HIGHLY CONNECTED MULTICOLOURED SUBGRAPHS OF MULTICOLOURED GRAPHS

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ABSTRACT. We consider the following question of Bollobás: given an r-colouring of $E(K_n)$, how large a k-connected subgraph can we find using at most s colours? In [3] the authors provided a partial solution to this problem when $s = 1$ (and n is not too small). Here we shall consider the case $s \geqslant 2$, proving in particular that when $s = 2$ and $r + 1$ is a power of 2 then the answer lies between $\frac{4n}{r+1} - 5kr(r + 2k + 1)$ and $\frac{4n}{r+1} + 4$, that if $r = 2s + 1$ then the answer lies between $(1-1/\binom{r}{s})n - 2\binom{r}{s}k$ and $(1-1/\binom{r}{s})n + 1$, and that phase transitions occur at $2s = r$ and $s = \Theta(\sqrt{r})$. We shall also mention some of the more glaring open problems relating to this question.

1. Introduction

A graph G on $n \geqslant k+1$ vertices is said to be *k-connected* if whenever at most $k - 1$ vertices are removed from G, the remaining vertices are still connected by edges of G. The following question is due to Bollobás: When we colour the edges of the complete graph K_n with at most r colours, how large a k-connected subgraph are we guaranteed to find using only at most s of the colours? In [3], the current authors considered the case $s = 1$ of this question, and proved fairly close bounds in the case that $r - 1$ is a prime power. In this paper we shall continue the investigations of [3] by considering the case $s \geqslant 2$, and in particular the case $s = 2$, and the cases $2s \approx r$ and $s = \Theta(\sqrt{r})$, where the function 'jumps'. The majority of the problem remains wide open however, and so we shall also discuss some open problems and conjectures.

Let us begin by recalling the results and notation of [3]. We note that a gentler introduction into the problem is provided in that paper. Suppose we are given $n, r, s, k \in \mathbb{N}$, and a function $f : E(K_n) \to [r]$, i.e., an r-colouring of the edges of K_n. We assume always that $n \geqslant 2$. Given a subgraph H of K_n, write $c_f(H)$ for the order of the image

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of $E(H)$ under f, i.e., $c_f(H) = |f(E(H))|$, the number of different colours with which f colours H. Now, define $M(f,n,r,s,k) = \max\{|V(H)| : H \subset K_n,\ c_f(H) \leqslant s\}$, the order of the largest k-connected subgraph of K_n using at most s colours from $[r]$. Finally, define $m(n,r,s,k) = \min_f\{M(f,n,r,s,k)\}$. Thus, the question of Bollobás asks for the determination of $m(n,r,s,k)$ for all values of the parameters. We shall state all of our results in terms of $m(n,r,s,k)$.

In [3] fairly tight bounds were given on the function $m(n,r,1,k)$. To be precise, it was shown that $m(n,2,1,k) = n - 2k + 2$ for every $n \geqslant 13k - 15$, that

$$\frac{n-k+1}{2} \ \leqslant\ m(n,3,1,k) \ \leqslant\ \frac{n-k+1}{2} \ +\ 1$$

for every $n \geqslant 480k$, and more generally that

$$\frac{n}{r-1} \ -\ 11(k^2-k)r \ \leqslant\ m(n,r,1,k) \ \leqslant\ \frac{n-k+1}{r-1} \ +\ r$$

whenever $r-1$ is a prime power.

In this paper we shall study the function $m(n,r,s,k)$ when $s \geqslant 2$; in other words, we are looking for large highly-connected multicoloured subgraphs of multicoloured graphs. When trying to work out what happens to $m(n,r,s,k)$ when $s > 1$, one quickly realises that new ideas are going to be needed. For example, none of the extremal examples from [3] are any help to us, since in each of them *any* two colours k-connect almost the entire vertex set! However, we shall find that a number of the tools developed in that paper are still useful to us. We shall recall these results as we go along.

Our main results are as follows. We begin with the case $s = 2$. When also $r+1$ is a power of 2, we have the following fairly tight bounds.

Theorem 1. *Let $n,r,k \in \mathbb{N}$, with $r \geqslant 3$, $r+1$ a power of 2 and $n \geqslant 16kr^2 + 4kr$. Then*

$$\frac{4n}{r+1} \ -\ 5kr(r+2k+1) \ \leqslant\ m(n,r,2,k) \ \leqslant\ \frac{4n}{r+1} \ +\ 4.$$

In particular, if also k and r are fixed, then $m(n,r,2,k) = \dfrac{4n}{r+1} + o(n)$.

We remark that the lower bound in Theorem 1 in fact holds for all $r \geqslant 3$, but the upper bound may increase by a factor of at most 2 when $r+1$ is not a power of 2. For $r = 3$ (and $n \geqslant 13k - 15$), however, we can solve the problem exactly.

Theorem 2. *Let $n,k \in \mathbb{N}$, with $n \geqslant 13k - 15$. Then*

$$m(n,3,2,k) \ =\ n-k+1.$$

Our next result shows that there is a jump at $2s = r$, from $(1-\varepsilon)n$ to $n - 2k + 2$.

Theorem 3. *Let $n, s, k \in \mathbb{N}$, with $n \geqslant \max\left\{2\binom{2s}{s}(k-1)+1, 13k-15\right\}$. Then*

$$m(n, 2s, s, k) \;=\; n - 2k + 2.$$

Moreover, if $2s < r \in \mathbb{N}$, then there exists $\varepsilon = \varepsilon(s, r) > 0$ such that

$$m(n, r, s, k) \;<\; (1-\varepsilon)n$$

for every sufficiently large $n \in \mathbb{N}$.

Moreover, we can determine the maximum possible ε exactly.

Theorem 4. *Let $n, s, k \in \mathbb{N}$ with $s \geqslant 2$ and $n \geqslant 100\binom{2s+1}{s}^2 k^2$, and let $\varepsilon = \varepsilon(s) = \binom{2s+1}{s}^{-1}$. Then*

$$(1-\varepsilon)n \;-\; 2\binom{2s+1}{s}k \;\leqslant\; m(n, 2s+1, s, k) \;\leqslant\; (1-\varepsilon)n \;+\; 1.$$

We have seen that (rather unsurprisingly) when r is very large compared with s, the function $m(n, r, s, k)/n$ is very close to 0, and when s and r are comparable the same function is close to 1. The next theorem shows that the function changes from one of these states to the other rather rapidly. This is another example of a phase transition with respect to s.

Theorem 5. *Let $n, r, s, k \in \mathbb{N}$, with $n \geqslant 16kr^2 + 4kr$. Then*

$$\left(1 - e^{-s^2/4r}\right)n \;-\; 2kr\binom{r}{\lceil s/2 \rceil} \;\leqslant\; m(n, r, s, k) \;\leqslant\; (s+1)\left\lceil \frac{n}{\lfloor \sqrt{2r} \rfloor} \right\rceil.$$

In particular, if $n = n(r) \gg kr\binom{r}{\lceil s/2 \rceil}$ as $r \to \infty$, then

$$\lim_{r\to\infty} \frac{m(n, r, s, k)}{n} = \begin{cases} 0 & \text{if} \quad s \ll \sqrt{r} \\ 1 & \text{if} \quad s \gg \sqrt{r} \end{cases}$$

The rest of the paper is organised as follows. In Section 2 we shall recall the tools developed in [3]; in Section 3 we shall prove Theorems 1 and 2; in Section 4 we shall prove Theorems 3 and 4, and discuss the jump at $2s = r$; in Section 5 we shall prove Theorem 5; and in Section 6 we shall look back on what we have learnt, and point out some of the most obvious questions of the many that remain.

2. Tools

In [3] various techniques were developed to find monochromatic k-connected subgraphs. Many of these tools will prove to be useful to us below, and for the reader's convenience we shall begin by stating them here. We start with the most crucial lemma from [3], which may be proved by induction on $m+n$.

Lemma 6. *Let $q, \ell, m, n \in \mathbb{N}$ with $m, n \geqslant \ell$ and $m+n \geqslant 2\ell+1$. Let G be a bipartite graph with parts M and N of size m and n, respectively. If G has no $(\ell+1)$-connected subgraph on at least q vertices, then*

$$e(G) \leqslant \frac{q(n-\ell)(m-\ell)}{m+n-2\ell} + (\ell^2+\ell)(m+n-2\ell).$$

We shall use Lemma 6 to prove Theorem 1 and Lemma 8, below. We shall use Lemma 8 to prove Theorem 4, but we also consider the result to be interesting in its own right. First however we note the following simple corollary of Lemma 6.

Corollary 7. *Let $c, d, k, m, n \in \mathbb{N}$ with $m, n > k$, and let G be a bipartite graph with parts M and N of size m and n, respectively. Let $f : E(G) \to \mathbb{N}$ be a colouring of the edges of G, and for each $i \in \mathbb{N}$, let $q_k(i)$ be the maximum order of an k-connected subgraph of G which has all edges coloured i. Suppose that all but at most d edges have colours from the set $[c]$. Then*

$$\sum_{i=1}^{c} q_k(i) \geqslant (m+n-2k)\left(1 - \frac{d}{mn} - \frac{ck^2(m+n)}{mn}\right).$$

Proof. Apply Lemma 6 with $\ell = k-1$ and $q = q_k(i)+1$ for each $i \in [c]$, and note that

$$\frac{q(n-\ell)(m-\ell)}{m+n-2\ell} + (\ell^2+\ell)(m+n-2\ell) \leqslant \frac{qmn}{m+n-2k} + k^2(m+n) - 1.$$

Adding the resulting inequalities gives

$$mn - d \leqslant \frac{mn}{m+n-2k}\sum_{i=1}^{c}(q_k(i)+1) + ck^2(m+n) - c.$$

Rearranging the inequality gives the desired result. □

We shall need the following observation of Bollobás and Gyárfás [2].

Observation 1. *For any graph G and any $d \in \mathbb{N}$, either*

(*a*) *G is k-connected, or*
(*b*) *$\exists\, v \in V(G)$ with $d_G(v) \leqslant d+k-3$, or*
(*c*) *$\exists\, K_{p,q} \subset \overline{G}$, with $p+q = |G|-k+1$, and $\min\{p,q\} \geqslant d$.*

Suppose we are given an r-colouring f of the edges of K_n. For each $S \subset [r]$, let $q_k(S)$ denote the maximum order of a k-connected subgraph of K_n, all of the edges of which have colours from S.

Corollary 7 and Observation 1 now allow us to prove the following result, which we shall use in the proof of Theorem 4 to show that *any* s-set of colours gives a large k-connected subgraph. It may be thought of as a stability result for 3-colourings.

Lemma 8. *Let $n,k,r,t \in \mathbb{N}$, with $n > 2t+k$, let f be an r-colouring of $E(K_n)$, and let $S,T,U \subset [r]$ be such that $S \cup T \cup U = [r]$. Then either*

(*a*) $q_k(U) \geqslant n - t$, *or*

(*b*) $q_k(S) + q_k(T) \geqslant n - 2t - 4k - \dfrac{2k^2n^2}{t(n-2t-k)}$.

In particular, if $q_k(U) < n - k\sqrt{n}$ and $n \geqslant 25k^2$, then

$$q_k(S) + q_k(T) \geqslant n - 9k\sqrt{n}.$$

Proof. Let $n,k,r,t \in \mathbb{N}$ with $n > 2t+k$, let f be an r-colouring of $E(K_n)$, and let $S,T,U \subset [r]$ be such that $S \cup T \cup U = [r]$. The result is trivial if $t \leqslant k$, so assume that $t > k$. We divide the problem into two cases, as follows.

Case 1: There exists a complete bipartite subgraph $K_{a,b}$, with $a+b \geqslant n-t-k$, $b \geqslant a \geqslant t$, and all edges having colours from the set $S \cup T$.

We apply Corollary 7 to $K_{a,b}$, with $k=k$, $c=2$ and $d=0$. The lemma says exactly that

$$\begin{aligned} q_k(S) + q_k(T) &\geqslant (a+b-2k)\left(1 - \frac{2k^2(a+b)}{ab}\right) \\ &> a + b - \frac{2k^2(a+b)^2}{ab} - 2k \\ &> n - 2t - 4k - \frac{2k^2n^2}{t(n-2t-k)} \end{aligned}$$

and so we are done in this case.

Case 2: No such bipartite subgraph of K_n exists, and $q_k(U) < n-t$.

Let G be the graph with $V(G) = V(K_n)$ and $E(G) = f^{-1}(U)$, and apply Observation 1 to G with $d = t$. Since $q_k(U) < n$, we know that G is not k-connected. Similarly there does not exist a complete bipartite subgraph $K_{a,b}$ of $\overline{G}$ with $a + b = |G| - k + 1$ and $b \geqslant a \geqslant t$, since

Case 1 does not hold. Hence there must exist a vertex $v_1 \in V(G)$ with $d_G(v_1) \leqslant t+k-3$.

Now let $G_1 = G - v_1 = G[V(G) \setminus \{v_1\}]$, and apply Observation 1 to G_1, again with $d = t$. Again (since $q_k(U) < n-1$, and $|G_1| - k + 1 \geqslant n - t$), there must exist a vertex $v_2 \in V(G_1)$ with $d_{G_1}(v_2) \leqslant t+k-3$. Let $G_2 = G_1 - v_2$, and continue until we have obtained a set $X = \{v_1, \ldots, v_t\} \subset V$ satisfying $|\Gamma_G(v_i) \setminus X| \leqslant t+k-3$ for every $i \in [t]$.

