On $K_{s,t}$-minors in graphs with given average degree

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Abstract

Let $D(H)$ be the minimum d such that every graph G with average degree d has an H-minor. Myers and Thomason found good bounds on $D(H)$ for almost all graphs H and proved that for 'balanced' H random graphs provide extremal examples and determine the extremal function. Examples of 'unbalanced graphs' are complete bipartite graphs $K_{s,t}$ for a fixed s and large t. Myers proved upper bounds on $D(K_{s,t})$ and made a conjecture on the order of magnitude of $D(K_{s,t})$ for a fixed s and $t \to \infty$. He also found exact values for $D(K_{2,t})$ for an infinite series of t. In this paper, we confirm the conjecture of Myers and find asymptotically (in s) exact bounds on $D(K_{s,t})$ for a fixed s and large t.

Keywords: Graph minors, average degree, complete bipartite graphs.

1 Introduction

Recall that a graph H is a *minor* of a graph G if one can obtain H from G by a sequence of edge contractions and vertex and edge deletions. In other words, H is a minor of G if there is $V_0 \subset V(G)$ and a mapping $f \ : \ (V(G) - V_0) \ \to V(H)$ such that for every $v \in V(H)$, the set $f^{-1}(v)$ induces a nonempty connected subgraph in G and for every $uv \in E(H)$, there is an edge in G connecting $f^{-1}(u)$ with $f^{-1}(v)$.

Mader [4] proved that for each positive integer t, there exists a $D(t)$ such that every graph with average degree at least $D(t)$ has a K_t-minor. Kostochka [1, 2] and Thomason [11] determined the order of magnitude of $D(t)$, and recently Thomason [12] found

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the asymptotics of $D(t)$. Furthermore, Myers and Thomason [9, 6], for a general graph H, studied the minimum number $D(H)$ such that every graph G with average degree at least $D(H)$ has an H-minor, i.e., a minor isomorphic to H. They showed that for almost all graphs H, random graphs are bricks for constructions of extremal graphs. On the other hand, they observed that for fixed s and very large t, the union of many K_{s+t-1} with $s-1$ common vertices does not have any $K_{s,t}$-minor and has a higher average degree than a construction obtained as a union of random subgraphs.

In view of this, Myers [8, 7] considered $D(K_{s,t})$ for fixed s and large t. The above example of the union of many K_{s+t-1} with $s-1$ common vertices shows that $D(K_{s,t}) \geq t+2s-3$. Myers proved

Theorem 1 ([8]) *Let $t > 10^{29}$ be a positive integer. Then every graph $G = (V, E)$ with more than $\frac{t+1}{2}(|V|-1)$ edges has a $K_{2,t}$-minor.*

This bound is tight for $|V| \equiv 1 \pmod{t}$. Myers noted that probably the average degree that provides the existence of a $K_{s,t}$-minor, provides also the existence of a $K^*_{s,t}$-minor, where $K^*_{s,t} = K_s + \overline{K_t}$ is the graph obtained from $K_{s,t}$ by adding all edges between vertices in the smaller partite set. In other words, $K^*_{s,t}$ is the graph obtained from K_{s+t} by deleting all edges of a subgraph on t vertices. Myers also conjectured that for every positive integer s, there exists $C = C(s)$ such that for each positive integer t, every graph with average degree at least $C\,t$ has a $K_{s,t}$-minor.

Preparing this paper, we have learned that Kühn and Osthus [3] proved the following refinement of Myers' conjecture.

Theorem 2 ([3]) *For every $\epsilon > 0$ and every positive integer s there exists a number $t_0 = t_0(s, \epsilon)$ such that for all integers $t \geq t_0$ every graph of average degree at least $(1+\epsilon)t$ contains $K^*_{s,t}$ as a minor.*

In this paper, we prove a stronger statement but under stronger assumptions: We find asymptotically (in s) exact bounds on $D(K_{s,t})$ for t much larger than s. Our main result is

Theorem 3 *Let s and t be positive integers with $t > (180s\log_2 s)^{1+6s\log_2 s}$. Then every graph $G = (V, E)$ with $|E| \geq \frac{t+3s}{2}(|V|-s+1)$ has a $K^*_{s,t}$-minor. In particular, $D(K^*_{s,t}) \leq t+3s$. On the other hand, for arbitrarily large n, there exist graphs with at least n vertices and average degree at least $t+3s-5\sqrt{s}$ that do not have a $K_{s,t}$-minor.*

This confirms the insight of Myers that $D(K^*_{s,t})$ and $D(K_{s,t})$ are essentially the same for fixed s and large t. It follows from our theorem that the above described construction giving $D(K_{s,t}) \geq t+2s-3$ is not optimal for $s > 100$.

In the next section we describe a construction giving the lower bound for $D(K_{s,t})$. In Section 3 we handle graphs with few vertices. Then in Section 4 we derive a couple of technical statements on contractions and in Section 5 we finish the proof of Theorem 3.

Throughout the paper, $N(x) = \{v \in V : xv \in E\}$ is the open neighborhood of the vertex x, and $N[x] = N(x) \cup \{x\}$ is the closed neighborhood of x. If $X \subseteq V$, then $N(X) = \bigcup_{x\in X} N(x) - X$ and $N[X] = \bigcup_{x\in X} N[x]$. We denote the minimum degree of G by $\delta(G)$.

2 Lower bound

We will need the following old result of Sauer [10]:

Lemma 1 [10] *Let $g \geq 5$ and $m \geq 4$. Then for every even $n \geq 2(m-1)^{g-2}$, there exists an n-vertex m-regular graph of girth at least g.*

If $2 \leq s \leq 18$, then $3s - 5\sqrt{s} < 2s - 3$ and the construction above described by Myers and Thomason gives the lower bound. Let $s \geq 19$.

