Generating Functions for Ferrers-like Diagrams

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Weight enumarating functions are calculated for some variants of Ferrers diagrams. These diagrams are generalizations of Propp's hexagonal and rhombic diagrams. Propp has conceived these variants from geometrical view point, but his method of enumeration is feasible of calculating some series of variations which contain his own. It turns out that these graphs are again closely related with Ferrers tableaux which satisfy some constraints.

1. Introduction

There are some ways of making variants of Ferrers diagrams (e.g.hexagonal, rhombic, tilted, punctured diagrams) and James Propp has calculated some generating functions for those diagrams. In this note we would like to generalize a little further Propp's hexagonal and rhombic diagrams and study their generating functions.

A Ferrers diagram and their variants may be regarded as lower ideals of partially ordered sets [2]. For a poset S, let $p(S;n)$ signify the number of diagrams on S of weight n, i.e. the number of lower ideals of S with exactly n elements. Then the weight enumerator

$$f(S;q)=\sum_{n=0}^{\infty}p(S;_n)q^n$$

is called the genarating function of the diagrams on S. Thus $f(S;q)$ is a formal power series in the variable q.

1. Hexagonal diagrams

Let $\boldsymbol{Hex}$ be the poset defined by Fig.1.

Theorem 1 (Propp [2, Theorem 2 (a)])

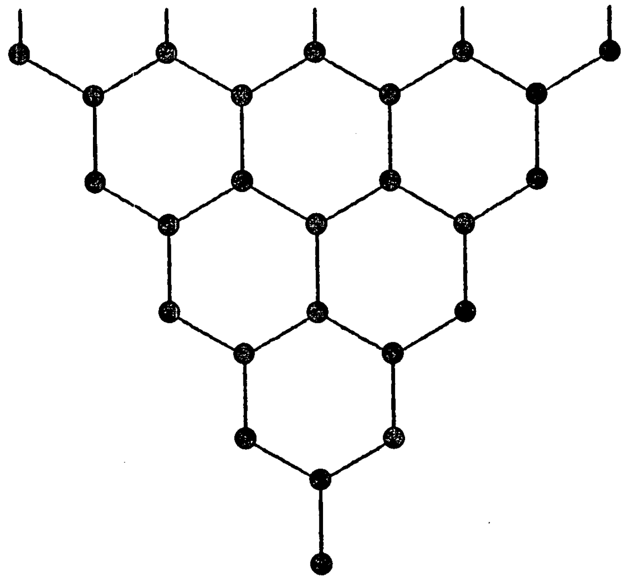

Fig. 1 The poset Hex.

$$f(\boldsymbol{Hex};q)=\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1-q^{2n}}$$

The poset $\boldsymbol{Hex}$ may be regarded as a special case (of d=2) of a wider class of posets $\boldsymbol{Hex}^d$. For example, the poset $\boldsymbol{Hex}^3$ is illustrated in Fig.2. Then we have the following result.

Theorem 2

$$f(\boldsymbol{Hex}^d;q)=$$
$$\prod_{n=1}^{\infty}\frac{1+q^{dn-d+1}+q^{dn-d+2}+\cdots+q^{dn-1}}{1-q^{dn}}$$

To prove this, we have only to mention that diagrams on $\boldsymbol{Hex}^d$ are in one-to-one correspon-

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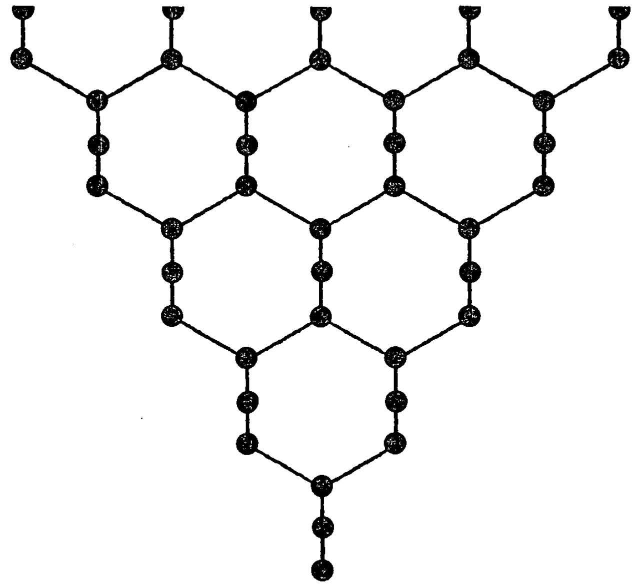

Fig. 2 The poset Hex³.

