2010年7月9日　統計数理研究所　オープンハウス

単純自己修正点過程における母数推定について Parametric Estimation for Simple Self-Correcting Point Processes

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Introduction

We consider the parametric estimation problem for the simple self-correcting point (SSCP) process which is first introduced by Inagaki and Hayashi (1990). Our main concern is the threshold estimation problem for the the SSCP process. This is one of the non-regular parametric estimation problems since the likelihood function is discontinuous with respect to the threshold parameter. In such non-regular estimation problems, the likelihood ratio approach is effective to study asymptotic properties of the MLE. Also we provide a simple equation that indicates the invariant density of the stress release process. This invariant density helps us to study the asymptotic properties of the maximum likelihood estimator for the SSCP process.

Stress Release Process

The self correcting point process N_t is a point process with the conditional intensity function $\lambda_t = \phi(t - N_{t-})$. To study some properties for the self-correcting point process, the stress release process $X_t = t - N_t$ plays an important role. When the function $\phi(\cdot)$ satisfies the suitable conditions, it was shown that the stress release process X_t is ergodic, see Hayashi (1986), Vere-Jones (1988). Let f be its invariant density, then we provide a simple formula that helps us to calculate the invariant density function f of the stress release process.

Theorem 1

The invariant density function $f(x)$ of the stress-release process X_t is given by the solution of the integral equation:

$$f(x) = \int_x^{x+1} \phi(y) f(y)\, dy \qquad (1)$$

with

$$\int_{-\infty}^{\infty} \phi(y) f(y)\, dy = 1. \qquad (2)$$

Simple Self-Correcting Point Process

The SSCP process has the conditional intensity function:

$$\lambda_t = \phi(X_{t-}) = \begin{cases} a, & X_{t-} < \tau \\ b, & X_{t-} \geq \tau \end{cases},$$

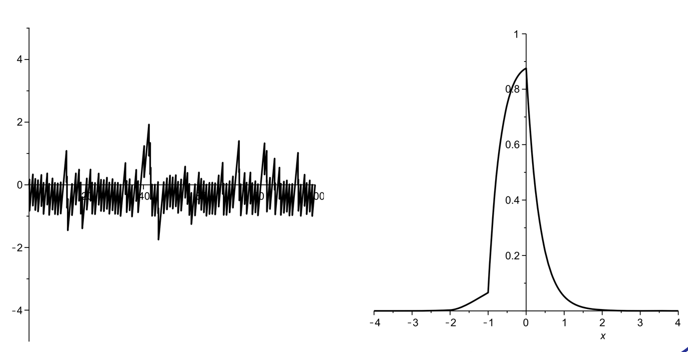
where a and b are some constants with $0 < a < 1 < b < \infty$. By Theorem 1, the invariant density $f(x)$ for $x \geq \tau$ is given by

$$f(x) = c(1-p)p^{x-\tau}, \qquad (3)$$

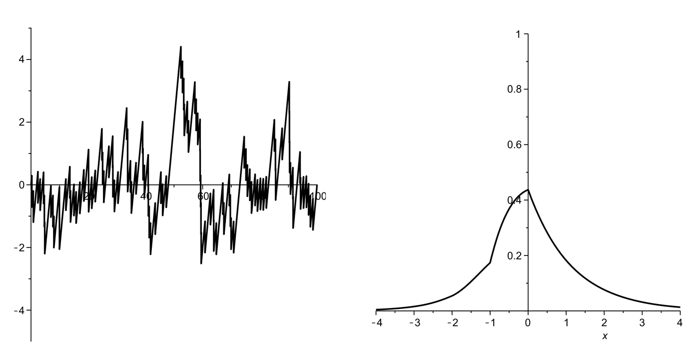
where p $(0 < p < 1)$ is the solution of the equation

$$\log p = b(p-1) \qquad \text{and} \qquad c = \frac{b(1-a)}{b-a}.$$

For $x < \tau$, it is difficult to obtain the invariant density $f(x)$ clearly, but it can be calculated by solving the equation (1) sequentially on each interval $[\tau-k, \tau-k+1)$, $k = 1, 2, \ldots$, see Figure.