We now apply Corollary 7 to the bipartite graph with parts X and $V \setminus X$, and edges from the set $S \cup T$. The lemma says that

$$\begin{aligned} q_k(S) + q_k(T) &\geqslant (n-2k)\left(1 - \frac{t(t+k-3)}{t(n-t)} - \frac{2k^2 n}{t(n-t)}\right) \\ &> n - 2k - 2(t+k-3) - \frac{2k^2n^2}{t(n-t)} \\ &> n - 2t - 4k - \frac{2k^2n^2}{t(n-2t-k)} \end{aligned}$$

and so we are done in this case as well.

The final part of the lemma follows by letting $t = \lfloor k\sqrt{n} \rfloor$, and noting that $(k\sqrt{n}-1)(n - 2k\sqrt{n} - k) \geqslant kn^{3/2}/3$ if $n \geqslant 25k^2$. $\square$

We shall frequently need to show that specific *bipartite* graphs have large k-connected subgraphs. The following observation from [3] is the basic tool we use to do this.

Lemma 9. *Let G be a bipartite graph with partite sets M and N such that $d(v) \geqslant k$ for every $v \in M$, and $|\Gamma(y) \cap \Gamma(z)| \geqslant k$ for every pair $y, z \in N$. Then G is k-connected.*

The next two lemmas now follow from Lemma 9 by removing a suitably chosen set of 'bad' vertices.

Lemma 10. *Let $a, b, k \in \mathbb{N}$, and let G be a bipartite graph with parts M and N such that $|M| \geqslant 4b + k$, $|N| \geqslant a \geqslant 2k$, and $d(v) \geqslant |M| - b$ for every $v \in N$. Then G has a k-connected subgraph on at least*

$$|G| - \frac{ab}{a-k+1} \; > \; |G| - 2b$$

vertices.

Proof. Let $a, b \in \mathbb{N}$ and G be as described. Let

$$U = \{v \in M : d_G(v) \leqslant k-1\},$$

and observe that each vertex of U sends at least $|N| - k + 1$ non-edges into N, and that there are in total at most $b|N|$ non-edges between M and N (since $d(v) \geqslant |M| - b$ for every $v \in N$). Hence

$$|U|(|N| - k + 1) \;\leqslant\; b|N|,$$

and so

$$|U| \;\leqslant\; \frac{b|N|}{|N| - k + 1} \;\leqslant\; \frac{ab}{a - k + 1} \;<\; 2b,$$

since the function $\dfrac{bx}{x - k + 1}$ is decreasing for $x > k - 1$, and $|N| \geqslant a \geqslant 2k$. Now, consider the bipartite graph $G' = G[M \setminus U, N]$. Each vertex of $M \setminus U$ has degree at least k in G', by the definition of U, and each pair of vertices of N have at least k common neighbours in M, since $|M| \geqslant 4b + k$, so $|M \setminus U| \geqslant 2b + k$, and each vertex of N has at most b non-neighbours in M. Thus, by Lemma 9, G' is k-connected, and has order

$$|G| - |U| \;\geqslant\; |G| - \frac{ab}{a - k + 1} \;>\; |G| - 2b.$$

$\square$

The following easy lemma is very similar to Lemma 15 of [3], but a little stronger. In particular, we have removed the requirement that $3|M| \geqslant |N|$.

Lemma 11. *Let $k \in \mathbb{N}$, and let G be the complete bipartite graph with parts M and N, where $|N| \geqslant |M| \geqslant 15k$. Let f be an r-colouring of $E(G)$, and let $S, T, U \subset [r]$ be such that $S \cup T \cup U = [r]$. Suppose that*

(a) $|\{v \in N : f(uv) \in S\}| \leqslant k$ for every $u \in M$, and
(b) $|\{u \in M : f(uv) \in T\}| \leqslant k$ for every $v \in N$.

Then there exists a k-connected subgraph of G, using only colours from U, and avoiding at most $5k$ vertices of M and $2k$ vertices of N. In particular, $q_k(U) \geqslant |G| - 7k$.

Proof. We may assume that $r = 3$, and that $S = \{1\}$, $T = \{2\}$ and $U = \{3\}$. Let $k, m, n \in \mathbb{N}$ with $n \geqslant m \geqslant 15k$, let $|M| = m$ and $|N| = n$, and let f be a 3-colouring of $E(G)$ satisfying the conditions of the lemma. Let

$S_M = \{v \in M : v$ sends at most $3n/5$ edges of colour 3 into $N\}$, and

$S_N = \{v \in N : v$ sends at most $6k$ edges of colour 3 into $M\}$

be sets of 'bad' vertices. We shall remove the bad sets and apply Lemma 9.

We need to bound $|S_M|$ and $|S_N|$ from above. Since each vertex of M has at most k incident edges of colour 1, we have $|f^{-1}(1)| \leqslant km$, and similarly $|f^{-1}(2)| \leqslant kn$. Also, since each vertex of S_M has at least $2n/5$ incident edges of colour 1 or 2, we have $|f^{-1}(1)| + |f^{-1}(2)| \geqslant |S_M|(2n/5)$. Finally, each vertex of S_N has at most $6k$ incident edges of colour 3, and at most k incident edges of colour 2, so has at least $(m - 7k)$ incident edges of colour 1. Hence $|f^{-1}(1)| \geqslant |S_N|(m - 7k)$. Thus

$$|S_M| \;\leqslant\; \frac{5}{2n}\Big(|f^{-1}(1)| + |f^{-1}(2)|\Big) \;\leqslant\; \frac{5k(m+n)}{2n} \;\leqslant\; 5k, \quad \text{and}$$

$$|S_N| \;\leqslant\; \frac{|f^{-1}(1)|}{m-7k} \;\leqslant\; \frac{km}{m-7k} \;\leqslant\; 2k,$$

since $m \geqslant 14k$.

Now, let $M' = M \setminus S_M$ and $N' = N \setminus S_N$, and let H be the bipartite graph with vertex set $M' \cup N'$, and edge set $f^{-1}(3)$. If $x \in N'$, then x sends at least $6k$ edges of colour 3 into M, so

$$d_H(x) \;\geqslant\; 6k - |S_M| \;\geqslant\; 6k - 5k \;=\; k,$$

and similarly if $y, z \in M'$, then

$$\begin{aligned} |\Gamma_H(y) \cap \Gamma_H(z)| \;&\geqslant\; 3n/5 \;+\; 3n/5 \;-\; n \;-\; |S_N| \\ &=\; n/5 - |S_N| \;\geqslant\; 3k - 2k \;=\; k, \end{aligned}$$

since $n \geqslant 15k$, so the conditions of Lemma 9 are satisfied. Thus by Lemma 9, H is k-connected. Since also $|M \setminus V(H)| = |S_M| \leqslant 5k$ and $|N \setminus V(H)| = |S_N| \leqslant 2k$, H is the desired subgraph. $\square$

The results above will be our main tools in the sections that follow. However, we shall also use the following well-known theorem of Mader [4] in the proofs of Theorems 1 and 5.

Mader's Theorem. *Let $\alpha \in \mathbb{R}$, and let G be a graph with average degree α. Then G has an $\alpha/4$-connected subgraph.*

Mader's Theorem implies that a monochromatic $(n-1)/4r$-connected subgraph exists in any r-colouring of $E(K_n)$ (to see this, simply consider the colour which is used most frequently). This subgraph is k-connected if $n \geqslant 4kr + 1$, and has at least $(n-1)/4r + 1$ vertices. It is this weak bound that we shall use.

We also state the following result from [3] here, so that we may refer to it more easily. We shall use Theorem 12 to prove the lower bounds in Theorems 2 and 3.

Theorem 12. *Let $n, k \in \mathbb{N}$, with $n \geqslant 13k - 15$. Then*

$$m(n, 2, 1, k) \;=\; n - 2k + 2.$$

Finally we make some simple observations about k-connected graphs.

Observation 2. *Let G be a graph, and $v \in V(G)$. If $G - v$ is k-connected and $d(v) \geqslant k$, then G is also k-connected.*

Given graphs G and H, define $G \cup H$ to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Observation 3. *Let $k \in \mathbb{N}$, and H_1 and H_2 be k-connected subgraphs of a graph G. If there exist k vertices $\{v_1, \ldots, v_k\} \subset V(H_1)$ such that $|\Gamma(v_i) \cap V(H_2)| \geqslant k$ for each $i \in [k]$, then $H_1 \cup H_2$ is k-connected.*

Observation 4. *Let $k \in \mathbb{N}$. If G and H are k-connected graphs, and $|V(G) \cap V(H)| \geqslant k$, then the graph $G \cup H$ is also k-connected.*

Throughout, we shall write V for $V(K_n)$. For any undefined terms see either [1] or [3].

3. The case $s = 2$

We begin at the bottom, with the case $s = 2$. We shall be able to give fairly tight bounds on $m(n, r, 2, k)$ for infinitely many value of r. We begin with a construction, which will give us our upper bound.

Lemma 13. *For every $n, r, s, k \in \mathbb{N}$, we have*

$$m(n, r, s, k) \;\leqslant\; m(n, r, s, 1) \;\leqslant\; 2^s \left\lceil \frac{n}{2^{\lfloor \log_2(r+1) \rfloor}} \right\rceil \;<\; 2^s \left\lceil \frac{2n}{r+1} \right\rceil.$$

In particular, if $r + 1$ is a power of 2 and a divisor of n, then

$$m(n, r, s, k) \leqslant \frac{2^s n}{r+1}.$$

Proof. Let $n, r, s, k \in \mathbb{N}$, and let $R = \lfloor \log_2(r + 1) \rfloor$. We shall define a $(2^R - 1)$-colouring $f : E(K_n) \to \{0, 1\}^R \setminus \mathbf{0}$ of the edges of K_n. First partition $V = V(K_n)$ into 2^R subsets $\{V_x : x \in \{0, 1\}^R\}$ of near-equal size (i.e., $|(|V_x| - |V_y|)| \leqslant 1$ for every pair x, y). Now if $i \in V_x$, $j \in V_y$ and $x \neq y$, then let $f(ij) = x - y \pmod 2$; for the remaining edges choose f arbitrarily.

Choose a subset $S \subset \{0, 1\}^R \setminus \mathbf{0}$ of size s, a vector $x \in \{0, 1\}^R$ and a vertex $v \in V_x$. Let G be the graph with vertex set V and edge set $f^{-1}(S)$, and let $\mathcal{P}(S) = \{y_1 + \ldots + y_t \pmod 2 \in \{0, 1\}^R : t \in \mathbb{N}$, and $y_i \in S$ for each $i \in [t]\}$. Note that $|\mathcal{P}(S)| \leqslant 2^s$.

Now, there is a path from v to a vertex $u \in V_y$ using only edges of S if and only if $x - y \in \mathcal{P}(S)$, since such a path corresponds to a sum of vectors from S. Hence the component of v in G is exactly $\bigcup\{V_y : x - y \in \mathcal{P}(S)\}$.

Since v and S were arbitrary, $|V_y| \leqslant \lceil n/2^R \rceil$ for each $y \in \{0,1\}^R$, and $|\mathcal{P}(S)| \leqslant 2^s$, it follows that in the colouring f, there is no 1-connected subgraph using at most s colours on more than

$$2^s \left\lceil \frac{n}{2^R} \right\rceil = 2^s \left\lceil \frac{n}{2^{\lfloor \log_2(r+1) \rfloor}} \right\rceil$$

vertices. This proves that $m(n,r,s,1) \leqslant 2^s \left\lceil \frac{n}{2^{\lfloor \log_2(r+1) \rfloor}} \right\rceil$; the remaining inequalities are trivial. □

The following corollary is immediate from the lemma; we state it just for emphasis.

Corollary 14. *If $n, r, k \in \mathbb{N}$, and $r+1$ is a power of 2, then*

$$m(n,r,2,k) \ \leqslant \ 4 \left\lceil \frac{n}{r+1} \right\rceil.$$

We shall prove an almost matching lower bound on $m(n,r,2,k)$. To help the reader (and because we can prove a stronger result in this case), we begin with the case $k=1$. We shall need the following result from [3].

Lemma 15. *The order of the largest monochromatic component of an r-colouring of $E(K_{m,n})$ is at least $\dfrac{m+n}{r}$.*

Theorem 16. *Let $n, r \in \mathbb{N}$, with $r \geqslant 3$. Then*

$$m(n,r,2,1) \geqslant \frac{4n}{r+1}.$$

Proof. Let $n, r \in \mathbb{N}$, with $r \geqslant 3$, and let f be an r-colouring of $E(K_n)$. We shall show that there exists a connected subgraph of K_n, on at least $\dfrac{4n}{r+1}$ vertices, using only at most two colours in the colouring f. For aesthetic reasons, we shall assume that $r+1 \,|\, 4n$ (otherwise the proof is almost the same, but slightly messier). Let G be the largest connected *monochromatic* subgraph of K_n, let $A = V(G)$, and let $|A| = \dfrac{cn}{r+1}$. If $c \geqslant 4$ then we are done, so assume that $c < 4$.

Let $B = V \setminus A$, and without loss of generality, assume that the edges of G all have colour 1. Then since G is maximal, no edge in the bipartite graph $H = K_n[A,B]$ has colour 1, so some colour occurs at least $\dfrac{|A||B|}{r-1}$ times in H. Again without loss let this colour be 2, and let $B_2 = \{u \in B : f(uv) = 2 \text{ for some } v \in A\}$ be the set of vertices of B which are incident to some edge of H of colour 2.