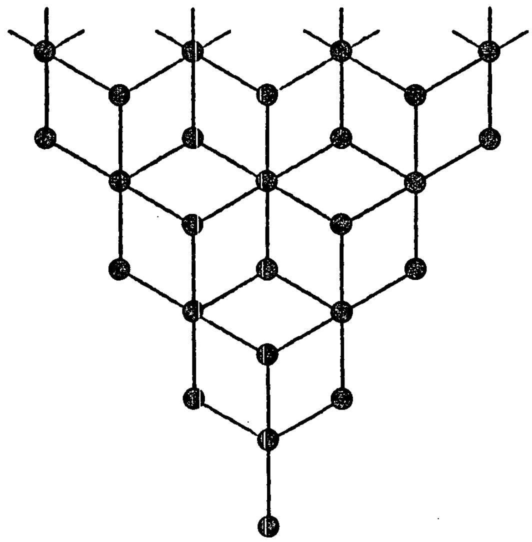

Fig. 3 The poset Rhomb.

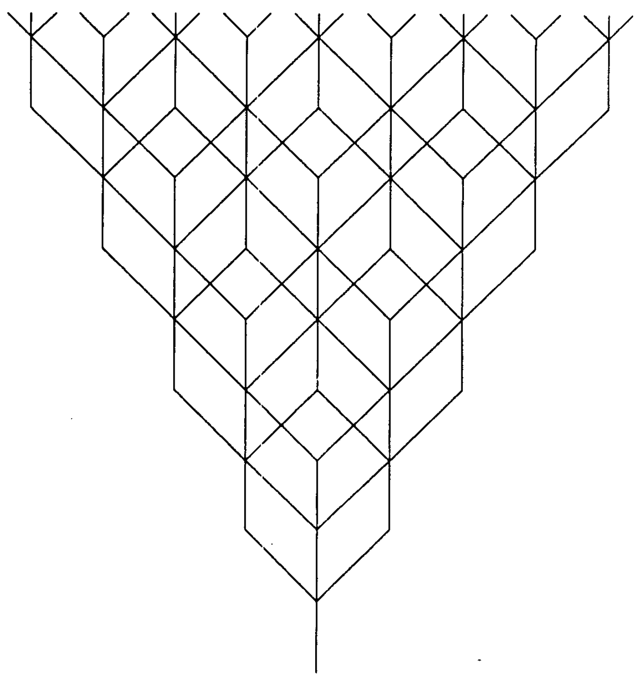

Fig. 4 The poset Rhomb⁴.

dence to partitions in which no parts are repeated except multiples of d and no two parts fall within the same segment between one multiple of d and the next. This correspondence is obtained by dividing $\boldsymbol{Hex}^d$ into chains of d elements and transforming the diagram on $\boldsymbol{Hex}^d$ into the 'occupancy tableau' [2] .

2. Rhombic diagrams

Let ***Rhomb*** be the poset defined by Fig.3.

Theorem 3 (Propp [2, Theorem 3]) .

$$f(\boldsymbol{Rhomb};q)=(1+q)\prod_{n=1}^{\infty}\frac{1+q^{3n-1}+q^{3n}+q^{3n+1}}{1-q^{3n}}$$

Again the poset ***Rhomb*** may be regarded as a special case (of $d=3$) of a wider class of posets $\boldsymbol{Rhomb}^d$. For example, the poset $\boldsymbol{Rhomb}^4$ is illustrated in Fig.2. Then we have the following result.

Theorem 4

$$f(\boldsymbol{Rhomb}^4;q)$$

$$=(1+q)\frac{\prod_{n=0}^{\infty}\{1+q^{4n}f_1(q)+q^{8n}f_2(q)\}}{\prod_{n=1}^{\infty}(1-q^{4n})}$$

where $f_1(q)=q^2+q^3+q^4+2q^5+q^6$ and $f_2(q)=q^6+q^7+2q^8+q^9$.

Proof. $\boldsymbol{Rhomb}^4$ may be divided into quadruples. A diagram can be represented by an occupancy tableau, which indicates how many of the vertical 4 dots in each quadruple are occupied by the diagram. Rotating 45° clockwise, the occupancy tableau is a Ferrers tableau with the constraint that every 3 must be supported by 4's, every 2 must be supported by 3's and 4's and every 1 must be supported by 2's, 3's and 4's.