Sample Path and Invariant Density $(a,b) = (0.1, 3.0)$



Sample Path and Invariant Density $(a,b) = (0.5, 1.5)$

Intensity Estimation

Provided that the threshold parameter is known, we estimate the intensity parameter $\theta = (a,b)^T$. The likelihood function is given by

$$\begin{aligned} L_T(\theta) &= \exp\left\{ \int_0^T \log \lambda_t\, dN_t - \int_0^T \lambda_t\, dt \right\} \qquad (4) \\ &= \exp\left\{ \log a \int_0^T 1_{\{X_{t-}<\tau\}}\, dN_t + \log b \int_0^T 1_{\{X_{t-}\geq\tau\}}\, dN_t - a\int_0^T 1_{\{X_{t-}<\tau\}}\, dt - b\int_0^T 1_{\{X_{t-}\geq\tau\}}\, dt \right\}. \end{aligned}$$

The likelihood estimating function is

$$dM_t = dN_t - \lambda_t\, dt$$

$$\begin{aligned} \frac{\partial}{\partial a} \log L_T(\theta) &= \frac{1}{a}\int_0^T 1_{\{X_{t-}<\tau\}}\, dN_t - \int_0^T 1_{\{X_{t-}<\tau\}}\, dt = \frac{1}{a}\int_0^T 1_{\{X_{t-}<\tau\}}\, dM_t, \\ \frac{\partial}{\partial b} \log L_T(\theta) &= \frac{1}{b}\int_0^T 1_{\{X_{t-}\geq\tau\}}\, dN_t - \int_0^T 1_{\{X_{t-}\geq\tau\}}\, dt = \frac{1}{b}\int_0^T 1_{\{X_{t-}\geq\tau\}}\, dM_t. \end{aligned}$$

Hence the maximum likelihood estimators of a and b are

$$\hat{a}_T = \frac{\int_0^T 1_{\{X_{t-}<\tau\}}\, dN_t}{\int_0^T 1_{\{X_{t-}<\tau\}}\, dt} \qquad \text{and} \qquad \hat{b}_T = \frac{\int_0^T 1_{\{X_{t-}\geq\tau\}}\, dN_t}{\int_0^T 1_{\{X_{t-}\geq\tau\}}\, dt}.$$

The asymptotic normality is proved by the martingale central limit theorem. The Fisher information matrix can be evaluated as follows:

$$\frac{I_T(\theta)}{T} = \frac{1}{T}\begin{pmatrix} E[\int_0^T 1_{\{X_{t-}<\theta\}}\, dt]/a & 0 \\ 0 & E[\int_0^T 1_{\{X_{t-}\geq\tau\}}\, dt]/b \end{pmatrix} \longrightarrow I(\theta) = \begin{pmatrix} \frac{b-1}{a(b-a)} & 0 \\ 0 & \frac{1-a}{b(b-a)} \end{pmatrix}.$$

Threshold Estimation

Next we assume that a and b are known. The likelihood (4) is not differentiable with respect to τ, so the MLE is defined by

$$\hat{\tau}_T = \arg\sup_{\tau\in\Theta} L_T(\tau),$$

where Θ is a parametric space. Consider the likelihood ratio process:

$$\begin{aligned} Z_T(h) &= \frac{L_T(\tau + \frac{h}{T})}{L_T(\tau)}, \qquad \tau, \tau + \frac{h}{T} \in \Theta \\ &= \begin{cases} \exp\left\{ \log\frac{a}{b} \int_0^T 1_{\{\tau\leq X_{t-}<\tau+\frac{h}{T}\}}\, dN_t - (a-b)\int_0^T 1_{\{\tau\leq X_{t-}<\tau+\frac{h}{T}\}}\, dt \right\}, & h > 0 \\ \exp\left\{ \log\frac{b}{a} \int_0^T 1_{\{\tau+\frac{h}{T}\leq X_{t-}<\tau\}}\, dN_t - (b-a)\int_0^T 1_{\{\tau+\frac{h}{T}\leq X_{t-}<\tau\}}\, dt \right\}, & h < 0 \end{cases}. \end{aligned}$$

Theorem 2

The likelihood ratio process $Z_T(h)$ weakly converges to

$$Z(h) = \begin{cases} \exp\left\{ \log\frac{a}{b}\, \mathcal{N}_{f(\tau)b}(h) - (a-b) f(\tau)\cdot h, \right\}, & h > 0 \\ \exp\left\{ \log\frac{b}{a}\, \mathcal{N}_{f(\tau)a}(|h|) - (b-a) f(\tau)\cdot |h| \right\}, & h < 0 \end{cases},$$

in the Skorohod space $D(\mathbf{R})$, where $\mathcal{N}_{f(\tau)b}$ and $\mathcal{N}_{f(\tau)a}$ are independent Poisson processes with the intensities $f(\tau)b$ and $f(\tau)a$, respectively.

This theorem provides the standardized asymptotic distribution of the MLE as follows:

$$\begin{aligned} P_{\tau_0}\left(T(\hat{\tau}_T - \tau_0) \in \Delta\right) &= P_{\tau_0}\left(\sup_{h\in\Delta} Z_T(h) > \sup_{h\notin\Delta} Z_T(h) \right) \\ &\to P_{\tau_0}\left(\sup_{h\in\Delta} Z(h) > \sup_{h\notin\Delta} Z(h) \right) \\ &= P_{\tau_0}\left(\hat{h} \in \Delta\right). \end{aligned}$$

$$\hat{h} = \arg\sup_{h\in\mathbf{R}} Z(h)$$

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