Suppose first that $|B_2| \geqslant \dfrac{(4-c)n}{r+1}$. Then the set $A \cup B_2$ is connected by colours 1 and 2, and $|A \cup B_2| \geqslant \dfrac{4n}{r+1}$, so we are done. So assume that $|B_2| < \dfrac{(4-c)n}{r+1}$, and choose a set B_2' such that $B_2 \subset B_2' \subset B$, and $|B_2'| = \dfrac{(4-c)n}{r+1}$.

We apply Lemma 15 to the bipartite graph H_2 with parts A and B_2', and edges of colour 2. We have

$$\begin{aligned} e(H_2) &\geqslant \frac{|A|\,|B|}{r-1} = \left(\frac{1}{r-1}\right)\left(\frac{cn}{r+1}\right)\left(n - \frac{cn}{r+1}\right) \\ &= \left(\frac{r+1-c}{(4-c)(r-1)}\right)\left(\frac{cn}{r+1}\right)\left(\frac{(4-c)n}{r+1}\right) \\ &= \left(\frac{r+1-c}{(4-c)(r-1)}\right)|A|\,|B_2'|, \end{aligned}$$

so Lemma 15 implies that there exists a connected subgraph of H_2 on at least

$$\frac{e(H_2)(|A| + |B_2'|)}{|A|\,|B_2'|} \geqslant \left(\frac{(r+1-c)}{(4-c)(r-1)}\right)\left(\frac{4n}{r+1}\right)$$

vertices, since $|A| + |B_2'| = \dfrac{4n}{r+1}$. This subgraph is monochromatic, and so, since G was chosen to be the largest monochromatic subgraph, we have

$$\frac{4(r+1-c)n}{(4-c)(r-1)(r+1)} \leqslant \frac{cn}{r+1},$$

which implies that

$$(r-1)c^2 - 4rc + 4(r+1) \leqslant 0,$$

since $c < 4$ and $r > 1$. The quadratic factorises as $(c-2)((r-1)c - 2(r+1)) \leqslant 0$, so we have $2 \leqslant c \leqslant \dfrac{2(r+1)}{r-1}$.

We shall only need that $c \geqslant 2$. For suppose some vertex $u \in B$ sends edges of only one colour into A, i.e., $|\{i \in [r] : f(uv) = i$ for some $v \in A\}| = 1$. Let that colour be j, and consider the star, centred at u, with edges of colour j. It is monochromatic, connected, and has order larger than G, a contradiction. Thus every vertex in B sends edges of at least two different colours into A, and so, by the pigeonhole principle, some colour (ℓ, say) is sent by at least $\dfrac{2|B|}{r-1}$ different vertices of B.

Let $D = \{u \in B : f(uv) = \ell \text{ for some } v \in A\}$. We have

$$\begin{aligned}|A \cup D| &\geqslant \frac{cn}{r+1} + \frac{2|B|}{r-1} = \frac{n}{r+1}\left(c + \frac{2(r+1-c)}{r-1}\right) \\ &= \frac{n}{r+1}\left(\frac{c(r-3)+2(r+1)}{r-1}\right) \\ &\geqslant \frac{n}{r+1}\left(\frac{2(r-3)+2(r+1)}{r-1}\right) = \frac{4n}{r+1},\end{aligned}$$

the last inequality following because $c \geqslant 2$ and $r \geqslant 3$. The vertices of $A \cup D$ are connected by edges of colour 1 and ℓ, so we are done. $\square$

We now modify the proof of Theorem 16 to prove Theorem 1. We shall use Mader's Theorem, and Lemmas 6 and 10.

Proof of Theorem 1. Let $n, r \in \mathbb{N}$, with $r \geqslant 3$ and $n \geqslant 16kr^2 + 4kr$. The upper bound follows by Lemma 13 if $r+1$ is a power of 2; it remains to prove the lower bound. If $n < kr(r+1)(r+2k+1)$ then the result is trivial, so assume $n \geqslant kr(r+1)(r+2k+1)$.

Let f be an r-colouring of the edges of K_n. We shall show that there exists a connected subgraph of K_n, on at least $\dfrac{4n}{r+1} - 5kr(r+2k+1)$ vertices, using only at most two colours in the colouring f. Let G be the largest k-connected *monochromatic* subgraph of K_n, let $A = V(G)$, and let $|A| = \dfrac{cn}{r+1}$. Following the proof of Theorem 16, we shall show that $c \geqslant 2 - \dfrac{17kr^2(r+2k)}{n}$. Suppose for a contradiction that $c < 2 - \dfrac{17kr^2(r+2k)}{n}$.

Let $B = V \setminus A$, and without loss of generality, assume that the edges of G all have colour 1. Now, since G is maximal, no vertex in B sends more than $k-1$ edges of colour 1 into A, by Observation 2. Thus in the bipartite graph $H = K_n[A, B]$, some colour occurs at least $\dfrac{(|A|-k+1)|B|}{r-1}$ times; without loss let this colour be 2. Let $B_2 = \{u \in B : |\{v \in A : f(uv) = 2\}| \geqslant k\}$ be the set of vertices of B which are incident to at least k edges of H of colour 2.

Suppose first that $|B_2| \geqslant \dfrac{(4-c)n}{r+1}$. Then the set $A \cup B_2$ is k-connected by colours 1 and 2 by Observation 2, and $|A \cup B_2| \geqslant \dfrac{4n}{r+1}$, so we are done. So assume that $|B_2| < \dfrac{(4-c)n}{r+1}$, and choose a set B_2' such that $B_2 \subset B_2' \subset B$, and $|B_2'| = \left\lfloor \dfrac{(4-c)n}{r+1} \right\rfloor$.

We shall apply Lemma 6 to the bipartite graph H_2 with parts A and B_2', and edges of colour 2. First note that

$$e(H_2) \;\geqslant\; \frac{(|A| - r(k-1))|B|}{r-1}, \tag{1}$$

since we discarded at most $(k-1)|B|$ edges of colour 2 from H when forming H_2. Let $\ell = k-1$; we must check that $|A|, |B_2'| \geqslant \ell$ and $|A| + |B_2'| \geqslant 2\ell + 1$. These bounds follow because $n > 4kr$, so

$$|A| \;\geqslant\; \frac{n-1}{4r} + 1 \;>\; k$$

by Mader's Theorem, and

$$|B_2'| \;\geqslant\; \frac{(4-c)n}{r+1} - 1 \;>\; \frac{2n}{r-1} - 1 \;>\; k,$$

since we assumed that $c < 2$.

So, by Lemma 6, if there does not exist a k-connected subgraph of H_2 on at least q vertices, then

$$\begin{aligned}
e(H_2) \;&\leqslant\; \frac{q(|A| - \ell)(|B_2'| - \ell)}{|A| + |B_2'| - 2\ell} \;+\; (\ell^2 + \ell)(|A| + |B_2'| - 2\ell) \\
&\leqslant\; \frac{q\Big(\dfrac{cn}{r+1}\Big)\Big(\dfrac{(4-c)n}{r+1}\Big)}{\dfrac{4n}{r+1} - 2k} \;+\; (\ell^2 + \ell)\left(\frac{4n}{r+1}\right) \\
&\leqslant\; \frac{qc(4-c)n^2}{(r+1)(4n - 2k(r+1))} \;+\; k^2\left(\frac{4n}{r+1}\right),
\end{aligned} \tag{2}$$

since $|A| + |B_2'| \geqslant \dfrac{4n}{r+1} - 1$ and $k = \ell + 1$. Combining (1) and (2), we get

$$\begin{aligned} q \;&\geqslant\; \frac{(r+1)(4n-2k(r+1))}{c(4-c)n^2}\left(\frac{\big(|A| - r(k-1)\big)|B|}{r-1} - k^2\left(\frac{4n}{r+1}\right)\right) \\ &\geqslant\; \frac{4n(r+1) - 2k(r+1)^2}{c(4-c)n^2}\left(\frac{\big(cn - kr(r+1)\big)(r+1-c)n}{(r-1)(r+1)^2} - \frac{4k^2 n}{r+1}\right) \\ &=\; \frac{4(r+1-c)n}{(4-c)(r-1)(r+1)} \;-\; \left(\frac{4kr(r+1-c)}{c(4-c)(r-1)} + \frac{16k^2}{c(4-c)} + \frac{2k(r+1-c)}{(4-c)(r-1)}\right) \\ &\qquad\qquad + \frac{1}{n}\left(\frac{2k^2 r(r+1)(r+1-c)}{c(4-c)(r-1)} + \frac{8k^3(r+1)}{c(4-c)}\right) \\ &>\; \frac{4(r+1-c)n}{(4-c)(r-1)(r+1)} \;-\; 17k(r+2k), \end{aligned} \tag{3}$$

since $\dfrac{r+1-c}{r-1} < 2$ and $\dfrac{1}{4} < c < 2$, so $c(4-c) > \dfrac{1}{2}$.

Inequality (3) holds if there does not exist a k-connected subgraph of H_2 on at least q vertices. Therefore, since G was chosen to be the largest k-connected monochromatic subgraph of K_n, and $|G| = \dfrac{cn}{r+1}$, we have

$$\frac{cn}{r+1} \;+\; 1 \;>\; \frac{4(r+1-c)n}{(4-c)(r-1)(r+1)} \;-\; 17k(r+2k),$$

which implies

$$(c-2)((r-1)c - 2(r+1)) \;<\; \frac{68kr^2(r+2k)}{n},$$

as in the proof of Theorem 16. Now, $c < 2$, so $(r-1)c - 2(r+1) < -4$, and thus

$$2 - c \;<\; \frac{17kr^2(r+2k)}{n},$$

so $c \geqslant 2 - \dfrac{17kr^2(r+2k)}{n}$, as claimed.

Now, for each $j \in [r]$ define

$$C_j = \{v \in B : |\{u \in A : f(uv) \neq j\}| \leqslant kr\},$$

and suppose that $|C_j| \geqslant 2kr$ for some colour $j \in [r]$, i.e., there are at least $2kr$ distinct vertices in B which each send at least $|A| - kr$ edges of colour j into A.

We shall apply Lemma 10 to obtain a contradiction. Let F be the bipartite graph with parts A and C_j, and edges of colour j, and let $a = 2kr$ and $b = kr$. Now, $|C_j| \geqslant 2kr \geqslant 2k$, and $|A| \geqslant \dfrac{n-1}{4r} + 1 \geqslant 4kr + k$ (by Mader's Theorem, and because $n \geqslant 16kr^2 + 4kr$), and $d_F(v) \geqslant |A| - kr$ for every $v \in C_j$, by the definition of C_j.

Thus by Lemma 10, there exists a k-connected subgraph F' of F on more than $|A| + |C_j| - 2b$ vertices. But $|C_j| \geqslant 2kr = 2b$, so $|V(F')| > |A|$, and F' is monochromatic. This is a contradiction, since G was chosen to be the largest monochromatic k-connected subgraph of K_n.

So for each colour $j \in [r]$, there are at most $2kr$ distinct vertices in B which send at least $|A| - kr$ edges of colour j into A. We remove these vertices from B to obtain

$$B' = \{v \in B : |\{u \in A : f(uv) \neq j\}| > kr \text{ for every } j \in [r]\},$$

with $|B'| \geqslant |B| - 2kr(r-1)$ (note that $|C_1| = 0$). Now, for each vertex $v \in B'$, we have $|\{i \in [r] : |\{u \in A : f(uv) = i\}| \geqslant k\}| \geqslant 2$, i.e., v sends at least k edges of at least two different colours into A. Therefore, by the pigeonhole principle, there must exist a colour, ℓ say, such that at least $\dfrac{2|B'|}{r-1}$ vertices of B' send at least k edges of colour ℓ into A.

Let $D = \{v \in B : |\{u \in A : f(uv) = \ell\}| \geqslant k\}$. We have

$$\begin{aligned}
|A \cup D| &\geqslant \frac{cn}{r+1} + \frac{2|B'|}{r-1} \geqslant \frac{cn}{r+1} + \frac{2|B|}{r-1} - 4kr \\
&= \frac{n}{r+1}\left(c + \frac{2(r+1-c)}{r-1}\right) - 4kr \\
&= \frac{n}{r+1}\left(\frac{c(r-3) + 2(r+1)}{r-1}\right) - 4kr \\
&\geqslant \frac{4n}{r+1} - \frac{17kr^2(r+2k)(r-3)}{4(r-1)(r+1)} - 4kr \\
&> \frac{4n}{r+1} - 5kr(r+2k+1)
\end{aligned}$$

since $c \geqslant 2 - \dfrac{17kr^2(r+2k)}{n}$ and $r \geqslant 3$. Now by Observation 2, the subgraph of K_n with vertex set $A \cup D$ and all edges of colour 1 or ℓ is k-connected, so we are done. $\square$

When $r = 3$ we can do better than Theorem 1; in fact we can determine the function $m(n, 3, 2, k)$ exactly when $n \geqslant 13k - 15$. The alert reader will have noticed that this is the same bound on n as we obtained in Theorem 12 – this is not coincidence, the bound is necessary because we shall use Theorem 12 in the proof of Theorem 2!

The following simple construction gives us our upper bound.

