Original Article

Certain properties of normaloid operators

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Abstract

In this paper we establish new conditions for contractivity of normaloid operators. We employ some results for contractivity due to Furuta, Nakomoto, Arandelovic and Dragomir. A particular generalization is also given.

Keywords:
Normaloid operators, Contractive operators, Cauchy-Schwarz inequality and Tensor product

1. Introduction

An interesting area in operator theory is the study of norm inequalities for Hilbert space operators. Many mathematicians have worked on this subject, for example in [2, 3 and 5]. On the other hand, contractive and normaloid operators have been considered separately by [1, 6, 7 and 8]. In this paper, we results on conditions for normaloidity and contractivity of Hilbert space operators. We begin by simple lemmas before we move to main results. Let H be a complex Hilbert space with an inner product $\langle .,. \rangle$ and $B(H)$ the algebra of all bounded linear operators on H. $\|\cdot\|$ denotes the usual operator norm and Dom(S) denotes the domain of S.

2. Basic Concepts and Preliminaries

In this section, we start by defining some key terms that are useful in the sequel.

Definition 2.1. An operator $S \in B(H)$ is said to be normaloid if $\|S\|$ =sup $\{|\langle Sx, x \rangle|: \|x\| =1\}$ and contractive if $\|S\| \leq 1$.

Definition 2.2. Let $S : H \rightarrow H$ the adjoint of S is S^*: $H \rightarrow H$ such that

$$\langle Sx, x \rangle = \langle x, S^* y \rangle \ \forall\, x, y \in H.$$

Definition 2.3. An operator S is said to be normal if $SS^* = S^*S$ and is a self-adjoint if $S = S^*$.

3. Main results

In this section we give the main results. We first discuss conditions for normaloidity and lastly we consider conditions for contractivity.

Lemma 3.1. Let $S \in B(H)$. Then S is normaloid if it is self-adjoint.

Proof. Since $S \in B(H)$, then without loss of generality we assume that S is normal i.e. $SS^* = S^*S$. Hence, S is normaloid if $\|S\| =$ sup $\{|\langle Sx, x\rangle|: \|x\| = 1\}$. But S is self-adjoint i.e. $S = S^*$. So $\|S\| = \|S^*\| =$ sup $\{|\langle x, S^*x\rangle|: \|x\| =1\}$, and this completes the proof.

Lemma 3.2. Let $S \in B(H)$ then S is normaloid if it is normal.

Proof. Suppose S is normal i.e. $SS^* = S^*S$, then $\|S\|^2 = \langle SS^*x, x \rangle = \langle S^*Sx, x\rangle =$

$\|S^*\|^2$, $\forall\, x \in Dom(SS^*) = Dom(S^*S)$. But the subspace $Dom(SS^*) = Dom(S^*S)$is a core of both S and S^*, therefore the norm of S and norm of S^* coincide with
$Dom(SS^*) = Dom(S^*S)$. Hence it follows that, $Dom(S) = Dom(S^*)$ and
$\|Sx\| = \|S^*x\| \ \forall\, x \in Dom(SS^*) = Dom(S^*S)$. By Lemma 3.1, S is self-adjoint so $SS^* = S^*S$.

Lemma 3.3. Let $S \in B(H)$ then S is normaloid if it is positive.

Proof. From Lemma 3.1, every positive operator is self-adjoint. This implies that; $\langle Sx, x \rangle = \overline{\langle Sx, x \rangle} = \langle x, S^* x\rangle. \ \forall\ x, y \in B(H)$. But $\langle x + y, S(x + y)\rangle = \langle S(x + y), x + y \rangle$ and $\langle x - y, S(x - y)\rangle = \langle S(x - y), x - y\rangle$.
So subtracting gives
$\langle S(x + y), x + y \rangle - \langle S(x - y), x - y \rangle = \langle x, Sy \rangle - \langle Sx, y \rangle = 0$. This implies that $\langle x, Sy \rangle = \langle Sx, y \rangle$. By Lemma 3.2. We have $SS^* \geq 0. \forall\, S \in B(H)$, since $\langle x, S^*Sx \rangle = \langle Sx, Sx\rangle = \|Sx\|^2$. But $S = S^*$, hence either $S \geq 0$ or $S^* \geq 0$ or both are ≥ 0. Clearly, S is positive.

Theorem 3.4. Let $S_1$and S_2 in $B(H)$ be normaloid then $S_1 + S_2$ is normaloid.