First, we describe the complement $\overline{G(s,t)}$ of a brick $G(s,t)$ for the construction. Let q be the number in $\{\lceil\sqrt{3s}\rceil, 1 + \lceil\sqrt{3s}\rceil\}$ such that $t - q$ is even. Observe that for $s \geq 18$,

$$2.5\sqrt{s} \geq 2 + \lceil\sqrt{3s}\rceil \geq q + 1, \tag{1}$$

and $q \geq \lceil\sqrt{3s}\rceil \geq 8$.

By Lemma 1, if $2s + t - q > (q-3)^{2s-1}$, then there exists a $(q-2)$-regular graph $F(s,t)$ of girth at least $2s+1$ with $2s+t-q$ vertices. Since $t > (180s\log_2 s)^{1+6s\log_2 s}$ and $2s > q$, the condition $2s + t - q > (q-3)^{2s-1}$ holds. Let $G(s,t) = \overline{F(s,t)}$.

Claim 2.1 $|E(G(s,t))| \geq 0.5(t + 3s - 2q)(|V(G(s,t))| - s + 1) + (s-1)^2/4.$

PROOF. Since $|V(G(s,t))| = 2s + t - q$ and $F(s,t)$ is $(q-2)$-regular, the statement of the claim is equivalent to the inequality

$$(2s + t - q)(2s + t - 2q + 1) \geq (t + 3s - 2q)(s + t - q + 1) + (s-1)^2/2.$$

Open the parentheses: all factors of t cancel out and we get the inequality $s^2 - s \geq q(s-1) + (s-1)^2/2$ which reduces to $s + 1 \geq 2q$. The last inequality holds for $s \geq 18$. ■

Claim 2.2 *$G(s,t)$ has no $K_{s,t}$-minor.*

PROOF. Suppose to the contrary that there exist $V_0 \subset V(G(s,t))$ and a mapping $f : (V(G(s,t)) - V_0) \to V(K_{s,t})$ as in the definition of a minor. Let X be the set of vertices $x \in V(K_{s,t})$ with $|f^{-1}(x)| \geq 2$ and let $V' = V_0 \cup f^{-1}(X)$. Since $|V(G(s,t))| = 2s + t - q$, we have $|V'| \leq 2(s-q)$.

Let S denote the partite set of s vertices in $K_{s,t}$ and $V'' = f^{-1}(S - X) = f^{-1}(S) - V'$. Then $|V''| \geq q$. Since every $v \in V''$ is adjacent in $G(s,t)$ to every vertex outside of $V'' \cup V'$, the subgraph F' of $F(s,t)$ on $V'' \cup V'$ contains all edges incident with V''. Since the girth of $F(s,t)$ is at least $2s+1$, F' has at most $|V''| - 1$ edges inside V''. Therefore, F' has at least $(q-2)|V''| - (|V''| - 1)$ edges of $F(s,t)$ incident with V''. If the subgraph F_0 of F' induced by these edges has a cycle, at least half of the vertices of this cycle should be in V'' and therefore, the length of this cycle should be at most $2|V''| \leq 2s$, a contradiction to the definition of $F(s,t)$. If F_0 has no cycles, then, by the above, $|V'' \cup V'| \geq 2 + (q-3)|V''|$. Recall that $|V'' \cup V'| \leq |V''| + 2(s-q)$, and therefore we have $2(s-q) \geq 2 + (q-4)|V''| \geq 2 + (q-4)q$, i.e. $2s \geq 2 + q(q-2)$. But this does not hold if $s \geq 18$ and $q \geq \sqrt{3s}$. ■

Claim 2.3 *$F(s,t)$ has an independent set of size $s-1$.*

PROOF. We can construct such a set greedily, since $F(s,t)$ is $(q-2)$-regular and the number of vertices of $F(s,t)$ is greater than $(s-1)(q-1)$. ■

Let I be a clique of size $s-1$ in $G(s,t)$ that exists by Claim 2.3. Define $G(s,t,1) = G(s,t)$ and for $r = 2, \ldots,$ let $G(s,t,r)$ be the union of $G(s,t,r-1)$ and $G(s,t)$ with the common vertex subset I. In other words, we glue every vertex of I in $G(s,t,r-1)$ with its copy in $G(s,t)$.

Claim 2.4 *For every $r \geq 1$,*
(a) $|V(G(s,t,r))| = s - 1 + r(s+t-q+1)$;
(b) $|E(G(s,t,r))| \geq 0.5(t+3s-2q)(|V(G(s,t,r))| - s + 1) + \binom{s-1}{2} - r\frac{s^2}{4}$;
(c) $G(s,t,r)$ has no $K_{s,t}$-minor.

PROOF. Statement (a) is immediate and we will prove (b) and (c) by induction on r. For $r = 1$, (b) is clear from Claim 2.1 and (c) is equivalent to Claim 2.2. Suppose that the claim holds for $r \leq r_0 - 1$.

Suppose first that $G(s,t,r_0)$ contains a $K_{s,t}$-minor G'. Since the common part of $G(s,t,r_0-1)$ and $G(s,t)$ is a clique of size $s-1$ and neither of these graphs has a $K_{s,t}$-minor, each of $G(s,t,r_0-1) - I$ and $G(s,t) - I$ must contain a branching vertex of $K_{s,t}$. But then there are no s internally disjoint paths between these vertices, a contradiction.