Lemma 17. *Let $n, k \in \mathbb{N}$. If $n \leqslant 3k-3$ then $m(n,3,2,k) = 0$. If $n \geqslant 3k-2$ then $m(n,3,2,k) \leqslant n-k+1$.*

Proof. Let $n, k \in \mathbb{N}$ with $n \geqslant 3k-2$. Let A, B and C be pairwise disjoint subsets of $V = V(K_n)$, each of size $k-1$, and let $W = V \setminus (A \cup B \cup C)$. Colour the edges between A and $B \cup W$ with colour 1, those between B and $C \cup W$ with colour 2, and those between C and $A \cup W$ with colour 3. Colour the edges inside the sets arbitrarily.

Let H be a k-connected subgraph on at least $n-k+1$ vertices, using at most two colours, and let these colours be 1 and 2 (the proof in the other cases is identical). Let $V(H) = X$. Since $n \geqslant 3k-2$, $|X| \geqslant 2k-1$, so the set $X \cap (A \cup W)$ is non-empty. Let $u \in X \cap (A \cup W)$. Now $X \cap C = \emptyset$, since if $v \in X \cap C$, then u and v are disconnected in $H[X \setminus B]$, which is a contradiction, since $|X \cap B| \leqslant |B| = k-1$. Since $X \cap C = \emptyset$ and $|C| = k-1$, we have $|X| \leqslant n-k+1$.

We have shown that in the colouring described above, there is no k-connected subgraph using only two colours on more than $n-k+1$ vertices. Therefore $m(n,3,2,k) \leqslant n-k+1$ when $n \geqslant 3k-2$.

Now let $n, k \in \mathbb{N}$ with $n \leqslant 3k-3$. Partition V into parts A, B and C, each of size at most $k-1$, and colour the edges between the parts as above: colour 1 between A and B, colour 2 between B and C, and colour 3 between C and A. This time, however, colour the edges inside A with colour 2, those inside B with colour 3, and those inside C with colour 1. Now it is easy to check that there is no k-connected subgraph using only two colours, so $m(n,3,2,k) = 0$ as claimed. $\square$

We now prove the matching lower bound when $n \geqslant 13k-15$. The argument is similar to the proof of Theorem 12 in [3] – just one extra idea is needed.

Proof of Theorem 2. The upper bound follows from Lemma 17, and for $k = 1$ the result is trivial, so let $n, k \in \mathbb{N}$ with $k \geqslant 2$ and $n \geqslant 13k-15$, and let f be a 3-colouring of the edges of K_n. We shall find a k-connected subgraph H of K_n, using at most 2 colours of f, on at least $n-k+1$ vertices.

For $i = 1,2,3$, let $G^{(i)}$ denote the graph with vertex set $V = V(K_n)$ and edge set $f^{-1}(i)$ (the edges of colour i), and for each pair $\{i,j\} \subset \{1,2,3\}$, let $G^{(i,j)}$ denote the subgraph with vertex set V and edge set $f^{-1}(i) \cup f^{-1}(j)$ (the edges of colour i or j).

We shall first find two k-connected subgraphs, using at most two colours each, which cover the vertex set V. Since $n \geqslant 13k - 15$, by Theorem 12 either $G^{(1,2)}$ or $G^{(3)}$ contains a k-connected subgraph H on at least $n - 2k + 2 \geqslant 11k - 13 > 2k - 1$ vertices. Suppose that H is in $G^{(3)}$, and let $V(H) = X$. Let A be the set of vertices of $V \setminus X$ which send at least k edges of colour 1 or 3 into X, and let B be the set of vertices of $V \setminus X$ which send at least k edges of colour 2 or 3 into X. Since $|X| \geqslant 2k-1$, we have $A \cup B = V \setminus X$. Without loss of generality, let $|A| \geqslant |B|$. Now $G^{(1,3)}[X \cup A]$ is k-connected, by Observation 2, and

$$|X \cup A| \;\geqslant\; |X| + \frac{n - |X|}{2} \;\geqslant\; n - k + 1,$$

since $|X| \geqslant n - 2k + 2$, so we have found the desired subgraph.

So we may assume that $G^{(1,2)}$ contains a k-connected subgraph on at least $n-2k+2$ vertices, and similarly for $G^{(1,3)}$ and $G^{(2,3)}$. Let Y be the vertex set of the largest k-connected subgraph in $G^{(1,2)}$, and let Z be the vertex set of the largest k-connected subgraph in $G^{(1,3)}$. Since $|Y|, |Z| \geqslant n - 2k + 2$, $n \geqslant 13k - 15$ and $k \geqslant 2$ we have

$$|Y \cap Z| \;\geqslant\; n - 4k + 4 \;\geqslant\; 9k - 11 \;>\; 2k - 1.$$

We claim that $Y \cup Z = V$. To see this, suppose there is a vertex $v \in V \setminus (Y \cup Z)$. Since $|Y \cap Z| \geqslant 2k-1$, v must send at least k edges of colour 1 or 2, or at least k edges of colour 1 or 3 into $Y \cap Z$. Without loss of generality, assume that v sends at least k edges of colour 1 or 2. Then $G^{(1,2)}[Y \cup \{v\}]$ is k-connected by Observation 2, contradicting the maximality of Y. So $Y \cup Z = V$, as claimed, and we have found two k-connected bichromatic subgraphs which cover V.

Now, let $C = Y \setminus Z$, and $D = Z \setminus Y$. If $|Z| \geqslant n-k+1$ then $G^{(1,3)}[Z]$ is the desired k-connected subgraph, so assume not. Therefore $|C| \geqslant k$, and similarly we may assume that $|D| \geqslant k$. We wish to apply Lemma 9 to the bipartite graph $G' = G^{(2,3)}[Y \cap Z, C \cup D]$, so let $M' = Y \cap Z$ and $N = C \cup D$. We must first remove the 'bad' vertices, of degree at most $k-1$ in G', from the graph. As in the proof of Lemma 10, define

$$U = \{v \in M' : d_{G'}(v) \leqslant k - 1\}.$$

We shall show that $|U| \leqslant k - 1$.

For each $i \in \{1, 2, 3\}$, let $r(i) = |f^{-1}(i) \cap E(C, D)|$ be the number of edges between C and D that are coloured i. Since Z is maximal, each vertex of C can send at most $k - 1$ edges of colour 1 or 3 into Z, so $G^{(1,3)}[C, Z]$ has at most $|C|(k-1)$ edges. Therefore, $G^{(1)}[C, Y \cap Z]$ has at most $|C|(k-1) - r(1) - r(3)$ edges. Similarly, $G^{(1)}[D, Y \cap Z]$ has at most $|D|(k-1) - r(1) - r(2)$ edges, so G' has at most

$$|N|(k-1) - |C||D| - r(1)$$

non-edges, since $|C| + |D| = |N|$ and $\sum_{i=1}^{3} r(i) = |C||D|$.

Now, by the definition of U, each vertex of U sends at least $|N|-k+1$ edges of colour 1 into $N = C \cup D$, so G' has at least $|U|(|N|-k+1)$ non-edges. Hence

$$\begin{aligned} |U|(|N|-k+1) &\leqslant |N|(k-1) - |C||D| - r(1) \\ &\leqslant |N|(k-1) - k^2, \end{aligned}$$

since $|C|, |D| \geqslant k$ and $r(1) \geqslant 0$. Thus

$$|U| \leqslant \frac{|N|(k-1) - k^2}{|N|-k+1} = k - 1 - \frac{2k-1}{|N|-k+1} < k-1,$$

since $|N| - k + 1 > 0$.

We complete the proof of Theorem 2 by setting $M = M' \setminus U$, and applying Lemma 9 to the graph $G = G^{(2,3)}[M,N]$. By the definition of U, $d_G(x) \geqslant k$ for every vertex $x \in M$, and

$$\begin{aligned} |M| &= |Y \cap Z| - |U| \geqslant (n - 4k + 4) - (k-1) \\ &= n - 5k + 5 \geqslant 8k - 10 \geqslant 3k - 2, \end{aligned}$$

since $|U| \leqslant k-1$, $n \geqslant 13k - 15$ and $k \geqslant 2$. Also $d_G(y) \geqslant |M| - k + 1$ for every $y \in N$, since each vertex of $C \cup D$ sends at most $k-1$ edges of colour 1 to $Y \cap Z$. Therefore,

$$|\Gamma_G(y) \cap \Gamma_G(z)| \geqslant |M| - 2k + 2 \geqslant k$$

for every pair $y, z \in N$, so by Lemma 9, G is k-connected.

Since $M \cup N = V \setminus U$ and $|U| \leqslant k-1$, G is the desired k-connected subgraph using at most two colours. □

Remark 1. We needed the bound $n \geqslant 13k - 15$ in order to apply Theorem 12 – the rest of the proof required only $n \geqslant 8k-7$. Therefore any improvement on the bound on n in Theorem 12 would give an immediate improvement here also.

When $r+1$ is not a power of 2, we can in general only determine $m(n, r, 2, k)$ up to a factor of 2. However, it follows from Theorem 4, which we shall prove in the next section, that $m(n, 5, 2, k) = \dfrac{9n}{10} - O(k)$, and we have the following conjecture for the case $r = 6$.

Conjecture 1. *Let $n, k \in \mathbb{N}$, with n sufficiently large compared to k. Then*

$$m(n, 6, 2, k) = \frac{3n}{4} - O(k).$$

We remark that the upper bound, $m(n, 6, 2, k) \leqslant \frac{3n}{4}$, follows from the construction in Lemma 23 below, with $R = 4$. We suspect that the following problem is not easy.

Problem 1. *Determine $m(n, r, 2, k)$ (up to an error term depending on r and k) for those $r \in \mathbb{N}$ such that $r + 1$ is not a power of 2.*

4. The jump at $2s = r$

We next turn to the range $2s \approx r$, where the function $m(n, r, s, k)$ 'jumps' from $(c + o(1))n$ with $c < 1$, to $n - f(k)$. We shall prove Theorems 3 and 4, which describe this transition quite precisely. We begin with an instant corollary of Theorem 12, which turns out to give the exact minimum when $2s = r$.

Lemma 18. *Let $n, s, k \in \mathbb{N}$, with $n \geqslant 13k - 15$. Then*

$$m(n, 2s, s, k) \geqslant n - 2k + 2.$$

Proof. Let $n, s, k \in \mathbb{N}$ with $n \geqslant 13k - 15$, let f be a $(2s)$-colouring of $E(K_n)$, and let $S \subset [2s]$ with $|S| = s$. Define the 2-colouring f_S induced by f and S by $f_S(e) = 1$ if $f(e) \in S$, and $f_S(e) = 2$ otherwise. By Theorem 12, since $n \geqslant 13k - 15$, there exists a monochromatic k-connected subgraph H of K_n (in the colouring f_S) on at least $n - 2k + 2$ vertices. H uses at most s colours in the colouring f, so $m(n, 2s, s, k) \geqslant n - 2k + 2$. $\square$

Next we prove the matching upper bound. The colouring which gives the bound is a generalization of the 2-colouring of Bollobás and Gyárfás [2].

Lemma 19. *Let $n, s, k \in \mathbb{N}$, with $n \geqslant 2\binom{2s}{s}(k-1) + 1$. Then*

$$m(n, 2s, s, k) \leqslant n - 2k + 2.$$

Proof. Let $n, s, k \in \mathbb{N}$, with $n \geqslant 2\binom{2s}{s}(k-1) + 1$. For each subset $T \subset [2s]$ with $|T| = s$, let A_T and B_T be subsets of $V = V(K_n)$ of size $k - 1$, with the sets $\{A_T, B_T : T \subset [2s], |T| = s\}$ pairwise disjoint.

Let $W = V \setminus \bigcup_T A_T \cup B_T$, so $|W| = n - 2\binom{2s}{s}(k-1) \geqslant 1$. Define a $(2s)$-colouring f of $E(K_n)$ as follows. Let

- $f(\{i,j\}) \in [2s] \setminus T$ if $i \in W$ and $j \in A_T \cup B_T$,
- $f(\{i,j\}) \in [2s] \setminus (T \cup T')$ if $i \in A_T \cup B_T$, $j \in A_{T'} \cup B_{T'}$ and $T' \neq T^c$,

and for each s-set T with $1 \notin T \subset [2s]$, let

- $f(\{i,j\}) = 1$ if $i \in A_T$ and $j \in A_{T^c}$, or $i \in B_T$ and $j \in B_{T^c}$, and
- $f(\{i,j\}) \in T$ if $i \in A_T$ and $j \in B_{T^c}$, or $i \in B_T$ and $j \in A_{T^c}$.

Now, suppose H is a k-connected subgraph of K_n using at most s colours; let $T \subset [2s]$, with $|T| = s$, be a fixed s-set containing every colour used in H. We claim that H contains no vertex of $A_T \cup B_T$. Indeed, suppose $u \in V(H) \cap A_T$ say (the proof if $u \in V(H) \cap B_T$ is identical), and let $v \in W$ (recall that $|W| \geqslant 1$).

Observe that since H used only colours from T, $\Gamma_H(u) \subset A_{T^c} \cup B_{T^c}$. Moreover, if $1 \in T$ then $\Gamma_H(u) \subset A_{T^c}$, and if $1 \notin T$ then $\Gamma_H(u) \subset B_{T^c}$. So, if we set $H' = H - A_{T^c}$ if $1 \in T$ and $H' = H - B_{T^c}$ if $1 \notin T$, it is clear that u and v are disconnected in H'. Since $|A_{T^c}| = |B_{T^c}| = k-1$, this contradicts the assumption that H is k-connected. This proves the claim.

