Value-at-Risk estimation: A novel GARCH-EVT approach dealing with bias and heteroscedasticity

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My Achievement

Consideration of a **new way** to estimate **Value-at-Risk (VaR)** based on GARCH-EVT (**Extreme Value Theory**) method.
- Our approach (**GARCH-UGH**) is asymptotically unbiased and yields VaR reflecting volatility background (more realistic).
- **GARCH-UGH** performed better than the original **GARCH-EVT** when appropriate **threshold** is selected.

Background

Purpose: **EVT** to estimate extreme quantiles (VaR in finance).
- Assumption of normality does not hold for real data.
Threshold selection is difficult unsolved problem in EVT.
- How do we select the number of top observations for VaR estimation?

Problems of EVT for VaR estimation:

1. **I.I.D. assumption**
2. **Not reflecting volatility**
3. **Bias due to threshold selection**

Our approach

Previous approach: **Two-step GARCH-EVT** (McNeil and Frey (2000))
→ GARCH for filtering, EVT for tail estimation
→ **Only** overcome 1st and 2nd problems

Our solution for VaR estimation:

Propose **NEW** GARCH-EVT called **GARCH-UGH**.
→ Same GARCH filtering but **update** EVT (de Haan *et al.* (2016))
→ Overcome all 3 problems
What's new? A compromise between unbiasedness and volatility.

GARCH-UGH

Consider $(X_t, t \in \mathbb{Z})$ a strictly stationary financial time series.
Assume dynamics of X are $X_t = \mu_t + \sigma_t Z_t$ where Z_t are I.I.D. innovations.
Aim: Estimate the 1-step ahead conditional VaR as follows:

$$\boldsymbol{x_k = \mu_{t+1} + \sigma_{t+1} z_k}. \tag{1}$$

GARCH step:

1. Fit AR(1)-GARCH(1,1) model to the return data,
2. Estimate 1-step ahead mean μ_{t+1} and volatility σ_{t+1} for (1),
3. Extract the residuals z_t (**should be I.I.D.**) for UGH step.

UGH step (**Closed-form solutions, Semiparametric**):
Use top k order statistics which need to be an upper intermediate.
- i.e. $k = k_n$ with $k \to \infty$ and $k/n \to 0$ as $n \to \infty$.
For $1 < k$(**sample fraction**) $< n$ and $\alpha = 1, ..., 4$,

$$M_k^{(\alpha)} = \frac{1}{k}\sum_{i=1}^{k}(\log Z_{n-i+1,n} - \log Z_{n-k,n})^{\alpha} \tag{2}$$

$$S_k^{(2)} = \frac{3}{4}\frac{(M_k^{(4)} - 24(M_k^{(1)})^4)(M_k^{(2)} - 2(M_k^{(1)})^2)}{M_k^{(3)} - 6(M_k^{(1)})^3}, \tag{3}$$

$$\hat{\rho}_k = \frac{-4 + 6S_k^{(2)} + \sqrt{3S_k^{(2)} - 2}}{4S_k^{(2)} - 3}, \quad \text{provided} \quad S_k^{(2)} \in \left(\frac{2}{3}, \frac{3}{4}\right) \tag{4}$$

$$\hat{\gamma}_{k,k_\rho} = \hat{\gamma}_k^H - \frac{M_k^{(2)} - 2(\hat{\gamma}_k^H)^2}{2\hat{\gamma}_k^H \hat{\rho}_{k_\rho}(1 - \hat{\rho}_{k_\rho})^{-1}} \quad \text{given} \quad M_k^{(1)} = \hat{\gamma}_k^H \quad \text{is Hill estimator} \tag{5}$$

$$\hat{z}_{k,k_\rho} = Z_{n-k,n}\left(\frac{k}{np}\right)^{\hat{\gamma}_{k,k_\rho}}\left(1 - \frac{[M_k^{(2)} - 2(\hat{\gamma}_k^H)^2][1 - \hat{\rho}_{k_\rho}]^2}{2\hat{\gamma}_k^H \hat{\rho}_{k_\rho}^2}\left(1 - \left(\frac{k}{np}\right)^{\hat{\rho}_{k_\rho}}\right)\right) \tag{6}$$

where $\hat{\rho}_{k_\rho}$ is the one optimal $\hat{\rho}$ selected following de Haan *et al.* (2016).
Substitute (6) into (1) resulting in **VaR estimates**, our aim.

