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Original Article

On norm preserving conditions for local automorphisms of commutative banach algebras

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Abstract

The history of commutative algebra first appeared in 1890 by David Hilbert which was then followed by Banach spaces in 1924 since localization reduces many problems of geometric special case into commutative algebra problems of local ring. So far, many studies on preserver problems have been focusing on linear preserver problems (LPPs) especially LPPs in matrix theory. Also in consideration has been the characterization of all linear transformation on given linear space of matrices that leave certain functions, subsets and relations invariant. Clearly, we also have spectrum preserver problem or transmission. Kadison and Sourour have also shown that the derivation of local derivation of Von Neumann algebra **R** are continous linear maps if it coincides with some derivation at each point in the algebra over **C**. We employ the concept of 2-local automorphisms introduced by Serml that if we let A be an algebra, then the transformation $\phi: A \to A$ is called a 2-local automorphism if for all x, y$\in$ A there is an automorphism ϕ(xy) of A for which $\phi(x) = \phi_{x,y}$(x) and $\phi(y) = \phi_{x,y}$(y). In this paper, we characterize commutativity of local automorphism of commutative Banach algebras, establish the norm preserver condition and determine the norms of locally inner automorphisms of commutative Banach algebras. We use Hahn-Banach extension theorems and the great ideas developed by Richard, and Sorour to develop the algebra of local automorphisms, then integrate it with norm preserver conditions of commutative Banach algebras. The results of this work have a great impact in explaining the theoritical aspects of quantum mechanics especially when determining the distance of physical quantities.

1. Introduction

There are many scholars who have worked on derivations of algebras, David and Sourour studied on local derivation of bounded operators on Banach space and found that in any infinite dimensional Banach space (BX) there is a local automorphism which is an automorphism. Sakai on the other hand proved that every derivation of W* algebra is inner while Semrl P. showed that every local automorphism of B(X) is an automorphism in an infinite-dimensional Hilbert space. Of great impact is this the paper published by Otterman, Holger and Michel on Inner derivation of alternative algebra over commutative rings and they gave three classes of inner derivations: associator, standard and commutator derivations in which we extend the results of sakai derivation and Semrl P. to show that "Every *automorphism of commutative Banach algebra is inner".

Preliminaries

Definition [2.1]: Let B be Banach algebra and a linear operator $\delta: B \to B$ is called a derivation $\delta(xy) = \delta(x)y + x\delta(y)\ \forall\ x, y \in B$.

Each element $b \in B$ generates a derivation $\delta_a : B \to B$ defined by $\delta_a(x) = ax - bx$ for all $x \in B$ is called an inner derivation.

***Definition [2.2]*:**

Let B be commutative Banach algebra and δ is a derivation on B and $\overline{B}$ the closure of B, then there exist a bounded operator $b \in \overline{B}$ s.t $\delta_b(x) = ax - bx$ for all $x, y \in B$.

Let $P \in B$ and δ a derivation on P and the commutant of P is P^c in B and A be the maximal abelian *subalgebra of P^c

***Definition [2.3]*:**

Every $N-$dimensional normed vector space is isomorphic to C^N and so complete.[3]

Definition [2]:

Every Banach algebra can be embedded as a closed algebra of B(X)

***Theorem 1*:** Gleason Kahane-Zelasko Theorem [1]

Let H be an infinite dimensional Hilbert space. If $f: B(H) \to B(H)$ is surjective function with property that $\delta(f(E)f(F) = \delta(EF)\ \forall\ E, F \in B(H)$ then f is algebra of automorphism of $B(H)$

Definition [2.4]:

For $A \in B(H)$ the inner derivation induced by A on the Operator $\delta(xy) = \delta(x)y - x\delta(y)\ \forall\ x, y \in B(H)$ then the norm of inner derivation δ on H has been computed by J. G Stampli as:

$\|\delta A|B(H)\| = 2\delta(A)$ Where $\delta(A) = \inf\{\|A - \lambda\| : \lambda \in C\}$

Lemma 1:

let P be a W* algebra on Hilbert spaces H and δ a derivation on P and the commutant of P is P″ in H and A is the maximal abelian *subalgebra of P″

There are two elements x_0 and a_0 in B(H) such that $x_0 \in Ps$ and $a_o \in K$ and $r = \|x_0 - a_0\|$ where $K = (b + Ps) \cap S$ is weakly compact and Ps self adjoint of P. [2]

Gleason kahane Zelasko theorem

Let X be unital commutative Banach algebra and Y be uniform algebras. If $P: X \to Y$ is a linear map $\delta(Pf) \subset \delta(f)$ for all $f \in X$ the P is multiplicative that is $P(fg) = P(f)P(g)$ for all $f, g \in X$.

