On a Computer- Aided Decomposition of the Complete Digraph into Orientations of K_4-e with a Double Edge

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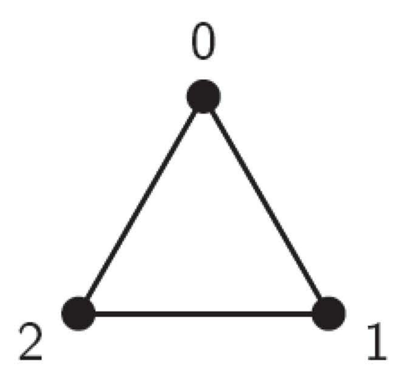
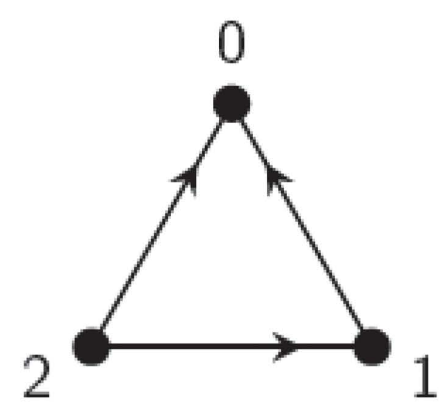
What is a graph?

An introduction to graphs. Graphs are mathematical objects comprising of a vertex set V and an edge set E, where the edges connect the vertices.

An introduction to directed graphs. In a directed graph, also known as a digraph, the edges are called arcs. The arcs are assigned directions and treated as ordered pairs. Different ways to assign directions to the edges are known as unique orientations.

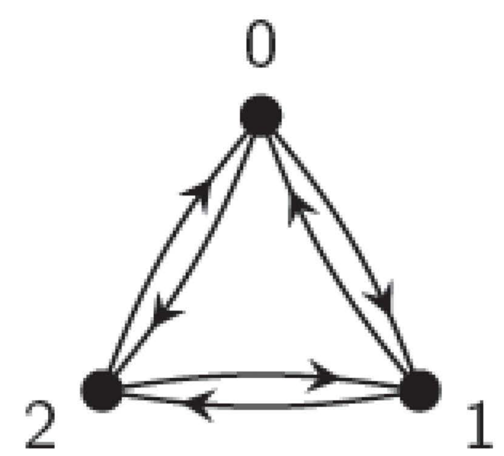


Here is an example of the graph K_3. It has vertex set V={0, 1, 2} and edge set E={(0,1), (0,2), (1,2)}.



This is an oriented K_3 digraph with same vertex set {0, 1, 2} and edge set {(1,0), (2,0), (2,1)}. Note that in this case, the order of the vertices in the edge set matters.

Complete digraphs. We call the complete digraph on *n* vertices K^*_n. It is the digraphs with arcs (a, b) and (b, a) for every pair of vertices a and b in the graph. a is always the vertex that the edge points toward. So if an edge points from 1 to 0, then the arc is denoted as (0, 1).
It follows that the number of arcs in K^*_n is $n(n-1)$.

