z(P')$. The invariant ring of A' by $G^Z(P')$ is said to be the decomposition ring of P'.*

Proposition 2.1. *Let $P_0 \subset P_1 \subset \cdots \subset P_r$ be a chain of prime ideals of A'. Then one obtains a chain of decomposition groups $G^Z(P_0) \supset G^Z(P_1) \supset \cdots \supset G^Z(P_r)$ of the group G.*

Proposition 2.2 (Prop.6 Ch.5[Bour]). *let A be an integrally closed ring, K its fractional field, K' a quasi-Galois extension (normal extension) of K, A' the integral closure of A in K'. Then*

1. *For each prime ideal P of A the group of K-automorphism of K' acts transitively upon the set of prime ideals of A' over P.*
2. *For each prime ideal P' of A' the fractional field K' of $A/A \cap P'$ and the canonical homomorphism $\sigma \to \overline{\sigma}$ of $G^Z(P')$ into the group Γ of k' gives a bijection of $G^Z(P')/G^T(P')$ onto Γ through a passage of quotient.*

We refer the concepts for toric varieties (resp. toroidal embeddings, resp. log smooth varieties) to [Ish] (resp. [KKMS], resp. [Kat]).

Definition 2.2. *Let $X = T_N emb(\Delta)$.*

$$K^j(X;p) = \begin{cases} 0 & j < 0 \ or \ p < j, \\ \oplus_{\sigma \in \Delta(j)} \Omega^{p-j}_{V(\sigma)} \langle D(\sigma) \rangle & 0 \le j \le p \end{cases} \tag{2.1}$$

$$\Omega^{p-j}_{V(\sigma)} \langle D(\sigma) \rangle = \mathcal{O}_{V(\sigma)} \otimes_{\mathbb{Z}} \wedge^{p-j} (M \cap \tau^{\perp})$$

The coboundary map $\delta : K^j(X;p) \to K^{j+1}(X;p)$ is defined todisplaystyle be $\delta = \oplus R_{\tau,\sigma} : \oplus \mathcal{O}_{V(\sigma)} \otimes_{\mathbb{Z}} \wedge^{p-j}(M \cap \sigma^{\perp}) \to \oplus \mathcal{O}_{V(\tau)} \otimes_{\mathbb{Z}} \wedge^{p-j}(M \cap \tau^{\perp})$ Here M is the group of characters of a torus.

Definition 2.3. 1. (a) *An etale covering of a toric variety is said to be an etale toric variety.*

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2. *(b) A toroidal embedding is said to be a variety which is locally etale toric, in other word, a log smooth variety.*

Lemma 2.1. *Let $j : U \subset X$ be a toroidal embedding with a polyhedral complex Δ. One can extend the local Ishida complex to the global Ishida $\mathcal{O}_X$-complex for $j_*\Omega^p_X|_U\ K^*(X;p)$ over X.*

Proof. One can patch local Ishida $\mathcal{O}_{X_i}$-complexes for $j_*\Omega^p_X|_U\ K^*(X_i;p)$ for locally etale toric open neighborhoods X_i with $\cup X_i = X$. □

Theorem 2.1. *Let X be a projective smooth variety over the complex number field. The canonical homomorphism $CH^p(X) \otimes \mathbb{C} \to H^p(X, \Omega^p_X)$ is surjective.*

We shall give the sketch of the proof by dividing several steps.

Lemma 2.2. *Let $\phi : X \to P$ be a projective morphism between projective smooth varieties. Assume*

P is a toric variety.

(a) D is a normal crossing divisor on P such that the restriction of the morphism ϕ to the inverse image outside D is etale.

(b) $\mu : X \to X', \pi : X' \to P$ are Stein factorization of ϕ. π is finite and $\pi|_{P-D}$ is etale. μ is birational. X and X' are toroidal embeddings without self-intersection with respect to the inverse image of D.

(c) $\phi^{-1}(D)$ is a divisor on X with the support in a normal crossing divisor.

(c) $R(X)$ is a Galois extension of $R(P)$.

Then $CH^p(X) \otimes \mathbb{C} \to H^p(X, \Omega^p_X)$ is surjective.

