

ISSN - 2277 - 0593  
© UNAAB 2009

Journal of Natural  
Sciences, Engineer-  
ing and Technology

## FOUR STEPS IMPLICIT METHOD FOR THE SOLUTION OF GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

\*<sup>1</sup>OSILAGUN, J. A., <sup>2</sup>ADESANYA, A.O. <sup>2</sup>ANAKE, T. A. AND <sup>2</sup>OGHOYON, G. J.

<sup>1</sup>Department of Mathematical Sciences, Olabisi Onabanjo University, Ago Iwoye, Ogun State, Nigeria.

<sup>2</sup>Department of Mathematics, Covenant University, Sango Ota, Ogun State, Nigeria.

\*Corresponding author: [torlar10@yahoo.com](mailto:torlar10@yahoo.com)

Tel: +2348086049694

### ABSTRACT

Four steps implicit scheme for the solution of second order ordinary differential equation was derived through interpolation and collocation method. Newton polynomial approximation method was used to generate the unknown parameters in the corrector. The method was tested with numerical examples and it was found to be efficient in solving second order ordinary differential equations.

**Keyword:** collocation, interpolation, continuous, implicit, Newton's polynomial, corrector.

### INTRODUCTION

The second order initial value ordinary differential equation of the form

$$y'' = f(x, y, y') \quad y(a) = \eta_0, y'(a) = \eta_1 \quad (1)$$

is considered in this paper. Differential equation is often used to model scientific and technological problems which most times, these equations do not have analytic solution, hence an approximate numerical method is required to solve these problems. Block methods for numerical solution of ordinary differential equations have been proposed by several researchers such as Milne (1953), Shampine and Watt (1969), Worland (1976) and Omar (1999). Rosser (1967) introduced the 3 point implicit block method based on integration formulae which is basically Newton's cote type. Zanariah *et al.* (2006) proposed 3 points im-

PLICIT block method based on Newton's backward divided difference formula. Adesanya *et al.* (2009) proposed a two step method for the general solution of second order which is self starting and adopted Newton's polynomial to generate the starting value. Awoyemi *et al.* (2009) recently proposed a self starting Numerov's method. This method solves both initial and boundary value problems of ordinary differential equation. Yahaya (2007) constructed a Numerov method from a quadratic continuous polynomial solution. This process led to method which is applied to both initial and boundary value problems.

In this research work, we propose a block method for step length of four. This method adopts Newton's polynomial approximation to generate the starting value and solves general second order ordinary differential equa-

tion directly.

### METHODOLOGY

We consider an approximate solution to (1) in power series

$$y(x) = \sum_{j=0}^k a_j \phi_j(x) \tag{2.2}$$

$$y(x) = \sum_{r=0}^{t-1} \phi_r(x) y_{n+r} + h^2 \sum_{r=0}^{m-1} \varphi_r(x) f_{n+r} \tag{2}$$

where  $x = [x_n, x_{n+r}]$ ,  $m$ =the distinct collocation point,  $t$  is the interpolation point, for our method, the step length  $k=4$

$\phi_j = \phi^j, a_j, j = 0(1)2k - 1$  are constants to be determined. We Consider a linear multistep method of the form

$$\phi_r(x) = \sum_{i=0}^{i+m-1} \phi_{i+1,r} p_i(x), \quad r = 0, 1, 2, \dots, m-1 \tag{3}$$

$$\varphi_r(x) = \sum_{i=0}^{i+m-1} \varphi_{i+1,r}(x) p_i(x), \quad r = 0, 1, 2, \dots, m-1 \tag{4}$$

$$y(x_{n+r}) = y_{n+r}, \quad r \in [0, 1, 2, \dots, t-1] \quad y''(x) = f_{n+r}, \quad r = 0, 1, m-1 \tag{5}$$

To get  $\phi_j(x)$  and  $\varphi_j(x)$ , According to Yahaya (2007), he derived a matrix of the form

$$DC = I$$

where:  $I$  is an identity matrix of dimension  $(t+m) \times (t+m)$ .

