Borel Complexity of the Isomorphism Relation for O-minimal Theories

BY

DAVENDER SINGH SAHOTA
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THESIS

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Defense Committee:

David E. Marker, Chair and Advisor
John T. Baldwin
Isaac Goldbring
Christian Rosendal
Michael C. Laskowski, University of Maryland

For Sheila, Darshan, and Ravi, with love.

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SUMMARY

In 1988, Mayer published a strong form of Vaught's Conjecture for o-minimal theories (1). She showed Vaught's Conjecture holds, and characterized the number of countable models of an o-minimal theory T if T has fewer than $2^{\aleph_0}$ countable models. Friedman and Stanley have shown in (2) that several elementary classes are Borel complete. This work addresses the class of countable models of an o-minimal theory T when T has $2^{\aleph_0}$ countable models, including conditions for when this class is Borel complete. The main result is as follows.

Theorem 1. *Let T be an o-minimal theory in a countable language having $2^{\aleph_0}$ countable models. Either*

i. For every finite set A, every $p(x) \in S_1(A)$ is simple, and isomorphism on the class of countable models of T is $\prod_3^0$ (and is, in fact, equivalence of countable sets of reals); or

ii. For some finite set A, some $p(x) \in S_1(A)$ is non-simple, and there is a finite set $B \supset A$ such that the class of countable models of T over B is Borel complete.

CHAPTER 1

INTRODUCTION

In this chapter, we build the context for this body of work. This is entirely expository and there are no new proofs.

Given a complete o-minimal theory T we assume the existence of a large saturated model called the monster model. Every model under consideration is an elementary submodel of the monster model, and every set is a subset of the universe of the monster model. Therefore we can refer to the closure of a set A with respect to $\emptyset$ without specifying a model. All languages are assumed to be countable.

We assume our language $\mathcal{L}$ contains a binary relation symbol $<$ that is a linear ordering in all considered structures, and interpret interval notation in the standard way. Unless otherwise stated, a theory T is assumed to be complete and in the language $\mathcal{L}$.

1.1 O-minimal Theories

Definition 2. (Pillay and Steinhorn (3)). A theory T is *o-minimal* if and only if for all models $\mathcal{M}$ of T, for all $\theta(x, y_1, \ldots, y_n)$ in $\mathcal{L}$ and for all n-tuples $\overline{m} \in M^n$, $\{x \in M : \mathcal{M} \models \theta(x, \overline{m})\}$ can be written as the union of finitely many points and intervals having endpoints in $M \cup \{\pm\infty\}$.

In (4), Knight, Pillay and Steinhorn show that we can replace "for all models $\mathcal{M}$ of T" by "there exists a model $\mathcal{M}$ of T such that".

If T is o-minimal and A and B are sets, the algebraic closure of A with respect to B is equal to the definable closure of A with respect to B.

The following theorems about o-minimal theories will be used throughout this work.

Theorem 3. *(Monotonicity Theorem for O-minimal Theories) (3). Let T be o-minimal and A be a set. Suppose that f is a unary function with domain (a, b), where possibly $a = -\infty$ and/or $b = \infty$, such that f is definable with parameters from A, and a, b are elements of $\mathrm{cl}_\emptyset(A) \cup \{\pm\infty\}$. Then there are $a_0 = a, a_1, \ldots, a_{n-1}, a_n = b \in \mathrm{cl}_\emptyset(A) \cup \{\pm\infty\}$ such that*

i. $a_0 < a_1 < \cdots < a_n$;

ii. f *is monotone or constant on each interval* (a_{i-1}, a_i), $i = 1, \ldots, n$; *and*

iii. if f is not constant on (a_{i-1}, a_i) then $f(a_{i-1}, a_i)$ is an interval and the restriction of f to (a_{i-1}, a_i) is an order preserving or reversing bijection onto $f(a_{i-1}, a_i)$.

As an application of Theorem 3, Pillay and Steinhorn showed the following result. We will rely on Theorem 4 in the main argument of Section 3.2.

Theorem 4. *(Exchange Principle for O-minimal Theories) (3). Let T be o-minimal with $\mathcal{M} \models T$ and $b, c, a_1, \ldots, a_n \in M$. If $b \in \mathrm{cl}_\emptyset(\{c, a_1, \ldots, a_n\})$ and $b \notin \mathrm{cl}_\emptyset(\{a_1, \ldots, a_n\})$ then $c \in \mathrm{cl}_\emptyset(\{b, a_1, \ldots, a_n\})$.*

1.1.1 Cell Decomposition

In this section, we generalize o-minimality to M^n. Cells are definable sets of a simple form, analogous to points and intervals in the 1-dimensional case. We show that a definable set in M^n can be split into finitely many cells, and for a function f on M^n, this can be done in such a way that f is continuous on each cell. This closely follows Chapter 3 in (5). For each definable set $X \subset M^n$ let

$$C(X) = \{f : X \to M : f \text{ is definable and continuous}\},$$

$$C_\infty(X) = C(X) \cup \{-\infty, +\infty\},$$

where we regard $-\infty$ and $+\infty$ as constant functions on X. For $f, g \in C_\infty(X)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case let

$$(f, g)_X = \{(x, y) \in X \times M : f(x) < y < g(x)\}.$$

Note that $(f, g)_X$ is a definable subset of M^{n+1}. We write (f, g) instead of $(f, g)_X$ when X is clear from the context.

Definition 5. Let $(i_1, \ldots, i_n)$ be a sequence of 0's and 1's of length n. An $(i_1, \ldots, i_n)$-*cell* is a definable subset of M^n defined inductively as follows:

(i) a (0)-cell is a one-element set $\{a\} \subseteq M$ (a "point"), a (1)-cell is an interval $(a, b) \subseteq M$, where both a and b are definable;

(ii) suppose $(i_1, \ldots, i_n)$-cells are already defined; an $(i_1, \ldots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a definable function $f \in C(X)$, where X is an $(i_1, \ldots, i_n)$-cell; an $(i_1, \ldots, i_n, 1)$-cell is a set $(f, g)_X$ where X is an $(i_1, \ldots, i_n)$-cell and $f, g \in C_\infty(X), f < g$.

For definable a and b, a $(0,0)$-cell is a "point" $\{(a,b)\} \subseteq M^2$, a $(0,1)$-cell is an "interval" on a vertical line $\{a\} \times M$, and a $(1,0)$-cell is the graph of a continuous definable function on a definable interval. A box in M^n is a $(1,\ldots,1)$-cell.

Definition 6. A *decomposition* of M^n is a special kind of partition of M^n into finitely many cells. This is defined inductively:

(i) a decomposition of $M^1 = M$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$$

where $a_1 < \ldots < a_k$ are definable points in M;

(ii) a decomposition of M^{n+1} is a finite partition of M^{n+1} into cells A such that the set of projections $\pi(A)$ is a decomposition of M^n.

A decomposition $\mathcal{D}$ of M^n is said to *partition* a set $A \subseteq M^n$ if each cell in $\mathcal{D}$ is either contained in A or disjoint from A. In other words, A is a union of cells in $\mathcal{D}$. This brings us to the main result of the section.

Theorem 7. *(Cell Decomposition Theorem) (5).*

(I) Given any definable sets $A_1, \ldots, A_k \subseteq M^n$ there is a decomposition of M^n partitioning each of the $A_1, \ldots, A_k$.

(II) For each definable function $f : A \to M, A \subseteq M^n$, there is a decomposition $\mathcal{D}$ of M^n such that the restriction $f \restriction B : B \to M$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.

The strength of this theorem is once we show a set is definable, we have an idea of what form it can take. This theorem will allow us to very specifically identify a definable set in Section 3.2.

1.2 Prime Models and Omitting Types

In several arguments in this work, we expand a model by a finite set, and use *the* prime model over the set. Naturally this depends on the existence and uniqueness of prime models. A useful feature of prime models is that realizing one *form* of type will not force a model to realize a different *form* of type, with the definition of *form* forthcoming. This is the crux of the final argument in Section 3.2.

Theorem 8. *(Existence and Uniqueness of Prime Models) (3). Let T be an o-minimal theory and let A be a set. There is a model of T, say $\mathcal{M}$, such that M contains A and $\mathcal{M}$ is prime over A. Furthermore, $\mathcal{M}$ is unique up to isomorphism over A.*

Theorem 8 allows us to refer to *the* prime model of an o-minimal theory T over a set A. We let $\mathrm{Pr}_T(A)$ denote the prime model of T over A. We omit T if the theory is clear from context. For a specific type $p(x) \in S_1(A)$ and a specific model $\mathcal{M} \supset A$, we let $p(\mathcal{M})$ denote the set of realizations of $p(x)$ in $\mathcal{M}$. Note that in an o-minimal setting, the realizations of a type form a convex set.

In an o-minimal setting, the isolated types over a set A are either the type of a point or an interval. Specifically, they are of the form $p(x) = \{\varphi(x) : x = a \vdash \varphi(x)\}$ for some $a \in \mathrm{cl}(A)$ or $p(x) = \{\varphi(x) : a < x < b \vdash \varphi(x)\}$ for $a < b \in \mathrm{cl}(A)$ and $(a, b) \cap \mathrm{cl}(A) = \emptyset$. We will refer to the latter as "atomic intervals." Marker (6) classifies the nonisolated elements of $S_1(A)$ as "cuts" and "noncuts".

Definition 9. "Cuts" and "noncuts" are defined as follows:

1. A nonisolated type $p(x) \in S_1(A)$ is a *cut* if and only if there exist $a_i, b_i \in \mathrm{cl}_\emptyset(A)(i \in \omega)$ such that for all $i \in \omega$, $a_i < a_{i+1}$ and $b_{i+1} < b_i$ and $p(x)$ is determined by $\{a_i < x < b_i : i \in \omega\}$.

2. A nonisolated type $p(x) \in S_1(A)$ is a *noncut* if and only if there exist $a_i \in \mathrm{cl}_\emptyset(A)(i \in \omega)$ and $b \in \mathrm{cl}_\emptyset(A) \cup \{\pm\infty\}$ such that either:

   a) For all $i \in \omega$, $a_i < a_{i+1}$ and $p(x)$ is determined by $\{a_i < x < b : i \in \omega\}$, or

   b) for all $i \in \omega$, $a_i > a_{i+1}$ and $p(x)$ is determined by $\{b < x < a_i : i \in \omega\}$.

If $p(x) \in S_1(A)$ is nonisolated then $p(x)$ has a unique extension in $S_1(M)$ whenever $\mathcal{M}$ is a model containing A and omitting $p(x)$. If $p(x)$ is a cut (noncut), so is its extension. Note that in a model containing a realization of $p(x)$, extensions of $p(x)$ are not unique, and an extension of a cut could be a noncut (or vice versa). Marker's omitting types results say that realizing a cut does not force us to realize a noncut (and vice versa).

"...but it takes a model theorist to omit one."

Theorem 10. *(Omitting Types) (6). Let $\mathcal{M} \models T$. Let $p(x) \in S_1(M)$ be a nonisolated cut and let $q(x) \in S_1(M)$ be a nonisolated noncut. Suppose a is a realization of $p(x)$ and b is a realization of $q(x)$. Then $p(x)$ is omitted in $\mathrm{Pr}(M \cup b)$ and $q(x)$ is omitted in $\mathrm{Pr}(M \cup a)$.*

1.3 Vaught's Conjecture for O-minimal Theories

In (1), Mayer shows the following strong form of Vaught's Conjecture for o-minimal theories.

Theorem 11. *(Vaught's Conjecture for o-Minimal Theories) (1). Let T be an o-minimal theory in a countable language. Either T has $2^{\aleph_0}$ countable models or T has exactly $6^a 3^b$ countable models, where a and b are non-negative integers. Moreover, for all $a, b \in \omega$ there exists an o-minimal theory T such that T has exactly $6^a 3^b$ countable models.*

Mayer's result proves Vaught's Conjecture for o-minimal theories and gives a further classification of the number of countable models in the case there are fewer than $2^{\aleph_0}$ countable models. This work will provide some classification of the countable models in the case there are $2^{\aleph_0}$ countable models. Mayer defined *independence* of types and *simple* types and proved several results from both. We define *independence* and *simple* and slightly generalize her results in Section 3.1.

CHAPTER 2

DESCRIPTIVE SET THEORY OF COUNTABLE MODELS OF A THEORY

This chapter is a brief survey of Descriptive Set Theory leading to the definition of Borel completeness. For more details, refer to (7). This is followed by a few key examples and the approach of reducing from one Borel complete class to another. Lastly, examples of reducing to the class of countable models of two specific o-minimal theories illustrates the technique for the result in Section 3.2.