Proof. By Quillen's theorem there exist the canonical homomorphisms $CH^p(X) = \mathrm{coker}(\coprod_{x\in X^{p-1}} K_1(k(x)) \to \coprod_{x\in X^p} K_0(k(x))) \otimes \mathbb{C} \to H^p(X, \Omega^p_X)$ and $CH^p(P) = \mathrm{coker}(\coprod_{x\in P^{p-1}} K_1(k(x)) \to \coprod_{x\in P^p} K_0(k(x))) \otimes \mathbb{C} \to H^p(P, \Omega^p_P)$. On has Ishida resolution sheaves $\Omega_P \to K^*_P(resp. \Omega_X \to K^*_X, resp. \Omega_{X'} \to K^*_{X'})$. Hence one obtains the spectral sequence $'E^{ab}_2 = H^a(H^b(P, K^*_P))$ (resp. $'E^{ab}_2 = H^a(H^b(X, K^*_X))$, resp. $'E^{ab}_2 = H^a(H^b(X^*, K^*_{X'})))$. Note that the supports of resolution sheaves of $'E^{ab}_2$ are greater than of codimension 1 if $a \neq 0$. One denotes by F the filtration of the spectral sequence above. Since $\oplus Gr_F H^p(P, \Omega^p_P) = \oplus_{a+b=p} E^{ab}_\infty$ (resp. $\oplus Gr_F H^p(X, \Omega^p_X) = \oplus_{a+b=p} E^{ab}_\infty$, resp. $\oplus Gr_F H^p(X', \Omega^p_{X'}) = \oplus_{a+b=p} E^{ab}_\infty$), by induction argument it suffices to prove the surjectivity of $\mathrm{coker}(\coprod_{x\in(X\setminus\phi^{-1}D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(X\setminus\phi^{-1}D)^p} K_0(k(x))) \otimes \mathbb{C} \to E^{0p}_\infty$. One knows the map $\mathrm{coker}(\coprod_{x\in(P\setminus\phi^{-1}D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(P\setminus\phi^{-1}D)^p} K_0(k(x))) \to E^{0p}_\infty(P)$ is surjective and both terms are zero if $p > 0$. One denotes by tr the trace map induced by Galois action $G = Gal(R(X)/R(P))$; $trx = \frac{1}{|G|}\sum_{\sigma\in G} x^\sigma$. Note that $\Omega^a_X\langle\phi^{-1}D\rangle = \phi^*\Omega^a_P\langle D\rangle$. Hence there exists a trace map $E^{ab}_\infty(X) \to E^{ab}_\infty(P)$. The latter one is a direct summand of the former one. It happens the same thing the canonical map $\mathrm{coker}(\coprod_{x\in(X\setminus\phi^{-1}D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(X\setminus\phi^{-1}D)^p} K_0(k(x))) \otimes \mathbb{C} \to \mathrm{coker}(\coprod_{x\in(P\setminus D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(P\setminus D)^p} K_0(k(x)))$. These actions are equivariant. Therefore the desired map is surjective. This completes the proof. □

Under the assumption of the lemma above one proceeds to prove taking the following lemma in mind.

Lemma 2.3. *One has the following commutative squares;*

$$\begin{array}{ccc} CH^p(X) & \longrightarrow & H^p(X, \Omega^p) \\ \downarrow & & \downarrow \\ CH^p(P) & \longrightarrow & H^p(P, \Omega^p), \end{array} \tag{2.2}$$

$$\begin{array}{ccc} Gr^*CH^p(X) & \longrightarrow & Gr^*H^p(X, \Omega^p_X) \\ \downarrow & & \downarrow \\ Gr^*CH^p(P) & \longrightarrow & Gr^*H^p(P, \Omega^p_P) \end{array} \tag{2.3}$$

Definition 2.4. *One can define the filtration associated to the polyhedral complex Δ.*

$$Gr^0CH^p(P) =$$

$$\mathrm{coker}(\coprod_{x\in(P\setminus D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(P\setminus D)^p} K_0(k(x)))$$

$$\text{modulo } \mathrm{coker}(\coprod_{x\in D^{p-1}} K_1(k(x)) \to \coprod_{x\in D^p} K_0(k(x))) = 0$$

,

$$Gr^0CH^p(X) =$$

$$\mathrm{coker}(\coprod_{x\in(X\setminus\phi^{-1}D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(X\setminus\phi^{-1}D)^p} K_0(k(x)))$$

$$\text{modulo } \mathrm{coker}(\coprod_{x\in(\phi^{-1}D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(\phi^{-1}D)^p} K_0(k(x))) =$$

$$\mathrm{coker}(\coprod_{x\in(X\setminus\phi^{-1}D)^{p-1}} K_1(k(x)) \to \coprod_{x\in(X\setminus\phi^{-1}D)^p} K_0(k(x))) \cap \ker(\mathrm{Gr}^0\mathrm{CH}^p(X) \to \mathrm{Gr}^0\mathrm{CH}^p(P))$$

One can see the following proposition easily.