Simulations

Simulation setting: Consider GARCH(1,1) model given by

$$X_t = \epsilon_t \sigma_t, \quad \sigma_t^2 = \lambda_0 + \lambda_1 X_{t-1}^2 + \lambda_2 \sigma_{t-1}^2,$$

with $\lambda_0 = 8.26 \times 10^{-7}$, $\lambda_1 = 0.052$, $\lambda_2 = 0.941$, ϵ_t follows standardized Student-t with d.o.f. $= 5.64$ and simulated theoretical VaR, $x_k = 0.0592$.

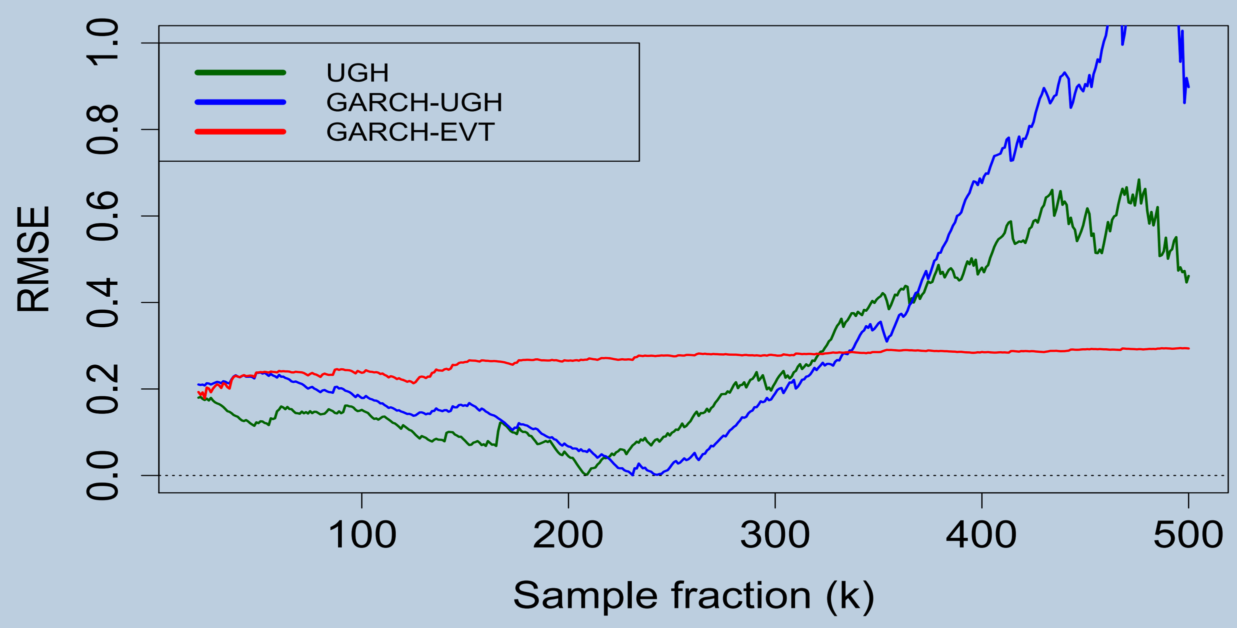

Figure 1: Simulated **RMSE** $\left(= \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\frac{\hat{x}_k^{(j)}}{x_k} - 1\right)^2}\right)$ of VaR based on UGH only, **GARCH-UGH** and **GARCH-EVT** at 99.9% level. $p = 0.001$, $N = 1000$, $n = 1000$.

Real data application

- Collected $n = 7796$ daily **Dow Jones Index** from 1985 to 2010.
- Computed daily negative log-returns (financial time series).
- Applied VaR estimators (1) for 1-step ahead VaR at 99.9% level.