2.1 Borel Completeness

It will be useful to limit our discussion to relational languages. Any language can be regarded as a relational language, by interpreting constant symbols and function symbols as relation symbols. For any language $\mathcal{L}$, we can instead consider a relational language $\mathcal{L}^*$ and for any $\mathcal{L}$-structure $\mathcal{M}$ we can instead consider $\mathcal{M}^*$, the corresponding $\mathcal{L}^*$ structure. First, any relation symbol in $\mathcal{L}$ is in $\mathcal{L}^*$ and for all n, for any n-ary relation symbol $R \in \mathcal{L}$ and any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models R(a_1, \ldots, a_n) \iff \mathcal{M}^* \models R(a_1, \ldots, a_n)$. For each constant symbol $c \in \mathcal{L}$, we have a unary relation $R_c(x) \in \mathcal{L}^*$ such that $\mathcal{M} \models c = a$ if and only if $\mathcal{M}^* \models R_c(a)$. Lastly, for each n-ary function $f \in \mathcal{L}$, we have an $(n+1)$-ary relation symbol $R_f \in \mathcal{L}^*$ such that $\mathcal{M} \models f(a_1, \ldots, a_n) = b$ if and only if $\mathcal{M}^* \models R_f(a_1, \ldots, a_n, b)$.

Having defined this interpretation, we will freely treat all languages as relational without further explanation.

Given a relational language $\mathcal{L} = \{R_i\}_{i \in I}$, where I is a countable set of indices and each R_i is an n_i-ary relation symbol, we let $\mathrm{Mod}(\mathcal{L})$ denote the space of all countable $\mathcal{L}$-structures with universe ω. Each element of $\mathrm{Mod}(\mathcal{L})$ can be viewed as an element of the product space $X_{\mathcal{L}} = \prod_{i \in I} 2^{\omega^{n_i}}$. $\mathrm{Mod}(\mathcal{L})$ is thus a compact Polish space, homeomorphic to the Cantor space, with the product topology on $X_{\mathcal{L}}$. In particular, for each $x \in X_{\mathcal{L}}$, $\mathcal{M}_x \in \mathrm{Mod}(\mathcal{L})$ denotes the countable $\mathcal{L}$-structure coded by x, with $R_i^{\mathcal{M}_x}(k_1, \ldots, k_{n_i}) \iff x_i(k_1, \ldots, k_{n_i}) = 1$. If φ is an $\mathcal{L}$-sentence, $\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\mathcal{L})$ is the class of countable $\mathcal{L}$-structures in which φ holds.

Definition 12. If G is a group and X is a set, an *action* of G on X is a map $a : G \times X \to X$ such that for all $x \in X$ and $g, h \in G$, $a(1_G, x) = x$ and $a(g, a(h, x)) = a(gh, x)$.

Definition 13. A *G-space* is a pair (X, a), where a is an action of the group G on the space X. If a is a Borel function, we say X is a *Borel G-space*.

Definition 14. The *logic action* of S_∞ on $\mathrm{Mod}(\mathcal{L})$ is defined by letting $g \cdot \mathcal{M} = \mathcal{N}$ iff $R_i^{\mathcal{N}}(k_1, \ldots, k_{n_i}) \iff R_i^{\mathcal{M}}(g^{-1}(k_1), \ldots, g^{-1}(k_{n_i}))$ for all $i \in I$ and all $(k_1, \ldots, k_{n_i}) \in \omega^{n_i}$.

Thus $g \cdot \mathcal{M} = \mathcal{N}$ iff g is an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$. The action of S_∞ on $\mathrm{Mod}(\mathcal{L})$ is continuous, and the orbit equivalence relation is the isomorphism relation on $\mathrm{Mod}(\mathcal{L})$, denoted $\cong_{\mathcal{L}}$.

Theorem 15. *(Dougherty (8)) Let $\mathcal{L}$ be a relational language containing symbols of unbounded arity. Let $\mathcal{L}'$ be an arbitrary relational language. There is a Borel embedding of $X_{\mathcal{L}'}$ into $X_{\mathcal{L}}$.*

Proof. Let $\mathcal{L} = \{R_i\}$ with each relation symbol R_i having arity n_i. By assumption, $\{n_i\}$ is unbounded in ω. Let $\mathcal{L}' = \{R_i'\}$ with each relation symbol R_i' having arity n_i'. Let $\varphi : \omega \to \omega$ be an injection such that $\varphi(i) = j \iff n_i' \leq n_j$. We now define $\pi : X_{\mathcal{L}'} \to X_{\mathcal{L}}$. Let $x \in X_{\mathcal{L}'}$ code $\mathcal{M}_x$. Let $\mathcal{M}$ be the $\mathcal{L}$-structure on ω such that for $j \notin \varphi(\omega)$, $R_j^{\mathcal{M}} = \omega^{n_j}$ and for $j = \varphi(i)$,

$$R_j^{\mathcal{M}}(a_1, \ldots, a_{n_i'}, \ldots, a_{n_j}) \iff R_i'^{\mathcal{M}_x}(a_1, \ldots, a_{n_i'})$$

Let $\pi(x) = y$ where y is the unique element of $X_{\mathcal{L}}$ with $\mathcal{M}_y = \mathcal{M}$. Clearly π is a Borel embedding of $X_{\mathcal{L}'}$ into $X_{\mathcal{L}}$. $\square$

Definition 16. Let $\mathcal{L}$ be a relational language. An *invariant Borel class* of countable $\mathcal{L}$-structures is an S_∞-invariant Borel subset of $\mathrm{Mod}(\mathcal{L})$.

Definition 17. Suppose E is an equivalence relation on a set X and F is an equivalence relation on a set Y. A function $f : X \to Y$ is called a *Borel reduction* from E to F if $x_1 E x_2 \iff f(x_1) F f(x_2)$ for all $x_1, x_2 \in X$ and f is Borel. We say E is Borel reducible to F and write $E \leq_B F$ if there is a Borel reduction from E to F.

Definition 18. Let $\mathcal{C}$ be an invariant Borel class. We say that $\mathcal{C}$ (or $\cong\restriction \mathcal{C}$) is *Borel complete* if $\cong\restriction \mathcal{C}$ is Borel bireducible with the universal S_∞-orbit equivalence relation.

2.2 Fundamental Examples

This section contains a collection of theories for which the class of countable models is Borel complete. This is proved for the class of countable graphs by specifically illustrating the universality of the class. For the examples that follow, we construct a Borel reduction from one to the next. We will need the following Lemma from (7).

Lemma 19. *Let G be a Polish group and X and Y Borel G-spaces. Let E_G^X and E_G^Y denote the orbit equivalence relation of G on X and Y respectively. If $f : X \to Y$ is a Borel G-embedding from X to Y then f is a Borel reduction from E_G^X to E_G^Y.*

2.2.1 Countable Graphs

Theorem 20. *The class of all countable graphs is Borel complete.*

Proof. This proof of this well-known result is given in (7). Let $\mathcal{L}$ be the relational language $\{R_n\}_{n\geq 2}$ with each R_n an n-ary relation symbol. By Theorem 15, $\mathrm{Mod}(\mathcal{L})$ is a universal Borel S_∞-space. By Lemma 19, $\mathrm{Mod}(\mathcal{L})$ is Borel complete. Let $\mathcal{L}^\Gamma = \{R\}$ be the language comprising one binary relation symbol and let γ be the $\mathcal{L}^\Gamma$-sentence $\forall x \forall y[\neg R(x,x) \wedge (R(x,y) \leftrightarrow R(y,x))]$. Every model $\mathcal{M} \in \mathrm{Mod}(\gamma)$ is essentially a countable graph with vertex set $V = \omega$ and edge set $E = \{\{a,b\} : \mathcal{M} \models R(a,b)\}$. Conversely, every countable graph is represented by an element of $\mathrm{Mod}(\gamma)$. We will refer to the elements of

Mod(γ) as *countable graphs* and the isomorphism relation $\cong_\gamma$ as the *graph isomorphism*. It is now sufficient to construct a Borel reduction from Mod$(\mathcal{L})$ to the class of all countable graphs Mod(γ). To each $\mathcal{L}$-structure $\mathcal{M} \in$ Mod$(\mathcal{L})$ we will associate a countable graph $\Gamma(\mathcal{M})$.

For $n \geq 1$ an n-tag is a graph T_n with the vertex set

$$\{a_1, \ldots, a_n\} \cup \{b_{1,1}, b_{2,1}, b_{2,2}, \ldots, b_{n,n}, c, d_1, d_2, d_3, f\} = A \cup B,$$

where the two displayed sets A and B are disjoint and the demonstrated elements of B are distinct, with the following set of edges:

$$\begin{aligned}
&\{a_1 b_{1,1}, b_{1,1} c, \\
&\ a_2 b_{2,1}, b_{2,1} b_{2,2}, b_{2,2} c, \\
&\ \cdots\cdots \\
&\ a_n b_{n,1}, b_{n,1} b_{n,2}, \ldots, b_{n,n-1} b_{n,n}, b_{n,n} c, \\
&\ c d_1, d_1 d_2, d_2 d_3, d_3 d_1, f d_2\}.
\end{aligned}$$

Figure 1 illustrates an n-tag. It is important that the n-tags have no symmetry; each vertex in an n-tag is uniquely determined by its properties. To be specific, note that in such a graph f has degree 1, d_1, d_2, d_3 form a 3-cycle, c has degree $n+1$, and each other vertex in B has degree 2. Also, d_1 is adjacent to c, d_2 is adjacent to f, and d_3 is adjacent to neither c nor f. Such an n-tag will be used to code the tuple $(a_1, \ldots, a_n)$, and we denote

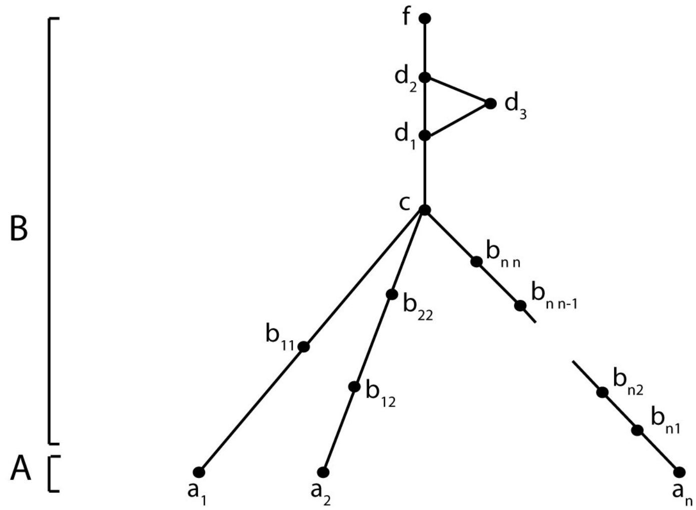

Figure 1. An n-tag $T_n(a_1, \ldots, a_n)$.

such an n-tag $T_n(a_1, \ldots, a_n)$. We call the vertex c the center of the n-tag. If $a_1, \ldots, a_n$ are (not necessarily distinct) vertices in some graph Γ_0 then by adding $T_n(a_1, \ldots, a_n)$ to to Γ_0 we mean to add fresh elements from B and the edges of T_n to form a new graph.

Given an $\mathcal{L}$-structure $\mathcal{M}$ we first let Γ_0 be the graph on $M = \omega$ with a 1-tag added for each $a \in M$. Then for each R_n, $n \geq 2$, and tuple $(a_1, \ldots, a_n)$, add an n-tag $T_n(a_1, \ldots, a_n)$ iff $\mathcal{M} \models R_n(a_1, \ldots, a_n)$. The resulting graph is denoted $\Gamma(\mathcal{M})$.

Note that in $\Gamma(\mathcal{M})$ each 3-cycle is created only by the addition of an n-tag. From this, the centers of n-tags can be identified by their adjacency to the 3-cycles. Let $C(\mathcal{M})$ denote the set of centers of all n-tags for $n \geq 1$. Let $\exists^{\neq} y_1 \ldots y_k \backslash x_1 \ldots x_l\ \varphi$ abbreviate the formula

$$\exists y_1, \ldots \exists y_k \left[\bigwedge_{\substack{1 \leq i < i' \leq k \\ 1 \leq j \leq l}} (\neg y_i = y_{i'} \wedge \neg y_i = x_j) \wedge \varphi \right].$$

Let $\theta(x)$ be the $\mathcal{L}^{\Gamma}$-formula

$$\exists^{\neq} y_1, y_2, y_3, z \backslash x\ [R(x, y_1), R(x, z), R(y_1, y_2), R(y_2, y_3), R(y_3, y_1)].$$

Then $\Gamma(\mathcal{M}) \models \theta(c)$ iff $c \in C(\mathcal{M})$.

For $i \geq 1$, let $\eta_i(x)$ be the $\mathcal{L}^{\Gamma}$-formula

$$\exists^{\neq} y_1 \ldots y_i \backslash x \bigwedge_{1 \leq j \leq i} R(x, y_j) \wedge \neg \exists^{\neq} y_1 \ldots y_{i+1} \backslash x \bigwedge_{1 \leq j \leq i+1} R(x, y_j).$$

Then $\Gamma(\mathcal{M}) \models \eta_i(c)$ iff c has degree i. Also, for $l \geq 1$ let $\lambda_l(x, y)$ be the $\mathcal{L}^{\Gamma}$-formula

$$\exists^{\neq} z_1 \ldots z_l \backslash x, y \left[\neg x = y \wedge R(x, z_1) \wedge R(z_l, y) \wedge \bigwedge_{1 \leq j \leq l} \eta_2(z_j) \wedge \bigwedge_{1 \leq j < l} R(z_j, z_{j+1}) \right].$$

Then $\Gamma(\mathcal{M}) \models \lambda_l(a, c)$ iff there is a path between a and c with l elements of degree 2 in between.

