Numerical Experiments on the Tricorn

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Dedicated to Professor H. Komatsu on his sixtieth birthday

Abstract

This note gives numerical experiments and an intuitive explanation of the non-pathwise connectivity of the tricorn, the connectedness locus of the family of antiquadratic polynomials.

1. Introduction

In this note, we will give a numerical explanation of the non-pathwise connectivity of the tricorn, the connectedness locus of the family of antiholomorphic polynomials of degree two of the form: $f_c(z)=\bar{z}^2+c, c\in\mathbf{C}$.

Though f_c itself is not holomorphic, its second iterate $f_c^{\circ 2}(z)=(z^2+\bar{c})^2+c$ is holomorphic. So, we can define its filled Julia set $K_c=K(f_c)$ and Julia set $J_c=J(f_c)$ analogously as in the polynomial case:

$$K_c=\{z\in\mathbf{C}\,;\text{ its forward orbit }\{f_c^{\circ n}(z)\}_{n=0}^{\infty}\text{ is bounded}\},$$
$$J_c=\partial K_c.$$

The connectedness locus of this family:

$$M_2^*=\{c\in\mathbf{C}\,;\,J_c\text{ is connected}\}.$$

is called in Milnor [Mil] the *tricorn* and in Rippon et. al. [Rip] the *Mandelbar set.* It is characterized also by

$$M_2^*=\{c\in\mathbf{C}\,;\,0\in K_c\}.$$

Note the 0 is the unique critical point of this family.

It can be regarded as an analogy with the *Mandelbrot set* M_2 for the polynomial family: $P_c(z)=z^2+c,\ c\in\mathbf{C}$. In fact, they share same properties to a certain extent. For example, Nakane [Nak1] showed that the tricorn M_2^* is connected.

The purpose of this note is not to give the proof of non-pathwise connectivity but rather to give numerical experiments suggesting it. Its proof will appear in Hubbard, Nakane and Schleicher [HNS]. See also Nakane and Schleicher[NS]. Note that the Mandelbrot set is conjectured to be locally connected. This difference is caused by lack of complex analyticity on the parameter of our family.

Our concern is where in the tricorn pathwise connectivity breaks. Actually we can show it on the boundary of every *maximally tuned* hyperbolic component of odd period off the *arcs of symmetry*. For details, see Theorem 2.8.

Though we can show non-pathwise connectivity only for a certain type of hyperbolic components, numerical experiments suggest that it is true for any hyperbolic component of

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odd period off the real axis. On the real axis, it is evidently pathwise connected. Suppose it is true. Then it follows that baby tricorns are not homeomorphic to the entire tricorn. This corresponds to the fact that the straightening map for the polynomial-like maps of degree 3 is not continuous, stated in Douady-Hubbard [DH2].

Our study is intimately related to the study of the dynamics of cubic polynomials. Milnor [Mil] classifies the dynamics of cubics into four types by the behaviours of two critical orbits. Our antipolynomial family gives a model for one of them, the bitransitive case, i. e. two critical points belong to a same cycle of attractive basins. The appearance of tricorn-like figures in its real slice suggests a strong evidence for this . In fact, our argument will show non-pathwise connectivity of its real slice.

This note is based on the joint work with Prof. D. Schleicher, which got started when I stayed at IHES. Prof. J. Milnor pointed us to this problem and Prof. J. H. Hubbard explained us a strategy towards non-local connectivity through the theory of Ecalle cylinder. We also thank Prof. M. Shishikura for helpful suggestions.

2. Numerical Experiments

The study of non-pathwise connectivity originates from the work of Milnor [Mil]. Actually he gives a numerical experiment of the real slice of the cubic connectedness locus there. See also Branner [Bra]. There appear a tricorn -like figure and a structure like the graph of the function: $\sin(1/x)$, a prototype of non-pathwise connected sets. In the summer of 1993, he gave the author the question of whether the tricorn is pathwise connected. That is the starting point of this study.

The proof of non-pathwise connectivity of the tricorn is based on the way Lavaurs [Lav] has proved the non-local connectivity of the cubic connectedness locus by using the *Ecalle cylinders*. That is, we translate the topology in the dynamical plane into parameter space by the *transit map* between cylinders in order to get a desired result. In this process, the following plays an important role.