Then we can define "inflation" and "deflation" operations as in [2], so that every tableau corresponds to a pair of tableaux, one of them "reduced", which cannot be deflated any more, and the other composed of 4's. Tableaux composed of 4's are enumerated by

$$\prod_{n=1}^{\infty}(1-q^{4n})^{-1}$$

It remains only to enumerate the reduced tableaux.

In a reduced tableau, a row consisting of just 4's cannot occur, since that would imply deflatability. Hence the non-empty rows that may occur are

1,
2,
2 1,
3 1,
3 2,
3 2 1,
4 1,
4 2,
4 2 1,
4 3 1,
4 3 2,
4 3 2 1,
4 4 1,
4 4 2,
4 4 2 1,

and so on. Legality implies that no row may appear twice and that the order in which they are stacked respects the order in which they are listed.

Now, legality implies moreover that some juxtapositions of rows cannot occur in a reduced tableau, and this constraint is illustrated conveniently in Fig.5, which divides these potentially occurring rows into groups of 6 rows. It is seen that (a) there is no forbidden juxtaposition between rows of different groups; and (b) juxtaposition constraint is of the same pattern in each single group. In Fig. 5, braces between two rows indicate that only those juxtapositions are allowed. These partitions are enumerated by

$(1+q)$
$(1+q^2+q^3+q^4+2q^5+2q^6+q^7+2q^8+2q^9)$
$(1+q^6+q^7+q^8+2q^9+q^{10}+q^{14}+q^{15}+2q^{16}+q^{17})\cdots$

and we are done. ■

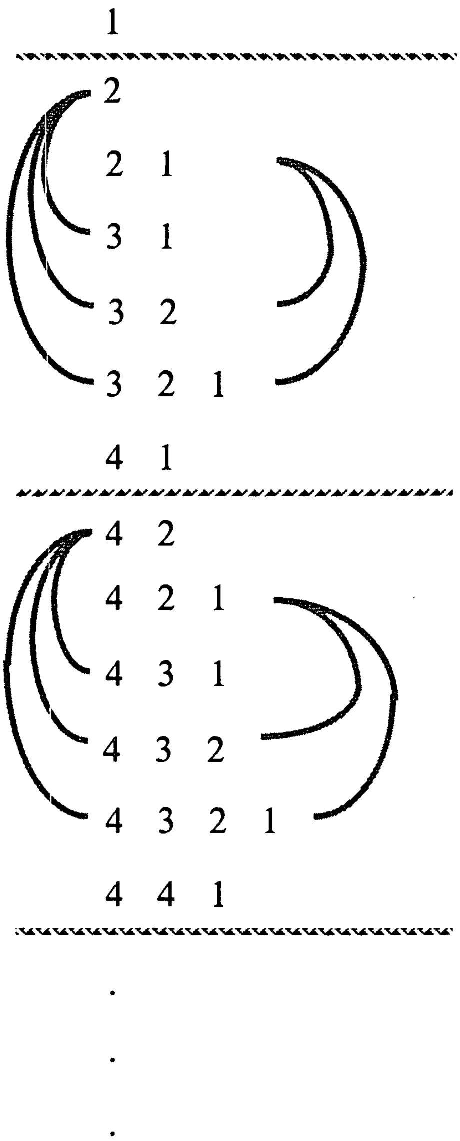

Fig. 5 Allowed juxtapositions in reduced tableaux.

3. Remarks

We can enumerate diagrams on ***Rhomb***^{5}, and so on, by the same technique, but the numerator will gradually become complicated.

The poset ***Rhomb*** has bilateral symmetry, as

Hex has, which is implicit in the pictorial definition of the poset. According to [2], the weight enumeration of symmetric diagrams on ***Rhomb*** is not yet done.

The idea of generalizing Ferrers graphs is not new, but there are undoubtedly many new identities waiting to be discovered in this broad area.

References

1) G.E. Andrews, The theory of partitions, in "Encyclopedia of Mathematics and Its Applications," Vol.2 (G.-C.Rota, Ed.), Addison-Wesley, Reading, Ma, 1976.

2) J. Propp, Some variants of Ferrers diagrams, Journal of Combinatorial Theory, Ser. A, 52 (1989), pp.98-128.