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \cdot & \cdot & x_n^{t+i-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdot & \cdot & x_{n+1}^{t+i-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdot & \cdot & x_{n+t-1}^{t+m-1} \\ 0 & 0 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 2 & \cdot & \cdot & (t+m-1)(t+m-2)x_{m-2}^{t+m-2} \end{pmatrix} \tag{6}$$

$$c = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdot & \alpha_{1,t-1} & h^2\varphi_{1,0} & \cdot & h^2\varphi_{1,m-1} \\ \alpha_{2,0} & \alpha_{2,1} & \cdot & \alpha_{2,t-1} & h^2\varphi_{2,0} & \cdot & h^2\varphi_{2,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{t+m,0} & \alpha_{t+m,1} & \cdot & \alpha_{t+m,t-1} & h^2\varphi_{t+m,0} & \cdot & h^2\varphi_{t+m,m-1} \end{pmatrix} \quad (7)$$

**Development value for the unknown**

Theorem (3.0):

Assuming that  $f \in C^{n+1}[a, b]$  and  $x_k \in [a, b]$  for  $k=0, 1, n$  are distinct values, then

$f(x) = y(x) + R_n(x)$ , where  $y(x)$  is a polynomial that can be used to approximate  $f(x)$

For Newton's polynomial

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad (8)$$

$f(x) \cong y(x) + R_n(x)$  is the remainder and has the form

$$R_n(x) = \frac{f^{(n+1)}}{(n+1)!} (x - x_0)(x - x_1)\dots(x - x_{n-1})(x - x_n) \quad (9)$$

(Awoyemi *et al.*,2009)

**Development of four steps method**

In developing the method with step length  $k=4$ , we consider

$$D = \begin{bmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 \end{bmatrix} \quad (10)$$

This gives a continuous scheme

$$\alpha_2 = -t$$

$$\alpha_3 = (t + 1)$$

$$\beta_0(t) = \frac{h^2}{1440} (2t^6 + 6t^5 - 5t^4 - 20t^3 + 11t)$$

$$\beta_1(t) = \frac{h^2}{360} (-2t^6 - 9t^5 + 5t^4 + 30t^3 - 18t)$$

$$\beta_2(t) = \frac{h^2}{240} (2t^6 + 12t^5 + 5t^4 - 60t^3 + 55t)$$

$$\beta_3(t) = \frac{h^2}{360} (-2t^6 - 15t^5 - 25t^4 + 50t^3 + 180t^2 + 118t)$$

$$\beta_4(t) = \frac{h^2}{1440} (2t^6 + 18t^5 + 55t^4 + 60t^3 - 21t)$$

(11)

where: 
$$t = \frac{x - x_{n+3}}{h}$$

Evaluating (11) at  $x = x_{n+4}$  i.e. when  $t = 1, -2, -3$ , gives

$$240y_{n+4} - 480y_{n+3} + 240y_{n+2} = h^2 (19f_{n+4} + 204f_{n+3} + 14f_{n+2} + 4f_{n+1} - f_n) \tag{12}$$

$$240y_{n+3} - 480y_{n+2} + 240y_{n+1} = h^2 (-f_{n+4} + 24f_{n+3} + 194f_{n+2} + 24f_{n+1} - f_n) \tag{13}$$

$$480y_{n+3} - 720y_{n+2} + 240y_{n+1} = h^2 (-3f_{n+4} + 52f_{n+3} + 402f_{n+2} + 252f_{n+1} + 17f_n) \tag{14}$$

Evaluating the first derivative of (4,1) at  $t = -3$ , gives

$$1440hy'_n - 1440y_{n+3} + 1440y_{n+2} = h^2 (33f_{n+4} - 284f_{n+3} - 966f_{n+2} - 1908f_{n+1} - 475f_n) \tag{15}$$

Solving (12)-(16) using block method gives

$$\begin{aligned}
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_n \end{bmatrix} = h^2 \begin{bmatrix} \frac{3}{8} & \frac{-47}{8} & \frac{29}{8} & \frac{-7}{8} \\ \frac{5}{117} & \frac{3}{27} & \frac{45}{3} & \frac{30}{-9} \\ \frac{40}{49} & \frac{80}{191} & \frac{8}{577} & \frac{160}{57} \\ \frac{12}{120} & \frac{120}{45} & \frac{80}{45} & \frac{160}{80} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} + \\
 & h^2 \begin{bmatrix} 0 & 0 & 0 & \frac{367}{1440} \\ 0 & 0 & 0 & \frac{53}{90} \\ 0 & 0 & 0 & \frac{147}{160} \\ 0 & 0 & 0 & \frac{869}{720} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y'_{n-1} \\ y'_{n-2} \\ y'_{n-3} \\ y'_n \end{bmatrix} \tag{16}
 \end{aligned}$$

Hence, from (16)

$$y_{n+1} = y_n + \frac{h^2}{1440} (540f_{n+1} + 282f_{n+2} + 116f_{n+3} - 21f_{n+4} + 367f) + hy'_n \tag{17}$$

$$y_{n+2} = y_n + \frac{h^2}{90} (144f_{n+1} - 30f_{n+2} + 16f_{n+3} + 3f_{n+4} + 53f) + 2hy'_n \tag{18}$$

$$y_{n+3} = y_n + \frac{h^2}{160} (468f_{n+1} + 54f_{n+2} + 60f_{n+3} - 9f_{n+4} + 147f) + 3hy'_n \tag{19}$$

$$y_{n+4} = y_n + \frac{h^2}{45} (192f_{n+1} + 48f_{n+2} + 64f_{n+3} + 56f) + 4hy'_n \tag{20}$$

