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A note on a semi continuous and Darboux continuous function

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Abstract
Present alternate proof for a function that is zero almost everywhere, not identically zero, of Baire class 1 (and indeed upper semi-continuous), and Darboux continuous.

1. Introduction

The purpose of this paper is to present alternate proof of well known result in real analysis [1].An alternate proof of a theorem provides a new way of looking at the theorem and this fresh perspective is often enough to justify the new approach. However, a new proof of an old result that is conceptually easier has obvious benefits.

2. The main result

Before giving the main result, the necessary definitions are stated.

Definition 2.1.

A function is said to be Darboux continuous if it takes every intermediate value between any two values that it takes, for some intermediate argument.

Definition 2.2.

A function is said to be of Baire class 1 if it is the limit function of a set of continuous functions in the interval concerned.

We now give the following main result:

Theorem 2.1.

There exists in the closed interval I = [0, 1], a function that is zero almost everywhere, not indentically zero, of Baire class 1 (and indeed upper semicontinuous), and Darboux continuous.

Proof.

Let $f(x) = 0$ if $x \neq \frac{1}{2}$ and $f(x) = 1$ if $x = \frac{1}{2}$.

Clearly f is a bounded upper semicontinuous function, thus by Theorem (A1) of [2], there exists a bounded Darboux upper semicontinuous function g such that $\{x \in I: f(x) \neq g(x)\}$ is a first category null subset of I, and $f(x) \leq g(x)$ for all x in I. Since $f(\frac{1}{2}) = 1$ and $f(\frac{1}{2}) \leq g(\frac{1}{2})$, thus g is not indentically zero. Also, $\{x \in I : g(x) \neq 0\} \subseteq \{x \in I: f(x) \neq g(x)\} \bigcup \{\frac{1}{2}\}$. Thus, $\lambda(\{x \in I; g(x) \neq 0\}) \leq \lambda(\{x \in I: f(x) \neq g(x)\}) + \lambda(\{\frac{1}{2}\}) = 0+0+0$;

where λ is the Lebesque measure.Therefore, g is zero almost everywhere. Also, since closed interval I = [0, 1] is a perfectly normal space, by Theorem 3 of [3] g is the limit of a monotonically decreasing sequence of continuous functions, thus g is of Baire class 1.

Theorem 2.2.

There exists in the closed interval I = [0, 1], a function that is zero almost everywhere, not indentically zero, of Baire class 1 (and indeed lower semicontinuous), and Darboux continuous.

Proof.

Since, a function f is zero almost everywhere, not indentically zero, of Baire class 1 (and indeed upper semicontinuous), and Darboux continuous if and only if $-f$ is zero almost everywhere, not indentically zero, of Baire class 1 (and indeed lower semicontinuous), and Darboux continuous. Therefore, by Theorem 2.1. there exists in the closed interval I = [0, 1], a function that is zero almost everywhere, not indentically zero, of Baire class 1 (and indeed lower semicontinuous), and Darboux continuous.

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