Theorem 2.1 ([NS]) *Let W be a hyperbolic component of odd period of M_2^*. Then the Ecalle height of the critical value parametrizes each parabolic arc (i. e. connected component of $\partial W - \{$cusp points$\}$) real analytically, along which f_c are quasiconformally conjugate to each other.*

Since we consider antipolynomials, there exists an invariant line (an *equator*) in the cylinder and it follows that the transit map between cylinders does not change the *Ecalle height*, the vertical coordinate on the cylinder. This makes the situation much simpler. Intuitively speaking, the theorem above then implies that we see, in a neighborhood of each parabolic arc, a "universal cover" of cylinders in the dynamical plane. Since, in the repelling cylinder, there is a projection of the Julia set, some of its topological properties directy reflect in the parameter space. Hence, to show non-pathwise connectivity, we have only to find appropriate topological properties of the Julia sets in the dynamical plane.

Now, we explain the numerical experiments. Fig. 1 is the entire tricorn. Theorem 2.1 means that hyperbolic components of odd periods correspond to baby tricorns. As for even period components, we can say that they correspond to the baby Mandelbrot sets. Fig. 2 is one of them. This suggests that even iterates $\{f_c^{\circ 2k}\}$ on some regions form a Mandelbrot-like family of polynomial-like mappings. Fig. 3 is a period 5 component. There appears a "zig-zag" structure in Fig. 4, an enlargement of Fig. 3. This is a numerical evidence of the non-pathwise connectivity of the tricorn. Fig. 5 is the Julia set of the center of the corresponding *principal parabolic arc*. Fig. 6 and 7 are its enlargements near a parabolic periodic point whose immediate basin contains the critical value. We can easily see a similar structure between Fig. 4 and 7.

Note that the "zig-zag" structure in the dynamical plane essentially comes from the *Hubbard tree*. That is, the "zig-zag" structure of the Hubbard tree in the dynamical plane reflects the similar structure in the parameter plane. Hence what we have to show is such property of the parabolic Hubbard tree on the principal arc of a hyperbolic component of odd period. Actually, we have only to show that it does not contain a real analytic arc.

We also note that this does not always happen. In fact, consider those components on the real axis. It is easy to see that they are pathwise connected to the main component by a real line segment. Fig. 8 and 9 are the period 3 component on the real axis and its enlargement.

It follows that every component of period greater that two lies on a *limb* $L_{p/q}(W_2)$ of a component W_2 of period two. Here p/q corresponds to the parabolic parameter on ∂W_2 with multiplier $e^{2\pi ip/q}$. In this case, the repelling 2-cycle plays an important role. Fig. 10 and 11 show W_2 and the limb $L_{1/4}(W_2)$ respectively.

Usually the Hubbard tree is defined only for critically finite polynomials as a tree obtained by connecting the critical orbit by regular arcs. For parabolic polynomials, we define it as a homotopy class of such regular arcs in each Fatou component. The following lemma shows that, at repelling 2-periodic points, real analytic arc on such Hubbard tree can be determined uniquely if it exists.

Lemma 2.2 *Let c be on a parabolic arc of a hyperbolic component of odd period $k \geq 3$ and H_c be its Hubbard tree. Then H_c containes a repelling 2-periodic point z_c of f_c and points on $J_c \cap H_c$ accumulate on z_c.*

Definition 2.3 (Maximal tunedness) *A hyperbolic component of odd period of the tricorn is called* maximally tuned, *if it is never expressed by a tuned image of a component of period greater than one. In other words, it is just once renormalizable.*

The following lemma assures that the local property of H_c near z_c is actually projected on the Ecalle cylinder.

Lemma 2.4 *Let c be on a principal arc of a maximally tuned component. Then the inverse orbit of z_c on H_c accumulates on the parabolic periodic points.*

As for the numerical experiments, see Fig. 5, 6 and 7. They are the Julia set for c, a parabolic parameter of period 5 in $L_{1/3}(W_2)$ and its enlargements. We can see H_c in Fig. 7. There is a sequence of branch points, which are the inverse orbit of z_c, accumulate on a cusp point, i. e. a parabolic periodic point. This is also an example of the following lemma.