**Developing the unknown for  $k=4$**

We evaluate the first derivative of (8), and neglecting  $a_6$  and higher values.

$$60hy'_n = 12y_{n+5} - 75y_{n+4} + 120y_{n+3} - 220y_{n+2} + 300y_{n+1} - 137y_n \quad (21)$$

$$60hy'_{n+1} = -3y_{n+5} + 20y_{n+4} - 60y_{n+3} + 120y_{n+2} - 65y_{n+1} - 12y_n \quad (22)$$

$$60hy'_{n+2} = 2y_{n+5} - 15y_{n+4} + 60y_{n+3} - 20y_{n+2} + 30y_{n+1} + 3y_n \quad (23)$$

$$60hy'_{n+3} = -3y_{n+5} + 30y_{n+4} + 20y_{n+3} - 60y_{n+2} + 15y_{n+1} - 2y_n \quad (24)$$

$$60hy'_{n+4} = 15y_{n+5} + 65y_{n+4} - 120y_{n+3} + 60y_{n+2} - 20y_{n+1} + 3y_n$$

Solving (24)-(28) for  $y_{n+i}$ ,  $i = 1(1)4$  gives,

$$y_{n+1} = y_n + \frac{h}{6348} (5386y'_{n+1} - 420y'_{n+2} + 2382y'_{n+3} - 247y'_{n+4} + 2259y'_n) \quad (25)$$

$$y_{n+2} = y_n + \frac{h}{529} (705y'_{n+1} + 288y'_{n+2} + 135y'_{n+3} - 12y'_{n+4} + 174y'_n) \quad (26)$$

$$y_{n+3} = y_n + \frac{h}{2116} (2598y'_{n+1} + 2520y'_{n+2} + 1578y'_{n+3} - 105y'_{n+4} + 729y'_n) \quad (27)$$

$$y_{n+4} = y_n + \frac{h}{1587} (2188y'_{n+1} + 1272y'_{n+2} + 2580y'_{n+3} + 476y'_{n+4} + 504y'_n) \quad (28)$$

Substituting for  $y_{n+2}$  in (15) and substituting in (16)-(19) and solving for  $f_{n+i}$ ,  $i = 1(1)4$  gives,

$$f_{n+1} = \frac{-1}{4} f_n + \frac{1}{h} \left( \frac{1}{6} y'_{n+1} + \frac{3}{4} y'_{n+2} - \frac{1}{6} y'_{n+3} + \frac{1}{48} y'_{n+4} - \frac{37}{48} y'_n \right) \quad (29)$$

$$f_{n+2} = \frac{1}{6} f_n + \frac{1}{h} \left( \frac{-4}{3} y'_{n+1} + \frac{1}{2} y'_{n+2} + \frac{4}{9} y'_{n+3} - \frac{1}{24} y'_{n+4} + \frac{31}{72} y'_n \right) \quad (30)$$

$$f_{n+3} = \frac{-1}{41} f_n + \frac{1}{h} \left( \frac{3}{2} y'_{n+1} - \frac{9}{4} y'_{n+2} + \frac{7}{6} y'_{n+3} + \frac{3}{16} y'_{n+4} - \frac{29}{48} y'_n \right) \quad (31)$$

$$f_{n+4} = f_n + \frac{1}{h} \left( \frac{-16}{3} y'_{n+1} + 6y'_{n+2} - \frac{16}{3} y'_{n+3} + \frac{7}{3} y'_{n+4} + \frac{7}{3} y'_n \right) \quad (32)$$

Substituting (25)-(28) into the second derivative of (8) and comparing with (29)-(32) give

$$y'_{n+1} = y'_n - 0.222193hf_n \quad (33)$$

$$y'_{n+2} = y'_n + 0.111768hf_n \quad (34)$$

$$y'_{n+3} = y'_n + 1.188485hf_n \quad (35)$$

$$y'_{n+4} = y'_n + 1.311393hf_n \quad (36)$$

**Analysis of the scheme**

**Convergence**

The convergence analysis of this scheme is determined using the approach of Fatunla (1992), where each block integrator is represented as a single step block r- point multistep method of the form

$$A^{(0)} Y_m = \sum_{i=1}^k A^{(i)} Y_{m-1} h^2 B^i F_{m-1}$$

where:  $A^{(i)}, B^{(i)}, i = 0(1)k$  are r by r matrix respectively with element