From the above theorem we can say that if P is multiplicative preserver map we can also say that P is locally inner then it preserves its local automorphisms

Again if we combine with Gelfand Mazur Theorem which claims that the only normed fields there exist up to Banach algebras isometries are Real R and complex C both equipped with the standard absolute value.

Theorem 2

Let B be infinite dimensional Banach space and δ is a linear map from B(X) into itself such that δ is a local automorphism the is an automorphism. [4]

Mapping principle: **[5]**

Let E,F be two algebras (lie, associative or jordan) and $f: E \to F$ a homoeomorphism, then every derivation δ of E admits an inner derivation δ_b^* of B that is *f* related to δ_b^*

Main results

Lemma

There is bounded operator $b \in B$ (P, F) generated by P and F such that $\delta_b(x) = bx - xb$ for all $b \in B$

Place $\delta_{b^*}^*(x) = b^* x - xb^*$ and $\delta_{b^*}^*(x) = -b(bx^* - x^* b)$ hence δ^* is a derivation on P and it follows that from involutive Banach $\delta = \frac{\delta + \delta^*}{2} + i\frac{\delta - \delta^*}{2i}$ hence δ^* and δ are self adjoint.

We can assume that $\|b = 1\|$, let B(O) be the set of all bounded operators of Banach space B(X) and U the unit ball of B(X) and $Q = (b + P_S^c) \cap U$ is totally/locally compact. Then the self-adjoint element h with $\|h \le 1\|$ in B(X) belong to Q iff:

$$\delta_h(x) = hx - xh = bx - xb = \delta_b(b)$$

***Theorem*:**

Every *automorphism of commutative Banach algebra is inner.

***Proof*:**

Let P^c is decomposable Banach algebra then we know from Zorn's Lemma that there is maximal element $m_{\alpha 0} = x_{\alpha 0} - a_{\alpha 0} = 0$ for all $x_{\alpha 0} \in P_S$ and $a_{\alpha 0} \in Q$ in the family of all elements $(m_\alpha | \alpha \in I)$ in B(O), where B(O) is the set of all bounded operators on Banach space B(X).

And $\|m_\alpha\| = \rho$ where $\rho = \|x_0 - a_o\|$ following Definition 1.2 that there are two elements x_0 and a_0 in B(O) such that $x_0 \in P_S$ and $a_0 \in Q$ then $\rho = \|x_0 - a_o\|$. We have $\delta_{x_{\alpha 0}}(x) = x_{\alpha 0} x - x x_{\alpha 0} = x x_{\alpha 0} - \alpha x_0 x = \delta_x(\alpha x_o) = \delta_b(x)$ where $x \in M$ hence δ is inner.

Now let P be arbitrary Banach algebra, take the countable decomposable projections $pc \in P^C$ and put $\delta_{p^c}^*(x) = p^c x - xp^c = xp^c - p^c x = \delta_x^*(p^c)$ and δ^* is a derivation on Banach algebra Pp^c on commutative Banach space $B^c(X)$ so that $(Pp^c)^c = p^c P p^c$ hence there is an element $y \in P$ such that $\delta_{yp^c}(xp^c) = yp^c(xp^c) - xp^c(yp^c) = p^c[xb - bx = p^c \delta_b(x)$ for all $x \in M$ and q is the central support of P^c. This implies that $bq = yq + m^c$ where $m^c \in M^c$ and $q \in Q$. But every banach algebra can be embedded in a subalgebra, then the compact set Q is non empty and the set R is center of P^c then we can say $R \in B(X)$

Take $aq \in Q \cap R$ where $q \in P$ $\forall$ $yq + aq \in P$ and the $\|yq + aq\| \le \|bq\| \le 1$ and $\delta_{yq+aq}(x) = \delta_{yq}(x)$ for all $x \in M$ and so we choose the family of orthogonal central projections $(q_\alpha | \alpha \in J)$ in M such that for all $\alpha \in J$ there is an element $y_\alpha \in P_{q\alpha}$ such that $\|y_\alpha\| \le 1$ and $\delta_{y\alpha}(x) = \delta_{yq}(x)$ for all $x \in P$ then $\sum_{\alpha \in J} y\alpha = 1$.

Take $y_0 = \sum_{\alpha \in J} y\alpha$ then $y_0 \in P$ and $\delta_{y0}(x) = \sum_{\alpha \in J}(\delta_{y\alpha}(x)) = \sum_{\alpha \in J}(\delta_{q\alpha}(x)) = \delta_b(x)$.

This completes the proof.

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