Lemma 2.5 *Let c be on a principal arc of a maximally tuned component sitting in a limb $L_{p/q}(W_2)$ with odd q. Then H_c does not contain a real analytic arc near z_c.*

The proof is done by a combinatorial argument around z_c. The figures above suggest that H_c is neither real analytic nor differentiable also at points in the inverse orbit of z_c. In case q is even, we have to assume one more assumption at this moment. We believe it is unnecessary.

Definition 2.6 (Arcs of symmetry) *Consider a limb $L_{p/q}(W_2)$. It has q branches at a Misiurewicz point c which satisfies $f_c^{\circ 2q+2}(c) = f_c^{\circ 2q}(c)$. That is, c is the endpoint of the baby Mandelbrot set of the Mandelbrot-like family $\{f_c^{\circ 2q}\}$*

corresponding to -2. *For even* q, *we call the* $(q/2-1)$*-th branch from the one contained in the baby Mandelbrot set, i. e. the central one, the* branch of symmetry *of the limb* $L_{p/q}(W_2)$. *If* q *is odd, there is no such branch. On that branch, there exists an arc which always runs the branch of symmetry at any one of its branch points. We call such an arc the* arc of symmetry.

The part of tricorn on the real axis or its rotaion is an example of the arc of symmetry. In Fig. 11, we can see the Misiurewicz point where four branches meet. We can also see baby tricorns on a branch of non-symmetry (the upper branch in Fig. 11).

Lemma 2.7 *Let* c *be on a principal arc of a maximally tuned component which sits on a limb* $L_{p/q}(W_2)$ *with even* q *but not on its arc of symmetry. Then* H_c *does not contain a real analytic arc near inverse orbits of* z_c.

Fig. 12 is an enlargement of the Julia set of a parameter $c\in L_{1/4}(W_2)$ not on the arc of symmetry. At its center, there is an immediate parabolic basin containing the critical value c. Its Hubbard tree seems neither real analytic nor differentiable at some points.

In case of arcs of symmetry, such a combinatorial argument does not work. See Fig. 13. This is an example of a Julia set for c on the arc of symmetry of $L_{1/4}(W_2)$. We cannot see any evidence that the Hubbard tree is not real analytic. But, if $q\neq 2$, numerical experiments suggest that, on such arcs, there is no component of odd period.

The most difficult case is $q=2$, i. e. the tuned images of the components on the real axis. At this moment, we have no effective idea to attack this case. In this case, the inverse orbit of a repelling 2-cycle on the Hubbard tree never accumulates on the parabolic periodic points. Of course, since the inverse orbit of a repelling cycle is dense in the Julia set, it accumulates on the parabolic periodic points. But, its restriction on the Hubbard tree does not. Hence the above argument does not work. Nevertheless, numerical experiments suggest that, even at such components, there exists a $\sin(1/x)$-like structure. See Fig. 14 and 15, which are near a component of period 15, a tuned image of the period 3 component on the real axis.

Now we sum up our argument.

Theorem 2.8 ([**HNS**]) *The tricorn is not pathwise connected near the principal arc of any maximally tuned hyperbolic component of odd period which never sits on an arc of symmetry.*

As for the non-maximally tuned components, some of them can be treated analogously as above if we consider repelling cycles in the baby Hubbard tree of appropriate periods instead of two. The period depends on the renormalization property. We omit the details. Our conjecture is:

Conjecture 2.9 *The tricorn is not pathwise connected near the principal arc of any hyperbolic component of odd period off the real axis and its rotations.*

A strategy for this conjecture, especially in the tuned image case, is to show that the multiplier of a repelling cycle on the baby Hubbard tree is non-real. We do not know how to do it. However, once this is verified, it follows that the baby Hubbard tree is not real analytic near it.

We can show non-pathwise connectivity of the real slice of the cubic connectedness locus analogously.

Finally, Fig. 16 shows a comb-like structure in the tricorn. This is a numerical evidence of the non-local connectivity of the tricorn, which can be shown in the same way.