We have shown that H contains no vertex of $A_T \cup B_T$. Since $|A_T \cup B_T| = 2k-2$, and H was an arbitrary k-connected subgraph using at most s colours, this completes the proof of the lemma. $\square$

Remark 2. Note that when $s = 1$ the bound $n \geqslant 2\binom{2s}{s}(k-1)+1$ reduces to $n \geqslant 4k-3$, and the construction reduces to that of Bollobás and Gyárfás [2]. Notice also that the construction may be altered slightly to give the bound $m(n,2s,s,k) \leqslant n - 2a$ if $n \geqslant 2\binom{2s}{s}a + 1$, for each $a \leqslant k-1$.

By Lemma 18, any r-colouring of $E(K_n)$ contains a k-connected subgraph, using at most s colours, on at least $n - 2k + 2$ vertices, if $r \leqslant 2s$. Suppose s is decreased a little, are similar statements are still true? In particular, for which s can we always find a k-connected subgraph on at least $n - g(k)$ vertices (for some function g)? Or on at least $n - o(n)$ vertices? It turns out that the answer is the same in each case: if and only if $2s \geqslant r$.

Lemma 20. *Let $n, r, s, k \in \mathbb{N}$. If $2s < r$, then*

$$m(n, r, s, k) \leqslant \left\lceil \left(1 - \binom{r}{s}^{-1}\right) n \right\rceil .$$

Proof. Let $n, r, s, k \in \mathbb{N}$, with $s < 2r$. Partition $V = V(K_n)$ into $t = \binom{r}{s} \geqslant 3$ subsets $\{A_T : T \subset [r], |T| = s\}$ of near equal size. Colour an edge between A_T and $A_{T'}$ with any colour from the set $[r] \setminus (T \cup T')$. Since $|T \cup T'| \leqslant 2s < r$, such a colour always exists. Colour the edges within the sets A_T arbitrarily.

Let T be any subset of $[r]$ with $|T| = s$. Note that

$$\left\lfloor \frac{n}{t} \right\rfloor \;\leqslant\; |A_T| \;\leqslant\; \left\lceil \frac{n}{t} \right\rceil \;\leqslant\; \frac{n}{2},$$

since the parts are of near equal size and $t \geqslant 3$. There are no edges of colour T between A_T and $V \setminus A_T$, so the largest k-connected subgraph using the colours of T has order at most

$$\max\{|A_T|, n - |A_T|\} \;\leqslant\; n - \left\lfloor \frac{n}{t} \right\rfloor \;=\; \left\lceil \left(1 - \frac{1}{t}\right) n \right\rceil .$$

Since T was an arbitrary subset of $[r]$ of size s, the lemma follows. □

Theorem 3 now follows immediately from Lemma 18, 19 and 20.

Proof of Theorem 3. Let $n, s, k \in \mathbb{N}$ with $n \geqslant 2\binom{2s}{s}(k-1) + 1$, and $n \geqslant 13k - 15$. Since $n \geqslant 13k - 15$, we have $m(n, 2s, s, k) \geqslant n - 2k + 2$ by Lemma 18, and since $n \geqslant 2\binom{2s}{s}(k-1)+1$, we have $m(n, 2s, s, k) \leqslant n - 2k + 2$ by Lemma 19. Hence $m(n, 2s, s, k) = n - 2k + 2$.

The moreover part of the theorem follows by Lemma 20. □

When $s = 1$ it is easy to modify the construction of Bollobás and Gyárfás [2] to give $m(n, 2, 1, k) = 0$ when $n \leqslant 4k - 4$, and they conjectured that the function jumps at this point, from 0 to $n - 2k + 2$. For $s \geqslant 2$ however, we have little clue how this transition occurs.

Problem 2. *Determine $m(n, 2s, s, k)$ when $n \leqslant 2\binom{2s}{s}(k-1)$.*

Theorem 3 implies that $2s = r$ marks a sort of 'threshold' for $m(n, r, s, k)$: when $2s < r$ there is a constant $\varepsilon(r, s, k) < 1$ such that $m(n, r, s, k) < (1 - \varepsilon(r, s, k))n$ for every (large) $n \in \mathbb{N}$; when $2s \geqslant r$, no such constant exists, and in fact $m(n, r, s, k) \geqslant n - 2k + 2$ (if $n \geqslant 13k$). Putting it concisely, the function 'jumps' from $n - \Omega(n)$ to $n - 2k + 2$.

It is natural to ask what how large $\varepsilon(r,s,k)$ can be, given $2s < r$. Theorem 4 shows that the maximum is exactly $\binom{2s+1}{s}^{-1}$.

Say that a set $A \subset V$ is (k-)*connected by the set* $X \subset [r]$ if the graph with vertex set A and edge set $f^{-1}(X)$ is (k-)connected. We shall now prove Theorem 4, beginning with the case $k = 1$.

Theorem 21. *Let $n, s \in \mathbb{N}$, with $s \geqslant 2$. Then*

$$m(n, 2s+1, s, 1) = \left\lceil \left(1 - \binom{2s+1}{s}^{-1} \right) n \right\rceil .$$

Proof. The upper bound follows from Lemma 20; we shall prove the lower bound. Let $n, s \in \mathbb{N}$, with $s \geqslant 2$, let $r = 2s+1$, and let f be an r-colouring of $E(K_n)$. Let $\mathcal{S} = \{S \subset [r] : |S| = s\}$, and for each subset $S \in \mathcal{S}$, let A_S be a set of maximum order which is connected by S, and let $B_S = V \setminus A_S$. Note that $|A_S| \geqslant 1$ for every S. We are required to show that there exists a set $S \in \mathcal{S}$ such that $|B_S| \leqslant \binom{2s+1}{s}^{-1} n$, and we shall do so by showing that either A_S is large for some $S \in \mathcal{S}$, or the sets $\{B_S : S \in \mathcal{S}\}$ must be pairwise disjoint. To make things easier to follow, we break the proof into several cases.

Case 1: There exist $S, T \in \mathcal{S}$ such that $A_S = A_T$ but $S \neq T$.

If $A_S = A_T = V$ then we are done, so assume that B_S is non-empty. Since A_S and A_T are maximal, every edge between A_S and B_S must have a colour from the set $U = \overline{S} \cap \overline{T}$. Note that $|U| = 2s+1-|S \cup T| \leqslant s$, since $S \neq T$. Thus $|A_W| = n$ for any $U \subset W \in \mathcal{S}$, and we are done.

Case 2: There exist $S, T \in \mathcal{S}$ such that $A_S \cap A_T = \emptyset$.

Since A_S and A_T are maximal, every edge between A_S and A_T must have some colour from $U = \overline{S} \cap \overline{T}$. Note again that $|U| \leqslant s$, since $S \neq T$. Now, let $B = B_S \cap B_T$, let

$$C = \{v \in B : \exists w \in A_S \cup A_T \text{ with } f(vw) \in U\},$$

and let $D = B \setminus C$. The set $V \setminus D$ is connected by U, so if $|D| = 0$ then $|A_W| = n$ for any $U \subset W \in \mathcal{S}$, and we are done.

So assume that $|D| \neq 0$ and let $u \in D$. Since $u \notin A_S$, edges between u and A_S do not have colours from S. Also, by the definition of D, these edges do not have colours from U. Thus they must have colours from $[r] \setminus (S \cup U) \subset T$. Similarly, edges between u and A_T must have colours from S. This is true for any vertex in D, so the set $A_S \cup D$ is connected by T, and the set $A_T \cup D$ is connected by S. But A_S and A_T have maximum order, so $|A_S| + |D| \leqslant |A_T|$, and $|A_T| + |D| \leqslant |A_S|$, which implies that $|D| = 0$, a contradiction.

Case 3: There exist $S, T \in \mathcal{S}$ such that $A_S \cap A_T \neq \emptyset$, $A_S \not\subset A_T \not\subset A_S$, $A_S \cup A_T \neq V$ and $|S \cup T| \geqslant s + 2$.

Let $u \in B = V \setminus (A_S \cup A_T)$, and let $C = A_S \cap A_T \neq \emptyset$. Since $u \notin A_S$ and $u \notin A_T$, the edges between u and C must all have colours from $U = \overline{S} \cap \overline{T}$. Similarly, all edges between $A_S \setminus A_T$ and $A_T \setminus A_S$ must have colours from U. Thus, the entire vertex set V is connected by $U \cup \{i\}$, where $f(vw) = i$ for some $v \in B \cup C$ and $w \in A_S \triangle A_T$. Now, since $|S \cup T| \geqslant s + 2$, it follows that $|U| \leqslant s - 1$, so $|U \cup \{i\}| \leqslant s$ and we are done.

Case 4: There exist $S, T \in \mathcal{S}$ such that $A_S \cap A_T \neq \emptyset$, $A_S \not\subset A_T \not\subset A_S$, $A_S \cup A_T \neq V$ and $|S \cup T| = s + 1$.

We shall show that Case 3 still holds. Indeed, let B, C and U be as in Case 3, and note that $|U| = s$, and that $U \cap S = U \cap T = \emptyset$. As before, the sets $B \cup C$ and $A_S \triangle A_T$ are each connected by U, and these sets partition the vertex set V. Hence, either $A_U = V$, in which case we are done, or $A_U = B \cup C$, or $A_U = A_S \triangle A_T$. It is simple to check that in either of the latter two cases S and U satisfy the conditions of Case 3, and so we are done as before. Note in particular that $|S \cup U| = 2s \geqslant s + 2$ since $s \geqslant 2$.

Case 5: There exist $S, T \in \mathcal{S}$ such that $A_S \supset A_T$ but $S \neq T$.

This case is a little more complicated than the first four. First we shall show that $A_R \supset A_T$ for *every* set $R \in \mathcal{S}$.

If $A_S = V$ we are done, so we may assume that B_S is non-empty. Since A_S and A_T are maximal, all edges between B_S and A_T must have colours from $U = \overline{S} \cap \overline{T}$. Choose $W \in \mathcal{S}$ such that $U \subset W$, and let C_W be the maximal set connected by W containing $B_S \cup A_T$. We shall show that $|C_W| > n/2$, and so $C_W = A_W$. If $C_W = V$ then we are done, so assume that $D_W = V \setminus C_W$ is non-empty. Now, no edge between $D_W \subset A_S$ and $B_S \subset C_W$ can have a colour from S or from U, so all of these edges have colours from T. But A_T was chosen to have maximal size, so $|A_T| \geqslant |D_W| + |B_S|$. Hence

$$|C_W| \;\geqslant\; |A_T| + |B_S| \;\geqslant\; |D_W| + 2|B_S| \;>\; |D_W|,$$

and so $C_W = A_W$ as claimed.

We have shown that $A_W \supset A_T$ for every $\overline{S} \cap \overline{T} \subset W \in \mathcal{S}$, so in particular we can choose W so that $T \cap W = \emptyset$. Let $\{i\} = [r] \setminus (T \cup W)$. Now, by the method of the previous paragraph, $A_R \supset A_T$ for any $R \in \mathcal{S}$ with $i \in R$. In particular, if $X = W \triangle \{i, j\}$ with $j \in W$, then $A_X \supset A_T$. Once again applying the method of the previous paragraph, we infer that $A_R \supset A_T$ for any $R \in \mathcal{S}$ with $j \in R$. Since j was an arbitrary member of $W = [r] \setminus (T \cup \{i\})$, we have proved that $A_R \supset A_T$ for *every* set $R \in \mathcal{S}$, as claimed.

We next claim that $B_Q \cap B_R = \emptyset$ for every $Q, R \in \mathcal{S} \setminus \{T\}$ with $Q \neq R$. Indeed, if $B_Q \cap B_R \neq \emptyset$ and $B_Q \not\subset B_R \not\subset B_Q$, then we are in either Case 3 or Case 4, since $B_Q \cup B_R \subset B_T \neq V$. But if, on the other hand, $B_Q \subset B_R$ say, then $A_Q \supset A_R$, so $A_P \supset A_R$ for every $P \in \mathcal{S}$ as above, and in particular $A_T \supset A_R$. But then $A_R = A_T$, and we are in Case 1. Hence $B_Q \cap B_R = \emptyset$ for every $Q, R \in \mathcal{S} \setminus \{T\}$ with $Q \neq R$, as claimed.

Now, simply observe that for any pair $W, X \in \mathcal{S}$ such that $\overline{T} \subset W, X$ and $W \neq X$, every edge between B_W and B_X must have a colour from T, since $[r] \setminus (W \cup X) \subset T$. Since B_W and B_X are disjoint, and A_T was chosen to be maximal, it follows that $|A_T| \geqslant |B_W| + |B_X|$.

Hence, recalling that $A_T \cap B_R = B_R \cap B_Q = \emptyset$ for every $Q, R \in \mathcal{S} \setminus \{T\}$ with $Q \neq R$, we obtain

$$n \;\geqslant\; \left(\sum_{T \neq R \in \mathcal{S}} |B_R| \right) + |B_W| + |B_X| \;\geqslant\; \left(\binom{2s+1}{s} + 1 \right) \min_{R \in \mathcal{S}} |B_R|,$$

and thus $\displaystyle\min_{R \in \mathcal{S}} |B_R| \leqslant \left(\binom{2s+1}{s} + 1 \right)^{-1} n$, as required.

Finally, suppose that none of Cases 1–5 hold. The only remaining possibility is that $A_S \cup A_T = V$ for every pair $S, T \in \mathcal{S}$ with $S \neq T$, and therefore that $B_S \cap B_T = \emptyset$ for every such pair. But now we have

$$n \;\geqslant\; \sum_{R \in \mathcal{S}} |B_R| \;\geqslant\; \binom{2s+1}{s} \min_{R \in \mathcal{S}} |B_R|,$$

and so $\displaystyle\min_{R \in \mathcal{S}} |B_R| \leqslant \binom{2s+1}{s}^{-1} n$, and we are done. $\square$

The proof for general k is similar, but we shall need some of the tools from Section 2: to be precise, we shall use Lemmas 8, 10 and 11.