Proof. From Lemmas 3.1, 3.2 and Lemma 3.3 if we suppose that $S_1 + S_2$ is densely defined then let $x \in Dom(S_1 + S_2)$, such that $Dom(S_1) \cap Dom(S_1)$ contains x Then we can find $y \in Dom(S_1^* + S_2^*)$contains y.
From Lemma 3.1 we have,
$\langle (S_1^* + S_2^*)x, y \rangle = \langle S_1^*x, y\rangle + \langle S_2^*x, y\rangle = \langle x, S_1y\rangle + \langle x, S_2y \rangle = \langle x, (S_1 + S_2)y \rangle$.
Hence,
$\| S_1 + S_2\| =$ sup $\{|\langle S_1x_1 \rangle + \langle S_2x_2, x_2 \rangle|: \| x_1\| = 1$ and $\| x_2\| = 1\}$
$=$ sup $\{|\langle S_1x_1 + S_2x_2, x_1 + x_2\rangle|: \| x_1\| = 1$ and $\| x_2\| = 1\}$
Therefore, $S_1 + S_2$is normaloid.

Corollary 3.5. Let $S_1, S_2, ..., S_n$ in $B(H)$ be normaloid. Then $\oplus_{i=1}^{n}$ is normaloid in $B(H)$

Proof. From Theorem 3.4 it follows that
$\| S_1 + S_2 + \cdots + S_n \| = \| \sum_{i=1}^{n} S_i \| \leq \sum_{i=1}^{n} \|S_i\|$

Let $x_n \in$ Dom $(S_1 + S_2 + \cdots + S_n)$, such that $Dom(S_1) \cap \ldots \cap Dom(S_n)$ contains x_n. Then we can find $y_n \in Dom(S_1 + S_2 + S_3 + \cdots + Sn)$, such that $DomS1 \cap \ldots \cap DomSn$ contains yn. Hence, from Theorem 3.4 we have

$\| S_1 + S_2 + \cdots + S_n \| = \sup \{ |\langle (S_1 + S_2 + \cdots + S_n)x_n, x_n \rangle +$

$\langle (S_1 + S_2 + \cdots + S_n)y_n, y_n \rangle| : \| x_n \| = 1$ and $\| y_n \| = 1\}$

$= \sup \{ |\langle (S_1 + S_2 + \cdots + S_n)x_n + (S_1 + S_2 + \cdots + S_n)y_n, y_n, x_n + y_n \rangle|$

$: \| x_n \| = 1$ and $\| y_n \| = 1\}$

Therefore $\oplus_{i=1}^{n}$ is normaloid.

Theorem 3.6. Let S_1, S_2 be normaloid then $S_1S_2 - S_2S_1$ is normaloid and

$$\|S_1S_2\| \leq \max\{\|S_1\|, \|S_2\|\} \max\{\| S_1 - S_2\|, \| S_1 + S_2\|\}. \quad (1)$$

Proof. Since S_1, S_2 are normaloid we have

$$\|S_1S_2\| = \max\{|\langle S_1S_2x, x\rangle| : \|x\| = 1\}.$$

So

$$\|S_1S_2 - S_2S_1\| = \| (S_1 - S_2)S_2 - S_2(S_1 - S_2)\| \leq 2 \|S_1 - S_2\| \| S_2\| \quad (2)$$

Similarly,

$$\|S_1S_2 - S_2S_1\| \leq 2 \|S_1 - S_2\| \|S_1\| \quad (3)$$

But $S_1S_2 - S_2S_1$ is normaloid. So using Equation 2 and Equation 3 we have

$$\| S_1S_2 - S_2S_1\| \leq 2 \max\{\| S_1\|, \|S_2\|\} \|S_1 - S_2\| \quad (4)$$

In Equation 4 replacing S_2 by $-S_2$, we get

$$\| S_1S_2 - S_2S_1\| \leq 2 \max\{\|S_1\|, \|S_2\|\} \|S_1 + S_2\| \quad (5)$$

From Equation 4 and Equation 5 we obtain the required result i.e.

$$\|S_1S_2\| \leq \max\{\|S_1\|, \|S_2\|\} \max\{\| S_1 - S_2\|, \| S_1 + S_2\|\}.$$

Corollary 3.7. Let S be normaloid then

$$\| SS^* - S^*S \| \leq 2\| S \| \max\{\| S - S^*\|, \| S + S^*\|\} \quad (6)$$

Proof. From Equation 4 and Equation 5. Let $S_1 = S$ and $S_2 = S^*$. This gives

$$\| SS^* - S^*S \| \leq 2\max\{\| S \|, \|S^*\| \} \| S - S^*\| \quad (7^*)$$

and

$$\| SS^* - S^*S \| \leq 2\max\{\| S \|, \|S^*\| \} \| S + S^*\| \quad (7^{**})$$

From Equation 7* and 7** we obtain

$$\| SS^* - S^*S \| \leq 2\| S \| \max\{\| S - S^*\|, \| S + S^*\|\}.$$

The proof is complete.