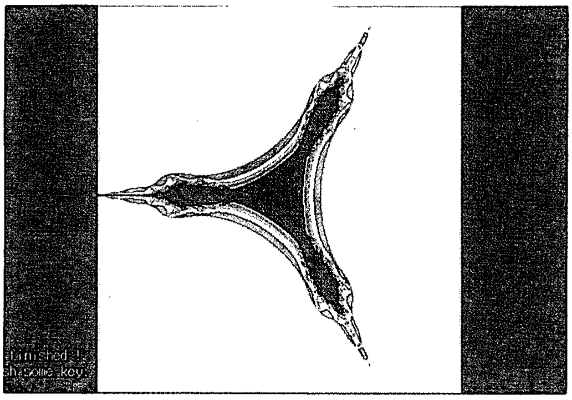

Fig. 1

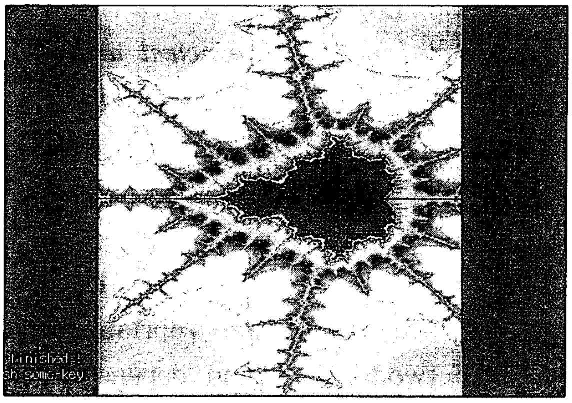

Fig. 2

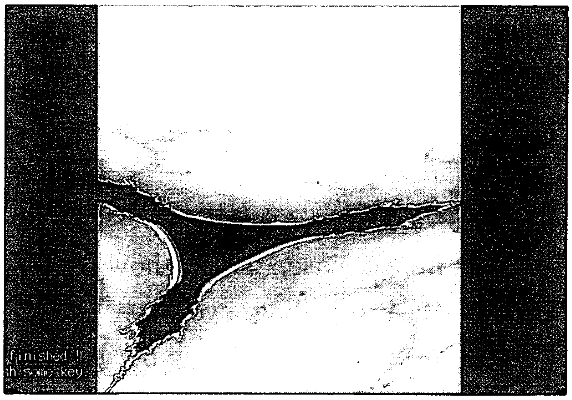

Fig. 3

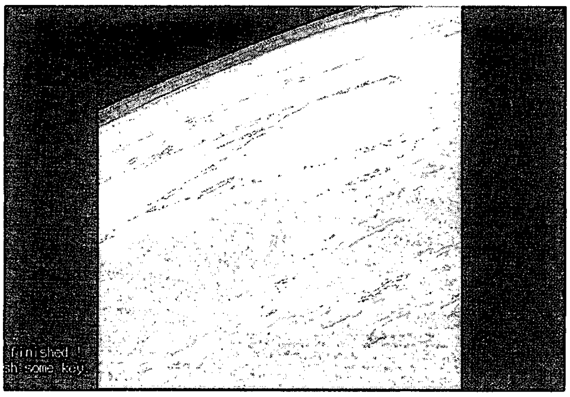

Fig. 4

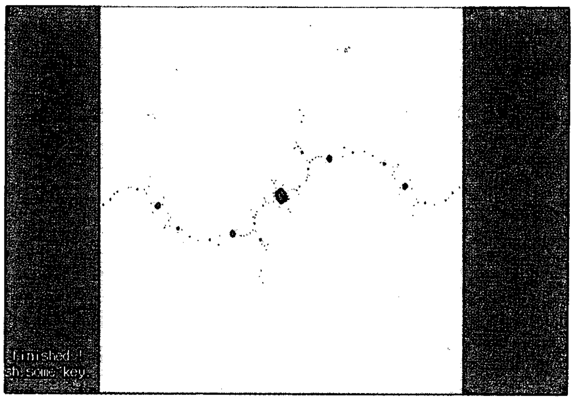

Fig. 5

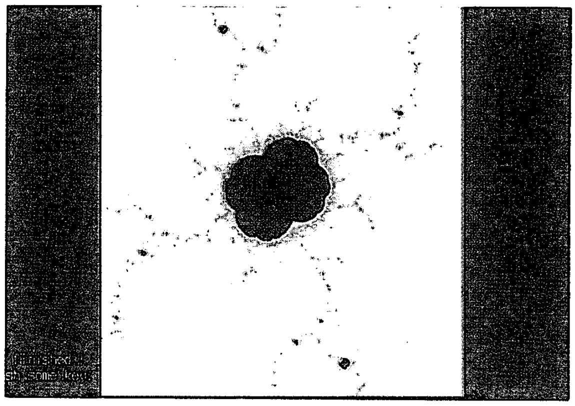