Proposition 2.3. *Let H be a hyperplane of X which associates to a ray of a polyhedral complex Δ.*

1. *The canonical map $Gr^0CH^{p-1}(H) \to Gr^1CH^p(X)$ is a surjection.*
2. *The canonical map $Gr^iCH^{p-1}(H) \cong Gr^{i+1}CH^p(X)$ for $i > 1$ is an isomorphism.*

Lemma 2.4. *Let H be a hyperplane of X which associates to a ray of a polyhedral complex Δ.*

$$\oplus_{a+b=p} Gr^a H^p(X, \Omega_X^p) = \oplus_{a+b=p} E^{ab}(X)$$

$$\oplus_{a+b=p-1} Gr^a H^{p-1}(H, \Omega_X^{p-1}) = \oplus_{a+b=p-1} E^{ab}(H)$$

1. *$E^{0,p-1}(H) \to E^{1,p-1}(X)$ is a surjection.*
2. *$E^{a,b}(H) \to E^{a+1,b}(X)$ for $a + b = p - 1$ and $a \geq 1$ is an isomorphism.*

Proof. Hodge decomposition implies the lemma. □

Note that the inverse cycles of arbitrary two rationally equivalent cycles on P by $\pi : X \to P$ are Drinfeld equivalent, hence homologically equivalent and that any cycle of codimension p is a multiple of one fixed cycle of codimension p.

Hence the decomposition group of a non exceptional cycle of codimension j is isomorphic. One denotes by $G(j)$ the isomorphism class of the decomposition group.

On the other hand one has

Remark 2.1.

$$tr_{G(0)}^{G(p)} = tr_{G(0)}^{G(1)} \circ tr_{G(0)}^{G(p)}$$

Lemma 2.5.

$$\begin{array}{ccc} Gr^0CH^p(X)\otimes\mathbb{C} & \xrightarrow{tr_{G(1)}^{G(p)}} & E^{0p}(X) \\ \Big\downarrow tr_{G(0)}^{G(p)} & & \Big\downarrow tr_{G(0)}^{G(1)} \\ Gr^0CH^p(P)\otimes\mathbb{C} & \xrightarrow[id]{} & E_\infty^{0p}(P) \end{array} \tag{2.4}$$

Proof. The commutativity is obtained by the remark above. □

Hence one has

Lemma 2.6. *There exists a cycle of $Gr^0CH^p(X)$ the canonical image of which is not zero in $Gr^0H^p(X, \Omega_X^p) = E^{0p}(X)$.*

Proof. There exists an cycle $cyc(x)$ such that $x \in X^p$ and $G^Z(x) \in G(p)$. Let $\xi = cyc(x) - cyc(x^\sigma)$ for $1 \neq \sigma \in G(0)/G(1)$.

1. $tr_{G(1)}^{G(p)}(\xi) \neq 0$
2. $tr_{G(0)}^{G(p)}(\xi) = 0$

□

We recall the following definition. It is the canonical homomorphism $G^Z(P') \to \mathrm{Aut}(A'/P')$, whose image is denoted by Γ_0. For $\sigma \in G^Z(P')$ the endomorphism $x \to \sigma x$ of A' induces $z \to \sigma z$ of A'/P'.

Definition 2.5. *The subgroup of $G^Z(P')$ which is the kernel of the canonical homomorphism is said to be the inertia group of P' and one denotes it by $G^T(P')$. The invariant ring of A' by $G^T(P')$ is said to be the inertia group of P'.*

Note that $(A'/P')^{\Gamma_0} = A^Z/(P' \cap A^Z)$.
We remind ourselves the following proposition and theorem.

Proposition 2.4. *Let k be a field , $S = Speck$ and Ω an algebraically closed extension of k. Let $a \in S$ be a geometric point $Spec\ \Omega \to S$. Let $\bar{k}$ be the algebraic closure of k in Ω. Then there exists the canonical isomorphism $\pi_1(S,a) \approxeq Gal(\bar{k}/k)$ as topological groups.*

Theorem 2.2. *Let X be a smooth variety over a field of characteristic 0 and C a non singular irreducible hyperplane of an ample divisor.*

1.

$$H^b(X, \Omega_X^a\langle C\rangle) = 0 \quad \textit{for } a+b > \dim X$$

2.

$$H^b(X, \Omega_X^a\langle C\rangle(-C)) = 0 \quad \textit{for } a+b < \dim X$$

Note that

Remark 2.2. $2p > \dim X$ *There exists no primitive element in $H^p(X, \Omega_X^p)$. Hence $E_2^{0p}(X) = 0$.*

One has

Theorem 2.3. *(a)* $2p > \dim X$

$$E_\infty^{0p}(X) = 0$$

(b) $2p < \dim X$

$$E_\infty^{0p}(X) = 0$$

Proof. 1. $2p > \dim X$ Since there exists no primitive element, $E_\infty^{0p}(X) = 0$.

$$H^p(C, \Omega_C^{p-1}) \to H^p(X, \Omega_X^p) \to H^p(X, \Omega X^p\langle C\rangle)$$

The former map is defined by Lefschetz map L.

2. $2p < \dim X$ See the canonical exact sequence

$$H^p(X, \Omega_X\langle C\rangle(-C)) \to H^p(X, \Omega_X^p) \to H^p(C, \Omega_C^p)$$

Hence the latter map is injective, which is Lefschetz theorem. By inductive argument, the following canonical map is surjective;

$$CH^p(C) \otimes \mathbb{C} \to H^p(C, \Omega_C^p)$$

Therefore $E_\infty^{0p}(X) = Gr_F^0 H^p(X, \Omega_X^p) = 0$.