By construction, $|V(G(s,t,r_0))| - |V(G(s,t,r_0-1))| = s+t-q+1$ and by Claim 2.1,

$$|E(G(s,t,r_0))| - |E(G(s,t,r_0-1))| = |E(G(s,t))| - \binom{s-1}{2} \geq$$

$$\geq 0.5(t+3s-2q)(s+t-q+1) - \frac{s^2}{4}.$$

This together with the induction assumption proves (b). ■

Now, by part (b) of Claim 2.4, if $|V(G(s,t,r))| \geq st + 4s^2$ (to be crude), then $|E(G(s,t,r))| > 0.5(t+s-2q-2)|V(G(s,t,r))|$. Since this happens whenever $r \geq s+1$, we conclude from (1) that for large r, $G(s,t,r)$ has average degree greater than

$$t + 3s - 2q - 2 \geq t + 3s - 5\sqrt{s}.$$

This proves the lower bound.

3 Graphs with few vertices

In this section, we prove the upper bound of Theorem 3 for graphs with at most $10t/9$ vertices.

Lemma 2 *Let m, s, and n be positive integers such that*

$$n > 10s(30m)^m. \tag{2}$$

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| \leq 0.5mn$ such that

$$\deg(v) \leq 0.6n \qquad \forall v \in V. \tag{3}$$

Then there exist an $L \subset V$ with $|L| \leq m - 1$ and s disjoint pairs (x_i, y_i) of vertices in $G - L$ such that $\operatorname{dist}_{G-L}(x_i, y_i) > 2$ for all $i = 1, \ldots, s$.

PROOF. For every two distinct vertices x, y in G, let $A(x, y)$ denote the set of common neighbors of x and y and $a(x, y) = |A(x, y)|$. For $a(G) = \sum_{x,y \in V} a(x, y)$, we have

$$a(G) \leq \sum_{v \in V} \binom{\deg(v)}{2} \leq \binom{0.6n}{2} \frac{mn}{0.6n} < 0.3n(n-1)m. \tag{4}$$

Let $V_0 = \{v \in V : \deg_G(v) \geq 0.1n/m\}$ and $V_1 = V - V_0$. For every two distinct vertices x, y in G and $i = 0, 1$, let $A_i(x, y) = A(x, y) \cap V_i$ and $a_i(x, y) = |A_i(x, y)|$. Also, for $i = 0, 1$, let $a_i(G) = \sum_{x,y \in V} a_i(x, y)$. Similarly to (4),

$$a_1(G) \leq \sum_{v \in V_1} \binom{\deg(v)}{2} \leq \binom{0.1n/m}{2} \frac{mn}{0.1n/m} < 0.05n(n-1). \tag{5}$$

Let $W = \{(x, y) \in \binom{V}{2} : xy \notin E,\ a_1(x, y) = 0, \text{ and } a_0(x, y) \leq m - 1\}$. Then $|W| \geq \binom{n}{2} - |E| - a_1(G) - a(G)/m$. Hence, by (5) and (4),

$$|W| \geq \binom{n}{2} - \frac{mn}{2} - \frac{n(n-1)}{20} - 0.3n(n-1) = \frac{n}{2}(0.3(n-1) - m) > \frac{n(n-1)}{9}. \tag{6}$$

Consider the auxiliary graph H with the vertex set V and edge set W. By (6), H has a matching M with $|M| \geq n/9$. Since the number of distinct subsets of V_0 of size at most $m - 1$ is $\sum_{k=0}^{m-1} \binom{10m^2}{k} < \binom{10m^2}{m} < (10em)^m$, there exists an $L \subset V_0$ with $|L| \leq m - 1$ such that for the set $M_L = \{xy \in M : A_0(x, y) = L\}$ we have (remembering (2))

$$|M_L| \geq \frac{n/9}{(10em)^m} > s.$$

But then L and the pairs in M_L are what we need. ∎

A graph G is (s, t)-*irreducible* if
(i) $v(G) \geq s$;
(ii) $e(G) \geq 0.5(t + 3s)(v(G) - s + 1)$;
(iii) G has no minor G' possessing (i) and (ii).

For an edge e of a graph G, $t_G(e)$ denotes the number of triangles in G containing e.

Lemma 3 *If G is an (s,t)-irreducible graph and $t \geq s^2$, then*
(a) $v(G) \geq t + 2s + 1$;
(b) $t_G(e) \geq 0.5(t + 3s - 1)$ for every $e \in E(G)$;
(c) if $W \subset V(G)$ and $v(G) - |W| \geq s$, then W is incident with at least $0.5(t + 3s)|W|$ edges; in particular, $\delta(G) \geq 0.5(t + 3s)$;
(d) G is s-connected;
(e) $e(G) < 0.5(t + 3s)v(G)$.

PROOF. The number n of vertices of G should satisfy the inequality $n(n-1)/2 \geq 0.5(t+3s)(n-s+1)$. The roots of the polynomial $f(n) = n^2 - n - (t+3s)(n-s+1)$ are

$$n_{1,2} = \frac{1}{2}\left(t + 3s + 1 \pm \sqrt{(t+3s+1)^2 - 4(t+3s)(s-1)}\right).$$

Observe that $(t+3s+1)^2 - 4(t+3s)(s-1) > (t+s+1)^2$ for $t \geq s^2$. Therefore, either $n < s$ or $n > t + 2s + 1$. This together with (i) proves (a).

Let G_e be obtained from G by contracting e. Then $e(G_e) = e(G) - t_G(e) - 1$. By (iii), $e(G_e) \leq 0.5(t+3s)(v(G_e) - s + 1) - 0.5 = 0.5(t+3s)(v(G) - s) - 0.5$. This together with (ii) yields

$$t_G(e) = e(G) - e(G_e) - 1 \geq 0.5(t+3s) + 0.5 - 1 = 0.5(t + 3s - 1),$$

i.e., (b) holds.

Observe that (c) follows from the fact that $G - W$ does not satisfy (ii).