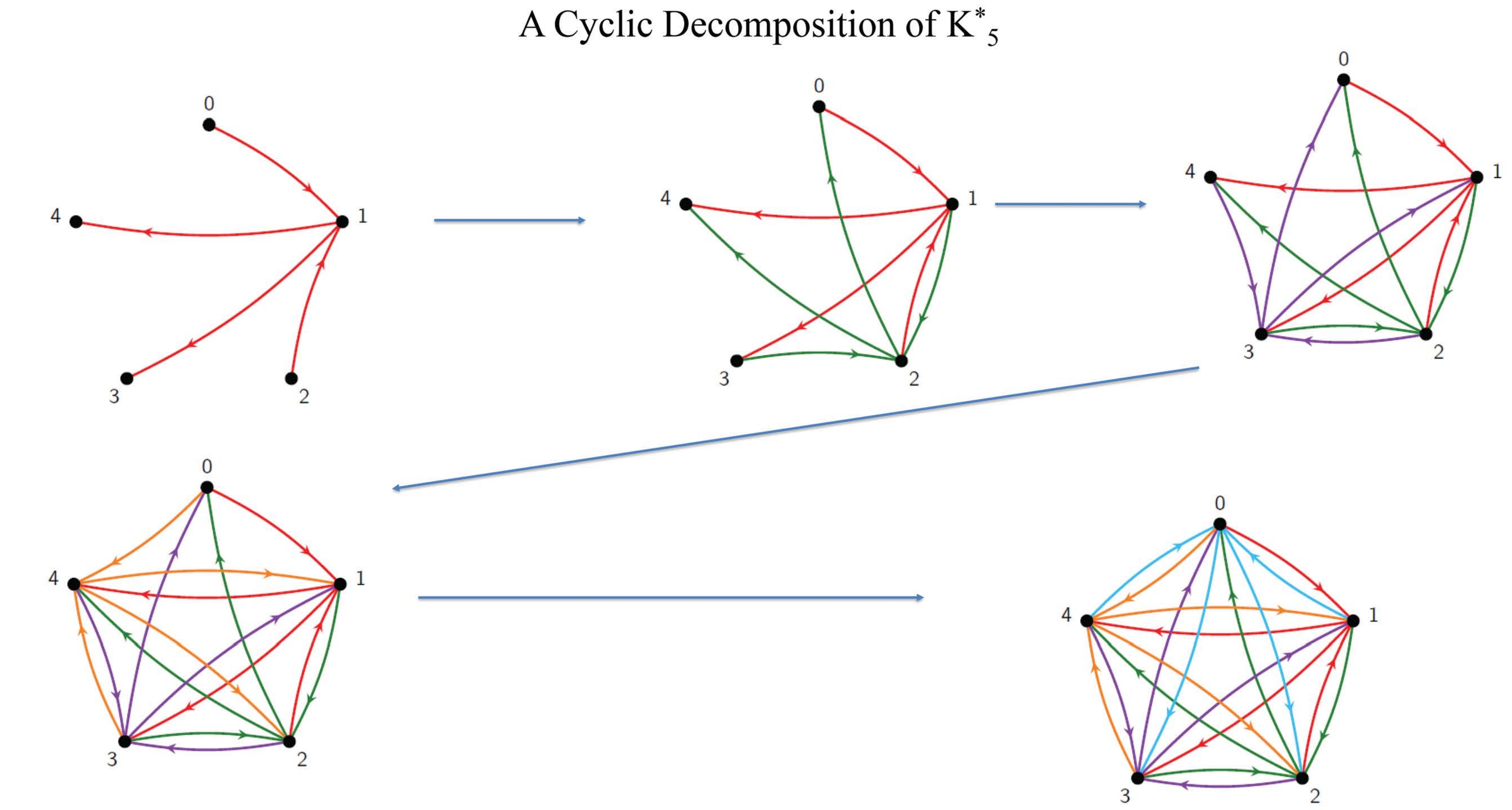


Here is an example of the complete digraph K^*_3.

How do graphs decompose?

Subgraphs and decompositions. If D and H are digraphs with D as a subgraph of H (if V(D) is a subset of V(H) and E(D) is a subset of E(H)), then a D- decomposition of H is a partition of the arc set of H into subgraphs that are isomorphic to D, called D-blocks.
Spectrum. The spectrum for a digraph D is the set of all *n* for which a D-decomposition of K^*_n exists.
Isomorphism. If $V(K^*_n)$ is all of the natural integers, and D is a subgraph of K^*_n then by rotating, or clicking D, we apply an isomorphism, where each element V(D) will go from *i* to *i* + 1.
Cyclic. A D- decomposition of K^*_n is called cyclic if clicking D preserves the D-blocks of the decomposition,

A Cyclic Decomposition of K^*_5

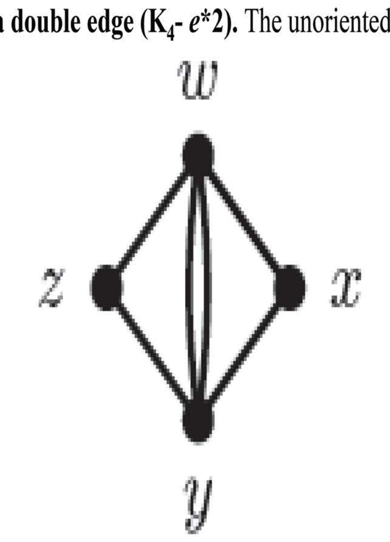


Abstract

Let D be any of the 5 non-symmetric digraphs obtained by orienting the edges of K_4- *e* with a double edge (denoted thereafter by K_4- *e**2). We obtain some (K^*_n, D) designs for small values of n where n < 36 aided by a C++ program. The C++ program was able to verify the nonexistence of results as well as construct new (K^*_n, D) designs. It also used a memorization technique, where previous runs were stored and referenced, in order to reduce runtime. Furthermore, we establish necessary and sufficient conditions for the existence of a (K^*_n, D) - design for some of the general constructions using the aforementioned small cases and a "blow-up" construction. Partial results as well as some nonexistence results are established for the remaining digraphs. Future work on this project may be done by developing more of the partial results and improve the code to reduce both memory usage and runtime, possibly by the use of parallel processing.