□

It remains to be proved when $2p = \dim X$.

Lemma 2.7. $2p = \dim X$ *The following canonical map is surjective;*

$$Gr^0 CH^p(X) \to E_\infty^{0p}(X)$$

Proof. Given a form $\omega \in E_\infty^{0p}(X)$, there exists a representative $\omega \in H^p(X, \Omega_X^p)$. By the long exact sequence

$$H^p(X, \Omega\langle C\rangle(-C)) \to H^p(X, \Omega_X^p) \to H^p(C, \Omega_C^p)$$

one has a representative $\omega \in H^p(X, \Omega\langle C\rangle(-C))$. Let $G^Z(\omega) = \{\sigma G | \sigma^* \omega = \omega\}$. Take a trace $tr\omega = \frac{\sum_\sigma \sigma^* \omega}{|G|}$. There exists a cycle corresponding to $tr\omega$ in P. Take a smooth hyperplane B which contains this cycle and an irreducible component A of the reciprocal image of B. By the canonical map $H^p(X, \Omega^p) \to H^p(A, \omega_A^p)$, the ω does not vanish. By inductive argument, the canonical map

$$CH^p(A) \otimes \mathbb{C} \to H^p(A, \Omega_A^p)$$

is surjective. The inclusion $A \subset X$ induces the toroidal embedding structure on A. Thus one can assume the following map is surjective

$$Gr^0 CH^p(A) \to E_\infty^{0p}(A)$$

The latter cohomology group has a non zero element ω. Hence one has a cycle x in $CH^p(X)$ such that $C_p(x) \in E_\infty^{0p}(X)$ maps to $C_p(x) = \omega \in E_\infty^{0p}(A)$. Hence $C_p(x) \in E_\infty^{0p}(X)$ is not zero. Therefore there exists $\sigma \in G$ such that

$$C_p(x)^\sigma = \omega \in E_\infty^{0p}(X)$$

□

3 H-T Conjectures

Let k be a field, $\bar{k}$ its algebraic closure, $G_k = Gal(\bar{k}/k)$, X a smooth projective variety, $\overline{X} = X \times_k \bar{k}$ and $CH^r(X)$ the Chow groups of algebraic cycles of codimension r on X modulo linear equivalence.

There exists the natural cycle map for $\ell \neq chark$

$$cl_\ell^r : CH^r(X) \to H_{et}^{2r}(\overline{X}, \mathbb{Q}_\ell(r)) = H_\ell^{2r}(X)(r)$$

This image lies in the fixed part

$$\Gamma_\ell(H_\ell^{2r}(X)(r)) := H_\ell^{2r}(\overline{X}, \mathbb{Q}(r))^{G_k}$$

under G_k. The T conjecture is the following statement([Ta], [Mot], [Jan]).

Conjecture 3.1. *The image of cl_ℓ^r generates $\Gamma_\ell(H_\ell^{2r}(X)(r))$, if k is finitely generated as a field.*

Let k be the field of the complex numbers. Let X be a smooth projective variety.@Then one obtains a cycle map

$$cl^r : CH^r(X) \to H^{2r}(X, \mathbb{Q}),$$

whose image consists of (r,r)-classes, or in the explicit form

$$H^{2r}(X, \mathbb{Q}) \bigcap H^{(r,r)}(X, \mathbb{Q}).$$

The H conjecture is the following statement([Hod], [Jan], [Mot], [Sh]).

Conjecture 3.2. *The image of $cl^r \otimes \mathbb{Q}$ is the whole of $H^{2r}(X, \mathbb{Q}) \bigcap H^{(r,r)}(X, \mathbb{Q})$.*

4 Local Lefschetz teory

In the following sections we recall the Lefschetz theory investigated by Grothendieck, Katz and Deligne ([Katz], [SGA], [Dix]). Let S be the spectre of a henselian discrete valuation A with an algebraically closed residue field, η its generic point and s its closed point. Let $f : X \to S$ be a proper morphism from a smooth variety of dimension n. Suppose that f is smooth except for a ordinary quadratic singular point x in the special fibre X_s. One has a specialization morphism

$$sp : H^i(X_s, \mathbb{Q}_\ell) \cong H^i(X, \mathbb{Q}_\ell) \to H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell).$$

The Galois group $\mathrm{Gal(k(\overline{k})/k(\eta)) = I}$ acts on $H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ by structure transportation:

$$\mathrm{Gal(k(\overline{k})/k(\eta)) = I \to GL(H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell))}.$$