Assume that there is a partition (V_1, V_0, V_2) of $V(G)$ such that $|V_0| \leq s - 1$ and G has no edges connecting V_1 with V_2. By (c), $|V_1|, |V_2| \geq 0.5(t+3s) - (s-1)$. Let G_i be the subgraph of G induced by $V_0 \cup V_i$, $n_i = v(G_i)$, and $e_i = e(G_i)$, $i = 1, 2$. Since G_1 and G_2 are minors of G, (iii) yields $e_i < 0.5(t+3s)(n_i - s + 1)$ for $i = 1, 2$. But then

$$e(G) \leq e_1 + e_2 < \frac{1}{2}(t+3s)\Big((n_1 - s + 1) + (n_2 - s + 1)\Big).$$

Since $n_1 + n_2 - s + 1 = v(G) + |V_0| - s + 1 \leq v(G)$, this contradicts (ii).

If (e) does not hold for G, then for any $e \in E(G)$, $G - e$ satisfies (ii), a contradiction to (iii). ■

Lemma 4 *Suppose that $t > (180 s \log_2 s)^{1 + 6s\log_2 s}$. If H satisfies (i) and (ii) and $v(H) \leq t + 6s\log_2 s + 2s$, then H has a $K^*_{s,t}$-minor.*

PROOF. Let H_0 be an (s,t)-irreducible minor of H. H_0 also has at most $t + 6s\log_2 s + 2s$ vertices. Suppose that $v(H_0) = n = t + 2s + m$. By Lemma 3(a) and conditions of our lemma, $1 \leq m \leq 6s\log_2 s$. Let G be the complement of H_0. By (ii), we have

$$e(G) \leq \binom{n}{2} - \frac{1}{2}(t+3s)(n-s+1) = \frac{1}{2}(n^2 - n - (n+s-m)(n-s+1) =$$

$$= \frac{\ }{2}((m-2)n + (s-1)(s-m)) < \frac{\ }{2}.$$

By (c) of Lemma 3, the degree of every vertex in G is at most $n-1-0.5(t+3s) = 0.5(t+s)+m-1 < 0.6n$. Applying Lemma 2 to G we find an $L \subset V(G)$ with $|L| \le m-1$ and s disjoint pairs of vertices (x_i, y_i), $i = 1, \ldots, s$ such that $\mathrm{dist}_{G-L}(x_i, y_i) > 2$ for all $i = 1, \ldots, s$. Then contracting the edges $x_i y_i$ in the graph $H_0' = H_0 - L$ we get a $K^*_{s,n-|L|-s}$-minor. ∎

Lemma 5 *Let $m, s, k,$ and n be positive integers such that $k \ge 10$, $s \ge 3$, $m \le 0.1n$*

$$n > 10sk^2, \quad \text{and} \quad (5/9)^{k-2} m < 1. \tag{7}$$

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| \le 0.5mn$ such that

$$\deg(v) \le \frac{5}{9} n \qquad \forall v \in V. \tag{8}$$

Then there exist s pairwise disjoint k-tuples $X_i = \{x_{i,1}, \ldots, x_{i,k}\}$ of vertices in G such that for every $i = 1, \ldots, s$,
(q1) no vertex is a common neighbor of all the vertices in X_i ;
(q2) $G(X_i)$ does not contain any complete bipartite graph $K_{j,k-j}$, $1 \le j \le k/2$.

PROOF. First, we count all k-tuples not satisfying (q1), i.e. all $X = \{x_1, \ldots, x_k\}$ having a common neighbor. This number q_1 is at most

$$\sum_{v \in V} \binom{\deg(v)}{k} \le \binom{\frac{5}{9}n}{k} \frac{mn}{5n/9} \le (5/9)^{k-1} \binom{n}{k} m.$$

Thus by (7), $q_1 < \frac{5}{9}\binom{n}{k}$.

Let $V_0 = \{v \in V : \deg_G(v) \ge n/3\}$ and $V_1 = V - V_0$. The number q_2' of k-tuples X that contain a complete bipartite graph $K_{j,k-j}$, $1 \le j \le k/2$ such that the partite set of size j contains a vertex in V_1 does not exceed

$$\sum_{v \in V_1} \binom{\deg(v)}{\lceil \frac{k}{2} \rceil} \binom{n}{\lfloor \frac{k}{2} \rfloor - 1} \le \binom{n}{\lfloor \frac{k}{2} \rfloor - 1} \binom{n/3}{\lceil \frac{k}{2} \rceil} \frac{mn}{n/3}.$$

Since $k \ge 10$, $m \le 0.1n$, and $n > 10sk^2 \ge 300k$, the last expression is at most

$$\binom{n}{k-1} 3^{-k/2} 3m \le \binom{n}{k} 3^{-0.5k+1} \frac{k}{n-k+1} m \le \frac{1}{80} \binom{n}{k}.$$

Similarly, the number q_2'' of k-tuples X that contain a complete bipartite graph $K_{j,k-j}$, $1 \le j \le k/2$ such that the partite set of size j contains only vertices in V_0 does not exceed

$$\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{|V_0|}{j} \binom{\frac{5}{9}n}{k-j} \le \binom{|V_0| + \frac{5}{9}n}{k} \le \left(\frac{3m + 5n/9}{n} \right)^k \binom{n}{k} \le$$

$$\le \left(\frac{11}{90}\right) \binom{n}{k} < 0.211\binom{n}{k}.$$

Hence the total number q of k-tuples X not satisfying (q1) or (q2) is at most

$$q_1 + q_2' + q_2'' < \binom{n}{k}\left(\frac{5}{9} + \frac{1}{80} + 0.211\right) < 0.78\binom{n}{k}.$$