Lemma 3.8. Let $a = \sum_{i=1}^{n} x_i \otimes y_i \in H_1 \otimes H_2$ and $b = \sum_{j=1}^{m} x_j \otimes y_j \in H_1 \otimes H_2$ and $\langle ., . \rangle$ be an inner product on $B(H_1 \otimes H_2)$. Then it is well defined.

Proof. Suppose $\langle a, b\rangle = 0$ when $a = 0$ in $B(H_1, H_2)$ and $\langle a, b\rangle = 0$ when $b = 0$. For each

$x \in H_1$ and $y \in H_2$ then

$$(x \otimes y)(x_1, y_1) = \langle x, x_1\rangle\langle y, y_1\rangle \quad (8)$$

For each $x_1 \in H_1$ and $y_1 \in H_2$

Let $a = \sum_{i=1}^{n} x_i \otimes y_i \in H_1 \otimes H_2$ and $b = \sum_{j=1}^{m} x_j \otimes y_j \in H_1 \otimes H_2$

$$\langle a, b\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i \otimes y_i)(x_j \otimes y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m}\langle x_i, x_j\rangle\langle y_i, y_j\rangle \quad (9)$$

From Equation 8 and Equation 9 we obtain

$0 = \sum_j a(x_j, y_j) = \sum_{i,j}(x_i \otimes y_i)(x_j, y_j)$
$= \sum_{i,j}\langle x_i, x_j\rangle_1 \langle y_i, y_j\rangle_2$

Similarly, $\langle a, b\rangle = 0$ when $b = 0$. Therefore $\langle a, b\rangle$ is well defined. But, $\langle a, b\rangle$ is a Hermitian sesquilinear form then $\langle a, b\rangle \geq 0$. Choosing orthonormal basis $\{e_1, \ldots, e_k\}$ for the linear span of $\{x_1, \ldots, x_p\}$ and $\{f_1, \ldots, f_q\}$ of $\{y_1, \ldots, y_k\}$ and by the bilinearity rules of elementary tensors, we get

$$a = \langle x_i, e_c\rangle_1\langle y_i, f_d\rangle_2\, e_c \otimes f_d \quad (10)$$

Inserting Equation 10 into Equation 9, we get

$$\langle a, a\rangle = \sum_{i,c,c',d,d'}\langle x_i, e_c\rangle_1 \overline{\langle x_i, e_{c'}\rangle}\langle y_i, f_d\rangle_2\overline{\langle y_i, f_d\rangle}\langle e_c, e_{c'}\rangle_1\langle f_d, f_{d'}\rangle_1$$

$$= \sum_i \sum_{c,d}|\langle x_i, e_c\rangle|^2\, |\langle y_i, f_d\rangle|^2 \geq 0.$$

Thus $\langle a, a\rangle$ is positive. Since $\langle a, a\rangle = 0$ then, $\langle x_i, e_c\rangle.\langle y_i, f_d\rangle = 0\ \ \forall\, i, c, d$ and so $a = 0$. Therefore $\langle a, b\rangle$ is well defined.

Lemma 3.9. Let $S_1, S_2 \in B(H_1 \otimes H_2)$ then $S_1 \otimes S_2$ is a well defined operator on $B(H_1 \otimes H_2)$ with domain $Dom(S_1 \otimes S_2)$.

Proof. Let $a = \sum_{i=1}^{n} x_i \otimes y_i,\ \forall\ x_i \in Dom(S_1),\ y_i \in Dom(S_2)$. Given an orthonormal basis $\{e_1, \ldots, e_k\}$ for the linear span of $\{x_1, \ldots, x_k\}$ and set $f_c = \sum_i\langle x_i, e_c\rangle y_i$ then

$$a = \sum_{i,c}^{n}\langle x_i, e_c\rangle\, e_c \otimes y_i = \sum_i e_c \otimes f_c \quad (11)$$

$$\|a\|^2 = \sum_{c,d}\langle e_c, e_d\rangle_1 \langle f_c, f_d\rangle_2 = \sum_d \|f_d\|^2. \quad (12)$$