Our Research Question

We want to find the spectrum for all D such that D is an orientation of K_4- *e* with a double edge (K_4- *e2).** The unoriented graph for this is shown below.

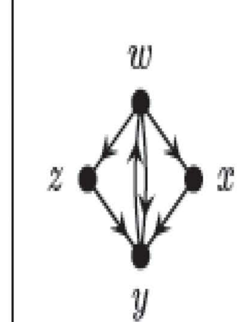
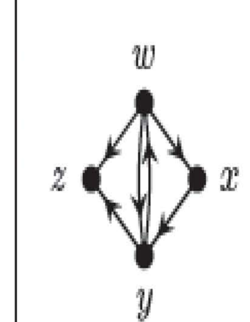
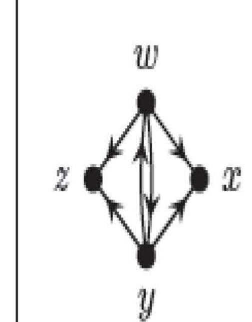
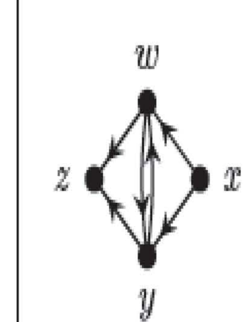
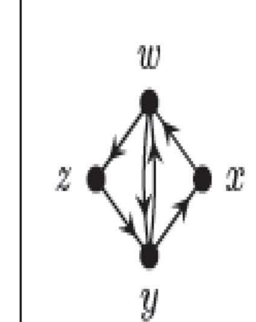


When we orient the graph, one of the two double edges from *w* to *y* must be directed towards *w* and the other must be directed toward *y*.
Digraphs are named using the conventions in *An Atlas of Graphs* by Read and Wilson.

Other Observations

Reversing digraphs. We can reverse the orientation of a digraph D by changing the direction of the arrow on each arc. We denote the reverse orientation as Rev (D).
Observation 1: We observe that if D and H are digraphs, then a D- decomposition of H exists if and only if a Rev (D)- decomposition of Rev (H) exists. Also find that K^*_n is congruent to Rev (K^*_n), since every possible directed arc in the vertex set is represented in K^*_n
Observation 2: If D is a digraph, then D decomposes K^*_n if and only if Rev (D) decomposes Rev (K^*_n).

Our Graphs

D125[w, x, y, z]	D110[w, x, y, z]	D106[w, x, y, z]	D118[w, x, y, z]	D135[w, x, y, z]
				

We eliminated any digraphs that were isomorphic or reverses of each other and ended with the above 5 digraphs of interest from the original 16 possible orientations.