Therefore, there are at least $0.22\binom{n}{k}$ *good* k-tuples, i.e., k-tuples satisfying (q1) and (q2). Now, we choose disjoint good k-tuples $X_1, \ldots, X_s$ one by one in a greedy manner. Let X_1 be any good k-tuple. Suppose that we have chosen $1 \le i \le s-1$ good k-tuples $X_1, \ldots, X_i$. The set $X = \bigcup_{j=1}^{i} X_j$ meets at most $\binom{n}{k} - \binom{n-k(s-1)}{k}$ good k-tuples. But by (7),

$$\binom{n}{k} - \binom{n-k(s-1)}{k} < \binom{n}{k}\left(1 - \left(\frac{n-sk}{n-k}\right)^k\right) <$$

$$< \binom{n}{k}\left(1 - \left(1 - \frac{sk^2}{n-k}\right)\right) < \frac{1}{10}\binom{n}{k}.$$

Thus, we can choose a good k-tuple X_{i+1} disjoint from X. ■

Lemma 6 *Suppose that $s \ge 3$, $t > (180 s \log_2 s)^{1+6s\log_2 s}$. If H satisfies (i) and (ii) and $v(H) \le 10t/9$, then H has a $K^*_{s,t}$-minor.*

PROOF. Let H_0 be an (s,t)-irreducible minor of H. H_0 also has at most $10t/9$ vertices.

Let $v(H_0) = n = t+m$. By Lemma 4 and conditions of our lemma, $6s\log_2 s + 2s \le m \le t/9$. Let G be the complement of H_0. We want to prove that G satisfies the conditions of Lemma 5 for $k = \max\{10, 2 + \lceil \log_{9/5} m \rceil\}$. Inequalities $k \ge 10$, $s \ge 3$, and $m \le 0.1n$ follow from the definitions under the conditions of our lemma. So does the second part of (7). The inequality $|E(G)| \le 0.5mn$ follows from (ii) as in the proof of Lemma 4. By (c) of Lemma 3, the degree of every vertex in G is at most

$$n - 1 - 0.5(t+3s) = 0.5(t-3s) + m - 1 < 0.5n + (m-3s)/2 < 5n/9.$$

Thus, we need only to verify the first part of (7), namely, $n > 10sk^2$. If $k = 10$, then this is implied by $n > t \ge (180 s\log_2 s)^{1+6s\log_2 s} > 1000s$.

Suppose now that $k = 2 + \lceil \log_{9/5} m \rceil$. Since $m \le t/9$,

$$k = 2 + \lceil \log_{9/5} m \rceil < 3 + \log_{9/5}(t/9) < \log_{9/5} t < 1.2\log_2 t,$$

in order to verify $n > 10sk^2$, it is sufficient to check that

$$t > 10s(1.2\log_2 t)^2. \tag{9}$$

Observe that the derivative of the RHS of (9) with respect to t is equal to $20s(1.2\log_2 t)\frac{\ }{t\ln 2}$ which is less than 1 for $t > (180s\log_2 s)^{1+6s\log_2 s}$. Therefore, it is enough to check (9) for $t = (180s\log_2 s)^{1+6s\log_2 s}$. Since $180s\log_2 s > 10s \cdot 1.2^2$, this would follow from

$$(180s\log_2 s)^{3s\log_2 s} > \log_2(180s\log_2 s)^{1+6s\log_2 s},$$

which is easy to verify. Thus we can apply Lemma 5 to G.

Let $X_1, \ldots, X_s$ be the k-tuples provided by Lemma 5. The conditions (q1) and (q2) mean that every X_i is a connected dominating set in H_0. Thus, H_0 has a $K^*_{s,n-sk}$-minor.

We need now only to check that $n - sk \geq t$, i.e., $sk \leq m$. Observe first that $m \geq 6s\log_2 s + 2s \geq s(6\log_2 3 + 2) > 11s$. This verifies $sk \leq m$ for $k \leq 10$. Let $k = 2 + \lceil \log_{9/5} m \rceil$. As above, $k < 1.2\log_2 m$ and it is enough to verify the inequality $1.2s < m/\log_2 m$ for $m = 6s\log_2 s$. In this case, the last inequality reduces to $1 < \frac{5\log_s}{\log_2(6s\log_2 s)}$ which in turn reduces to $s^5 > 6s\log_2 s$. This is true for $s \geq 3$. ∎

4 Auxiliary statements

Lemma 7 *Let G be a connected graph. If $\delta(G) \geq k$, $|V(G)| = n$, then there exists a partition $V(G) = W_1 \cup W_2 \cup \ldots$ of $V(G)$ such that for every i,*
(a) the subgraph of G induced by $\bigcup_{j=1}^{i} W_j$ is connected;
(b) $|W_i| \leq 3$;
(c)

$$V(G) - \bigcup_{j=1}^{i} N[W_j]| \leq n\left(\frac{n-k-1}{n}\right)^i. \tag{10}$$

Furthermore, one can have $|W_1| = 1$.

PROOF. For $i = 1$, $n\left(\frac{n-k-1}{n}\right)^i = n-k-1$, so we can take $W_1 = \{w_1\}$, where w_1 can be any vertex. Suppose that the lemma holds for $i = m-1$ and let $X_m = V(G) - \bigcup_{j=1}^{m-1} N[W_j]$. Then

$$\sum_{v \in X_m} |N[v]| \geq (k+1)|X_m|$$

and hence there exists some w_m that belongs to at least $(k+1)|X_m|/n$ sets $N[v]$ for $v \in X_m$. We can choose w_m as close to $\bigcup_{j=1}^{i-1} W_j$ as possible. Since every vertex on distance 3 from $\bigcup_{j=1}^{i-1} W_j$ dominates at least $k+1$ vertices in $V(G) - \bigcup_{j=1}^{i-1} W_j$, the distance from $\bigcup_{j=1}^{i-1} W_j$ to w_m is at most 3. Therefore, we can form W_m from w_m and the vertices of a shortest path P_m from $\bigcup_{j=1}^{i-1} W_j$ to w_m. ∎