For each $c \in C(\mathcal{M})$, if the degree of c is 2 then the tag it is a center of is a 1-tag. Therefore a vertex a of $\Gamma(\mathcal{M})$ is in M iff there is a path of length 2 from a to a center of a 1-tag. Let $\delta(x)$ be the $\mathcal{L}^\Gamma$-formula

$$\exists y(\lambda_1(x, y) \wedge \theta(y) \wedge \eta_2(y)).$$

Then $\Gamma(\mathcal{M}) \models \delta(a)$ iff $a \in M$.

If $c \in C(\mathcal{M})$ then the degree of c is one more than the arity of the tuple it codes. Thus, if c has degree $n + 1$, then c codes a unique n-tuple $(a_1, \ldots, a_n)$. For $n \geq 2$, let $\rho_n(x_1, \ldots, x_n)$ be the $\mathcal{L}^\Gamma$-formula

$$\exists y \bigwedge_{1 \leq i \leq n} \left[\delta(x_i) \wedge \theta(y) \wedge \eta_{n+1}(y) \wedge \lambda_i(x_i, y)\right].$$

Then $\Gamma(\mathcal{M}) \models \rho_n(a_1, \ldots, a_n)$ iff there is $c \in C(\mathcal{M})$ and c codes the tuple $(a_1, \ldots, a_n)$ in the sense that the unique n-tag $T_n(a_1, \ldots, a_n)$ has center c. Clearly $\Gamma(\mathcal{M}) \models \rho_n(a_1, \ldots, a_n)$ iff $\mathcal{M} \models R_n(a_1, \ldots, a_n)$.

We can now show that for $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{L})$, $\mathcal{M} \cong \mathcal{N}$ iff $\Gamma(\mathcal{M}) \cong \Gamma(\mathcal{N})$. Clearly Γ is invariant. Suppose $\pi : \Gamma(\mathcal{M}) \cong \Gamma(\mathcal{N})$. For all $a \in \Gamma(\mathcal{M})$, $a \in M$ iff $\Gamma(\mathcal{M}) \models \delta(a)$ iff $\Gamma(\mathcal{N}) \models \delta(\pi(a))$ iff $\pi(a) \in \mathcal{N}$. Therefore π restricted to M is a bijection between M and N. For any $n \geq 2$, $a_1, \ldots, a_n \in M$, $\mathcal{M} \models R_n(a_1, \ldots, a_n)$ iff $\Gamma(\mathcal{M}) \models \rho_n(a_1, \ldots, a_n)$ iff $\Gamma(\mathcal{N}) \models \rho_n(\pi(a_1), \ldots, \pi(a_n))$ iff $\mathcal{N} \models R_n(\pi(a_1), \ldots, \pi(a_n))$. Thus $\pi \restriction M$ is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. With any reasonable coding of $\Gamma(\mathcal{M})$ by a genuine element of $\mathrm{Mod}(\gamma)$, Γ is clearly a Borel map. □

2.2.2 Countable Trees

In descriptive set theory a tree T on ω is a subset of $\omega^{<\omega}$ closed under initial segments. That is, if $s \subseteq t$ and $t \in T$ then $s \in T$. We denote by $\mathrm{lh}(s)$ the length of s, regarding $s \in \omega^{<\omega}$ as a finite sequence. By definition, every tree on ω contains the empty sequence $\emptyset$, with $\mathrm{lh}(\emptyset) = 0$. For trees S, T on ω, an isomorphism is a bijection $\pi : S \to T$ preserving initial segments, that is, for all $s_1, s_2 \in S$, $s_1 \subseteq s_2$ iff $\pi(s_1) \subseteq \pi(s_2)$. For any $s \in S$, $\mathrm{lh}(\pi(s)) = \mathrm{lh}(s)$. In particular, $\pi(\emptyset) = \emptyset$. Every tree on ω is an element of $2^{(\omega^{<\omega})}$. Let Tr be the set of all trees on ω. Then Tr is a closed subset of the Polish space $2^{(\omega^{<\omega})}$ and is itself Polish.

In graph theory, a tree is an acyclic connected graph. Here, a tree on ω would be called a rooted tree. Conversely, a graph theoretic rooted tree can be coded by a tree on ω. By this correspondence the class of all trees on ω becomes an invariant Borel class.

Theorem 21. *(Friedman-Stanley (2)) The class of all countable trees on ω is Borel complete.*

Proof. We will define a Borel reduction from the class of all countable graphs to the class of countable trees on ω. To each countable graph Γ we will associate $T(\Gamma)$, a tree on ω.

Fix a countable graph Γ with underlying set ω and edge relation R. Let T_0 be the full tree of nonrepeating finite sequences in $\omega^{<\omega}$. $T(\Gamma)$ will be obtained by adding at most one new, terminal, immediate successor to each node in T_0. For $m, n \in \omega$, let $\langle m, n\rangle = 2^m 3^n$. Then for all $m, n > 0$, if $s = (x_1, \ldots, x_{\langle m,n\rangle}) \in T_0$, then add $s^\frown x_1$ to T_0 iff $R(x_m, x_n)$. The resulting tree is $T(\Gamma)$.

The map $T : \Gamma \to T(\Gamma)$ is continuous. We show that it is a reduction. First, let $\pi : \Gamma \cong \Gamma'$. Let $\pi^* : \omega^{<\omega} \to \omega^{<\omega}$ be the automorphism induced by π:

$$\pi^*(x_1, \ldots, x_k) = (\pi(x_1), \ldots, \pi(x_k)).$$

Then $\pi^*(T_0) = T_0$. Now

$$\begin{aligned} & (x_1, \ldots, x_k, x_1) \in T(\Gamma) \\ \Longleftrightarrow & \exists m, n > 0\ (k = \langle m, n\rangle \wedge R^\Gamma(x_m, x_n)) \\ \Longleftrightarrow & \exists m, n > 0\ (k = \langle m, n\rangle \wedge R^{\Gamma'}(\pi(x_m), \pi(x_n))) \\ \Longleftrightarrow & (\pi(x_1), \ldots, \pi(x_k), \pi(x_1)) \in T(\Gamma'). \end{aligned}$$

Thus $\pi^*(T(\Gamma)) = T(\Gamma')$ and $T(\Gamma) \cong T(\Gamma')$.

Conversely, suppose $\sigma : T(\Gamma) \cong T(\Gamma')$. By a back-and-forth argument we find two permutations π and π' of ω such that for all $l \in \omega$,

$$\sigma(\pi(0), \ldots, \pi(l)) = (\pi'(0), \ldots, \pi'(l)).$$

This is done by induction on l. For the base case, let $\pi(0) = 0$, $s_0 = (\pi(0))$, and define $\pi'(0)$ so that $\sigma(s_0) = (\pi'(0))$. Next let $\pi'(1)$ be the least element of $\omega - \{\pi'(0)\}$, and $t_1 = (\pi'(0), \pi'(1))$. Define $\pi(1)$ so that $\sigma^{-1}(t_1) = (\pi(0), \pi(1))$. Since t_1 is not a terminal node of $T(\Gamma')$, neither is $\sigma^{-1}(t_1)$, and hence, $\pi(1) \neq \pi(0)$. In general, suppose distinct $\pi(0), \ldots, \pi(l)$ and distinct $\pi'(0), \ldots, \pi'(l)$ have been defined. Suppose l is odd. Let $\pi(l+1)$ be the least element of $\omega - \{\pi(0), \ldots, \pi(l)\}$ and $s_{l+1} = (\pi(0), \ldots, \pi(l+1))$. Then $\sigma(s_{l+1}) = (\pi'(0), \ldots, \pi'(l), y)$ for some $y \notin \{\pi'(0), \ldots, \pi'(l)\}$ since s_{l+1} and $\sigma(s_{l+1})$ are not terminal nodes. Define $\pi'(l+1) = y$ and continue the construction. If l is even, then the definition is similar to the case $l = 0$.

Now we claim that $\pi' \circ \pi^{-1}$ is an isomorphism between Γ and Γ'. To see this suppose $R^\Gamma(a, b)$. Let $m = \pi^{-1}(a)$, $n = \pi^{-1}(b)$ and $k = \langle m, n \rangle$. Then the node $(\pi(0), \ldots, \pi(k-1), \pi(0))$ is a terminal node of $T(\Gamma)$. It follows that $\sigma(\pi(0), \ldots, \pi(k-1), \pi(0))$ is a terminal note of $T(\Gamma')$. Hence $\sigma(\pi(0), \ldots, \pi(k-1), \pi(0)) = (\pi'(0), \ldots, \pi'(k-1), \pi'(0)) \in T(\Gamma')$. This implies that $R^{\Gamma'}(\pi'(m), \pi'(n))$ or $R^{\Gamma'}(\pi' \circ \pi^{-1}(a), \pi' \circ \pi^{-1}(b))$. By symmetry, we have $R^\Gamma(a, b)$ iff $R^{\Gamma'}(\pi' \circ \pi^{-1}(a), \pi' \circ \pi^{-1}(b))$ for any $a, b \in \omega$. $\square$

2.2.3 Countable Linear Orderings

Our next objective is to prove the Borel completeness of the class of countable linear orderings. Let $\mathcal{L} = \{<\}$ be the language with one binary relation symbol, and let ρ be the conjunction of the axioms of linear orders:

$$\forall x \ \neg(x < x)$$

$$\forall x \forall y \ (x < y \ \vee \ x = y \ \vee \ y < x)$$

$$\forall x \forall y \forall z \ [(x < y \ \wedge \ y < z) \rightarrow x < z].$$

Every element of $\mathrm{Mod}(\rho)$ is a linear ordering of ω.

The proof that the class of linear orderings is Borel complete will be similar to that of Theorem 21 in that a base linear ordering will be defined and the orders coding other structures will be obtained by a uniform operation on the base linear order. We will be working with dense linear orders without endpoints. Since there is only one such order up to isomorphism, we will use the natural linear order $(\mathbb{Q}, <)$ on the set of rational numbers. We begin with a definition.

Definition 22. If $\mathcal{P} = \{P_m : m \in \omega\}$ is a partition of $\mathbb{Q}$, we say that $\mathcal{P}$ is *mutually dense* if for any $p < q \in \mathbb{Q}$ and any $m \in \omega$, there is $r \in P_m$ with $p < r < q$.

In particular, if $\mathcal{P}$ is a mutually dense partition of $\mathbb{Q}$, then every $(P_m, <)$ is a dense linear order without endpoints.

Lemma 23. *There exists a mutually dense partition of* $(\mathbb{Q}, <)$.

Proof. Let $q_0, q_1, \ldots$ be an enumeration of $\mathbb{Q}$. We define a function $f : \mathbb{Q} \to \omega$ so that the partition $\mathcal{P} = \{f^{-1}(m) : m \in \omega\}$ is mutually dense. For any $i, j, m \in \omega$, let $\langle i, j, m \rangle = 2^i 3^j 5^m$. Then $\langle \cdot, \cdot, \cdot \rangle$ is an injection from ω^3 into ω. By induction on $n \in \omega$ we define a finite set $D_n \subseteq \mathbb{Q}$ and $f(q)$ for each $q \in D_n$ so that the following properties hold:

(i) $q_n \in D_n$ and $D_n \subseteq D_{n+1}$ for all $n \in \omega$;

(ii) if $n = \langle i, j, m \rangle$ for some $i, j, m \in \omega$ with $i \neq j$, then there is $r \in D_n$ with $f(r) = m$ and either $q_i < r < q_j$ or $q_j < r < q_i$.

For the base step of the induction let $D_0 = \{q_0\}$ and $f(q_0) = 0$. For the inductive step, let $n > 0$ and assume D_{n-1} has been defined and $f(q)$ has been defined for all $q \in D_{n-1}$.

If $n = \langle i, j, m \rangle$ for $i, j, m \in \omega$ and $i \neq j$, we have either $q_i < q_j$ or $q_j < q_i$. In either case, by the density of $\mathbb{Q}$ there is $r \notin D_{n-1}$ such that $q_i < r < q_j$ or $q_j < r < q_i$. Let k be least so that q_k has this property. Let $D_n = D_{n-1} \cup \{q_n, q_k\}$. Let $f(q_k) = m$. If $q_n \in D_{n-1} \cup \{q_k\}$ then $f(q_n)$ is already defined, otherwise let $f(q_n) = 0$. We have that (i) and (ii) are satisfied in this case.

If $n \notin \{\langle i, j, m \rangle : i \neq j, m \in \omega\}$, then let $D_n = D_{n-1} \cup \{q_n\}$. If $q_n \in D_{n-1}$ then $f(q_n)$ is already defined; otherwise let $f(q_n) = 0$. This completes the inductive definition.