Fig. 6

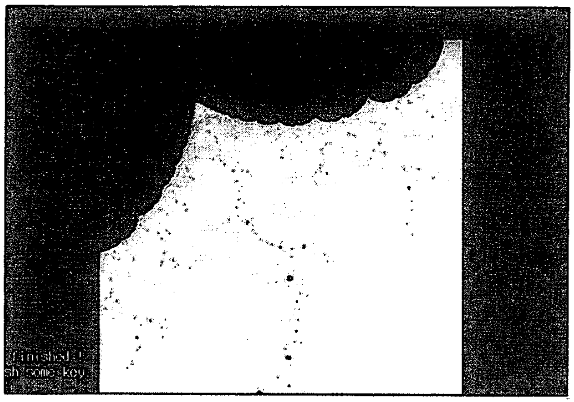

Fig. 7

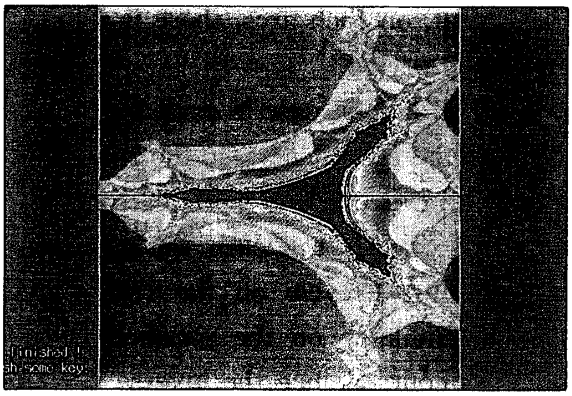

Fig. 8

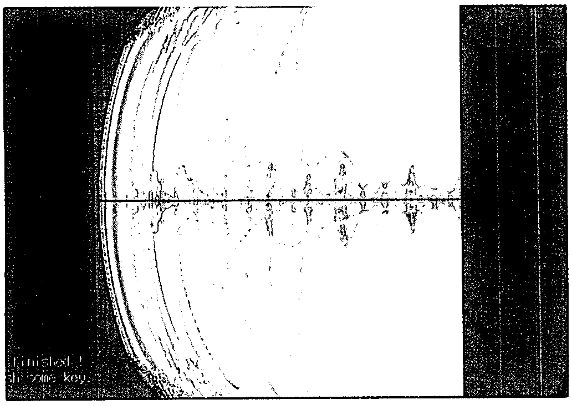

Fig. 9

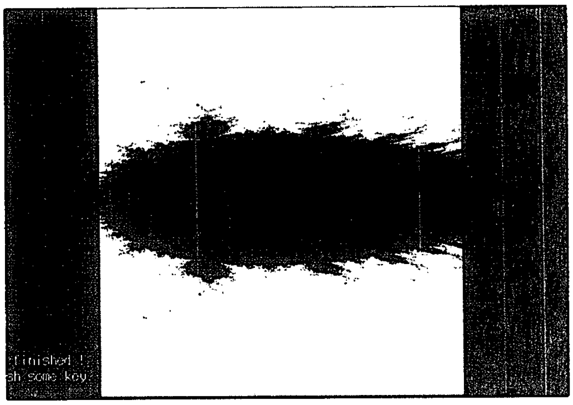

Fig. 10

Fig. 11

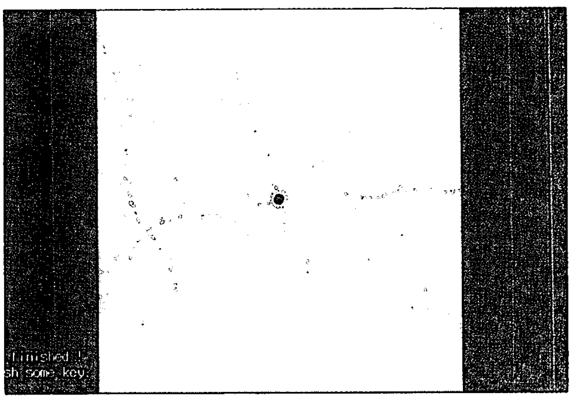

