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Algorithmic analogies to Kamae-Weiss theorem on normal numbers

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1 Selection function

Example

$$\begin{array}{rcccccc} x = & 0 & 1 & 0 & 0 & 1 & \cdots \\ y = & 1 & 0 & 1 & 0 & 1 & \cdots \\ x/y = & 0 & & 0 & & 1 & \cdots \end{array}$$

$x, y \in \{0,1\}^\infty$.

2 Kamae-Weiss theorem

Definition 1. $x = x_1 x_2 \cdots$ is called normal number if

$$\forall s \in \{0,1\}^* \lim_{n\to\infty} \#\{1 \le i \le n \mid x_i \cdots x_{i+|s|-1} = s\}/n = 2^{-|s|}.$$

Let $\mathcal{N}$ be the set of normal numbers.

Definition 2. p is called cluster point if there is a sequence $\{n_i\}$

$$\forall s\ p(s) = \lim_{i\to\infty} \#\{1 \le j \le n_i \mid x_j \cdots x_{j+|s|-1} = s\}/n_i.$$

Let $V(x)$ be the set of cluster point of x.
$V(x) \neq \emptyset$ for all x.

Kamae entropy is defined by

$$h(x) = \sup\{h(p) \mid p \in V(x)\}.$$

Theorem 1 (Kamae[1]). *Suppose that* $\liminf \frac{1}{n}\sum_{i=1}^n y_i > 0$ *then (i) and (ii) are equivalent:*
(i) $h(y) = 0$.
(ii) $\forall x \in \mathcal{N}\ x/y \in \mathcal{N}$,

Note: The part (i)⇒ (ii) is appeared in Weiss [3].

3 van Lambalgen's conjecture

K : prefix Kolmogorov complexity.
$\mathcal{R}$: the set of Martin-Löf random sequences with respect to (1/2,1/2)-i.i.d. process.

In van Lambalgen [2] the following equivalence is conjectured:

(i) $\lim_{n\to\infty} K(y_1^n)/n = 0$.

(ii) $\forall x \in \mathcal{R}\ x/y \in \mathcal{R}$.

4 Proposition 1

Proposition 1. *Suppose that y is Martin-Löf random with respect to some computable probability P and $\sum_{i=1}^\infty y_i = \infty$. Then the following two statements are equivalent:*
(i) y is computable.
(ii) $\forall x \in \mathcal{R}\ \ x/y \in \mathcal{R}^y$.

$\mathcal{R}^y$: the set of Martin-Löf random sequences with respect to (1/2,1/2)-i.i.d. process relative to y.

5 Weak randomness

Definition 3. y is called weakly random with respect to a computable P if

$$\lim_{n\to\infty} K(y_1^n)/n = \lim_{n\to\infty} -\frac{1}{n}\log P(y_1^n).$$

y is weakly random with respect to (1/2,1/2)-i.i.d. process if

$$\lim_{n\to\infty} K(y_1^n)/n = 1.$$

Note

$$y \in \mathcal{R} \to \lim_{n\to\infty} K(y_1^n)/n = 1 \to y \in \mathcal{N}.$$

None of the converse is true.

6 Proposition 2

Proposition 2. *Suppose that y is weakly random with respect to a computable measure and $\lim_n \frac{1}{n}\sum_{i=1}^n y_i > 0$.*
Then the following two statements are equivalent:
(i) $\lim_{n\to\infty} K(y_1^n)/n = 0$.
(ii) $\forall x\ \ \lim_{n\to\infty} K(x_1^n)/n = 1 \to \lim_{n\to\infty} \frac{1}{|x_1^n/y_1^n|} K(x_1^n/y_1^n|y_1^n) = 1$.

Example: computable sequences and sturmian sequences satisfy (i).

References

[1] T. Kamae. Subsequences of normal numbers. *Israel J. Math.*, 16:121–149, 1973.

[2] M. van Lambalgen. *Random sequences.* PhD thesis, Universiteit van Amsterdam, 1987.

[3] B. Weiss. Normal sequences as collectives. In *Proc. Symp. on Topological Dynamics and Ergodic Theory.* Univ. of Kentucky, 1971.