Now by (i) we have that $\bigcup_n D_n = \mathbb{Q}$, hence f is defined on all of $\mathbb{Q}$. To see that f has the required property, let $p < q \in \mathbb{Q}$ and $m \in \omega$. For some unique i, j we have $p = q_i$ and $q = q_j$. Also, $i \neq j$. Then for $n = \langle i, j, m \rangle$, by (ii) we have some $r \in D_n$ with $f(r) = m$ with $p = q_i < r < q_j = q$, as required. $\square$

A standard back-and-forth argument shows that any two mutually dense partitions of $\mathbb{Q}$ are isomorphic, that is, if $\mathcal{P}_1 = \{P_{m,1} : m \in \omega\}$ and $\mathcal{P}_2 = \{P_{m,2} : m \in \omega\}$ are mutually dense partitions of $\mathbb{Q}$, then there is an order-preserving bijection $\pi : \mathbb{Q} \to \mathbb{Q}$ such that for any $m \in \omega$ and $q \in \mathbb{Q}$, $q \in P_{m,1}$ iff $\pi(q) \in P_{m,2}$.

Fix a mutually dense partition $\mathcal{P} = \{P_m : m \in \omega\}$ for $\mathbb{Q}$. We define a labelled linear order $\mathcal{Q}_{<\omega}$ with labels $\omega^{<\omega}$. $\mathcal{Q}_{<\omega}$ will be the union of a sequence of inductively defined linear orders $\mathcal{Q}_n$ with labels in $\omega^{\leq n}$ so that $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$ for all $n \in \omega$. The labelling function will be denoted $\lambda : \mathcal{Q}_{<\omega} \to \omega^{<\omega}$. We also define $l : \mathcal{Q}_{<\omega} \to \omega$ by $l(x) = \mathrm{lh}(\lambda(x))$. Thus $l(x)$ represents the level of $\lambda(x)$, and $l(x) = n$ iff $\lambda(x) \in \omega^n$. We will say that $x \in \mathcal{Q}_{<\omega}$ is of level n if $l(x) = n$.

To begin the inductive definition, let $\mathcal{Q}_0 = (\mathbb{Q}, <)$, and for every $x \in \mathcal{Q}_0$, define $\lambda(x) = \emptyset$ and $l(x) = 0$. Suppose $\mathcal{Q}_n$ has been defined and λ and l have been defined for elements of $\mathcal{Q}_n$. Let $\mathcal{Q}_{n+1}$ be the linear order obtained by adding adding a copy of $(\mathbb{Q}, <)$ to the immediate right of each element of $\mathcal{Q}_n$. Formally, $\mathcal{Q}_{n+1} = \mathcal{Q}_n \times (\{-\infty\} \cup \mathbb{Q})$ with the lexicographic order, where $(\{-\infty\} \cup \mathbb{Q}, <)$ is an extension of $(\mathbb{Q}, <)$ with $-\infty < q$ for all $q \in \mathbb{Q}$. In this formal definition we identify each $x \in \mathcal{Q}_n$ with $(x, -\infty) \in \mathcal{Q}_{n+1}$, thus maintaining $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$. Note that the new elements of $\mathcal{Q}_{n+1}$ form the product set $\mathcal{Q}_n \times \mathbb{Q}$. Thus for each $x \in \mathcal{Q}_n$ and $q \in \mathbb{Q}$, we define $l(x, q) = n + 1$ and $\lambda(x, q) = \lambda(x)^\frown m$, where m is the unique number such that $q \in P_m \in \mathcal{P}$. Then λ on $\mathcal{Q}_n \times \mathbb{Q}$ has the properties:

(i) for each $x \in \mathcal{Q}_n$ and $q \in \mathbb{Q}$, $\lambda(x, q) \supseteq \lambda(x)$ and $\lambda(x, q) \in \omega^{n+1}$; and

(ii) for each $x \in \mathcal{Q}_n$, the partition $\{\lambda^{-1}(\lambda(x)^\frown m) : m \in \omega\} = \{\{x\} \times P_m : m \in \omega\}$ is mutually dense in $\{x\} \times \mathbb{Q}$.

This finishes the inductive definition of $\mathcal{Q}_n$ and also of $\mathcal{Q}_{<\omega}$.

Theorem 24. *(Friedman-Stanley (2)) The class of all countable linear orderings is Borel complete.*

Proof. Following (7), we will construct a Borel reduction from the class of all binary relations on ω to the class of countable linear orderings. This is sufficient, since the Borel complete class of all countable graphs is a subclass.

Let $\mathcal{L}^R = \{R\}$ where R is a binary relation symbol. If

$$\varphi_0^{\mathcal{M},\vec{a}} = \bigwedge\{\theta(\vec{v}) : \theta \text{ is atomic or negated atomic and } \mathcal{M} \models \theta(\vec{a})\}$$

is the atomic type of $\vec{a}$ over $\mathcal{M}$, for each $n \in \omega$ the set

$$\Phi_n = \{\varphi_0^{\mathcal{M},\vec{a}} : \mathcal{M} \in \mathrm{Mod}(\mathcal{L}^R), \vec{a} \in \omega^n\}$$

of formulas with n free variables $v_0, \ldots, v_{n-1}$ is finite since the language is finite. Let $\Phi = \bigcup_n \Phi_n$. Fix a bijection $c : \Phi \to \omega$ so that for all $\varphi, \psi \in \Phi$ if $\varphi \in \Phi_n$, $\psi \in \Phi_m$ and $n < m$, then $c(\varphi) < c(\psi)$. Thus the function c gives a coding of atomic types of tuples by natural numbers.

For each $n \in \omega$ define a linear order $(B_n, <)$ by

$$B_n = D_1 \cup F_n \cup D_2$$

where D_1 and D_2 are dense linear orders without endpoints, F_n contains $n+2$ elements, and for any $p \in D_1$, $r \in F_n$, and $q \in D_2$, $p < r < q$.

Let $\theta_n(u, v)$ be the conjuction of the following formulas:

$$\exists c_1 \dots c_{n+2}\ u < c_1 < \dots < c_{n+2} < v$$

$$\nexists x \bigvee_{i=1}^{n+1} c_i < x < c_{i+1}$$

$$\forall xy\ u < x < y < c_1 \rightarrow \exists z_1 z_2 z_3\ u < z_1 < x < z_2 < y < z_3 < c_1$$

$$\forall xy\ c_{n+2} < x < y < v \rightarrow \exists z_1 z_2 z_3\ c_{n+2} < z_1 < x < z_2 < y < z_3 < v$$

Note that for any $n \in \omega$ and any linear ordering $\mathcal{N} \in \mathrm{Mod}(\rho)$ and $a, b \in N$, $\mathcal{N} \models \theta_n(a, b)$ iff $a < b$ and the linear order $\{x \in \mathcal{N} : a < x < b\}$ is isomorphic to B_n. We let $\psi_n(u)$ be the formula

$$\exists v\, \theta_n(u, v) \vee \exists v\, \theta_n(v, u).$$

We can now define for each $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}^R)$ a countable linear order $\mathcal{Q}(\mathcal{M})$. Note that each $x \in \mathcal{Q}_{<\omega}$ gives rise to a tuple $\lambda(x)$, which in turn is coded by a natural number

$c(\varphi_0^{\mathcal{M},\lambda(x)})$. Define $c_x = c(\varphi_0^{\mathcal{M},\lambda(x)})$. Now $\mathcal{Q}(\mathcal{M})$ is obtained from $\mathcal{Q}_{<\omega}$ by replacing each element x of $\mathcal{Q}_{<\omega}$ by a copy of the linear order B_{c_x}. This finishes the definition of the map $\mathcal{Q} : \mathcal{M} \mapsto \mathcal{Q}(\mathcal{M})$. The map is clearly Borel.

We check that $\mathcal{Q}$ is a reduction. First suppose $\pi : \mathcal{M} \mapsto \mathcal{M}'$, where $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}(\mathcal{L}^R)$. Since π is a bijection from ω onto ω, it induces an automorphism π^* of the tree $\omega^{<\omega}$, where $\pi^*(a_1, \ldots, a_n) = (\pi(a_1), \ldots, \pi(a_n))$. Furthermore, π^* induces an automorphism π^*_n of $\mathcal{Q}_n$ as labelled linear orders such that $\pi^*_{n+1} \restriction \mathcal{Q}_n = \pi^*_n$. Let $\pi^*_{<\omega} = \bigcup_n \pi^*_n$. Then $\pi^*_{<\omega}$ is an automorphism of $\mathcal{Q}_{<\omega}$. Now if $x \in \mathcal{Q}_{<\omega}$ then $\pi^*(\lambda(x)) = \lambda(\pi^*_{<\omega}(x))$ and $c_x = c_{\pi^*_{<\omega}(x)}$. Thus for any $x \in \mathcal{Q}_{<\omega}$, the copy of B_{c_x} in $\mathcal{Q}(\mathcal{M})$ replacing x is isomorphic to the copy of $B_{c_{\pi^*_{<\omega}(x)}}$ in $\mathcal{Q}(\mathcal{M}')$ replacing $\pi^*_{<\omega}(x)$. This shows that $\mathcal{Q}(\mathcal{M}) \cong \mathcal{Q}(\mathcal{M}')$.

Conversely, assume $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}(\mathcal{L}^R)$ and $\sigma : \mathcal{Q}(\mathcal{M}) \cong \mathcal{Q}(\mathcal{M}')$. Note that for any $a \in \mathcal{Q}(\mathcal{M})$ there is some $n \in \omega$ such that $\mathcal{Q}(\mathcal{M}) \models \psi_n(a)$, implying $\mathcal{Q}(\mathcal{M}') \models \psi_n(\sigma(a))$. It follows that σ induces an order-preserving bijection σ' from the copy of $\mathcal{Q}_{<\omega}$ in the construction of $\mathcal{Q}(\mathcal{M})$ to the copy of $\mathcal{Q}_{<\omega}$ in the construction of $\mathcal{Q}(\mathcal{M}')$. Moreover, for each $x \in \mathcal{Q}_{<\omega}$, $c_x = \sigma'(x)$, and hence $\varphi_0^{\mathcal{M},\lambda(x)} = \varphi_0^{\mathcal{M}',\lambda(\sigma'(x))}$. Also by construction, if $x, y \in \mathcal{Q}_{<\omega}$ then $\lambda(x) \subseteq \lambda(y)$ iff $x < y$ and for all z with $x < z < y$ we have $l(x) < l(z)$. Since this property is preserved by σ', we have that for all $x, y \in \mathcal{Q}_{<\omega}$, $\lambda(x) \subseteq \lambda(y)$ iff $\lambda(\sigma'(x)) \subseteq \lambda(\sigma'(y))$. Similar to the argument in Theorem 21, we can construct two permutations π and π' of ω such that for all $n \in \omega$,

$$\varphi_0^{\mathcal{M},(\pi(0),\ldots,\pi(n))} = \varphi_0^{\mathcal{M}',(\pi'(0),\ldots,\pi'(n))}.$$

Then $\pi' \circ \pi^{-1}$ is an isomorphism from $\mathcal{M}$ to $\mathcal{M}'$, as in the proof of Theorem 21. $\square$

We will use the fact that the class of countable linear orderings is Borel complete in Sections 2.3 and 3.2, and construct Borel reductions from this class.

2.3 O-minimal Examples

This section contains two examples of o-minimal theories for which the class of countable models is Borel complete. In both cases, we construct a Borel reduction from the class of countable linear orderings which is Borel complete by Theorem 24. The approach in each of these examples (by Marker) is generalized in 3.2.

Theorem 25. *The class of countable divisible ordered abelian groups is Borel complete.*

Proof. Let $(G, +, <)$ be an ordered abelian group. A pair of positive elements $a, b \in G$ are *comparable* if there are $n, m \in \mathbb{N}$ such that $a < nb$ and $b < ma$. We denote this $a \sim b$. Let $L(G)$ be the set of comparability classes of G. The ordering of G restricts to an ordering $<_L$ of the comparability classes. We call $(L(G), <_L)$ the *ladder* of G. Isomorphic groups have isomorphic ladders.

We construct a Borel reduction from the class of countable linear orderings to the class of countable divisible ordered abelian groups. Let $(L, <_L)$ be a countable linear order. Let $G(L) = \{f : L \to \mathbb{Q} : f(l) = 0 \text{ for all but finitely many } l\}$. $G(L)$ is a divisible abelian group under coordinate wise addition. We order $G(L)$ by $f < g$ if and only if $f(l) < g(l)$ where l is greatest such that $f(l) \neq g(l)$. This makes $G(L)$ a divisible ordered abelian group.

$L(G(L)) = L$. This follows since for any $l \in L$, we define $f_l \in G(L)$ by

$$f_l(x) = \begin{cases} 1 & \text{if } x = l \\ 0 & \text{otherwise} \end{cases}$$

Clearly, any $f \in G(L)$ is comparable to some f_l, and $f_j \nsim f_k$ for $j \neq k$.