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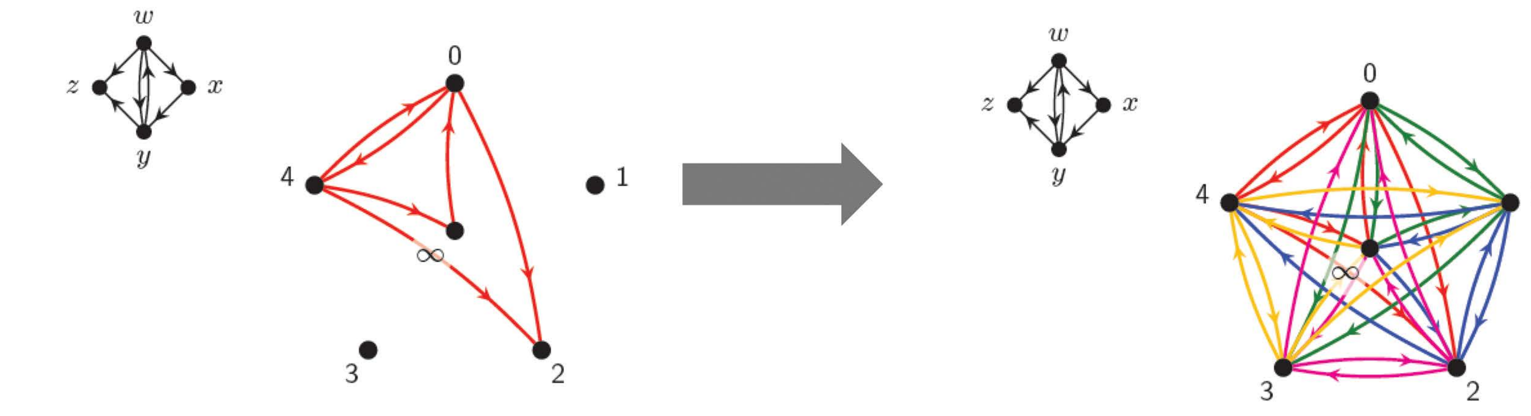
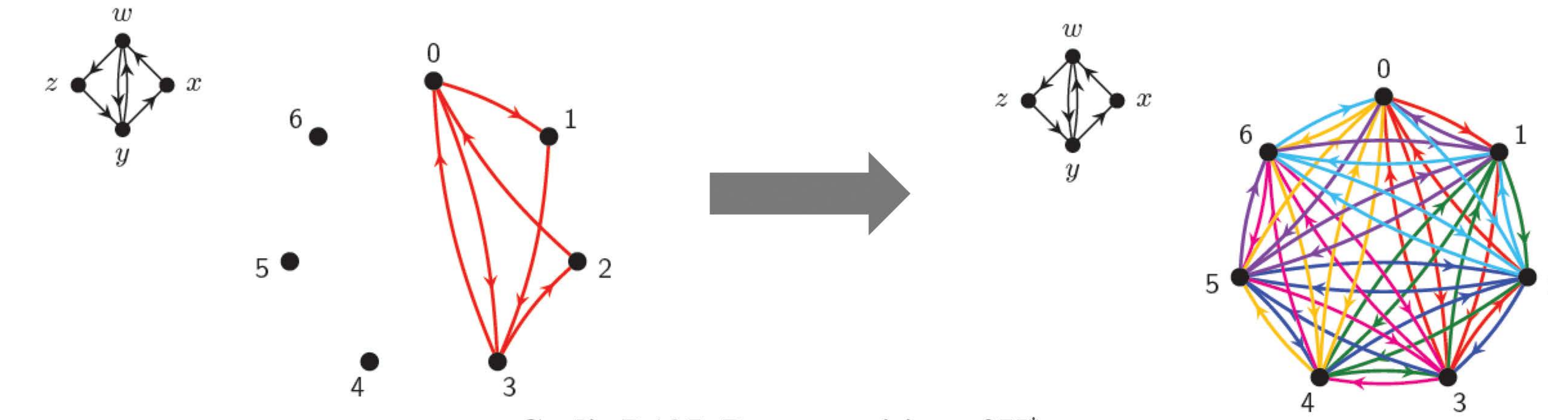
Examples of Our Decompositions