Lemma 8 *Let $\alpha \geq 2$. If G is a connected graph, $\delta(G) \geq k$, and $n \leq \alpha(k+1)$, then there exists a dominating set $A \subseteq V(G)$ such that $G[A]$ is connected and*

$$|A| \leq 3\log_{\alpha/(\alpha-1)} n. \tag{11}$$

PROOF. Let $V(G) = W_1 \cup W_2 \cup \ldots$ be a partition guaranteed by Lemma 7. Let $m = \lfloor \log_{\alpha/(\alpha-1)} n \rfloor$. Then $A' = \bigcup_{j=1}^m W_j$ does not dominate at most

$$n\left(1-\frac{1}{\alpha}\right)^m = \left(\frac{\alpha}{\alpha-1}\right)^x$$

vertices, where x is the fractional part of $\log_{\alpha/(\alpha-1)} n$. Since $\alpha \geq 2$, we have $\left(\frac{\alpha}{\alpha-1}\right)^x < 2$. Thus, A' dominates all but at most one vertices in G. Suppose that the non-dominated vertex (if exists) is w_0. Since G is connected, there is a common neighbor y_0 of w_0 and A'. Then $A = A' + y_0$ is a connected dominating set in G and $|A| = |A'| + 1 \leq 1 + 3(m-1) + 1 < 3\log_{\alpha/(\alpha-1)} n$. ■

Lemma 9 *Let s, k, and n be positive integers and $\alpha \geq 2$. Suppose that $n \leq \alpha(k+1)$. Let G be a $(3s\log_{\alpha/(\alpha-1)} n)$-connected graph with n vertices and $\delta(G) \geq k + 3(s-1)\log_{\alpha/(\alpha-1)} n$. Then $V(G)$ contains s disjoint subsets $A_1, \ldots, A_s$ such that for every $i = 1, \ldots, s$,*
(i) $G[A_i]$ is connected;
(ii) $|A_i| \leq 3\log_{\alpha/(\alpha-1)} n$;
(iii) A_i dominates $G - A_1 - \ldots - A_{i-1}$.

PROOF. Apply Lemma 8 s times. ■

A subset X of vertices of a graph H is *k-separable* if $X \cup N(X) \neq V(H)$ and $|N(X) - X| \leq k$.

Lemma 10 *Let H be a graph and k be a positive integer. If C is an inclusionwise minimal k-separable set in H and $S = N(C) - C$, then the subgraph of H induced by $C \cup S$ is $(1 + \lceil \frac{k}{2} \rceil)$-connected.*

PROOF. Assume that there is $D \subseteq S \cup C$ with $|D| \leq \lceil \frac{k}{2} \rceil$ that separates $H[S \cup C]$ into H_1 and H_2. Let H_1 be those of the two parts with fewer (or equal) vertices in S. Then the set $S_1 = D \cup (S \cap V(H_1))$ has at most k vertices and is a separating set in H. Moreover, a component of $H - S_1$ is a proper part of C, a contradiction. ■

Lemma 11 *Let G be a $100s\log_2 t$-connected graph. Suppose that G contains a vertex subset U with $t + 100s\log_2 t \leq |U| \leq 3t$ such that $\delta(G[U]) \geq 0.4t + 100s\log_2 t$. Then G has a $K^*_{s,t}$-minor.*

PROOF. Run the following procedure. Let S_1 be a smallest separating set in $G[U]$. If $|S_1| \geq 20s\log_2 t$, then stop. Otherwise, let $U'_1, U'_2, \ldots$ be the components of $G[U] - S_1$. If some of these components has a separating set S_2 with $|S_2| < 20s\log_2 t$, then let $U_1^2, U_2^2, \ldots$ be the components of $G[U] - S_1 - S_2$ and so on. Consider the situation after four such steps (if we did not stop earlier).

Claim 4.1 *If we did not stop after Step 3, then at most two components of $G[U]-S_1-S_2-S_3-S_4$ are not $20s\log_2 t$-connected.*

Proof. Let $H=G[U]-S_1-S_2-S_3-S_4$. By construction, H has at least 5 components and

$$\delta(H)\geq\delta(G[U])-4\cdot 20s\log_2 t\geq 0.4t+20s\log_2 t. \tag{12}$$

It follows that each component of H has more than $0.4t+20s\log_2 t$ vertices. Moreover, if a component H' of H has fewer than $0.8t$ vertices, then each two vertices in H' have at least $40s\log_2 t$ common neighbors, and thus H' is $40s\log_2 t$-connected. Therefore, if some three components of H are not $20s\log_2 t$-connected, then $|U|\geq|V(H)|\geq 3\cdot 0.8t+2\cdot 0.4t=3.2t$, a contradiction.

Claim 4.2 *For some $1\leq m\leq 3$, there are m vertex disjoint subgraphs $H_1,\ldots,H_m$ of $G[U]$ such that*
1) H_i is $20s\log_2 t$-connected for $i=1,\ldots,m$;
2) $\delta(H_i)\geq 0.4t+20s\log_2 t$ for $i=1,\ldots,m$;
3) $|V(H_1)|+\ldots+|V(H_m)|\geq t+m20s\log_2 t$.

Proof. Note that we stopped immediately after Step 4 or earlier. This implies 2). If we stopped before Step 4, then each component of $G[U]-S_1-\ldots$ is $20s\log_2 t$-connected. By Claim 4.1, if we stopped after Step 4, then at least three of the components are $20s\log_2 t$-connected. If we have at least three such components, then together they contain more than $3(0.4t+20s\log_2 t)>t+60s\log_2 t$ vertices. If we have at most two components, then we stopped before Step 2 and the total number of vertices in them is at least $|U|-20s\log_2 t\geq t+80s\log_2 t$. This proves the claim.