Therefore, $G(L_1) \cong G(L_2)$ as groups if and only if $L_1 \cong L_2$ as orderings. Since the class of countable linear orderings is Borel complete, and $L \mapsto G(L)$ is a Borel reduction, the class of countable divisible ordered abelian groups is Borel complete. □

In (2), Friedman and Stanley show that for p prime or $p = 0$, the class of countable fields of characteristic p is Borel complete. The following example strengthens this result and is proved with the same approach as Theorem 25.

Theorem 26. *The class of countable real closed fields is Borel complete.*

Proof. If R is a real closed field, we say positive infinite elements a and b are *comparable* if there are $m, n \in \mathbb{N}$ such that $a < b^n$ and $b < a^m$. We denote comparability by $\sim$ and the set of comparability classes of R by $L(R)$. Again, note that the ordering of R restricts to an ordering $<_L$ of the set of comparability classes, and call $(L(R), <_L)$ the *ladder* of R. Isomorphic real closed fields have isomorphic ladders.

If $(L, <)$ is a linear order, let $F(L) = \mathbb{Q}(X_l : l \in L)$. We order $F(L)$ such that each $X_l > \mathbb{Q}$ and $X_i^n < X_j$ for all pairs $i < j \in \mathbb{N}$ and all $n \in \mathbb{N}$. Let $R(L)$ be the real closure of $F(L)$.

We will prove $L(R(L)) = L$ by showing every positive infinite element of $R(L)$ is comparable to some X_l. If $a \in R(L)$ then a is in the real closure of $\mathbb{Q}(X_{l_1}, \ldots, X_{l_n})$ where $l_1 < \ldots < l_n$ for some n. We proceed by induction on n.

Let K be the real closure of $\mathbb{Q}(X_{l_1}, \ldots, X_{l_{n-1}})$. We can identify the real closure of $K(X_{l_n})$ as a subfield of the field of Puiseux series over K. Thus we can write $a = \alpha X_{l_n}^q(1+\epsilon)$ where $\alpha \in K$, $q \in \mathbb{Q}$ and ϵ is infinitesimal. Since a is positive infinite, we have $\alpha > 0$ and $q \geq 0$. If $q = 0$, then $a \sim \alpha$ and we are done by induction. Otherwise, choose $0 < r < q < s$ in $\mathbb{Q}$. Clearly $X_{l_n}^r < a < X_{l_n}^s$ so $a \sim X_{l_n}$.

Therefore, $R(L_1) \cong R(L_2)$ as real closed fields if and only if $L_1 \cong L_2$ as orderings. Since the class of countable linear orderings is Borel complete, and $L \mapsto R(L)$ is a Borel reduction, the class of countable real closed fields is Borel complete. □

In Section 3.2, we will generalize the notion of comparability to countable models of an o-minimal theory T (satisfying certain conditions), and construct a similar reduction from the class of countable linear orderings to the class of countable models of T.

CHAPTER 3

A CLASSIFICATION OF O-MINIMAL THEORIES HAVING $2^{\aleph_0}$ COUNTABLE MODELS

In (9), Vaught proposed the following conjecture.

Conjecture 27. *(Vaught's Conjecture). Let T be a complete theory in a countable language $\mathcal{L}$ having infinite models. There are at most $\aleph_0$ countable models of T or there are $2^{\aleph_0}$ countable models of T.*

Obviously, Vaught's Conjecture holds under the assumption of the Continuum Hypothesis. Vaught's Conjecture has been studied for many classes of theories having specific properties. As outlined in the Summary, Mayer proved Vaught's Conjecture for o-minimal theories, and furthermore, classified the theories having countably many models as having 6^a3^b countable models for non-negative integers a and b. This chapter explores the class of countable models of an o-minimal theory T when that class has cardinality $2^{\aleph_0}$. These theories are classified into those lacking non-simple types and those having them. This distinction results in the following dichotomy.

Theorem 28. *Let T be an o-minimal theory in a countable language having $2^{\aleph_0}$ countable models. Either* *i. Every $p(x) \in S_1(A)$ for every finite set A is simple, and isomorphism on the class of countable models of T is $\prod_3^0$ (and is, in fact, equivalence of countable sets of reals); or*

ii. Some $p(x) \in S_1(A)$ is non-simple for some finite set A, and there is a finite set $B \supset A$ such that the class of countable models of T over B is Borel complete.

In the first section, we define simple and address the case where all types over all finite sets are simple. In the next section, we handle the second case, where we have a non-simple type.

3.1 No Non-simple Types

In the first case, every type is simple and the class of countable models of T has a fairly basic structure. This section follows Mayer (1) very closely. We begin with a few definitions.

Definition 29. A function is *trivial* if it is a finite union of constant and projection functions. A *non-trivial* function is a function that is not trivial.

Definition 30. A one-type $p(x) \in S_1(A)$ is *simple* if and only if for all $n \in \omega$ whenever $f(x_1, \ldots, x_n)$ is a non-trivial A-definable n-ary function and $a_1, \ldots, a_n$ are realizations of $p(x)$ then $f(a_1, \ldots, a_n)$ is not a realization of $p(x)$.

Essentially, if a type is simple, we can understand very specifically what the realizations can look like. There are no functions mapping realizations to other realizations, and there cannot be a great deal of complexity within the realizations of a specific type. The following lemma is an example of this lack of complexity.

Lemma 31. *Let T be an o-minimal theory and let $p(x) \in S_1(A)$ be a simple type. If $a_1 < a_2$ are realizations of $p(x)$ then there exists a_3 such that a_3 realizes $p(x)$ and $a_1 < a_3 < a_2$.*

Proof. Let a_1 and a_2 be as in the lemma and suppose there is no a_3 satisfying $p(x)$ and $a_1 < a_3 < a_2$. Let $f(x)$ be the A-definable unary function $\{\langle x, y\rangle : x < y$ and whenever $x \leq z \leq y$ then $x = z$ or $y = z\}$. In this case, $a_1 \in \mathrm{dom}(f)$ and $p(f(a_1))$. This contradicts $p(x)$ is simple. □

The following lemma about realizations of noncuts will allow us to completely characterize the realizations of simple types.

Lemma 32. *Let T be an o-minimal theory and let $\mathcal{M}$ be a model of T containing A. Let $p(x) \in S_1(A)$ be a noncut. Then $\mathcal{M}$ does not contain both a least realization of $p(x)$ and a greatest realization of $p(x)$.*

Proof. We will show that depending on the form of $p(x)$, either a least realization or a greatest realization will result in $p(x)$ being isolated, which is a contradiction. Consequently, $p(x)$ cannot have both a least and greatest realization. Suppose $p(x)$ is of the form $\{a < x < b_i : i \in \omega\}$ with $\{a, b_i\} \subset \mathrm{Pr}(A)$. Suppose $\mathcal{M}$ is a model of T and $\mathcal{M}$ contains a least realization of $p(x)$, say c. In this case, c is A-definable by

$$(a < x) \wedge \forall y\, y > a \rightarrow (y \geq x).$$

Similarly, if $p(x)$ is of the form $\{x < b_i\}$ with $\{b_i\} \subset \Pr(A)$, c is A-definable by $\forall y\ x \leq y$. If $p(x)$ is of the form $\{a_i < x < b : i \in \omega\}$ with $\{a_i, b\} \subset \Pr(A)$ and has a largest realization c, c is A-definable by

$$(x < b) \wedge \forall y\ y < b \rightarrow (y \leq x).$$

Similarly, if $p(x)$ is of the form $\{a_i < x\}$ with $\{a_i\} \subset \Pr(A)$, c is A-definable by $\forall y\ y \leq x$. In any case, $p(x)$ is isolated, as desired. □

Lemma 31 requires simple types to be realized by dense linear orders, and Lemma 32 restricts when the set of realizations can have endpoints. Therefore, simple types can only be realized in very specific ways.

Theorem 33. *Let T be an o-minimal theory and let $p(x) \in S_1(A)$.*

1. *If $p(x)$ is a simple cut, then whenever $\mathcal{M}$ is a model of T one of the following six possibilities holds.*

   i. *$p(x)$ is omitted in $\mathcal{M}$,*

   ii. *$p(x)$ has exactly one realization in $\mathcal{M}$,*

   iii. *$p(\mathcal{M})$ is order-isomorphic to $(0,1) \cap \mathbb{Q}$ in $(\mathbb{Q}, <)$,*

   iv. *$p(\mathcal{M})$ is order-isomorphic to $[0,1) \cap \mathbb{Q}$ in $(\mathbb{Q}, <)$,*

   v. *$p(\mathcal{M})$ is order-isomorphic to $(0,1] \cap \mathbb{Q}$ in $(\mathbb{Q}, <)$,*

   vi. *$p(\mathcal{M})$ is order-isomorphic to $[0,1] \cap \mathbb{Q}$ in $(\mathbb{Q}, <)$.*

Moreover, for each of the six possibilities there is a model $\mathcal{M}$ *of* T *in which this possibility occurs.*

2. *If* $p(x)$ *is a simple noncut, then whenever* $\mathcal{M}$ *is a model of* T*, there are three possibilities for the order type of* $p(\mathcal{M})$*.*

   i. $p(x)$ *is omitted in* $\mathcal{M}$*,*

   ii. $p(\mathcal{M})$ *is order-isomorphic to* $(0,1) \cap \mathbb{Q}$ *in* $(\mathbb{Q}, <)$*,*

   iii. *a)* $p(\mathcal{M})$ *is order-isomorphic to* $(0,1] \cap \mathbb{Q}$ *in* $(\mathbb{Q}, <)$*,*

   b) $p(\mathcal{M})$ *is order-isomorphic to* $[0,1) \cap \mathbb{Q}$ *in* $(\mathbb{Q}, <)$*.*

with possibility a) if $p(x)$ *is of the form* $\{a < x < b_i : i \in \omega\}$ *or* $\{x < b_i\}$ *and possibility b) if* $p(x)$ *is of the form* $\{a_i < x < b : i \in \omega\}$ *or* $\{a_i < x\}$*. Moreover, for each of the three possibilities there is a model* $\mathcal{M}$ *of* T *in which this possibility occurs.*

Proof. Suppose $p(x)$ has more than one realization. Note that if $p(x)$ is a noncut, $p(x)$ cannot have exactly one realization, since this would contradict Lemma 32. We will construct an order isomorphism from $p(\mathcal{M})$ to $((0,1) \cap \mathbb{Q}) \cup E$ where $E \subseteq \{0,1\}$. If $p(\mathcal{M})$ has a least element, we name it α and let $(\alpha, 0) \in f_0$. If $p(\mathcal{M})$ has a greatest element, we name it β and let $(\beta, 1) \in f_0$. By Lemma 31, $p(\mathcal{M}) \backslash \{\alpha, \beta\}$ is a dense linear order without endpoints, and so is $(0,1) \cap \mathbb{Q}$. Let g be a standard order-isomorphism between two dense linear orders without endpoints, from $p(\mathcal{M}) \backslash \{\alpha, \beta\}$ to $(0,1) \cap \mathbb{Q}$. Then $f = f_0 \cup g$ is the desired order-isomorphism. $\square$

In the remainder of this section, we show that if all types over all finite sets are simple, the isomorphism class of a countable model is determined by the order types of the realizations of types.

Definition 34. Let $\mathcal{M}$ and $\mathcal{N}$ be models of a theory T. Suppose $A \subset \mathcal{M}$, $B \subset \mathcal{N}$ and f is an elementary unary function mapping A onto B. If $p(x) \in S_1(A)$ then $p_f(x) \in S_1(B)$ is given by $\{\theta(x, f(a_0), \ldots, f(a_n)) : \theta(x, a_0, \ldots, a_n) \in p(x)\}$.

The next few lemmas follow from o-minimality and no non-simple types. Together they allow us to prove order-isomorphism of realizations of types implies isomorphism.