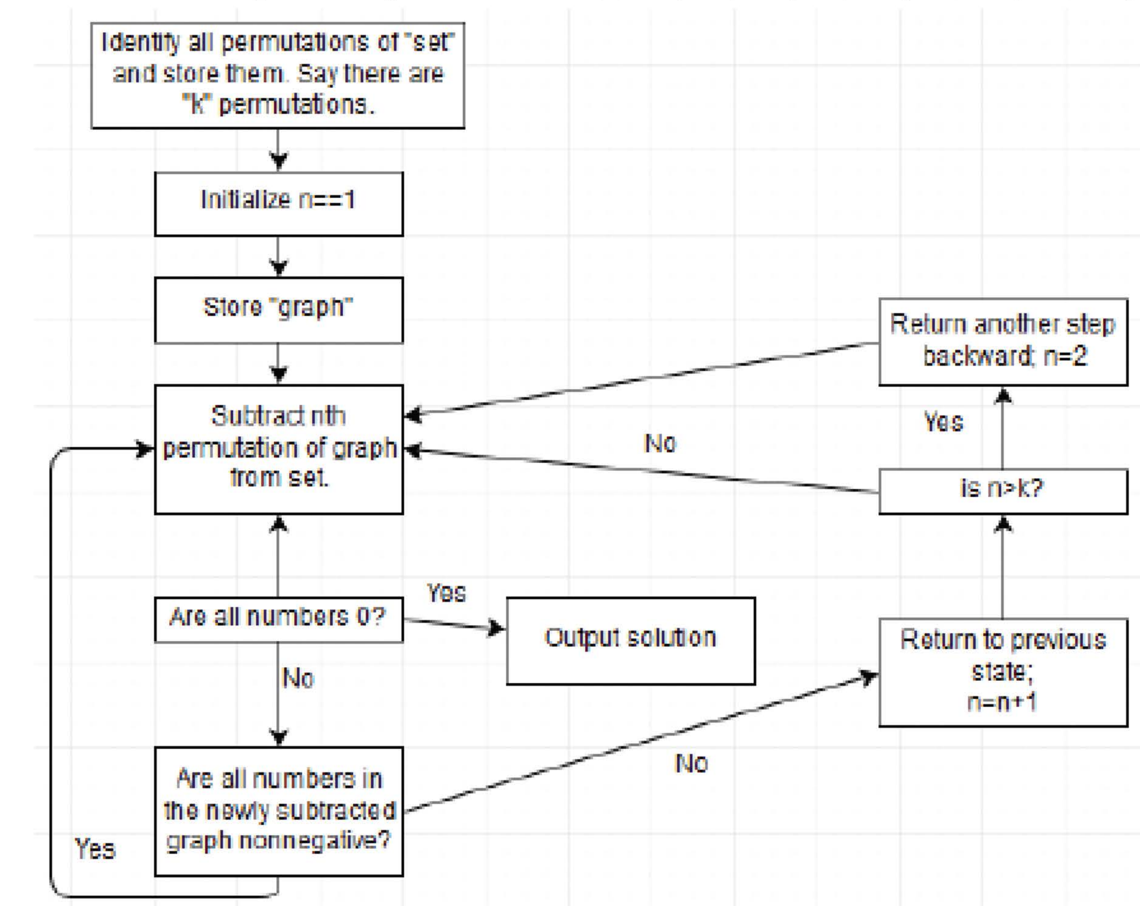
Cyclic D110-Decomposition of K^*_6



Cyclic D135- Decomposition of K^*_7

Computer Algorithm

Graphs and D-blocks become sets of (indegree, outdegree) ordered pairs.
For example, K^*_6 becomes {(5,5), (5,5), (5,5), (5,5), (5,5), (5,5)}.



Example computer-aided decomposition of K^*_6

$$\{(0,0),(0,0),(0,0),(0,0),(0,0),(0,0)\}$$
$$\{(0,0),(0,0),(1,1),(1,1),(2,2),(2,2)\}$$
$$\{(2,2),(1,1),(1,1),(1,1),(4,4),(3,3)\}$$
$$\{(3,3),(1,1),(3,3),(1,1),(5,5),(5,5)\}$$
$$\{(4,4),(3,3),(4,4),(3,3),(5,5),(5,5)\}$$
$$\{(5,5),(5,5),(5,5),(5,5),(5,5),(5,5)\}$$

General "Blow-up" Method

Uses the following theorems:

If n is odd, then a $\{K_3, K_5\}$-decomposition of K_n exists.
The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{u\times m}$ are
(i) $u \geq 3$,
(ii) $(u-1)m \equiv 0 \pmod 2$, and
(iii) $u(u-1)m^2 \equiv 0 \pmod 6$.
If $u \geq 3$ and $u \equiv 0 \pmod 3$, then there exists a K_3-decomposition of $K_{u\times 2,4}$.
Let m, r, s, t, u_1, u_2, …, u_m all be positive integers. If there exists a $\{K_r, K_s\}$-decomposition of $K_{u_1,u_2,\ldots,u_m}$, then there exists a $\{K_{r\times t}, K_{s\times t}\}$-decomposition of $K_{tu_1,tu_2,\ldots,tu_m}$. In particular, if there exists a $(K_{u_1,u_2,\ldots,u_m}, K_r)$-design, then there exists a $(K_{tu_1,tu_2,\ldots,tu_m}, K_{r\times t})$-design.

Conclusions and Future Work

-A majority of our cases yielded partial results, as some specific building blocks were not found and so the general blow-up method failed.
We want to:
-Optimize code to reduce runtime and memory usage.
-Investigate specific decomposition attempts to see why they fail.