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Original Article

Norms and numerical radii inequalities for (α, β) - Normal transaloid operators

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Abstract

The studies on Hilbert spaces for the last decade has been of great interest to many mathematicians and researchers, especially on operator inequalities related to operator norms and numerical radii for a family of bounded linear operators acting on a Hilbert spaces. Results on some inequalities for normal operators in Hilbert spaces for instance numerical ranges W(T), numerical radii w(T) and norm ||.|| obtained by Dragomir and Moslehian among others due to some classical inequalities for vectors in Hilbert spaces. The techniques employed to prove the results are elementary with some special vector inequalities in inner product spaces due to Buzano, Goldstein, Ryff and Clarke as well as some reverse Schwarz inequalities. Recently, the new field of operator theory done by Dragomir and Moslehian on norms and numerical radii for (α, β) - normal operators developed basic concepts for our Statement of the problem on normal transaloid operators. M. Fujii and R. Nakamoto characterize transaloid operators in terms of spectral sets and dilations and other non-normal operators such as normaloid, convexoid and spectroid. Furuta did also characterization of normaloid operators. Since none has done on norms and numerical radii inequalities for (α, β) – normal transaloid operators, then our aim is to characterize (α, β)- normal transaloid operators, characterize norm inequalities for (α, β)- normal transaloid operators and to characterize numerical radii for (α, β)- normal transaloid operators. We use the approach of the Cauchy-Schwarz inequalities, parallelogram law, triangle inequality and tensor products. The results obtained are useful in applications in quantum mechanics.

1. Introduction

Studies on the properties of Hilbert space operators including spectrum, numerical ranges, numerical radii and norms are fundamental in various fields of mathematics including operator theory, trigonometry, numerical analysis, fluid dynamics among others. Operator inequalities related to operator norms and numerical radii for a family of bounded linear operators acting on a Hilbert space have been studied for instance by Dragomir and Moslehian [2, 6 and 7]. Recently Dragomir and Moslehian [1 and 4] studied norms and numerical radii for (α, β)- normed operators also D. Senthilkumar [3] studied on $\rho-(\alpha, \beta)$- normal operators.

From the research done by D. Senthilkumar, Dragomir and Moslehian and many other authors clearly shows that much has been done on numerical radii and numerical ranges in Hilbert space operators but so far little has been done on norms and numerical radii inequalities for (α, β)-normal transaloid operators in Hilbert spaces, however, in this paper we determine norms and numerical radii inequalities for (α, β)-normal transaloid operators in Hilbert spaces.

2. Preliminaries

In this section, the following definition of terms will gives a prerequisite knowledge in getting our main results, these are:

Definition 2.1. Let X be a linear space. A function $\|\cdot\|: X \to \mathbb{R}$ satisfies the following properties

a) $\|x+y\| \leq \|x\| + \|y\|$

b) $\|\alpha x\| = |\alpha|\|x\|$

c) $\|x\| = 0$ iff $x = 0$

Is called a norm on X $\forall x, y \in X$.

The ordered pair $(X, \|\cdot\|)$ is called a normed space

Definition 2.2. Inner product space. Let X be a vector space over $\mathbb{R}$ and a map $\langle \cdot,\cdot \rangle : X \times X \to \mathbb{R}$ satisfies the following properties

a. $\langle \lambda x + \mu y \rangle = \lambda(x,z) + \mu(y,z)$ for all $x, y \in X$

b. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$

c. $\langle x, x \rangle \geq 0$ for $x \in X$ with equality iff $x = 0$

d. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$

Definition 2.3. An operator is a structure preserving map. For instance;

Let V be a vector space. If $T: V \to V$ then T is considered to be an operator.

Definition 2.4. A linear operator $T: H \to H$ $\forall T \in H$ is said to be normal if $T^*T = T^*T$

Definition 2.5. An operator $T: H \to H$ is said to be normaloid if $\|T\| = w(T)$ and is said to be transaloid operator if $\lambda \in \mathbb{C}$ such that $T_\lambda = T - \lambda I$ is normaloid.

Definition 2.6. Numerical Radius. let $(H; \langle \cdot,\cdot \rangle)$ be a complex Hilbert space . Then the numerical range of an operator T is the subset of the complex numbers $\mathbb{C}$ given by
$W(T) = \{\langle Tx, x \rangle, x \in H, \|x\| = 1\}$

3. Main results

Under this section we give the main result.

Theorem 3.1 Let $A = \delta|A| \in B(H)$ be the polar decomposition of A.

Then $A \in k-(\alpha, \beta)$ normal transaloid operators if and only if there exist a positive number α, β such that

$$\alpha\left\||A|^k x\right\|\left\||A|^k y\right\| \leq \left|\left\langle \delta|A|^{2k} x, y\right\rangle\right| \leq \beta\left\||A|^k x\right\|\left\||A|^k y\right\|$$

Proof;

Assuming A is $k-(\alpha,\beta)$ -normal transaloid operator. Then

$$\left|\left\langle \delta|A|^{2k} x, y\right\rangle\right| = \left|\left\langle |A|^{2k} x, \delta^* y\right\rangle\right|$$

$$= \sup\left\{ \left|\left\langle |A|^{2K} x, \delta^* y\right\rangle\right|, x, y \in H, \|x\|=1, \|y\|=1 \right\}$$

$$\leq \sup\left\{ \left\||A|^k x\right\|\left\||A|^k \delta^* y\right\|, x, y \in H, \|x\|=1, \|y\|=1 \right\}$$

$$\leq \sup\left\{ \left\||A|^k x\right\|\left\||A^*| y\right\|, x, y \in H, \|x\|=1, \|y\|=1 \right\}$$

$$= \beta\left\||A|^k x\right\|\left\||A|^k y\right\| \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly

Let $\varepsilon > 0$ be given, then

$$\left|\left\langle |A|^k x, \delta^* y\right\rangle\right| + \varepsilon > \left|\left\langle \delta|A|^{2k} x, y\right\rangle\right|$$

$$> \left\||A|^k x\right\|\left\||A|^k \delta^* y\right\|$$

$$> \left\||A|^k x\right\|\left\||A^*|^k y\right\|$$

$$= \alpha\left\||A|^k x\right\|\left\||A|^k y\right\|$$

But since $\varepsilon > 0$ is arbitrary, this implies that

$$\alpha\left\||A|^k x\right\|\left\||A|^k y\right\| \leq \left|\left\langle \delta|A|^{2k} x, y\right\rangle\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

By (1) and (2) we can have

$$\alpha\left\||A|^k x\right\|\left\||A|^k y\right\| \leq \left|\left\langle \delta|A|^{2k} x, y\right\rangle\right| \leq \beta\left\||A|^k x\right\|\left\||A|^k y\right\|$$

4. Conclusion

The results we have determined are useful for for (α,β)-normal transaloid operators.

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