Lemma 35. *Let T be an o-minimal theory such that $S_1(A)$ does not contain any non-simple types for any finite set A. Let a and b be such that $\mathrm{cl}_\emptyset(\{a, b\}) \cap (a, b) = \emptyset$. Then there is no nontrivial $\{a, b\}$-definable unary function which maps the interval (a, b) into itself.*

Proof. Suppose, for contradiction, that f is a non-trivial $\{a, b\}$-definable unary function mapping (a, b) into itself. The domain of f and the range of f restricted to (a, b) include all of (a, b) by o-minimality. Without loss of generality, we may assume $f(x) > x$ for all $x \in (a, b)$. Choose $c \in (a, b)$ and let $p(x)$ be any non-isolated type which includes the set of formulas $\{a < x < b\} \cup \{d < x : d \in \mathrm{cl}_\emptyset(\{a, b, c\}) \cap (a, b)\}$. If α realizes $p(x)$, so does $f(\alpha)$, contradicting $p(x)$ is simple. $\square$

Lemma 36. *Let T be an o-minimal theory such that $S_1(A)$ does not contain any non-simple types for a finite set A and let $\mathcal{M}$ be a model of T. Let $p(x), q(x) \in S_1(A)$ be* *non-isolated. Then there is at most one A-definable function f which maps $p(\mathcal{M})$ into $q(\mathcal{M})$.*

Proof. Suppose, for contraction, that f and g are distinct, non-trivial A-definable functions mapping $p(\mathcal{M})$ into $q(\mathcal{M})$. Then $g^{-1}(f(x))$ maps $p(\mathcal{M})$ non-trivially into itself. By Lemma 36, this contradicts $p(x)$ is simple. □

Lemma 37. *Let T be an o-minimal theory such that $S_1(A)$ does not contain any non-simple types for a finite (possibly empty) set A, and let $\mathcal{M}$ and $\mathcal{N}$ be countable models of T over A such that whenever $p(x) \in S_1(A)$ then $p(\mathcal{M})$ is order-isomorphic to $p(\mathcal{N})$. Let B be a finite subset of M disjoint from A. Then there exists an elementary map f from B into N such that for all $p(x) \in S_1(A \cup B)$, $p(\mathcal{M})$ is order-isomorphic to $p_f(\mathcal{N})$.*

Proof. We proceed by induction on $n = |B|$. By hypothesis, the result holds for $n = 0$. Let $B = \{b_1, \ldots, b_{k+1}\}$ and let $\widehat{B} = B \backslash \{b_{k+1}\}$. By induction, there is an elementary map f from $\widehat{B}$ into N such that for all $p(x) \in S_1(A \cup \widehat{B})$, $p(\mathcal{M})$ is order isomorphic to $p_f(\mathcal{N})$. We note $p(x) = \operatorname{tp}(b_{k+1}/A \cup \widehat{B})$ is non-isolated. If $p(x)$ is an atomic interval, the result follows from Lemma 35.

Suppose $p(x)$ is a cut, determined by $\{c_i < c_{i+1} < x < d_{i+1} < d_i : i \in \omega\}$ for some $c_i, d_i \in \operatorname{cl}_A(\widehat{B})$. The noncut case is similar. Let $p(x)^+ \in S_1(A \cup B)$ be the noncut determined by $\{b_{k+1} < x < d_{i+1} < d_i : i \in \omega\}$ and let $p(x)^- \in S_1(A \cup B)$ be the noncut determined by $\{c_i < c_{i+1} < x < b_{k+1} : i \in \omega\}$. By hypothesis, we can find $f(b_{k+1}) \in N$ such that $p(\mathcal{M})^+$ is order-isomorphic to $p_f(\mathcal{N})^+$ and $p(\mathcal{M})^-$ is order-isomorphic to $p_f(\mathcal{N})^-$.

Let F be the partial isomorphism from $\mathrm{cl}_A(B)$ into N which is induced by f. We must show $q(\mathcal{M})$ is order-isomorphic to $q_F(\mathcal{N})$, for all $q(x) \in S_1(A \cup B)$.

Let α be an arbitrary element of $\mathrm{cl}_A(B) \backslash \mathrm{cl}_A(\widehat{B})$, say $\alpha = h(b_{k+1})$ where h is an $A \cup \widehat{B}$-definable function. Clearly, α realizes $p_h(x)$. By Lemma 36, α is the unique element of $\mathrm{cl}_A(B)$ realizing $p_h(x)$. It follows that every cut in $S_1(A \cup B)$ is the unique extension of a cut in $S_1(A \cup \widehat{B})$, and if $q(x)$ is a noncut in $S_1(A \cup B)$ which is not equal to its restriction to $S_1(A \cup \widehat{B})$, then there is an $A \cup \widehat{B}$-definable function h such that either $q(x) = p_h(x)^+$ or $q(x) = p_h(x)^-$. By o-minimality, the restriction of h to $\{x : x \text{ realizes } p(x)\}$ is either an order preserving map (in both M and N) or an order reversing map (in order M and N). In either case, $q(\mathcal{M})$ is order-isomorphic to $q_F(\mathcal{N})$, for all $q(x) \in S_1(A \cup B)$ as required. $\square$

The following theorem shows that order-isomorphism of realizations of types is a sufficient condition for isomorphism of models, when all types are simple.

Theorem 38. *Let T be an o-minimal theory such that $S_1(A)$ does not contain any non-simple types for any finite set A. Let B be a finite (possibly empty) set and let $\mathcal{M}$ and $\mathcal{N}$ be countable models of T such that whenever $p(x) \in S_1(B)$ then $p(\mathcal{M})$ is order-isomorphic to $p(\mathcal{N})$. Then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic over B.*

Proof. We construct an isomorphism using Lemma 37. Let $\{m_i : i \in \omega\}$ and $\{n_i : i \in \omega\}$ be enumerations of M and N respectively. We define an isomorphism f from M onto N and f^{-1} from N onto M by a back-and-forth construction defining a series of elementary maps from subsets of M (A_k^M) onto subsets of N (A_k^N) and vice versa.

Step 0. Let $A_0 = \mathrm{cl}_B(\emptyset)$ and define f_0 from $A_0^M = \{x \in M : x \in A_0\}$ onto $A_0^N = \{x \in N : x \in A_0\}$ in the obvious way. f_0^{-1} is then implicitly defined.

Step $2k{+}1$. Assume f_{2k}, A_{2k}^M and A_{2k}^N are defined so that f_{2k} is an elementary map from A_{2k}^M onto A_{2k}^N, A_{2k}^M is a finitely generated algebraically closed set and whenever $p(x) \in S_1(A_{2k}^M)$ then $p(\mathcal{M})$ is order-isomorphic to $p_{f_{2k}}(\mathcal{N})$. Choose i to be the least integer such that $m_i \notin A_{2k}^M$. Define A_{2k+1}^M to be $\{x \in M : x \in \mathrm{cl}_\emptyset(A_{2k}^M \cup \{m_i\})\}$. Applying Lemma 37 to the theory obtained from T by adding constants to the language of T for the finitely many generators of A_{2k}^M, we can extend f_{2k} to an elementary map f_{2k+1} with domain A_{2k+1}^M such that whenever $p(x) \in S_1(A_{2k+1}^M)$ then $p(\mathcal{M})$ is order-isomorphic to $p_{f_{2k+1}}(\mathcal{N})$. Let A_{2k+1}^N be $\{x \in N : c \in \mathrm{cl}_\emptyset(A_{2k}^N \cup f_{2k+1}(m_i))\}$.

Step $2k+2$. Assume f_{2k+1}, A_{2k+1}^M and A_{2k+1}^N are defined so that f_{2k+1} is an elementary map from A_{2k+1}^M onto A_{2k+1}^N, A_{2k+1}^M is a finitely generated algebraically closed set and whenever $p(x) \in S_1(A_{2k+1}^M)$ then $p(\mathcal{M})$ is order-isomorphic to $p_{f_{2k+1}}(\mathcal{N})$. Reversing the roles of $\mathcal{M}$ and $\mathcal{N}$ in step $2k+1$, define A_{2k+2}^N, A_{2k+2}^M, and an elementary map f_{2k+2}^{-1} from A_{2k+2}^N onto A_{2k+2}^M. f_{2k+2} is then implicitly defined.

Let $f = \bigcup_{n \in \omega} f_n$. □

Above we have shown that in the absence of non-simple types, nonisolated types can be realized in one of 6 (or 3) ways, and two models having isomorphic realizations of types are isomorphic.

Let $Y = S_1^T(\emptyset) \times \{0,1,2,3,4,5\}$ and let $\phi : \mathcal{M} \mapsto Y^\omega$ by $\phi(\mathcal{M})_n = (\mathrm{tp}_{\mathcal{M}}(n), i_{\mathcal{M}}(n))$, coding for each $n \in \omega$ the type of n in $\mathcal{M}$ and the form of the realization of this type in $\mathcal{M}$. Then,

$$\begin{aligned} \mathcal{M} \cong \mathcal{N} &\iff \{\phi(\mathcal{M})_n\}_{n\in\omega} = \{\phi(\mathcal{N})_n\}_{n\in\omega} \\ &\iff \forall n \exists m, k\, \phi(\mathcal{M})_n = \phi(N)_k \wedge \phi(\mathcal{M})_m = \phi(\mathcal{N})_n. \end{aligned}$$

This shows $\cong$ on $\mathrm{Mod}(T)$ is $\prod_3^0$ and furthermore, is equivalence of countable sets of reals.

3.2 A Non-simple Type

In the second case, we have a finite set A, $p(x) \in S_1(A)$, and a non-trivial, A-definable function $f(x_1, \ldots, x_n)$ such that $a_1, \ldots, a_n$ and $f(a_1, \ldots, a_n)$ are realizations of p. In this section, we show there is a finite set $B \supset A$ such that the class of countable models of T over B is Borel complete. We need the following definition and lemma.

Definition 39. Let A and B be sets. We write $A > B$ if and only if whenever $a \in A$ and $b \in B$ then $a > b$. If $A = \{a\}$ then we write $a > B$. Similarly for $<$.

Lemma 40. *Let T be o-minimal, and let $p(x) \in S_1(A)$. Let $\mathcal{M} \models T$ and suppose $\{a_1, a_2\} \subset p(\mathcal{M})$ such that $a_2 > \mathrm{cl}_A(\{a_1\}) \cap p(\mathcal{M})$. Then $\mathrm{cl}_A(\{a_2\}) \cap p(\mathcal{M}) > \mathrm{cl}_A(\{a_1\}) \cap p(\mathcal{M})$.*

Proof. For contradiction, assume f and g are A-definable functions mapping $p(\mathcal{M})$ into $p(\mathcal{M})$ such that $g(a_2) < f(a_1) < a_2$. O-minimality implies $g(x) < x$ for all x realizing p, and therefore $g(g(x)) < g(x)$. O-minimality also implies $g(x)$ is order-preserving or order-reversing on $p(\mathcal{M})$. If $g(x)$ was an order-reversing function, $g(x) < g(g(x))$, so g

must be order-preserving on p. $g(f(a_1)) < g(a_2) < f(a_1)$, so using g^{-1} we have $f(a_1) < a_2 < g^{-1}(f(a_1))$, contradicting $a_2 > \mathrm{cl}_A(\{a_1\}) \cap p(\mathcal{M})$. $\square$

We need the following reduction to assume our type $p(x)$ is non-simple due to a unary function f.

Lemma 41. *Suppose T is an o-minimal theory in a countable language $\mathcal{L}$, and there is a finite set A such that $p(x) \in S_1(A)$ is non-simple, due to an n-ary function f, with f chosen such that n is minimal among all such functions. Suppose $n > 1$. Then there is a finite set $B \supset A$ and $\widehat{p}(x) \in S_1(B)$ extending $p(x)$, such that $\widehat{p}(x)$ is a non-simple type due to a unary function. Furthermore, If $p(x)$ is a cut or noncut, $\widehat{p}(x)$ is a noncut, and if $p(x)$ is an atomic interval, so is $\widehat{p}(x)$.*

Proof. Suppose $\alpha_1, \ldots, \alpha_n, f(\alpha_1, \ldots, \alpha_n)$ all realize $p(x)$, where f is an A-definable function, and n is smallest among all such functions. By hypothesis, $n > 1$. By reordering, we may assume $\alpha_1 < \ldots < \alpha_n$.

There are three cases to consider:

Case 1. $p(x)$ is of the form $\{a_i < a_{i+1} < x : i \in \omega\}$, $\{a_i < a_{i+1} < x < b : i \in \omega\}$ or $\{a_i < a_{i+1} < x < b_{i+1} < b_i : i \in \omega\}$.

Note that we may assume $f(\alpha_1, \ldots, \alpha_n) < \alpha_1$. If not, by the monotonicity theorem, there is an A-definable function g such that $g(\alpha_2, \ldots, \alpha_n, f(\alpha_1, \ldots, \alpha_n)) = \alpha_1$, so we may rename the realizations of $p(x)$ to achieve this. Also note that $\mathrm{cl}_A(\alpha_2, \ldots, \alpha_n) \cap p(\mathrm{Pr}(A)) = \{\alpha_2, \ldots, \alpha_n\}$ by minimality of n. Let $B = A \cup \{\alpha_2, \ldots, \alpha_n\}$, $\widehat{p}(x) = \{a_i < x < \alpha_2\}$, and

$F(x) = f(x, \alpha_2, \ldots, \alpha_n)$. Then $\widehat{p}(\alpha_1)$, $\widehat{p}(F(\alpha_1))$, and $\widehat{p}(x) \in S_1(B)$ is a non-simple noncut due to the unary B-definable function F as desired.

Case 2. $p(x)$ is of the form $\{x < b_{i+1} < b_i : i \in \omega\}$ or $\{a < x < b_{i+1} < b_i : i \in \omega\}$.