The sheaf $R^i f_* \mathbb{Q}_\ell$ over S is completely determined by the two conditions above. One can explain them by a vanishing cycle $\delta \in H^{n-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)(m)$, which is well defined up to sign. Here $n-1 = 2m, n-1 = 2m+1$. One has

$$H^i(X_s, \mathbb{Q}_\ell) \cong H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$$

for $i \neq n-1, n$. For $i = n-1, n$, one obtains an exact sequence

$$0 \to H^i(X_s, \mathbb{Q}_\ell) \to H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell) \stackrel{x \mapsto Tr(x \cup \delta)}{\longrightarrow} \mathbb{Q}_\ell(m-n+1) \to H^i(X_s, \mathbb{Q}_\ell) \to H^i(X_{\overline{\eta}}, \mathbb{Q}_\ell) \to 0$$

The action of the local monodromy I is trivial if $i \neq n-1$. For $i = n-1$ it is described in the following

1. $n-1$ *odd* The action of $\sigma \in I$ is $x \mapsto x \pm t_\ell(\sigma)(x, \delta)\delta$, where $t_\ell : I \to \mathbb{Z}_\ell(1)$ is a canonical homomorphism.
2. $n-1$ *even* Excluding $p \neq 2$, there exists a unique character of order 2 $\epsilon : I \to \{\pm\}$. Then one has
   $\sigma x = x$ if $\epsilon(\sigma) = 1$
   $\sigma x = x \pm (x, \delta)\delta$ if $\epsilon(\sigma) = -1$.

Here the sign $\pm$ is defined to be $-(-1)^{\frac{(n-1)(n-2)}{2}} = -(-1)^m$. The (δ, δ) is 2 if $n-1$ mod $4 = 0$ (resp. 0 if $n-1$ mod $4 = 1$, resp. -2 if $n-1$ mod $4 = 2$, resp. 0 if $n-1$ mod $4 = 3$.) Hence one obtains the property for $R^i f_* \mathbb{Q}_\ell$.

(a) If $\delta \neq 0$

1. For $i \neq n-1$, the sheaf $R^i f_* \mathbb{Q}_\ell$ is constant.
2. Let $j : \eta \hookrightarrow S$. One has $R^{n-1} f_* \mathbb{Q}_\ell = j_* j^* \mathbb{Q}_\ell$.

(b)

1. For $i \neq n$, the sheaf $R^i f_* \mathbb{Q}_\ell$ is constant.
2. One has an exact sequence

$$0 \to \mathbb{Q}_{\ell s} \to R^n f_* \mathbb{Q}_\ell \to j_* j^* R^n f_* \mathbb{Q}_\ell \to 0$$

, where $j_* j^* R^n f_* \mathbb{Q}_\ell$.

5 Global Lefschetz theory

Let $\mathbb{P}$ be a projective space of dimension more than 1 over an algebraically closed field k of characteristic p and X a projective smooth subvariety of $\mathbb{P}$ of dimension n. For a linear subspace A of $\mathbb{P}$ of codimension 2, one can define a pencil $(H_t)_{t \in D}$ and $\widetilde{X}$ by blow-up with center $A \cap X$, which one denotes by $\rho : \widetilde{X} \to D$. Here D is a line.

Definition 5.1. *A pencil $(H_t)_{t \in D}$ is said to be a Lefschetz pencil of hyperplane sections if the following conditions are satisfied*

A) The ax A intersects transversally with X. The $\widetilde{X}$ is smooth.

B) There exists a finite set D of such points of D that for every $s \in S$ there is a point $x_s \in X_s$ such that $\rho | X_s$ is smooth outside x_s.

C) x_s is a quadratic singular point of X_s.

Let r be an integer ≥ 1, N the dimension of $\mathbb{P}$ and ι_r the embedding of $\mathbb{P}$ into the projective space of dimension $\binom{N+r}{r} - 1$

If $p > 0$ it happens that no pencil of hyperplane sections of X is Lefschetz pencil. A very general pencil however becomes Lefschetz pencil if one replaces the embedding $X \hookrightarrow \mathbb{P}$ by the composition of the embedding ι_r above for $r \geq 2$,i.e., a very general pencil of hypersurface sections of degree ≥ 2 is always Lefschetz pencil.

6 Lefschetz pencil-1

We consider a Lefschetz pencil except $p = 2, n-1$ even. Put $U = \mathbb{D}\backslash S$. Let $u \in U$ and $\ell \neq p$. By local Lefschetz theory, $R^{n-1}\rho_*\mathbb{Q}_\ell$ is tamely ramified at every $s \in S$. The tame fundamental group of U is the quotient of the pro-finite completion of the fundamental group as a transcendental analogue. The transcendental situation can translate in the algebraic situation.

(a) If there exists no vanishing cycle, one has $R^i\rho_*\mathbb{Q}_\ell$ is constant.

1. For $i \neq n$, the sheaf $R^i\rho_*\mathbb{Q}_\ell$ is constant.
2. One has an exact sequence

$$0 \to \oplus_{s\in S}\mathbb{Q}_\ell(m-n)_s \to R^n\rho_*\mathbb{Q}_\ell \to F \to 0,$$

where F is constant.