To finish the proof of the lemma, we consider 3 cases according to the smallest value of m for which Claim 4.2 holds.

CASE 1. $m=1$. Since $|V(H_1)|\leq|U|\leq 3t$, we have $|V(H_1)|/0.4t\leq 7.5$ and

$$3\log_{\frac{7.5}{6.5}}3t=\frac{3}{\log_2\frac{75}{65}}\log_2 3t<15\log_2 3t\leq 20\log_2 t$$

whenever $t\geq 27$. It follows that we can apply Lemma 9 to H_1. By this lemma, there are s disjoint subsets $A_1,\ldots,A_s$ of $V(H_1)$ such that for every $i=1,\ldots,s$,
(i) $G[A_i]$ is connected;
(ii) $|A_i|\leq 3\log_{\frac{75}{65}}3t\leq 20\log_2 t$;
(iii) A_i dominates $H_1-A_1-\ldots-A_{i-1}$.

Since $|V(H_1)-A_1-\ldots-A_s|\geq t+20s\log_2 t-s\cdot 20\log_2 t=t$, H_1 has a $K^*_{s,t}$-minor.

CASE 2. $m=2$. Since Case 1 does not hold, we know that Statement 3) of Claim 4.2 fails for both H_i, so $|V(H_i)|\leq t+20s\log_2 t\leq 1.2t$ for $i=1,2$. We can apply Lemma 9 to each of H_1 and H_2 with $\alpha=\frac{1.2t}{0.4t}=3$. Hence, there exist disjoint subsets $A^1_1,\ldots,A^1_s$ of $V(H_1)$ and disjoint subsets $A^2_1,\ldots,A^2_s$ of $V(H_2)$ such that for every $i=1,\ldots,s$ and every $j=1,2$,

(i) $G[A_i^j]$ is connected;
(ii) $|A_i^j| \le 3\log_{3/2} 1.2t \le 7\log_2 t$;
(iii) A_i^j dominates $H_j - A_1^j - \ldots - A_{i-1}^j$.

For $j = 1, 2$, let $A_j = \bigcup_{i=1}^s A_i^j$ and $V_j = V(H_j) - A_j$. Since the connectivity of $G - A_1 - A_2$ is at least $100s\log_2 t - 14s\log_2 t$, there are s vertex disjoint V_1, V_2-paths $P_1, \ldots, P_s$ in $G - A_1 - A_2$. We may assume that every P_i has exactly one vertex in V_1 and one vertex in V_2. For $i = 1, \ldots, s$, define $A_i^0 = A_i^1 \cup A_i^2 \cup V(P_i)$. Then by (i), $G[A_i^0]$ is connected for every i. By (iii), each A_i^0 dominates $U_0 = (V_1 \cup V_2) - \bigcup_{j=1}^s V(P_j)$ and A_k^0 for $k > i$. Note that

$$|U_0| \ge |V_1 \cup V_2| - 2s \ge |V(H_1)| + |V(H_2)| - |A_1| - |A_2| - 2s \ge$$

$$\ge t + 40s\log_2 t - 14\log_2 t - 2s > t.$$

Hence $G[V(H_1) \cup V(H_2) \cup \bigcup_{j=1}^s V(P_j)]$ has a $K_{s,t}^*$-minor.

CASE 3. $m = 3$. Since Cases 1 and 2 do not hold, we can assume that $|V(H_i)| \le 0.8t$ for $i = 1, 2, 3$. To see this, suppose without loss of generality that $|V(H_1)| \ge 0.8t$. Then $|V(H_2)| \ge \delta(H_2) > 0.4t$, so

$$|V(H_1)| + |V(H_2)| \ge 1.2t > t + 40s\log_2 t,$$

and Case 2 would apply, a contradiction.

Now we can apply Lemma 9 to each of H_1, H_2, and H_3 with $\alpha = 2$. Hence, there exist disjoint subsets $A_1^j, \ldots, A_s^j$ of $V(H_j)$, $j = 1, 2, 3$ such that for every $i = 1, \ldots, s$ and every $j = 1, 2, 3$,
(i) $G[A_i^j]$ is connected;
(ii) $|A_i^j| \le 3\log_2 0.8t < 3\log_2 t$;
(iii) A_i^j dominates $H_j - A_1^j - \ldots - A_{i-1}^j$.

For $j = 1, 2, 3$, let $U_j = V(H_j) - \bigcup_{i=1}^s A_s^j$. Then

$$|U_1 \cup U_2 \cup U_3| \ge 3(0.4t + 20s\log_2 t) - 3s(3\log_2 t) = 1.2t + 51s\log_2 t.$$

For $j = 1, 2, 3$, choose $X_j \subset U_j$ with $|X_1| = 2s$ and $|X_2| = |X_3| = s$. The connectivity of the graph $H_0 = G - \bigcup_{j=1}^3 \bigcup_{i=1}^s A_s^j$ is at least $100s\log_2 t - 9s\log_2 t = 91s\log_2 t$. Hence there are $2s$ vertex disjoint $(X_1, X_2 \cup X_3)$-paths $P_1, \ldots, P_{2s}$ in H_0. Let us renumber the P_i-s so that every P_i for an odd i is an (X_1, X_2)-path (and every P_i for an even i is an (X_1, X_3)-path). Then we can find $2s$ subpaths $Q_1, \ldots, Q_{2s}$ of $P_1, \ldots, P_{2s}$ such that for every $k = 1, \ldots, s$,
(a) $Q_{2k-1} \cup Q_{2k} \subseteq P_{2k-1} \cup P_{2k}$;
(b) $|V(Q_{2k-1} \cup Q_{2k}) \cap (U_1 \cup U_2 \cup U_3)| \le 4$;
(c) $V(Q_{2k-1} \cup Q_{2k}) \cap U_j \ne \emptyset$ for every $j = 1, 2, 3$.