Similar to Case 1, we can assume $f(\alpha_1, \ldots, \alpha_n) > \alpha_n$. Again, $\mathrm{cl}_A(\alpha_1, \ldots, \alpha_{n-1}) \cap p(\mathrm{Pr}(A)) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ by minimality of n. Let $B = A \cup \{\alpha_1, \ldots, \alpha_{n-1}\}$, $\widehat{p}(x) = \{\alpha_{n-1} < x < b_i\}$, and $F(x) = f(\alpha_1, \ldots, \alpha_{n-1}, x)$. Then $\widehat{p}(\alpha_n)$, $\widehat{p}(F(\alpha_n))$, and $\widehat{p}(x) \in S_1(B)$ is a non-simple noncut due to the unary B-definable function F as desired.

Case 3. $p(x)$ is of the form $\{a < x < b\}$.

As in Case 1, we may assume $f(\alpha_1, \ldots, \alpha_n) < \alpha_1$. Again, $\mathrm{cl}_A(\alpha_2, \ldots, \alpha_n) \cap p(\mathrm{Pr}(A)) = \{\alpha_2, \ldots, \alpha_n\}$ by minimality of n. Let $B = A \cup \{\alpha_2, \ldots, \alpha_n\}$, $\widehat{p}(x) = \{a < x < \alpha_2\}$, and $F(x) = f(x, \alpha_2, \ldots, \alpha_n)$. Then $\widehat{p}(\alpha_1)$, $\widehat{p}(F(\alpha_1))$, and $\widehat{p}(x) \in S_1(B)$ is a non-simple atomic interval due to the unary B-definable function F as desired. $\square$

This reduction allows us to work over a type that is non-simple due to a unary function. The following Definitions and Lemmas give us a way to describe the realizations of a non-simple type.

Definition 42. Let $\mathcal{M}$ be a model of T. Suppose $A \subset M$ is a finite set and $p(x) \in S_1(A)$. If a and b are realizations of $p(x)$ we say a and b are A-*comparable* if there are c and d realizing $p(x)$ with $c \in \mathrm{cl}_A(a)$ and $d \in \mathrm{cl}_A(b)$ with $d > a$ and $c > b$ and write $a \sim_A b$.

Lemma 43. *Let $\mathcal{M}$ be a model of T. Suppose $A \subset M$ is a finite set and $p(x) \in S_1(A)$ is a non-simple type due to a unary function $f(x)$. Then $\sim_A$ is an equivalence relation on $p(\mathcal{M})$.*

Proof. Since $f(a), f^{-1}(a) \in \mathrm{cl}_A(a)$, $a \sim_A a$. By definition, if $a \sim_A b$ then $b \sim_A a$. We must show $\sim_A$ is transitive.

Suppose $a \sim_A b$ and $b \sim_A c$. There are two possibilities. First, suppose $a < b < c$. There are A-definable functions $h_1(x)$ and $h_2(x)$ such that $a < b < h_1(a)$ and $b < c < h_2(b)$. Note that $h_2(x)$ is order-preserving on $p(\mathcal{M})$, since $x < h_2(x) \in p$ and if $h_2(x)$ was order-reversing, $h_2(h_2(b)) < h_2(b)$, which is a contradiction. Then $b < h_1(a)$ implies $h_2(b) < h_2(h_1(a))$. Since $c < h_2(b)$, $a \sim_A c$.

Instead, suppose $a < c < b$. In this case, since $a \sim_A b$, there is an A-definable function $h(x)$ such that $a < b < h(a)$. Since $c < b$, $c < h(a)$ and so $a \sim_A c$. $\square$

Definition 44. Let $\mathcal{M}$ be a model of T. Suppose $A \subset M$ is a finite set and $p(x) \in S_1(A)$ is a non-simple type due to a unary function $f(x)$. Let $[x]$ denote the $\sim_A$-class of x, and note that each class is a convex set. Let $\mathrm{Lad}^p_A(\mathcal{M})$ be the set of A-comparability classes of $p(\mathcal{M})$. The ordering of $\mathcal{M}$ restricts to an ordering $<$ of A-comparability classes. We call $(\mathrm{Lad}^p_A(\mathcal{M}), <)$ the A^p-*ladder* of $\mathcal{M}$.

Note that isomorphic models of T have isomorphic A^p-ladders. Similar to Theorem 25 and Theorem 26, we will use ladders to recover the order coded by a model. The following Definition and Lemmas will ensure our type does just that.

Definition 45. Suppose A is a finite set and $p(x) \in S_1(A)$ is a non-simple type due to a unary function $f(x)$. We say $p(x)$ is a *faithful* type if whenever g is an n-ary A-definable function, $c_1, \ldots, c_n$ realize $p(x)$, $c_i \nsim_A c_j$ for $i \neq j$, and $\alpha = g(c_1, \ldots, c_n)$ realizes $p(x)$, then $\alpha \sim_A c_i$ for some $i \in \{1, \ldots, n\}$.

Lemma 46. *Suppose A is a finite set and $p(x) \in S_1(A)$ is a non-simple type due to a unary function $f(x)$. If $p(x)$ is a non-cut, then $p(x)$ is faithful.*

Proof. For contradiction, suppose $\alpha = g(c_1, \ldots, c_n)$, $c_i \nsim_A c_j$ for $i \neq j$, $\alpha \nsim_A c_1, \ldots, \alpha \nsim_A c_n$, and n is minimal such that this occurs.

We know $n \neq 0$, since $p(x)$ is non-isolated over $\mathcal{M}_0$.

Also, $n \neq 1$, since if $\alpha = g(c_1)$, $\alpha \in \text{cl}(c_1)$ and $\alpha \notin \mathcal{M}_0$. By exchange, $c_1 \in \text{cl}(\alpha)$ and $\alpha \sim c_1$.

Suppose $n > 1$. Let $C = \{c_1, \ldots, c_n\}$, $\widehat{C} = C \cup \{\alpha\}$, $s = \min(\widehat{C})$, $t = \max(\widehat{C})$ and $\widetilde{C} = \widehat{C} \backslash \{s, t\}$. By assumption $\alpha \notin \text{cl}(C \backslash c_i)$ for all $i \in \{1, \ldots, n\}$. By exchange, $c_i \in \text{cl}(\widehat{C} \backslash c_i)$ for all $i \in \{1, \ldots, n\}$.

By assumption, $\alpha < [c_1]$, $[c_k] < \alpha < [c_{k+1}]$ with $1 \leq k \leq n-1$, or $[c_n] < \alpha$. Let $\sigma = \text{tp}(s)/\widetilde{C}$ and let $\tau = \text{tp}(t)/\widetilde{C}$. If $p(x) = \{a_i < x < b : a_i, b \in M \cup \{\infty\}, a_i < a_{i+1}\}$, then over $\text{cl}(\widetilde{C})$, σ is a cut and τ is a noncut. If $p(x) = \{a < x < b_j : a, b_j \in M \cup \{-\infty\}, b_{i+1} < b_i\}$, then σ is a noncut and τ is a cut. In either case, over $\widetilde{C}$, realizing σ forces a model to realize τ, which contradicts the Omitting Types Theorem, Theorem 10. Therefore there is no such α, and $p(x)$ is faithful. $\square$

Lemma 47. *Suppose A is a finite set and $p(x) \in S_1(A)$ is a non-simple type due to a unary function $f(x)$. If $p(x)$ is an atomic interval, then either $p(x)$ is faithful, or there is a point b such that if $B = A \cup \{b\}$, there is $\widehat{p}(x) \in S_1(B)$ extending $p(x)$ which is a non-simple noncut due to a unary function. By Lemma 46, this extension is a faithful type.*

Proof. This proof is very similar to the proof of Lemma 46. Suppose $p(x)$ is an atomic interval, $\{a < x < b\}$ and suppose $p(x)$ is not faithful. This will either give a contradiction, or we will construct B and $\widehat{p}$. Suppose $\alpha = g(c_1, \dots, c_n)$, $c_i \nsim_A c_j$ for $i \neq j$, $\alpha \nsim_A c_1, \dots, \alpha \nsim_A c_n$, and n is minimal such that this occurs. As in Lemma 46, we know $n > 1$. Again we have $\alpha < [c_1]$, $[c_k] < \alpha < [c_{k+1}]$ with $1 \leq k \leq n-1$, or $[c_n] < \alpha$.

Suppose $n > 2$. In this case we will get a contradiction. Let C and $\widehat{C}$ be defined as in the proof of Lemma 46, let $s = \min(\widehat{C})$ and let $t \in \widehat{C} \backslash s$ such that $t \neq \min(\widehat{C} \backslash s)$ and $t \neq \max(\widehat{C})$. This is possible since $n > 2$. Let $\widetilde{C} = \widehat{C} \backslash \{s, t\}$, $\sigma = \mathrm{tp}(s)/\widetilde{C}$, and $\tau = \mathrm{tp}(t)/\widetilde{C}$. Note that σ is a noncut and τ is a cut, and over $\widetilde{C}$, realizing σ forces a model to realize τ, contradicting the Omitting Types Theorem, Theorem 10. Therefore there is no such α and $p(x)$ is faithful as desired.

If $n = 2$, we will add one constant to A to get B and extend $p(x)$ to $\widehat{p}(x)$ which will be a non-simple noncut. Without loss of generality, we may assume $c_1 < c_2 < g(c_1, c_2)$. In this case, let $B = A \cup \{c_1\}$ and $\widehat{p}(x) = p(x) \cup \{\{p \cap \mathrm{cl}(c_1)\} < x < b\}$. Note that $\widehat{p}(x)$ is a noncut and $G(x) = g(c_1, x)$ maps $\widehat{p}$ to $\widehat{p}$. This $\widehat{p}$ is a non-simple noncut due to a unary function as desired. $\square$

Lemma 48. *Suppose A is a finite set and $p(x) \in S_1(A)$ is a non-simple type due to a unary function $f(x)$. If $p(x)$ is a cut, then either $p(x)$ is faithful, or there is a finite set $B \supset A$, and $\widehat{p}(x) \in S_1(B)$ extending $p(x)$ which is a non-simple noncut due to a unary function. By Lemma 46, this extension is a faithful type.*

Proof. Suppose $p(x)$ is a cut. If $p(x)$ is not faithful, we construct B and $\widehat{p}$ as follows. Suppose $\alpha = g(c_1, \ldots, c_n)$, $c_i \nsim_A c_j$ for $i \neq j$, $\alpha \nsim_A c_1, \ldots, \alpha \nsim_A c_n$, and n is minimal such that this occurs. As in the proof of Lemma 46, we know $n > 1$.

To illustrate the argument, we first consider the $n = 2$ case. This is followed by the general case.

Suppose $n = 2$. By o-minimality, we may assume, without loss of generality, that $[c_1] < \alpha < [c_2]$. Note that there is a c_1-definable interval containing c_2 such that $g(c_1, v)$ is increasing in v, since otherwise there is $c \in \mathrm{cl}(c_1)$ realizing $p(x)$ such that $g(c_1, c) > \alpha$, contradicting $[c_1] < \alpha$. Similarly, there is a c_2-definable interval containing c_1 such that $g(u, c_2)$ is increasing in u since otherwise there is $c \in \mathrm{cl}(c_2)$ realizing $p(x)$ such that $g(c, c_2) < \alpha$, contradicting $\alpha < [c_2]$.

Let

$$\begin{aligned} X = &\{(u, v) : u < g(u, v) < v\} \\ &\cap \{(u, v) : u_1 < u_2 \rightarrow g(u_1, v) < g(u_2, v)\} \\ &\cap \{(u, v) : v_1 < v_2 \rightarrow g(u, v_1) < g(u, v_2)\}. \end{aligned}$$

Note that $(c_1, c_2) \in X$ and that X is A-definable. By the Cell Decomposition Theorem (Theorem 7), there is a cell $\mathcal{A}$ containing (c_1, c_2) contained in X, and by the preceding paragraph, this must be a $(1,1)$-cell. Since there are no A-definable points in p, and $[c_1] < [c_2]$, there is an A-definable unary function $h : p \to p$ such that if x and y realize p, and $y > h(x)$, then $(x, y) \in X$. Furthermore, note that $h(x) = x$ or $h(x) > x$, and h is increasing and order preserving on an interval containing p.

If $h(x) = x$, let $B = A \cup \{c_1\}$, let $\widehat{p} = \{c_1 < x < b_k : b_k \in \mathrm{cl}(B) \cap p, b_k > c_1\}$ and let $\widehat{f}(x) = g(c_1, x)$. Note that $\widehat{p}$ is a non-simple noncut due to the unary function $\widehat{f}$, as desired.

If $h(x) > x$, first note that for $x > c_1$,

$$g(h^{-1}(c_1), h(c_1)) < g(h^{-1}(c_1), h(x)) < g(h^{-1}(x), h(x))$$

and that for $x < c_1$,

$$g(h^{-1}(x), h(x)) < g(h^{-1}(x), h(c_1)) < g(h^{-1}(c_1), h(c_1)).$$

Let $G_1(x) = g(h^{-1}(x), h(c_1))$, $G_2(x) = g(h^{-1}(c_1), h(x))$, and $\widehat{G}(x) = g(h^{-1}(x), h(x))$. Then the inequalities above can be rewritten as: if $x > c_1$, then $\widehat{G}(c_1) < G_2(x) < \widehat{G}(x)$ and if $x < c_1$, then $\widehat{G}(x) < G_1(x) < \widehat{G}(c_1)$.