3. $E = 0$.

Note that this case is exceptional and that $n-1$ is odd.

If the vanishing cycles are all non zero,

1. For $i \neq n-1$ the sheaf $R^i\rho_*\mathbb{Q}_\ell$ is constant.
2. Let $j : U \hookrightarrow D$. One has

$$R^{n-1}\rho_*\mathbb{Q}_\ell = j_*j^*R^{n-1}\rho_*\mathbb{Q}_\ell$$

3. Let $E \subset H^{n-1}(X_u, \mathbb{Q}_\ell)$ denote the vector subspace generated by the vanishing cycles.

7 Lefschetz pencil-2

We work over an algebraically closed field k of characteristic p. Let ℓ a prime number different from p. Deligne proved the hard Lefschetz theorem([Del]).

Theorem 7.1. *Let X be a smooth projective variety of dimension n over k, L an ample invertible sheaf over X and $\eta = c_1(L) \in H^2(X, \mathbb{Q}_\ell)$. Then for every integer j*

$$\eta^i : H^{n-i}(X, \mathbb{Q}_\ell(j)) \to H^{n+i}(X, \mathbb{Q}_\ell(i+j))$$

is an isomorphism for any $i \geq 0$.

One has its relative version ([BBD]).

Theorem 7.2. *(relative hard Lefschetz Theorem) If F_0 is a pure perverse sheaf over X_0, the homomorphism*

$$\ell^i : \ {}^pH^{-i}f_*F_0 \to \ {}^pH^if_*F_0(i)$$

is an isomorphism for any $i \geq 0$.

Let $D = \{H_{\tau\in\mathbb{P}^1}\}$ a pencil of hyperplanes whose ax Δ cuts X transversally in Δ. We describe by (λ, μ) the homogeneous coordinates of $\mathbb{P}^1$. The ax of the pencil is defined by $F = G = 0$ where F, G are two linear forms. The pencil is determined by $\lambda F = \mu G$. Let $\widetilde{X} = \{(x, (\lambda, \mu)) \in X \times \mathbb{P}^1 | \lambda F - \mu G = 0\}$. This is the closure of the graph of the map

$$X \setminus \Delta \to \mathbb{P}^1$$

where $x \mapsto (G(x), F(x))$. One denotes by $X_t = \rho^{-1} = X \cdot H_t$. Note that $\widetilde{X}$ is a smooth projective over k since Δ is so.

Definition 7.1. *The Lefschetz pencil D satisfies the condition (A) if the group of inertia on any point of $D \bigcap \check{X}$ acts non trivially on $H^{n-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$.*

Note that If $n-1$ is even, the condition (A) holds.

Lemma 7.1. *([Katz]) When $n-1$ is odd ($p \neq 2$), one can find an integer d_0 such that for $d \geq d_0$ every Lefschetz pencil of hypersurfaces of degree d satisfies the condition (A).*

Proof. If the condition (A) is not satisfied, $R^{n-1}\rho_*\mathbb{Q}_\ell$ is constant and of rank $\dim H^{n-1}(X, \mathbb{Q}_\ell)$. On the other hand, one has $\dim H^{n-1}(H(d), \mathbb{Q}_\ell)$ tends to the infinity as the degree d of a smooth hypersurface section $H(d)$ of X grows larger. □

Deligne has proved the following statement in ([Del]).

Lemma 7.2. *For $p = 2$, $n-1$ even, suppose the Lefschetz pencil $(X_{t \in D})$ is very general.*

(a) $n-1$ even The reflections $x \mapsto x - (-1)^{\frac{n-1}{2}}(x\delta)\delta$ are conjugates among them.

(b) $n-1$ odd The homomorphisms of $\mathbb{Z}(1)$ into the monodromy group given by the Picard-Lefschetz formula $x \mapsto x + \lambda(x, \delta)\delta$, for a vanishing cycle δ are conjugates among them.

Hence one obtains the following lemma.

Lemma 7.3. *The vanishing cycles modulo sign are conjugates one another in $H^{n-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$.*

Note that if one neglects the torsion, the vanishing cycle $\pm\delta$ is determined up to sign by the corresponding Picard-Lefschetz transformation.

Applying the following theorem to the very general Lefschetz pencil, one has the degeneration of Leray spectral sequence which proved Deligne([Katz],[Del]).

One denotes by π the Galois group of $k(\overline{\eta})/k(\eta)$.

Lemma 7.4. *For $q \neq n-1$, one obtains the following canonical isomorphisms.*

1. $E^{0,q} = H^0(\mathbb{P}^1, R^q\rho_*\mathbb{Q}_\ell) \cong H^q(X_{\overline{\eta}}, \mathbb{Q}_\ell)$
2. $E_2^{2,q} = H^2(\mathbb{P}^1, R^q\rho_*\mathbb{Q}_\ell) \cong H^q(X_{\overline{\eta}}, \mathbb{Q}_\ell(-1))$
3. $E_2^{p,q} = H^p(\mathbb{P}^1, R^q\rho_*\mathbb{Q}_\ell) = 0$ *for* $p \neq 0, 2$.