Proof of Theorem 4. The upper bound again follows from Lemma 20; we shall prove the lower bound. Let $n, s, k \in \mathbb{N}$, with $s \geqslant 2$ and $n \geqslant 100\binom{2s+1}{s}^2 k^2$. Let $r = 2s+1$, and let f be an r-colouring of $E(K_n)$. Let $\mathcal{S} = \{S \subset [r] : |S| = s\}$ as before, and for each subset $S \in \mathcal{S}$, let A_S be a set of maximum order which is k-connected by S, and let $B_S = V \setminus A_S$. We are required to show that there exists a set $S \in \mathcal{S}$ such that $|B_S| \leqslant \binom{2s+1}{s}^{-1} n + 2\binom{2s+1}{s}k$.

Let us assume, for a contradiction, that no such set S exists. We begin by using Lemma 8 to show that $|A_S| \geqslant (6s+78)k$ for *every* set $S \in \mathcal{S}$. Indeed, let $S \in \mathcal{S}$, and let $T, U \subset [r]$ satisfy $S \cup T \cup U = [r]$. We have

$$q_k(W) \;\leqslant\; \left(1 - \binom{2s+1}{s}^{-1}\right) n - 2\binom{2s+1}{s}k \;<\; n - k\sqrt{n}$$

for each $W \in \{T, U\}$, and $n \geqslant 25k^2$, so by Lemma 8,

$$\begin{aligned}
q_k(S) \;&\geqslant\; n \;-\; 9k\sqrt{n} \;-\; q_k(T)\\
&\geqslant\; \binom{2s+1}{s}^{-1} n \;+\; 2\binom{2s+1}{s}k \;-\; 9k\sqrt{n}\\
&=\; \binom{2s+1}{s}^{-1}\sqrt{n}\left(\sqrt{n} - 9k\binom{2s+1}{s}\right) \;+\; 2\binom{2s+1}{s}k\\
&\geqslant\; 10\binom{2s+1}{s}k^2 \;+\; 2\binom{2s+1}{s}k \;>\; (6s+78)k
\end{aligned}$$

since $\sqrt{n} \geqslant 10k\binom{2s+1}{s}$. Thus $|A_S| \geqslant (6s+78)k$ for every $S \in \mathcal{S}$, as claimed. We must once again consider five cases.

Case 1: There exist $S, T \in \mathcal{S}$ such that $|A_S \triangle A_T| \leqslant (4s+62)k$.

If $|V \setminus (A_S \cup A_T)| \leqslant 2k$, then $|A_S| \geqslant n - (4s+64)k$ and we are done, so assume that $|B| \geqslant 2k$, where $B = B_S \cap B_T$. Since A_S is maximal, a vertex $v \in B$ can send at most $k-1$ edges with colours from S into A_S, and similarly v can send at most $k-1$ edges with colours from T

into A_T. Therefore, each vertex of B sends at most $2k-2$ edges with colours not in the set $U = \overline{S} \cap \overline{T}$ into $A = A_S \cap A_T$.

Now, simply apply Lemma 10 to the bipartite graph G with parts A and B, and edges with colours from the set U, and with $a = b = 2k$. Note that

$$|A| \;=\; \frac{|A_S| + |A_T| - |A_S \triangle A_T|}{2} \;\geqslant\; 9k \;=\; 4b + k,$$

and $|B| \geqslant 2k$, so G contains a k-connected subgraph on more than $|G| - 4k$ vertices. But this subgraph uses only colours from U, and $|G| = n - |A_S \triangle A_T| \geqslant n - (4s+62)k$, so in this case $q_k(U) \geqslant n - (4s+66)k$ and we are done.

Case 2: There exist $S, T \in \mathcal{S}$ such that $|A_S \cap A_T| \leqslant (2s+16)k$.

Since A_S and A_T are maximal, each vertex of $A_S \setminus A_T$ sends at most $k-1$ edges with colours from T into $A_T \setminus A_S$, and similarly each vertex of $A_T \setminus A_S$ sends at most $k-1$ edges with colours from S into $A_S \setminus A_T$. We also have $|A_S \setminus A_T| = |A_S| - |A_S \cap A_T| \geqslant 15k$, and similarly $|A_T \setminus A_S| \geqslant 15k$, so we may apply Lemma 11 to obtain a set $X \subset A_S \triangle A_T$ with $|X| \geqslant |A_S \triangle A_T| - 7k$, which is k-connected by $U = \overline{S} \cap \overline{T}$.

Let $B = B_S \cap B_T$, let

$$C = \{v \in B : |\{w \in X : f(vw) \in U\}| \geqslant k\},$$

and let $D = B \setminus C$. Note that the set $X \cup C$ is k-connected by U, so $|A_W| \geqslant n - 12k - |D|$ for any $U \subset W \in \mathcal{S}$. Therefore we may assume that $|D| > (4s+50)k$.

Now, each vertex $u \in D$ sends at most $k-1$ edges with colours from S into A_S (since $u \notin A_S$), and at most $k-1$ edges with colours from U into X (by the definition of D). Apply Lemma 10 to the bipartite graph with parts D and $X \cap A_S$, and edges with colours from T, and with $a = b = 2k$. Note that $|D| \geqslant 2k$ and

$$|X \cap A_S| \;\geqslant\; |A_S| - |A_S \cap A_T| - 5k \;\geqslant\; 9k \;=\; 4b + k,$$

so the conditions of the lemma hold. Thus there exists a k-connected subgraph of K_n, using colours only from T, with at least

$$|D| + |X \cap A_S| - 4k \;\geqslant\; |D| + |A_S| - (2s+25)k$$

vertices. Similarly, there exists a k-connected subgraph of K_n, using colours only from S, with at least $|D| + |A_T| - (2s+25)k$ vertices. But A_S and A_T have maximum order, so $|A_S| + |D| - (2s+25)k \leqslant |A_T|$, and $|A_T| + |D| - (2s+25)k \leqslant |A_S|$, which implies that $|D| \leqslant (4s+50)k$, a contradiction.

Case 3: There exist $S,T \in \mathcal{S}$ such that $|A_S \cap A_T| \geqslant (2s+5)k$, $|A_S \setminus A_T|, |A_T \setminus A_S| \geqslant (s+13)k$, $|V \setminus (A_S \cup A_T)| \geqslant 2k$ and $|S \cup T| \geqslant s+2$.

Let $B = V \setminus (A_S \cup A_T)$, $C = A_S \cap A_T$, $X = A_S \setminus A_T$ and $Y = A_T \setminus A_S$, so $|B| \geqslant 2k$, $|C| \geqslant (2s+5)k \geqslant 9k$, and $|X|, |Y| \geqslant (s+13)k \geqslant 15k$. Since A_S and A_T are maximal, a vertex of B can send at most $2k-2$ edges with colours from $S \cup T$ into C, thus by Lemma 10 there exists a k-connected subgraph H_1 of $K_n[B \cup C]$, using only colours of $U = \overline{S} \cap \overline{T}$, on at least $|B| + |C| - 4k$ vertices. Furthermore, each vertex of X sends at most $k-1$ edges with colours from T into Y, and each vertex of Y sends at most $k-1$ edges with colours from S into X, so by Lemma 11 there exists a k-connected subgraph H_2 of $K_n[X \cup Y]$, using only colours of $U = \overline{S} \cap \overline{T}$, on at least $|X| + |Y| - 7k$ vertices. Thus $|V(H_1) \cup V(H_2)| \geqslant n - 11k$.

It remains only to 'k-connect' H_1 and H_2 using Observation 3. To be precise, for each $i \in S \cup T$ let

$$D_i = \{v \in V(H_1) : |\{w \in V(H_2) : f(vw) \in U \cup \{i\}\}| \geqslant k\},$$

and note that $\bigcup_i D_i = V(H_1)$, since $|H_2| \geqslant |X| + |Y| - 7k \geqslant 2sk$. Choose $j \in S \cup T$ such that $|D_j| \geqslant |H_1|/2s$, and note that $|H_1| \geqslant |B| + |C| - 4k \geqslant 2sk$, so $|D_j| \geqslant k$.

Now, by Observation 3, $V(H_1) \cup V(H_2)$ is k-connected by $U \cup \{j\}$, and so $q_k(U \cup \{j\}) \geqslant n - 11k$. But $|S \cup T| \geqslant s+2$, so $|U \cup \{j\}| \leqslant s$, and we are done.

Case 4: There exist $S,T \in \mathcal{S}$ such that $|A_S \cap A_T| \geqslant (2s+16)k$, $|A_S \setminus A_T|, |A_T \setminus A_S| \geqslant (2s+19)k$, $|V \setminus (A_S \cup A_T)| \geqslant (s+17)k$ and $|S \cup T| = s+1$.

We shall show that either Case 3 still holds, or $|A_U| \geqslant n - 11k$, where as usual $U = \overline{S} \cap \overline{T}$ (note that now $|U| = s$, since $|S \cup T| = s+1$). Let B, C, X and Y be as in Case 3, so $|B| \geqslant (s+17)k$, $|C| \geqslant (2s+16)k$, and $|X|, |Y| \geqslant (2s+19)k$. As before, we can find disjoint k-connected subgraphs H_1 and H_2, which use only edges with colours from U, such that $V(H_1) \subset B \cup C$, $V(H_2) \subset X \cup Y$, $|V(H_1)| \geqslant |B| + |C| - 4k$ and $|V(H_2)| \geqslant |X| + |Y| - 7k$.

Consider A_U. Clearly $|A_U| \geqslant \max\{|H_1|, |H_2|\} \geqslant (n-11k)/2 \geqslant 13k$, but any set of order $13k$ intersects either $V(H_1)$ or $V(H_2)$ (or both) in at least k vertices. Hence, by Observation 4, either $A_U \supset V(H_1)$ and $|A_U \cap V(H_2)| \leqslant k-1$, or $A_U \supset V(H_2)$ and $|A_U \cap V(H_2)| \leqslant k-1$, or $A_U \supset V(H_1) \cup V(H_2)$. In the third case we have $|A_U| \geqslant n - 11k$; we claim that in either of the first two (sub)cases, S and U satisfy the conditions of Case 3.

Subcase (a): If $A_U \supset V(H_1)$ and $|A_U \cap V(H_2)| \leqslant k - 1$, then

$$\begin{aligned} |A_S \cap A_U| &\geqslant |V(H_1) \cap C| \geqslant |C| - 4k \geqslant (2s+5)k, \\ |A_S \setminus A_U| &\geqslant |V(H_2) \cap X| - k \geqslant |X| - 8k \geqslant (s+13)k, \\ |A_U \setminus A_S| &\geqslant |V(H_1) \cap B| \geqslant |B| - 4k \geqslant (s+13)k, \text{ and} \\ |V \setminus (A_S \cup A_U)| &\geqslant |V(H_2) \cap Y| - k \geqslant |Y| - 8k \geqslant 2k. \end{aligned}$$

Subcase (b): Similarly, if $A_U \supset V(H_2)$ and $|A_U \cap V(H_1)| \leqslant k-1$, then

$$\begin{aligned} |A_S \cap A_U| &\geqslant |V(H_2) \cap X| \geqslant |X| - 7k \geqslant (2s+5)k, \\ |A_S \setminus A_U| &\geqslant |V(H_1) \cap C| - k \geqslant |C| - 5k \geqslant (s+13)k, \\ |A_U \setminus A_S| &\geqslant |V(H_2) \cap Y| \geqslant |Y| - 7k \geqslant (s+13)k, \text{ and} \\ |V \setminus (A_S \cup A_U)| &\geqslant |V(H_1) \cap B| - k \geqslant |B| - 5k \geqslant 2k. \end{aligned}$$

Also $|S \cup U| = 2s \geqslant s + 2$ since $s \geqslant 2$, and so S and U satisfy the conditions of Case 3, as claimed. Hence we are done as in that case.

Case 5: There exist $S, T \in \mathcal{S}$ such that $|A_T \setminus A_S| \leqslant (2s + 19)k$ but $S \neq T$.

This case is once again a little more complicated than the first four. First we shall show that $|A_T \setminus A_R| \leqslant (2s + 31)k$ for *every* set $R \in \mathcal{S}$.

Let $B = B_S \cap B_T$ and $C = A_S \cap A_T$, as in Cases 3 and 4. If $|A_S| \geqslant n - (4s + 48)k$ we are done, so we may assume that $|B_S| \geqslant (4s + 48)k$, and hence that

$$|B| = |B_S \cap B_T| = |B_S| - |A_T \setminus A_S| \geqslant (2s + 29)k.$$

Also recall that $|A_T| \geqslant (6s + 78)k$, so

$$|C| = |A_S \cap A_T| = |A_T| - |A_T \setminus A_S| \geqslant 9k.$$

Now, since A_S and A_T are maximal, each vertex of B sends at most $2k - 2$ edges with colours from $S \cup T$ into C. Let $U = \overline{S} \cap \overline{T}$, and apply Lemma 10 in the usual way (with $a = b = 2k$) to obtain a k-connected subgraph H of $K_n[B \cup C]$, using only colours from the set U, on at least $|B| + |C| - 4k$ vertices. Choose $W \in \mathcal{S}$ such that $U \subset W$, and let C_W be a set of maximum size, containing $V(H)$, which is k-connected by W. We shall show that $|C_W| \geqslant (n + k)/2$, and deduce that $C_W = A_W$.