We can now construct $\widehat{p}$. Let $c = \widehat{G}(c_1)$, and let $B = A \cup \{c\}$. If $c \geq c_1$, then let, $\widehat{p} = \{c < x < b_k : b_k \in \mathrm{cl}(B) \cap p, b_k > c\}$, and $\widehat{f}(x) = G_2(\widehat{G}^{-1}(x))$. If $c < c_1$, let $\widehat{p} = \{b_k < x < c : b_k \in \mathrm{cl}(B) \cap p, b_k < c\}$, and $\widehat{f}(x) = G_1(\widehat{G}^{-1}(x))$. In either case $\widehat{p}$ is a non-simple noncut due to the unary function $\widehat{f}$, as desired.

For the general case, we may assume $[c_1] < \ldots < [c_{n-1}] < \alpha < [c_n]$ by o-minimality. By minimality of n, we can assume $\mathrm{cl}(c_1, \ldots, c_{n-1}) < \alpha$. There is a $\{c_1, \ldots, c_{n-1}\}$-definable interval containing c_n such that $g(c_1, \ldots, c_{n-1}, v)$ is increasing in v, since otherwise there is c in $\mathrm{cl}(c_{n-1})$ realizing $p(x)$ such that $g(c_1, \ldots, c_{n-1}, c) > \alpha$. By minimality of n, this $g(c_1, \ldots, c_{n-1}, c)$ is comparable to one of $c_1, \ldots, c_{n-1}$, contradicting $[c_1] < \ldots < [c_{n-1}] < \alpha$. Similarly, there is a $\{c_1, \ldots, c_{n-2}, c_n\}$-definable interval containing c_{n-1} such that $g(c_1, \ldots, c_{n-2}, u, c_n)$ is increasing in u, since otherwise there is c in $\mathrm{cl}(c_n)$ realizing $p(x)$ such that $g(c_1, \ldots, c_{n-2}, c, c_n) < \alpha$. By minimality of n, this $g(c_1, \ldots, c_{n-2}, c, c_n)$ is comparable to one of $c_1, \ldots, c_{n-1}, c_n$ and in fact, $g(c_1, \ldots, c_{n-2}, c, c_n) \sim c_n$ since $g(c_1, \ldots, c_{n-2}, c, c_n) > \mathrm{cl}(\{c_1, \ldots, c_{n-2}\})$. This contradicts $\alpha < [c_n]$.

Let $A' = A \cup \{c_1, \ldots, c_{n-2}\}$, let $p'(x) = p(x) \cup \{p \cap \mathrm{cl}(c_1, \ldots, c_{n-2}) < x\}$, let $g'(u, v)$ denote $g(c_1, \ldots, c_{n-2}, u, v)$, and let

$$\begin{aligned} X = &\{(u, v) : u < g'(u, v) < v\} \\ &\cap \{(u, v) : u_1 < u_2 \to g'(u_1, v) < g'(u_2, v)\} \\ &\cap \{(u, v) : v_1 < v_2 \to g'(u, v_1) < g'(u, v_2)\}. \end{aligned}$$

Note that $(c_{n-1}, c_n) \in X$ and that X is A'-definable. By the Cell Decomposition Theorem (Theorem 7), there is a cell $\mathcal{A} \subset X$ containing the point (c_{n-1}, c_n), and by the preceding paragraph, this must be a $(1,1)$-cell. Since there are no A'-definable points in p', and $[c_{n-1}] < [c_n]$, there is an A'-definable unary function $h : p' \to p'$ such that if x and y realize p', and $y > h(x)$, then $(x, y) \in X$. Furthermore, note that $h(x) = x$ or $h(x) > x$, and h is increasing on an interval containing p'.

If $h(x) = x$, let $B = A' \cup \{c_{n-1}\}$, let $\widehat{p} = \{c_{n-1} < x < b_k : b_k \in \mathrm{cl}(B) \cap p, b_k > c_{n-1}\}$ and let $\widehat{f}(x) = g'(c_{n-1}, x)$. Note that $\widehat{p}$ is a non-simple noncut due to the unary function $\widehat{f}$, as desired.

If $h(x) > x$, first note that for $x > c_{n-1}$,

$$g'(h^{-1}(c_{n-1}), h(c_{n-1})) < g'(h^{-1}(c_{n-1}), h(x)) < g'(h^{-1}(x), h(x))$$

and that for $x < c_{n-1}$,

$$g'(h^{-1}(x), h(x)) < g'(h^{-1}(x), h(c_{n-1})) < g'(h^{-1}(c_{n-1}), h(c_{n-1})).$$

We now let $G_1(x) = g'(h^{-1}(x), h(c_{n-1}))$, $G_2(x) = g'(h^{-1}(c_{n-1}), h(x))$, and $\widehat{G}(x) = g'(h^{-1}(x), h(x))$. Then the inequalities above can be rewritten as: if $x > c_{n-1}$, then $\widehat{G}(c_{n-1}) < G_2(x) < \widehat{G}(x)$ and if $x < c_{n-1}$, then $\widehat{G}(x) < G_1(x) < \widehat{G}(c_{n-1})$.

We can now construct $\widehat{p}$. Let $c = \widehat{G}(c_{n-1})$. If $c \geq c_{n-1}$, then let $B = A' \cup \{c_{n-1}\}$, $\widehat{p} = \{c < x < b_k : b_k \in \mathrm{cl}(B) \cap p, b_k > c\}$, and $\widehat{f}(x) = G_2(\widehat{G}^{-1}(x))$. In this case $\widehat{p}$ is a non-simple noncut due to the function $\widehat{f}$. The case $c < c_{n-1}$ is similar. In either case, we have extended A to B and $p(x)$ to a non-simple noncut $\widehat{p}(x)$ as desired. □

The above Lemmas combine to give the following result.

Proposition 49. *Suppose T is an o-minimal theory in a countable language $\mathcal{L}$, and there is a finite set A such that $p(x) \in S_1(A)$ is non-simple. Then there is a finite set $B \supset A$ and $\widehat{p} \in S_1(B)$ such that $\widehat{p}$ is a faithful type.*

Proof. The proof follows from Lemmas 41, 46, 47, and 48. □

Theorem 50. *Suppose T is an o-minimal theory in a countable language $\mathcal{L}$, and there is a finite set A such that $p(x) \in S_1(A)$ is a faithful type. Then for every countable linear order L, there is a countable model $\mathcal{M}_L \models T$ with $\mathrm{Lad}^p_A(\mathcal{M}_L) = L$.*

Proof. Let $(L, <_L)$ be a non-empty countable linear order. Our goal is to construct countable $\mathcal{M}_L \models T$ such that $\mathrm{Lad}^p_A(\mathcal{M}_L) = L$.

Let $\mathcal{L}^* = \mathcal{L} \cup \{z_i : i \in L\}$ and let

$$T^* = T \cup \{\mathrm{cl}_A(z_{i_1}, \ldots, z_{i_n}) \cap \{x : x \models p(x)\} < z_j : \{i_1, \ldots, i_n\} <_L j\}$$

$$= T \cup \{f(z_{i_1}, \ldots, z_{i_n}) < z_j : \{i_1, \ldots, i_n\} <_L j, f \text{ an } n\text{−ary function}\}.$$

"Any fool can realize a type, ..."

We first show T^* is consistent. Suppose $\Delta \subset T$ is finite, invoking $z_{i_1}, \ldots, z_{i_k}$. There is a natural model of Δ constructed as follows.

Let $\mathcal{M}_0$ denote $\mathrm{Pr}_T(A)$. Without loss of generality, we may assume $i_1 <_L \ldots <_L i_k$. Extend $\mathcal{M}_0 \models T$ to $\mathcal{M}_1$ by adding z_{i_1} realizing $p(x) \cup \{x > a : a \in \mathcal{M}_0, p(a)\}$ and taking $\mathcal{M}_1 = \mathrm{Pr}_T(\mathcal{M}_0 \cup \{z_{i_1}\})$. Then construct $\mathcal{M}_{n+1}$ by adding $z_{i_{n+1}}$ to $\mathcal{M}_n$ realizing $p(x) \cup \{x > a : a \in \mathcal{M}_n, p(a)\}$ and letting $\mathcal{M}_{n+1} = \mathrm{Pr}_T(\mathcal{M}_n \cup \{z_{i_{n+1}}\})$. Stop after constructing $\mathcal{M}_k$ and interpret the remaining constants in $\mathcal{L}^*$ in any way. $\mathcal{M}_k \models \Delta$ showing Δ is consistent.

Let $\mathcal{M}_L$ be the prime model of T^*. Clearly $\mathcal{M}_L$ is a countable model of T. Since prime models are atomic, we start by showing any atom is comparable to some z_i. Since $p(x)$ is faithful, any $\alpha \in \mathrm{cl}(z_i : i \in L)$ realizing p is comparable to some z_i. On the other hand, an atom of the form $\phi(v) = a < v < b$, is either comparable to a or b: if a realizes p, $f(a) \geq b$ or $f^{-1}(a) \geq b$; if b realizes p, $f(b) \leq a$ or $f^{-1}(b) \leq a$. By Lemma 40 $z_i \not\sim z_j$ for $i \neq j$. Every realization of $p(x)$ is comparable to some z_i and $z_i \not\sim z_j$ for $i \neq j$. Therefore $\mathrm{Lad}^p_A(M_L) = L$, as desired. $\square$

Corollary 51. *Suppose T is an o-minimal theory in a countable language with a non-simple type over a finite set A. There is a finite set $B \supset A$ such that the class of countable models of T over B is Borel complete.*

Proof. By Proposition 49, there is a faithful type. By Theorem 50, we can map countable linear orders to countable models of T such that non-isomorphic orders are mapped to non-isomorphic models. It remains to show that $L \mapsto \mathcal{M}_L$ is Borel.

We show that construction of prime models is explicit. Let $F_n = \{f_i^n : i \in \omega\}$ denote the set of A-definable n-ary functions. We will define sets M_k inductively, with $\bigcup_{k\in\omega} M_k = M_L$, the set underlying $\mathcal{M}_L$.

First, note that:

$$T^* = \{\varphi : T \cup \{c_{l_{n+1}} > f_i^n(c_{l_1}, \ldots, c_{l_n}) : n, i \in \omega, l_1, \ldots, l_n < l_{n+1}\} \vdash \varphi\},$$

$$\mathrm{dcl}(A) = \{(n, i, a_1, \ldots, a_n) : n, i \in \omega, a_1, \ldots, a_n \in A\}, \text{ and let}$$

$$\mathcal{D}(A) = \{(a, b) : a, b \in A, a < b, \nexists c \in A \ a < c < b\}.$$

By o-minimality, $\mathcal{D}(A)$ denotes the set of atoms over a definably closed set A. Let $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ such that $\langle i, j \rangle \geq i, j$. Let $M_0 = \mathrm{dcl}(z_l : l \in L)$, and let $\mathcal{D}(M_i) = \{d_i^j : j \in \omega\}$ index the set of atoms over M_i. Then having defined M_k, if $k = \langle i, j \rangle$ we define M_{k+1} as follows:

$$M_{k+1} = \begin{cases} \mathrm{dcl}(M_k \cup \{d_i^j\}) & \text{if } d_i^j \in \mathcal{D}(M_k) \cap (M_k)^c \\ M_k & \text{otherwise} \end{cases}$$

and note that $M_L = \bigcup_{i\in\omega} M_i$ since prime models are atomic. Since our theory is o-minimal the relation symbols in $\mathcal{L}$ can be characterized in terms of the order. Therefore, they are naturally defined on $\mathcal{M}_L$. The construction of $\mathcal{M}_L$ from L is explicit and this reduction is clearly Borel.

Therefore, the class of countable models over B is Borel complete. $\square$

CHAPTER 4

CONCLUSION

This result is an extension of Mayer's proof of Vaught's Conjecture. For an o-minimal theory having $2^{\aleph_0}$ countable models, we give a condition for isomorphism to be $\prod_3^0$ (and, in fact, equivalence of countable sets of reals), and a condition for which the class of models over some finite set is Borel complete.

Question 52. *If T is an o-minimal theory having $2^{\aleph_0}$ countable models and a non-simple type, what is the Borel complexity of the isomorphism relation over $\emptyset$? More generally, if the class of countable models of T over some finite set B is Borel complete, what is the Borel complexity of isomorphism for the class of countable models over some set $A \subsetneq B$?*

The arguments in this work depend on adding the set over which the type is defined, and additional points to make the type faithful. While the former seems necessary, some improvement may be found on the latter. In either case, reducing this set and retaining Borel complexity is not trivial.

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VITA

I. Education

- Ph.D. in Mathematics

  University of Illinois at Chicago, August 2013.

- M.S. in Mathematics

  University of Illinois at Chicago, December 2003.

- B.S. in Mathematics

  University of Illinois at Chicago, December 2002.

  Highest Departmental Distinction.