For $q = n-1$, one has

1. $E_2^{0,n-1} = H^0(\mathbb{P}^1, R^{n-1}\rho_*\mathbb{Q}_\ell) \cong H^{n-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)^\pi$
2. $E_2^{2,n-1} = H^2(\mathbb{P}^1, R^{n-1}\rho_*\mathbb{Q}_\ell) \cong H^{n-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)^\pi \cong H^{n-1}(X, \mathbb{Q}_\ell(-1))$

Proof. One refers to [Katz]. □

There remains only the following part. One denotes by $E^{n-2}(\Delta, \mathbb{Q}_\ell(j))$ the orthogonal part of the image of $H^{n-2}(X, \mathbb{Q}_\ell(j))$ in $H^{n-2}(\Delta, \mathbb{Q}_\ell(j))$, which is said to be the vanishing part of the cohomology of Δ.

Lemma 7.5. *([Katz]) One obtains a direct sum decomposition*

$$E_2^{1,n-1} = H^1(\mathbb{P}^1, R^{n-1}\rho_*\mathbb{Q}_\ell) \cong \mathrm{Prim^n(X, \mathbb{Q}_\ell) \oplus E^{n-2}(\Delta, \mathbb{Q}_\ell(-1))}.$$

There exists an isomorphism (p.261[Katz])

$$H^q(\tilde{X}, \mathbb{Q}_\ell) \cong H^q(X, \mathbb{Q}_\ell) \oplus H^{q-2}(\Delta, \mathbb{Q}_\ell(-1)).$$

It suffices to obtain the proofs of H-T conjectures that one considers the primitive part of cohomologies by taking Lefschetz decomposition.

8 Griffiths map

Let $f : T \to S$ be a morphism. The Leray spectral sequence $E^{p,q} = H^p(S, R^q f_* \mathbb{Q}_\ell(j)) \Longrightarrow H^{p+q}(T, \mathbb{Q}_\ell(j))$ defines the edge maps $E_\infty^{0,q} \hookrightarrow E_2^{0,q}$. One defines by $\mathrm{Prim^n(T/S, \mathbb{Q}_\ell(j))}$ $F^1 H^n(T, \mathbb{Q}_\ell(j))$. Then $\mathrm{Prim^n(T/S} < \mathbb{Q}_\ell) = \mathrm{Ker(H^n(T, \mathbb{Q}_\ell(j)) \to H^0(S, R^n f_* \mathbb{Q}_\ell(j))}$. One also has $E_\infty^{1,n} \hookrightarrow E_2^{1,n}$. Hence

$$\mathrm{Prim^n(T/S, \mathbb{Q}_\ell(j)) \to E_\infty^{1,n-1} \to E_2^{1,n-1}},$$

i.e., $\mathrm{Prim^n(T/S, \mathbb{Q}_\ell(j)) \to H^1(S, R^{n-1} f_* \mathbb{Q}_\ell(j))}$. Apply this to a Lefschetz pencil. Let $\rho : \widetilde{X} \to \mathbb{P}^1$ be a projection of Lefschetz pencil. Choose a non void open set $\nu : U \hookrightarrow \mathbb{P}^1$ such that the restriction $\rho|U$ is projective and smooth. By the proper base change theorem, one has $\mathrm{Prim^n(\widetilde{X}|U/U, \mathbb{Q}_\ell(j)) \to Ker(H^n(\widetilde{X}|U, \mathbb{Q}_\ell(j)) \to H^n(X_{\overline{\eta}}, \mathbb{Q}_\ell(j))}$. Note that if $\dim X = n$,

$$x \in \mathrm{Prim^n(X, \mathbb{Q}_\ell(j))} \Longleftrightarrow \mathrm{x} \in \mathrm{Prim^n(\widetilde{X}|U/U, \mathbb{Q}_\ell(j))}$$

and that when the condition (A) is valid, it is equivalent to $x \in \mathrm{Prim^n(\widetilde{X}/\mathbb{P}^1, \mathbb{Q}_\ell(j))}$. One has the following definitions ([Grif], [Katz]).

Definition 8.1. *The composite map*

$$\mathrm{Prim^n(X, \mathbb{Q}_\ell(j)) \to Prim^n(\widetilde{X}|U/U, \mathbb{Q}_\ell(j)) \to H^1(U, \nu^* R^{n-1} \rho_* \mathbb{Q}_\ell(j))}$$

is said to be Griffiths map.