Indeed, we are done if $|C_W| \geqslant n-(2s+38)k$, since $|A_W| \geqslant |C_W|$, so let $D_W = V \setminus C_W$ and assume that $|D_W| \geqslant (2s+38)k$. Then

$$\begin{aligned} |D_W \cap A_S| &\geqslant |D_W| - |A_T \setminus A_S| - |B \cap D_W| \\ &\geqslant |D_W| - (2s+19)k - 4k \;\geqslant\; 15k, \end{aligned}$$

and also

$$|B_S \cap V(H)| \;\geqslant\; |B| - 4k \;\geqslant\; 15k.$$

Now, a vertex of D_W can send at most $k-1$ edges with colours from W into $V(H)$, and a vertex of B_S can send at most $k-1$ edges with colours from S into A_S. Therefore, by Lemma 11, there exists a k-connected subgraph of $(D_W \cap A_S) \cup (B_S \cap V(H))$, using only colours from the set T, on at least

$$|D_W \cap A_S| + |B_S \cap V(H)| - 7k \;\geqslant\; |D_W| + |B| - (2s+34)k$$

vertices. But A_T was chosen to have maximal size, so

$$|A_T| \;\geqslant\; |D_W| + |B| - (2s+34)k.$$

Hence

$$\begin{aligned} |C_W| &\geqslant |B| + |C| - 4k \;=\; |B| + |A_T| - |A_T \setminus A_S| - 4k \\ &\geqslant |D_W| + 2|B| - (4s+57)k \;\geqslant\; |D_W| + k, \end{aligned}$$

since $|B| \geqslant (2s+29)k$. Therefore $|C_W| \geqslant (n+k)/2$, as claimed. But now any subset of V of size at least $|C_W|$ must intersect C_W in at least k vertices, and so by Observation 4, any k-connected subgraph on at least $|C_W|$ vertices must contain C_W. In particular, $A_W \supset C_W$, since $|A_W| \geqslant |C_W|$ by definition, But C_W was chosen to have maximum size, so we must have $C_W = A_W$.

Since $V(H) \subset A_W$, we have shown that

$$|A_T \setminus A_W| \;\leqslant\; |A_T \setminus A_S| + |C \setminus V(H)| \;\leqslant\; (2s+23)k$$

for every set $W \in \mathcal{S}$ with $\overline{S} \cap \overline{T} \subset W \in \mathcal{S}$. In particular, we may choose W so that $T \cap W = \emptyset$. Now applying the method of the previous paragraphs to the sets T and W, we deduce that $|A_T \setminus A_R| \leqslant (2s+27)k$ for any $R \in \mathcal{S}$ with $i \in R$, where $\{i\} = \overline{T} \cap \overline{W}$. In particular, if $X = W \triangle \{i,j\}$ with $j \in W$, then $|A_T \setminus A_X| \leqslant (2s+27)k$. Once again applying the method of the previous paragraphs, we infer that $|A_T \setminus A_R| \leqslant (2s+31)k$ for any $R \in \mathcal{S}$ with $j \in R$. Since j was an arbitrary member of $[r] \setminus (T \cup \{i\})$, we have proved that $|A_T \setminus A_R| \leqslant (2s+31)k$ for *every* set $R \in \mathcal{S}$, as claimed.

We shall next show that either we are in Case 1, 3 or 4, or $|B_Q \cap B_R| < (s+17)k$ for every $Q, R \in \mathcal{S} \setminus \{T\}$ such that $Q \neq R$, and moreover $|B_Q \cap B_R| < 2k$ if $|Q \cup R| \geqslant s+2$. Indeed, let $Q, R \in \mathcal{S} \setminus \{T\}$ with $Q \neq R$, and let $|B_Q \cap B_R| \geqslant 2k$. Suppose first that $|B_Q \setminus B_R| \leqslant (2s+19)k$. Then $|A_R \setminus A_Q| \leqslant (2s+19)k$, and so $|A_R \setminus A_P| \leqslant (2s+31)k$ for every $P \in \mathcal{S}$, as above, and in particular $|A_R \setminus A_T| \leqslant (2s+31)k$. But now $|A_R \triangle A_T| \leqslant (4s+62)k$, and we are in Case 1.

So suppose next that $|B_Q \setminus B_R|, |B_R \setminus B_Q| \geqslant (2s+19)k$. Note that

$$\begin{aligned} |A_Q \cap A_R| &\geqslant |A_T| - |A_T \setminus A_R| - |A_T \setminus A_Q| \\ &\geqslant |A_T| - 2(2s+31)k \geqslant (2s+16)k, \end{aligned}$$

since $|A_T| \geqslant (6s+78)k$, that $|A_Q \setminus A_R|, |A_R \setminus A_Q| \geqslant (2s+19)k$, and that

$$|V \setminus (A_Q \cup A_R)| = |B_Q \cap B_R|.$$

Thus if $|Q \cup R| \geqslant s+2$ we are in Case 3, and if $|Q \cup R| = s+1$ and $|B_Q \cap B_R| \geqslant (s+17)k$ then we are in Case 4. Hence either we are done as before, or $|B_Q \cap B_R| < (s+17)k$ for every $Q, R \in \mathcal{S} \setminus \{T\}$ with $Q \neq R$, and moreover $|B_Q \cap B_R| < 2k$ if $|Q \cup R| \geqslant s+2$, as claimed.

Now, let W and X be as described above, so $[r] \setminus (W \cup X) \subset T$, and observe that a vertex of $B_W \setminus B_X$ sends at most $k-1$ edges with colours from W into $B_X \setminus B_W$, and similarly a vertex of $B_X \setminus B_W$ sends at most $k-1$ edges with colours from X into $B_W \setminus B_X$. Note also that $|B_W|, |B_X| \geqslant (s+32)k$, else we are done, so

$$|B_W \setminus B_X| \geqslant |B_W| - |B_W \cap B_X| \geqslant 15k,$$

and similarly $|B_X \setminus B_W| \geqslant 15k$. Hence we may apply Lemma 11 to the bipartite graph with parts $B_W \setminus B_X$ and $B_X \setminus B_W$ to obtain a k-connected subgraph on at least

$$|B_W \triangle B_X| - 7k \geqslant |B_W| + |B_X| - (s+24)k$$

vertices, using only colours from the set T.

Since A_T was chosen to be maximal, we have

$$|A_T| \geqslant |B_W| + |B_X| - (s+24)k.$$

Now, recall that for every $Q, R \in \mathcal{S} \setminus \{T\}$, we have

$$|B_R \setminus A_T| = |B_R| - |A_T \setminus A_R| \geqslant |B_R| - (2s+31)k,$$

$|B_Q \cap B_R| \leqslant 2k$ if $|Q \cup R| \geqslant s+2$ and $|B_Q \cap B_R| \leqslant (s+17)k$ if $|Q \cup R| = s+1$. There are exactly $\binom{s+1}{2}$ pairs $Q, R \in \mathcal{S}$ such that

$|Q \cap R| = s+1$. Thus, by inclusion-exclusion, we obtain

$$\begin{aligned} n \;&\geqslant\; |A_T| \;+ \sum_{T \neq R \in \mathcal{S}} |B_R \setminus A_T| \;- \sum_{\substack{Q,R \in \mathcal{S}\setminus\{T\},\\ Q \neq R}} |B_Q \cap B_R| \\ &>\; |B_W| \;+\; |B_X| \;-\; (s+24)k \;+ \sum_{T \neq R \in \mathcal{S}} \Big(|B_R| - (2s+31)k\Big) \\ &\qquad\qquad\qquad - \binom{2s+1}{s}^2 k \;-\; \binom{s+1}{2}(s+17)k \\ &>\; \left(\binom{2s+1}{s} + 1\right) \min_{R \in \mathcal{S}} |B_R| \;-\; \left(\binom{2s+1}{s}^2 + \binom{2s+1}{s}(2s+40)\right) k. \end{aligned}$$

Now,

$$\frac{n + \left(\binom{2s+1}{s}^2 + \binom{2s+1}{s}(2s+40)\right) k}{\binom{2s+1}{s} + 1} \;\leqslant\; \frac{n \;+\; 2\binom{2s+1}{s}^2 k}{\binom{2s+1}{s}}$$

reduces to $n \geqslant \binom{2s+1}{s}^2 \left(2s + 38 - \binom{2s+1}{s}\right) k$, which is true, so

$$\min_{R \in \mathcal{S}} |B_R| \;<\; \binom{2s+1}{s}^{-1} n \;+\; 2\binom{2s+1}{s} k,$$

as required.

Finally, suppose that none of Cases 1–5 hold. The only remaining possibility is that $|A_S \cup A_T| \geqslant n - 2k$ for every pair $S, T \in \mathcal{S}$ with $|S \cup T| \geqslant s+2$, and $|A_S \cup A_T| \geqslant n - (s+17)k$ for every pair $S, T \in \mathcal{S}$ with $|S \cup T| = s+1$. But $|B_S \cap B_T| = n - |A_S \cup A_T|$, so we have

$$\begin{aligned} n \;&\geqslant\; \sum_{R \in \mathcal{S}} |B_R| \;- \sum_{\substack{Q,R \in \mathcal{S}\setminus\{T\},\\ Q \neq R}} |B_Q \cap B_R| \\ &\geqslant\; \binom{2s+1}{s} \min_{R \in \mathcal{S}} |B_R| \;-\; \binom{2s+1}{s}^2 k \;-\; \binom{s+1}{2}(s+17)k, \end{aligned}$$

So $\min_{R \in \mathcal{S}} |B_R| \leqslant \binom{2s+1}{s}^{-1} n + 2\binom{2s+1}{s} k$, and we are done. □

Setting $s = 2$ we obtain the following corollary.

Corollary 22. *Let $n, k \in \mathbb{N}$ with $n \geqslant (100k)^2$. Then*

$$\frac{9n}{10} \; - \; 20k \; \leqslant \; m(n,5,2,k) \; \leqslant \; \frac{9n}{10} \; + \; 1.$$

Remark 3. In fact one can do a little better in both directions. By taking a little more care in the proof of Theorem 4, one easily obtains

$$m(n,5,2,k) \geqslant \frac{9n - 157k}{10},$$

and a simple modification of the construction in Lemma 20 gives

$$m(n,5,2,k) \leqslant \frac{9n - k + 1}{10},$$

and more generally

$$m(n,2s+1,s,k) \leqslant \left(1 - \binom{2s+1}{s}^{-1} \right) n - \binom{2s+1}{s}^{-1} (k-1).$$

We are sure that neither of these bounds is sharp.

5. The jump at $s = \Theta(\sqrt{r})$

Perhaps the most basic question one can ask about the function $m(n,r,s,k)$ is the following: for which values of s is $m(n,r,s,k)$ close to 0, and for which is it close to 1? Theorem 5 gives an asymptotic answer to this question. We begin with an easy lemma, which gives the upper bound in the theorem.

Lemma 23. *For every $n, r, s, k \in \mathbb{N}$, we have*

$$m(n,r,s,k) \; \leqslant \; (s+1) \left\lceil \frac{n}{\lfloor \sqrt{2r} \rfloor} \right\rceil.$$

Proof. Let $n, r, s, k \in \mathbb{N}$, let $V = V(K_n)$, and partition V into $R = \lfloor \sqrt{2r} \rfloor$ sets $V_1, \ldots, V_R$, each of size either $N = \lceil n/R \rceil$ or $N - 1$. Noting that $\binom{R}{2} < r$, assign to each pair $\{i,j\} \subset [R]$ a distinct colour $c(\{i,j\}) \in [r]$.

Let f be the following r-colouring of $E(K_n)$: if $x \in V_i$ and $y \in V_j$, and $i \neq j$, then set $f(xy) = c(\{i,j\})$. If $x, y \in V_i$, then $f(xy)$ may be chosen arbitrarily. Thus f is a 'blow-up' of a completely multicoloured complete graph.

Now, let $S \subset [r]$ be any subset of size s, and let G be the subgraph of K_n with vertex set V and edge set $f^{-1}(S)$. Each component of G

intersects at most $s+1$ of the sets $\{V_j : j \in [R]\}$, so since S was chosen arbitrarily, we have $m(n,r,s,k) \leqslant M(f,n,r,s,1) \leqslant (s+1)N$. $\square$

Lemma 23 shows that if $s \ll \sqrt{r}$, then $\dfrac{m(n,r,s,k)}{n} \to 0$ as $r \to \infty$. Somewhat surprisingly, this simple construction turns out to be asymptotically optimal. Once again, we begin with the case $k = 1$, and prove a slightly stronger result.

Theorem 24. *Let $n, r, s \in \mathbb{N}$. Then*

$$m(n,r,s,1) \geqslant \left(1 - e^{-s^2/3r}\right) n.$$

Proof. Let $n, r, s \in \mathbb{N}$, and let f be an r-colouring of the edges of K_n. If $s = 1$ then the result is trivial, since $e^{-x} > 1 - x$ if $x > 0$, and $m(n,r,1,1) \geqslant n/r$ (consider the largest monochromatic star centred at any vertex). So let $s \geqslant 2$, and assume the result holds for all smaller values of s. Let $t \in [s-1]$ (we shall eventually set $t = \lceil s/2 \rceil$, but we shall delay making this choice until it is clear why it is optimal), and let G be a connected subgraph of K_n, using at most t colours, of maximum order. Let $V = V(K_n)$, $A = V(G)$, $B = V \setminus A$, and $T = f(E(G))$, the set of colours used by G. Thus (assuming $|A| < n$), $|T| = t$. By the induction hypothesis, $|A| \geqslant \left(1 - e^{-t^2/3r}\right) n$.