From Lemma 3.8, to proof that $S_1 \otimes S_2$ is well defined, then $\sum_i S_1 x_i \otimes S_2 y_i = 0$ whenever $a = 0$. If $a = 0$, then all f_c are zero by Equation 12. Therefore

$$\sum_i S_1 x_i \otimes S_2 y_i = \sum_{i,e}\langle x_i, e_c\rangle S_1 e_c \otimes S_2 y_i = \sum_c S_1 e_c \otimes S_2 f_c = 0.$$

Theorem 3.10. Let S_1 and S_2 be normaloid then $S_1 \otimes S_2$ is normaloid under $\|\cdot\|_{CB}$ and $\|S_1 \otimes S_2\| = \|S_1\|\, \|S_2\|$.

Proof. Let $a \in Dom(S_1 \otimes S_2)$ as in Equation 11. Suppose $I_1 = I|\, Dom(S_1)$ and $I_2 = I|\, Dom(S_2)$. Using Equation 12 twice, i.e. for the element $(I_1 \otimes S_2)\, a$ and then for a, we get

$$\|(I_1 \otimes S_2)\, a\|^2 = \|\textstyle\sum_c e_c \otimes S_2\|^2 = \sum_c \|S_2 f_c\|^2 \leq \|S_2\|^2 = \|S_2\|^2\, \|a\|^2.$$

This implies that $\|I_1 \otimes S_2\| \leq \|S_2\|$. Similarly, $\|S_1 \otimes I_2\| \leq \|S_1\|$ therefore

$$\|S_1 \otimes S_2\| = \|(S_1 \otimes I_2)\, (I_1 \otimes S_2)\| \leq \|S_1\|\, \|I_1 \otimes S_2\| \leq \|S_1\|\, \|S_2\|$$
(13)

To prove the reverse inequality we let $\varepsilon > 0$ and the unit vectors $x \in Dom(S_1)$ and

$y \in Dom(S_2)$ Such that $\|S_1\| \leq \|S_1 x\|_1 + \varepsilon$ and $\|S_2\| \leq \|S_2 y\|_2 + \varepsilon$.

Then

$$(\|S_1\| - \varepsilon)(\|S_2\| - \varepsilon) \leq \|S_1 x\|_1 \|S_2 y\|_2 = \|S_1 x \otimes S_2 y\|$$
$$\leq \|(S_1 \otimes S_2)(x \otimes y)\| \leq \|S_1 \otimes S_2\| \|x \otimes y\|$$

So, $\|S_1 \otimes S_2\| \geq \|S_1\|\|S_2\|$ (14)

Since ε is arbitrary so small, now letting $\varepsilon \to 0$, thus from Equation 13 and 14, we get

$\|S_1 \otimes S_2\| = \|S_1\|\|S_2\|$.

4 Conditions for Contractivity

Lemma 4.1. Let S be normaloid positive then S is contractive.

Proof. Take S as in Lemma 3.2 and from [3 Theorem 1.2], $r(S) = \|S\|$. S is contractive if $\|S\| \leq 1$. Since $\|S\| = \sup\{|\langle Sx, x\rangle| : \|x\| = 1\}$, then it follows from [4] Theorem A, an indempotent numerical radius contraction is a projection. It follows that the idempotency of S that

$$\left\| I + aS + \frac{a^2}{2!} S^2 + \cdots \right\| = \|I + (e^a - I)S\| \leq e^{|a|}$$

where a is an arbitrary complex number. Let $a = t$, where t is a real number. Then

$\|e^{-t} I + (1 - e^{-t})S\| \leq 1$ as $t \to \infty$ we get $\|S\| \leq 1$.

Lemma 3.13. Let S be normaloid then S is contractive if and only if it is the identity.