Fig. 12

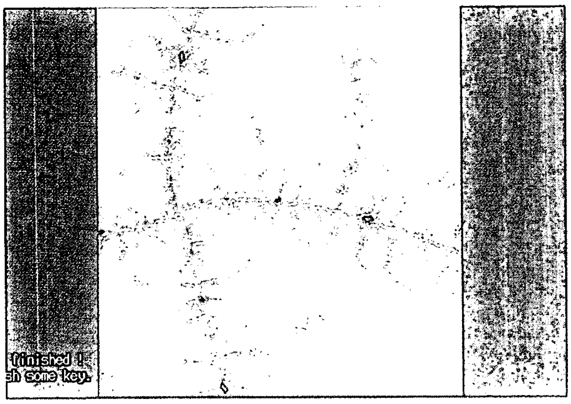

Fig. 13

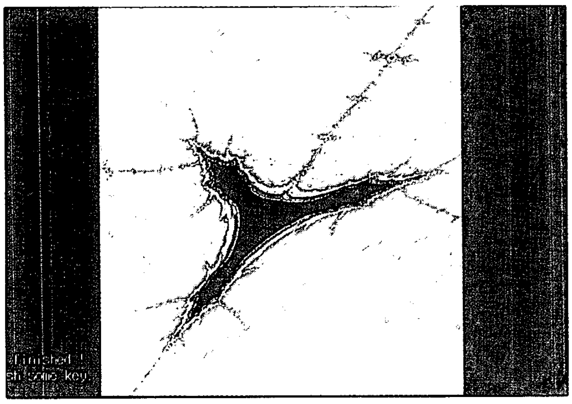

Fig. 14

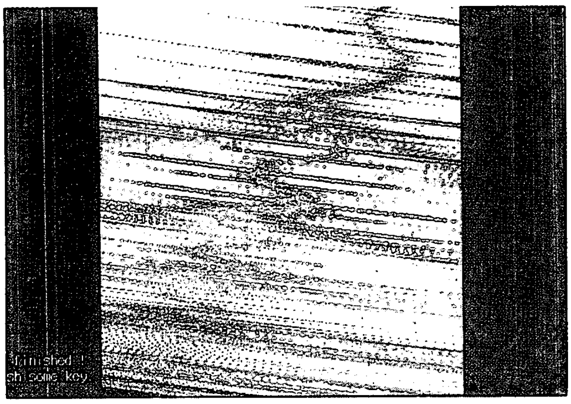

Fig. 15

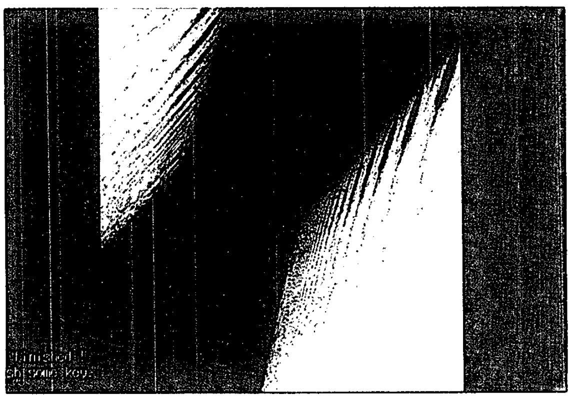

Fig. 16

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