For $i = 1, \ldots, s$ let $F_i = Q_{2i-1} \cup Q_{2i} \cup A_i^1 \cup A_i^2 \cup A_i^3$. Then
(i) $G[F_i]$ is connected for every i;
(ii) F_i-s are pairwise disjoint;
(iii) F_i dominates $U_1 \cup U_2 \cup U_3 - \bigcup_{k=1}^{2s} Q_k$ and F_j for $j > i$.

Since $|U_1 \cup U_2 \cup U_3 - \bigcup_{k=1}^{2s} Q_k| \ge 1.2t + 91s\log_2 t - 4s$, G has a $K_{s,t}^*$-minor. ■

5 Final argument

Below, $G = (V, E)$ is a minimum counterexample to Theorem 3. In particular, G is (s, t)-irreducible.

CASE 1. G is $200s \log_2 t$-connected. If G has a vertex v with $t + 100s \log_2 t \leq \deg(v) \leq 3t - 1$, then G satisfies Lemma 11 with $U = N[v]$ and we are done. Thus, we can assume that every vertex in G has either 'small' ($< t + 100s \log_2 t$) or 'large' ($\geq 3t$) degree. Let V_0 be the set of vertices of 'small' degree. If $|V_0| > t + 100s \log_2 t$, then there is some $V_0' \subseteq V_0$ such that

$$t + 100s \log_2 t \leq |\bigcup_{v \in V_0'} N[v]| \leq 3t - 1.$$

In this case, we can apply Lemma 11 with $U = \bigcup_{v \in V_0'} N[v]$.

Now, let $|V_0| \leq t + 100s \log_2 t$. By Lemma 3(e), the average degree of G is less than $t + 3s$. Since every vertex outside of V_0 has degree at least $3t$, we get

$$0.5t|V_0| + 3t(n - |V_0|) < (t + 3s)n$$

and hence $n < \frac{2.5|V_0|}{2 - 3s/t} < 3t$. If $n > t + 100s \log_2 t$, then we apply Lemma 11 with $U = V(G)$. If $n \leq t + 100s \log_2 t$, then we are done by Lemma 6.

CASE 2. G is not $200s \log_2 t$-connected. Let S be a separating set with less than $k = \lceil 200s \log_2 t \rceil$ vertices and $V(G) - S = V_1 \cup V_2$ where vertices in V_1 are not adjacent to vertices in V_2. Then each of V_1 and V_2 is a k-separable set. For $j = 1, 2$, let W_j be an inclusion minimal k-separable set contained in V_j and $S_j = N(W_j) - W_j$. By Lemma 10, the graph $G_j = G[W_j \cup S_j]$ is $100s \log_2 t$-connected.

CASE 2.1. $|W_j \cup S_j| \geq t + 100s \log_2 t$ for some $j \in \{1, 2\}$. Then we essentially repeat the argument of Case 1 with the restriction that the vertices v are taken only in W_j. Since by the minimality of G, the number of edges incident to W_j is less than $0.5(t + 3s)|W_j| + 200s \log_2 t|W_j|$, the argument goes through.

CASE 2.2. $|W_j \cup S_j| < t + 100s \log_2 t$ for both $j \in \{1, 2\}$. By Lemma 3(c), we need $|W_j| \geq t - 400s \log_2 t$. Let $H_j = G(W_j)$.

Claim 5.1 *(a)* $\delta(H_j) \geq 0.5t - 200s \log_2 t$;
(b) H_j *is* $400s \log_2 t$*-connected.*

PROOF. The first statement follows from Lemma 3(c). If S_0 is a separating set in H_j with $|S_0| < 400s \log_2 t$, then the smaller part, say, H_0, of $H_j - S_0$ has at most $0.5t + 50s \log_2 t$ vertices and $|S_0 \cup S_j| \leq 600s \log_2 t$. This contradicts Lemma 3(c). ∎

By the above claim and Lemma 9 (for $k = 0.4t$ and $\alpha = 3$), $V(H_j)$ contains s disjoint subsets $A_1^j, \ldots, A_s^j$ such that for every $i = 1, \ldots, s$,
(i) $G[A_i^j]$ is connected;
(ii) $|A_i^j| \leq 3 \log_{3/2} |W_j| < 6 \log_2 |W_j|$;
(iii) A_i^j dominates $W_j - A_1^j - \ldots - A_{i-1}^j$.

Since G is s-connected, $|S_j| \le s$, $j = 1, 2$, and there are s pairwise vertex disjoint S_1, S_2-paths $P_1, \ldots, P_s$. We may assume that the only common vertex of P_i with S_j is p_{ij}. By Lemma 3(b), each p_{ij} has at least $0.5t - 200s \log_2 t$ neighbors in W_j. Thus, we can choose $2s$ distinct vertices q_{ij} such that $q_{ij} \in W_j - \bigcup_{k=1}^{s} A_k^j$ and $p_{ij}q_{ij} \in E(G)$.

Define $F_i = A_i^1 \cup A_i^2 \cup V(P_i) + q_{ij}$, $i = 1, \ldots, s$. Then for every $i = 1, \ldots, s$,
(i) $G[F_i]$ is connected;
(ii) F_i-s are pairwise disjoint;
(iii) F_i dominates $\bigcup_{j=1}^{2} W_j - F_1 \ldots - F_{i-1}$.

Since

$$|\bigcup_{j=1}^{2} W_j - F_1 \ldots - F_{i-1}| \ge 2(t - 400s \log_2 t) - 12s \log_2 2t - 2s > t,$$

G has a $K_{s,t}^*$-minor. ■

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