Proof. Suppose S is a contractive i.e. $\|S\| \leq 1$, then $\|S^n\| \leq \|S\|^n$ for all $n \geq 0$ and the geometric series $1 + \|S\| + \|S^2\| + \cdots$ is convergent. It therefore follows that the infinite series $I + S + S^2 + \cdots +$ converges to some $S_1 \in B(H)$. Hence; $(I - S)S_1 = \lim_{n\to\infty}(I - S)(I + S + S^2 + \cdots + Sn = \lim n \to \infty I - Sn + 1 =$
$I - \lim_{n\to\infty} S^{n+1} = I$. Therefore, $S^{n+1} \to 0$ since $\|S\|^{n+1} \to 0$ as $n \to \infty$. Similarly, $S_1(I - S_1 : X \to Y) = I$ this shows that $I - S$ is invertible and

therefore S is an identity. Conversely, assume that S is an identity. Let S_1 and S_2 be normaloid and S_1 , $S_2 \in B(H)$ with S_1 and S_2 also invertible and $S_1 : X \to Y$ and $S_2 : Y \to X$ such that $S_1 S_2 = I_y$ and $S_2 S_1 = I_x$. Then the equality $S_1 S_2 v = V \quad \forall\, v \in Y$ implies that $KerS_2 = 0$ and $RanS_1 = Y$. Similarly, $S_2 S_1 u = U \quad \forall\, u \in X$ implies $KerS_1 = 0$ and $RanS_2 = X$. This implies that S_1 and S_2 are both invertible and $S_2 = S_1^{-1}$. Thus $S_1 S_2 = I$ and therefore S is contractive.

Theorem 4.3. Let S_1 and S_2 be normaloid then $S_1 S_2$ is also contractive.

Proof. Since S_1 and S_2 are normaloid, then $S_1 S_2$ is also normaloid. Now

$$\|S_1 S_2\| = \sup\{|\langle S_1 S_2 x, x\rangle|,\ x \in H : \|x\| = 1\}$$
$$= \sup\{|\langle S_1 x, S_2 x\rangle| : \|x\| = 1\}$$

The condition $\|S\| \le 1$ is equivalent to $\langle S_1 x, S_2 x\rangle \le \langle x, x\rangle$. Then

$$\| S_1 S_2\| \le \sup\{|\langle x, x\rangle| : \|x\| = 1\}$$
$$\le \sup\{\|x\|^2 : \|x\| = 1\}$$

Taking the supremum $\|S_1 S_2\| \le 1$, therefore $S_1 S_2$ is contractive.

Corollary 4.4. Let S_1 and S_2 be normaloid contractive, then the following are equivalent;

i. $S_1 - S_2$ is contractive.
ii. $S_1 - S_2$ is positive.
iii. $S_1 S_2$ is positive.
iv. $SS^* - S^*S$ is normal.

Proof. $(i \Rightarrow ii)$ From Theorem 4.3, it follows that $S_1 - S_2$ is also normaloid contractive. Let S_1 and S_2 be normaloid positive operators, then $S_1 - S_2$ is positive.

$(ii \Rightarrow iii)$ Suppose $S_1 - S_2$ is positive. Since S_1 and S_2 are positive, it follows that their product is positive. Hence $S_1 S_2$ is positive since multiplication is defined point wise and commutative.

$(iii \Rightarrow iv)$ An operator is said to be positive if it is self adjoint i.e. $S = S^*$. This implies that S^*S is positive and hence SS^* is also positive. Therefore, $SS^* - S^*S = 0$ this implies that $SS^* = S^*S$ thus S is normal and therefore $SS^* - S^*S = 0$ is also normal.

$(iv \Rightarrow i)$ From 4.3, let $S_1 = S$ and $S_2 = S^*$ this implies that $SS^* \le 1$.

Theorem 4.5. Let S be normaloid then S is contractive bounded linear operatorif for each $z \in K \subset H$ and any $r \in IntK$ there exist a positive integer c_o such that $S^n(z) < r$ for all $n > c_o$.

Proof. By the prove of [1 Theorem 3.5], let $(1 - S) \circ (1 + S + \cdots + S^n) =$
$1 - S^{n+1}$. Then it implies that; $(1 - S) \circ (1 + S + \cdots + S^n) \ge$
$(1 - S) \circ (n+1)S^n(z) = (1 - S^{n+1})z = z - S^{n+1}(z) \le z$. For each $z \in K \subset H$, since $S^n(z) \le S^a(z)$ for each $a = 0, \dots, n$ then, $(1 - S)\circ(n + 1Snz \le z$. Therefore, $Snz \le 1 - S - 1n + 1z$ given $0 < r$, there exist a positive integer $c_\circ$ such that $n > c_\circ$, implying that; $\frac{1}{n+1}(1 - S)^{-1}(z) < r$. Since ; $\frac{1}{n+1}(1 - S)^{-1}(z)$ being a convergent sequence, then $n > c_\circ$ implies that; $S^n(z) < r$.

5. Conclusion

These results are properties of Hilbert space operators are when they are normaloid and contractive. It would be interesting to give generalizations which thus will help in further classification of these operators.

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