Definition 8.2. *One denotes by $E^{n-1}\rho_*\mathbb{Q}_\ell(j) = \nu_*$(the orthogonal of the constant subbundle $H^{n-1}(X, \mathbb{Q}_\ell(j))_U$ in $\nu^* R^{n-1}\rho_* \mathbb{Q}_\ell(j)$),i.e., $\nu^* R^{n-1}\rho_*\mathbb{Q}_\ell(j) = E^{n-1}\rho_*\mathbb{Q}_\ell(j) \oplus H^{n-1}(X, \mathbb{Q}_\ell(j))_U$ This is called the vanishing cohomology sheaf over $\mathbb{P}^1$.*

Lemma 8.1. *The canonical map*

$$\mathrm{Prim^n(X, \mathbb{Q}_\ell(j)) \to H^1(U, \nu^* E^{n-1} \rho_* \mathbb{Q}_\ell(j))}$$

is an injection.

Proof. Since

$$E_2^{1,n-1} = H^1(\mathbb{P}^1, R^{n-1}\rho_*\mathbb{Q}_\ell) \cong \mathrm{Prim^n(X, \mathbb{Q}_\ell) \oplus E^{n-2}(\Delta, \mathbb{Q}_\ell(-1))},$$

one has $\mathrm{Prim^n(X, \mathbb{Q}_\ell(j)) \hookrightarrow H^1(\mathbb{P}^1, R^{n-1}\rho_*\mathbb{Q}_\ell(j))}$. Note that $E^{n-1}\rho_*\mathbb{Q}_\ell(j)) \cong \nu_*\nu^* E^{n-1}\rho_{ast}\mathbb{Q}_\ell(j))$. Hence one completes the proof.

□

9 Observation

In this section we give an observation.

Deligne generalizes Lefschetz theory described above to the case of any base field for $\mathbb{Q}_\ell$-cohomology((4.3)II [Del]). In local Lefschetz theory one has an epimorphism $\mathrm{Gal}(\overline{\eta}/\eta) \twoheadrightarrow \mathrm{Gal}(\overline{s}/s)$.

We recall the T conjecture. The image of the natural cycle map for $\ell \neq$ chark

$$cl_\ell^r : CH^r(X) \to H_{et}^{2r}(\overline{X}, \mathbb{Q}_\ell(r)) = H_\ell^{2r}(X)(r)$$

generates

$$\Gamma_\ell(H_\ell^{2r}(X)(r)) := H_\ell^{2r}(\overline{X}, \mathbb{Q}(r))^{G_k}$$

if k is finitely generated as a field.

Here $G_k = \mathrm{Gal(\overline{k}/k) = Gal(\overline{s}/s)}$.

It suffices to prove it in the case of $2r = n = \dim X$. We shall investigate this case elsewere. One refers to the following lemma.

Lemma 9.1. *The canonical map*

$$\mathrm{grif : Prim^n(X, \mathbb{Q}_\ell) \to H^1(U, \nu^* E^{n-1} \rho_* \mathbb{Q}_\ell)}$$

is an injection.

Secondly, we recall the H conjecture. Let k be the field of the complex numbers. Let X be a smooth projective variety. The image of the canonical cycle map

$$cl^r : CH^r(X) \to H^{2r}(X, \mathbb{Q})$$

generates $H^{2r}(X, \mathbb{Q}) \bigcap H^{(r,r)}(X, \mathbb{Q})$.

One fixes an isomorphism $\iota : \mathbb{Q}_\ell \cong \mathbb{C}$. By Lefschetz principle one can translate the results into the analytic case each other.

Lemma 9.2. $\mathrm{Prim}^n(X, \overline{\mathbb{Q}_\ell}) \bigcap H^n(X, \mathbb{Q}) \bigcap H^{r,r}(X)$ *is invariant under* $\mathrm{Gal}(\overline{\eta}/\eta)$.

Proof. $\mathrm{Gal}(\overline{\eta}/\eta)$ acts trivially on the image of the intersection with $H^n(X, \mathbb{Q})$ by $grif$. □

One therefore obtains the following observation.

10 Application

Assuming the H conjecture, one has the canonical cycle map

$$\gamma_{X \times X} : CH^{n-1}(X \times X) \to H^{2n-2}(X \times X, \mathbb{Q}_\ell(n-1)).$$

The homogeneous linear map of degree -2 Λ is an element of $H^{2n-2}(X \times X, \mathbb{Q}_\ell(n-1))$, which is invariant under the action of $\mathrm{Gal}(\overline{\eta}/\eta)$. Hence it is algebraic.

For $i \le \frac{n}{2}$, the $\mathbb{Q}$-valued pairing on $A^i(X) \bigcap \mathrm{Prim}^{2i}(X, \overline{\mathbb{Q}_\ell})$ one has

$$(x, y) \mapsto (-1)^i Tr(\ell^{n-2i} xy)$$

is positive definite. By induction if n is odd, it is by hypothesis. If n is even, one cannot find no primitive algebraic element corresponding for $i = \frac{n}{2}$.

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