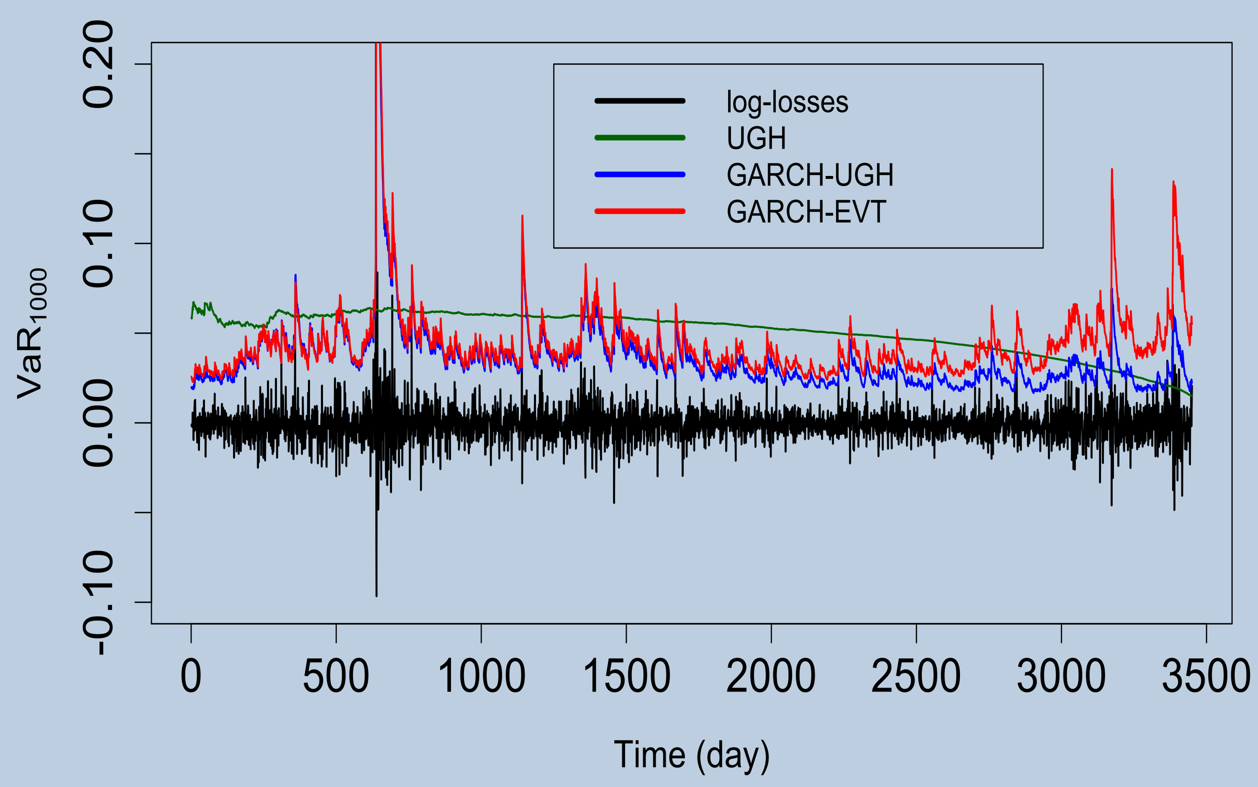

Figure 2: 13 years of negative log-returns (**black**) beginning in April 1985 with superimposed 99.9% VaR estimators with effective sample fraction $k = 1000$.

Length of Test (n)	3450							
Threshold Selection (k)	250	500	750	1000	1250	1500	1750	2000
0.999 Quantile								
Expected	3	3	3	3	3	3	3	3
UGH	6 (0.21)	9 (0.01)	13 (0.00)	12 (0.00)	13 (0.00)	16 (0.00)	21 (0.00)	24 (0.00)
GARCH-UGH	1 (0.12)	4 (0.77)	4 (0.77)	4 (0.77)	8 (0.04)	7 (0.09)	12 (0.00)	12 (0.00)
GARCH-EVT	1 (0.12)	1 (0.12)	1 (0.12)	1 (0.12)	1 (0.12)	1 (0.12)	1 (0.12)	1 (0.12)

Table 1: Backtesting based on Kupiec's test for several sample fractions k. The table reports number of violations (**observations > VaR**) and p-values.

Conclusion: Our new approach **GARCH-UGH** performed better than original **GARCH-EVT** when approx top 5% to 15% of data are used.

Kaibuchi, H., Kawasaki, Y. and Stupfler, G. (2019). VaR estimation: A novel GARCH-EVT approach dealing with bias and heteroscedasticity (in preparation)