Now, each vertex in B must send at least $t+1$ different colours into A, as otherwise the star centred at that vertex would be a connected component, using at most t colours, larger than G. Also, a vertex of B sends no edges with colours from T into A, since G was chosen to be maximal. For each vertex $v \in B$, choose a list $L(v)$ of $t+1$ colours $\{\ell_1, \ldots, \ell_{t+1}\} \subset [r] \setminus T$ which it sends into A. So for each $v \in B$ and $\ell \in L(v)$, there exists a vertex $u \in A$ such that $f(uv) = \ell$.

Let $\varepsilon > 0$, and let $\mathcal{T} = \{S \subset [r] \setminus T : |S| = s - t\}$. Suppose that $m(n,r,s,1) \leqslant n - \varepsilon|B|$. This means that for every set $S \in \mathcal{T}$, the largest connected component in K_n, using only the colours $S \cup T$, and containing G, avoids at least $\varepsilon|B|$ vertices of B. Hence, for each $S \in \mathcal{T}$ there are at least $\varepsilon|B|$ vertices $v \in B$ such that $S \cap L(v) = \emptyset$. For each $S \in \mathcal{T}$, let $M(S) = \{v \in B : S \cap L(v) = \emptyset\}$.

Now, observe that for each vertex $v \in B$, there are exactly $\dbinom{r-2t-1}{s-t}$ sets $S \in \mathcal{T}$ with $v \in M(S)$. So, summing over $\mathcal{T}$, we obtain

$$\varepsilon|B|\binom{r-t}{s-t} \;\leqslant\; \sum_{S \in \mathcal{T}} |M(S)| \;=\; \sum_{v \in B} \sum_{S \in \mathcal{T}} I[v \in M(S)] \;=\; |B|\binom{r-2t-1}{s-t},$$

where $I[T]$ denotes the indicator function of the event T, and therefore

$$\begin{aligned} \varepsilon &\leqslant \frac{(r-2t-1)!\,(r-s)!}{(r-t)!\,(r-s-t-1)!} = \frac{(r-2t-1)}{(r-t)} \cdots \frac{(r-s-t)}{(r-s+1)} \\ &\leqslant \left(\frac{r-2t-1}{r-t}\right)^{s-t} < \exp\left(\frac{-(t+1)(s-t)}{r-t}\right). \end{aligned}$$

Now, set $t = \lceil s/2 \rceil$ to (approximately) maximize $\dfrac{(t+1)(s-t)}{r-t}$, and note that $\dfrac{(\lceil s/2\rceil + 1)\lfloor s/2 \rfloor}{r - \lceil s/2 \rceil} > \dfrac{s^2}{4r}$. Recalling that $|B| \leqslant e^{-t^2/3r} n \leqslant e^{-s^2/12r} n$, we obtain

$$\varepsilon |B| \leqslant e^{-s^2/4r - s^2/12r} n = e^{-s^2/3r},$$

since $s \geqslant 2$. Hence

$$M(f,n,r,s,1) \geqslant |A| + (1-\varepsilon)|B| = n - \varepsilon|B| \geqslant \left(1 - e^{-s^2/3r}\right) n.$$

Since f was arbitrary, this proves the theorem. $\square$

The proof for general k is, in this case, very similar. All that is necessary is to throw out some 'bad' vertices.

Proof of Theorem 5. Let $n, r, s, k \in \mathbb{N}$, with $n \geqslant 16kr^2 + 4kr$, and let f be an r-colouring of the edges of K_n. If $s = 1$ then the result follows by Mader's Theorem (since $n \geqslant 4kr + 1$), and the fact that $e^{-x} > 1 - x$ if $x > 0$, so assume that $s \geqslant 2$. Let $t \in [s-1]$ (we shall again eventually set $t = \lceil s/2 \rceil$, but we again delay making this choice to emphasize the similarities with the previous proof), and let G be a k-connected subgraph of K_n, using at most t colours, of maximum order. Let $V = V(K_n)$, $A = V(G)$, $B = V \setminus A$, and $T = f(E(G))$, the set of colours used by G. Thus (assuming $|A| \leqslant n - rk$), $|T| = t$. By Mader's Theorem, we have $|A| \geqslant \dfrac{n}{4r} \geqslant 4kr + k$.

Now, suppose there are at least $2kr\dbinom{r-t}{t}$ vertices in B which send at least k edges of no more than t colours into A. To be more precise, given $v \in B$, let

$$L_k(v) = \{\ell \in [r] : |\{u \in A : f(uv) = \ell\}| \geqslant k\},$$

let $D = \{v \in B : |L_k(v)| \leqslant t\}$, and suppose that $|D| \geqslant 2kr\dbinom{r-t}{t}$. Note that by Observation 2, since G is maximal, $L_k(v) \cap T = \emptyset$ for every $v \in B$. Thus, by the pigeonhole principle, there exists a subset

$S \subset [r]\backslash T$ of size t, and a subset $C \subset B$ of size $2kr$ such that $L_k(v) \subset S$ for every $v \in C$.

Consider the bipartite graph H, with parts A and C, and edges with colours from S. Note that, by the definition of C, each vertex of C sends at most kr edges with colours from $\overline{S}$ into A, so $d_H(v) \geqslant |A| - kr$ for every $v \in C$. Let $a = 2kr$ and $b = kr$, and recall that $|A| \geqslant 4kr + k = 4b + k$, and that $|C| \geqslant a \geqslant 2k$.

We apply Lemma 10 to H, with $a = 2kr$ and $b = kr$, to obtain a k-connected subgraph of H on at least

$$|A| + |C| - \frac{2k^2r^2}{2kr - k + 1} \ > \ |A| + |C| - 2kr \ \geqslant \ |A|$$

vertices. This subgraph uses at most t colours, and so this contradicts the maximality of G. Thus $|D| \leqslant 2kr\binom{r-t}{t}$.

Let $B' = B \setminus D$, so each vertex in B' sends at least k edges of at least $t+1$ different colours into A, i.e., $|L_k(v)| \geqslant t+1$ for every $v \in B'$. For each vertex $v \in B'$, choose a list $L(v) \subset L_k(v)$ of size $t+1$. So for each $v \in B'$ and $\ell \in L(v)$, there exist at least k vertices $u \in A$ such that $f(uv) = \ell$.

The remainder of the proof now goes through exactly as before, since by Observation 2, for each vertex $v \in B'$ the vertices $A \cup \{v\}$ are k-connected by the colours $T \cup \{\ell\}$ if $\ell \in L(v)$. The reader who feels comfortable with this fact may therefore safely 'jump' to the end of the proof. For the remaining readers, and for completeness, we shall repeat the argument.

So let $\varepsilon > 0$, and let $\mathcal{T} = \{S \subset [r] \setminus T : |S| = s - t\}$. Suppose that $m(n, r, s, k) \leqslant n - \varepsilon|B'|$. This means that for every set $S \in \mathcal{T}$, the largest k-connected component in K_n, using only the colours $S \cup T$, and containing G, avoids at least $\varepsilon|B'|$ vertices of B'. Hence, for each $S \in \mathcal{T}$ there are at least $\varepsilon|B'|$ vertices $v \in B'$ such that $S \cap L(v) = \emptyset$, by Observation 2. For each $S \in \mathcal{T}$, let $M(S) = \{v \in B' : S \cap L(v) = \emptyset\}$.

Now, observe that for each vertex $v \in B'$, there are exactly $\binom{r-2t-1}{s-t}$ sets $S \in \mathcal{T}$ with $v \in M(S)$. So, summing over $\mathcal{T}$, we obtain

$$\varepsilon|B'|\binom{r-t}{s-t} \ \leqslant \ \sum_{v \in B'} \sum_{s \in \mathcal{T}} I[v \in M(S)] \ = \ |B'|\binom{r-2t}{s-t},$$

as before, and therefore

$$\varepsilon \ \leqslant \ \left(\frac{r-2t-1}{r-t}\right)^{s-t} \ < \ \exp\left(\frac{-(t+1)(s-t)}{r-t}\right).$$

Now, setting $t = \lceil s/2 \rceil$ to (approximately) maximize $\dfrac{(t+1)(s-t)}{r-t}$, and noting that $\dfrac{(\lceil s/2 \rceil + 1)\lfloor s/2 \rfloor}{r - \lceil s/2 \rceil} > \dfrac{s^2}{4r}$, we obtain

$$\begin{aligned} M(f,n,r,s,k) \geqslant\; |A| + (1-\varepsilon)|B'| \;\geqslant\; |A| + (1-\varepsilon)\left(|B| - 2kr\binom{r}{\lceil s/2 \rceil}\right) \\ \geqslant\; \left(1 - e^{-s^2/4r}\right) n - 2kr\binom{r}{\lceil s/2 \rceil}. \end{aligned}$$

Since f was arbitrary, this proves the theorem. $\square$

Remark 4. Using induction, as in the proof of Theorem 24, one can slightly improve this bound.

6. Further Problems

There is a great deal about the function $m(n,r,s,k)$ that we do not know. In this section we shall discuss some of the most obvious and intriguing of these open questions. We begin with the following corollary of Theorem 5 and Lemma 13. It demonstrates the rather embarrassing state of our knowledge in the range $2 < s \ll \sqrt{r}$.

Corollary 25. *There exist constants $C, C' \in \mathbb{R}$ such that*

$$Cs^2 \;\leqslant\; \frac{r}{n}\, m(n,r,s,k) \;\leqslant\; C' \min\left\{2^s, s\sqrt{r}\right\}$$

for every $r, s, k \in \mathbb{N}$ with $s^2 < r$, and n sufficiently large.

In particular, we do not know whether the function

$$g(s) = \liminf_{k\to\infty} \liminf_{r\to\infty} \left(r \liminf_{n\to\infty} \left(\frac{1}{n}\, m(n,r,s,k) \right)\right)$$

grows like a polynomial or an exponential function (or something in between!). We conjecture that the upper bound is correct in the range $s \ll \log(r)$.

Conjecture 2. *Let $2 \leqslant s, k \in \mathbb{N}$ be fixed. If $r > 4^s$, and n is sufficiently large, then*

$$m(n,r,s,k) \geqslant \frac{2^s n}{r+1} - O(k).$$

We suspect that Conjecture 2 is not easy, and pose the following much weaker statements as open problems.

Problem 3. *Prove any of the following.*

(i) $g(s) > (1+\varepsilon)^s$ for some $\varepsilon > 0$ and every $s \in \mathbb{N}$.

(ii) *$g(s) < (1+\varepsilon)^s$ for every $\varepsilon > 0$ and sufficiently large s.*
(iii) *$g(s) = O(s^t)$ for some $t \in \mathbb{N}$.*
(iv) *$g(s) = \Omega(s^t)$ for every $t \in \mathbb{N}$.*

When $r \gg s\sqrt{r} \gg 2^s$, we suspect that the upper bound in Theorem 5 becomes optimal, and $m(n,r,s,k) = \Theta\left(\frac{sn}{\sqrt{r}}\right)$, but at present we seem a long way from proving such a result.

We proved that the function $m(n,r,s,k)$ is 'small' when $s \ll \sqrt{r}$ and 'big' when $s \gg \sqrt{r}$. But what about when $s = \Theta(\sqrt{r})$? What is the exact nature of this phase change? Theorem 5 gives us (roughly) the bounds

$$\left(1 - e^{c/4}\right) n \;\leqslant\; m(n,r,\lfloor c\sqrt{r}\rfloor,k) \;\leqslant\; \frac{cn}{\sqrt{2}}$$

when n is sufficiently large compared to r. Again we conjecture that the upper bound is correct.

Conjecture 3. *Let $c \in (0,\sqrt{2}]$. Then*

$$h(c) \;=\; \liminf_{k\to\infty}\liminf_{r\to\infty}\liminf_{n\to\infty}\left(\frac{1}{n}m(n,r,\lfloor c\sqrt{r}\rfloor,k)\right) \;=\; \frac{c}{\sqrt{2}}.$$

Although we would really like to determine $h(c)$ exactly, we would in fact be very happy with an answer to either of the following, more basic questions.

Question 1. *Does there exist a constant $c \in \mathbb{R}$ such that $h(c) = 1$?*

Question 2. *Is $\lim_{c\to 0}\dfrac{h(c)}{c} > 0$?*

Finally, we have a question about the phase transition at $2s = r$. We would like to know the value of $m(n, 2s-1, s, k)$; in other words, what does the function jump *to* when r is odd? For $s = 2$ we showed that the answer is $n - k + 1$, and it is tempting to guess that this is always the correct answer, but we believe this to be false. More precisely we make the following conjecture. The rather strange right-hand side is derived from a fairly complicated construction, which we found and then lost! Since we cannot prove it, we state it as a conjecture.

Conjecture 4. *Let $n, s, k \in \mathbb{N}$, with $s \geqslant 3$ and n and k sufficiently large. Then*

$$m(n,2s-1,s,k) \leqslant n - \left(\frac{2s(r-2)}{r^2-2r+2}\right)k = n - \left(1 + \frac{2s-5}{4(s-1)^2+1}\right)k,$$

where $r = 2s-1$.

Problem 4. *Determine the value of $m(n, 2s-1, s, k)$ for every $s, k \in \mathbb{N}$ and all sufficiently large n.*

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