$a^i_{l,j}, b^i_{l,j}$  for  $l, j = 1(1)r$ . specifically,  $A^{(0)}$  is an  $r \times r$  identity matrix,

$Y_m, Y_{m-1}, F_m$  and  $F_{m-1}$  are vectors of numerical estimate describe below. With r-vector

$Y_m$  and  $F_m$  (for  $n = mr, m = 0, 1, \dots$ )

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+r} \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+r} \end{bmatrix}, Y_{m-i} = \begin{bmatrix} y_{n-r} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-r} \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}$$

Awoyemi (2009) for details

**Analysis of the block**

$$\det \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right) = 0$$

$$\det \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda - 1 \end{pmatrix} = 0$$

$$\lambda^3(\lambda - 1) = 0$$

$$\lambda = 0, 0, 0, 1$$

Hence the block is zero stable.

**Numerical example**

We test the efficiency of our scheme on linear and non linear second order differential equation.

**Problem 1:**

$$y'' - x(y')^2 = 0$$

$$y(0) = 1, y'(0) = \frac{1}{2}, h = 0.1 / 40$$

Exact solution

$$y(x) = 1 + \frac{1}{2} \ln \left( \frac{2+x}{2-x} \right)$$



S/N	Expected solution	New method	error
1	1.00125000065104	1.001250000186	4.650D-10
2	1.00250000520835	1.0025000039971	1.211D-09
3	1.00375001757828	1.003750013474	4.030D-09
4	1.00500004166729	1.005000047982	6.314D-09
5	1.00625008138212	1.0062500803586	8.462D-09
6	1.00750014062974	1.00750002579038	1.148D-09
7	1.00875022331755	1.008750239662	1993D-09
8	1.01000033335333	1.0100000783829	2.550D-09
9	1.01125047464542	1.0112504890376	4.256D-09
10	1.01250065110271	1.0125006101014	4.100D-08

**Problem 11**

$$y'' = y + xe^{3x}$$

$$y(0) = \frac{-3}{32}, y'(0) = \frac{-5}{32}, h = 0.1/40$$

Exact solution

$$y(x) = \frac{4x - 3}{32 \exp(-3x)}$$

S/N	Expected result $y(x)$	New method $y_n(x)$	Error $y(x)-y_n(x)$
1	-0.094140915761848	-0.0941409131568	2.61D-09
2	-0.094532404142338	-0.0945324074753	3.33D-09
3	-0.094924451608388	-0.0949244224215	2.92D-08
4	-0.095317044390700	-0.0953170247449	2.02D-08
5	-0.095710168480980	-0.0957101793552	1.08D-08
6	-0.096103809629113	-0.0961039982252	1.88D-07
7	-0.09649533403163	0.0964952920353	4.23D-08
8	-0.096892584872264	-0.0968923659413	2.18D-07
9	-0.097289689232184	-0.0972874625827	2.22D-06
10	-0.097683251173919	-0.0976830958236	1.55D-07

## REFERENCES

- Adesanya, A.O., Anake, T.A., Bishop S. A., Osilagun, J.A.** 2009. Two steps block method for the solution of general second order initial value problems of ordinary differential equation. *Asset International Journal*, University of Agriculture, Abeokuta
- Awoyemi, D.O., Adesanya, A.O., Ogunyebi, S.N.** 2009. Construction of self Starting Numerov method for the solution of initial value problems of general second order ordinary differential equation. *Int. j. Num. Math*, University of Ado Ekiti, 4(2): 269-278
- Milne, W.E.** 1953. Numerical solution of differential equation, Wiley, New York.
- Omar, Z.** 1999. Developing parallel block method for solving higher orders ODES directly. Thesis, University putra Malaysia
- Rosser, J.B.** 1967. A Runge Kutta for all season, SIAM, P. 417-452.
- Sham pine, L.F., Watts, H.** 1969. Block implicit one step method, *international journal of Math-comp* 23: 731-740.
- Worland, P.B.** 1976. Parallel method for the numerical solution of ordinary differential equation, *IEEE trans. Computer C-25*: 1045-1048.
- Yahaya, Y. A.,** 2007. A note on the construction of Numerov method through a quadratic continuous polynomial for the general second order ordinary differential equation. *Journal of Nigerian Association of mathematical physics*, II: 253-260
- Zanariah Abdul, Mohamed Bin, Zurni Omar** 2006. 3 point implicit block method for solving ordinary differential equation. *Bulletin of the Malaysian Mathematical Science* 2, 29(1):23-31

(Manuscript received: 12th January, 2009; accepted: 22nd April, 2009)