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Original Article

On convergences of contractive maps in metric spaces

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Abstract

In this paper, we introduce a new class of contraction maps, called *A* – contractions in fuzzy metric space. Under different sufficient conditions, existence of common fixed point for a pair of maps, four maps and also for a sequence of maps will be established here. Also it is shown that *A* – contractions is more generalized than TS – Contraction, B – Contraction in FM-space. If two fuzzy metrics are given on a set X, which are related, a pair of self map can have common fixed point though the contractive condition with respect one fuzzy metric is given. Our result extends, generalized and fuzzifies several fixed point theorems with *A* – contractions on metric space. We give generalizations and convergences of these maps.

1. Introduction

Ever since the concept of fuzzy sets was introduced by Zadeh [5] in 1965 to describe the situation in which data are imprecise or vague or uncertain. It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc. Kramosil and Michalek [7] introduced the concept of fuzzy metric spaces (briefly, FM- spaces) in 1975, which opened an avenue for further development of analysis in such spaces.

The study of common fixed points with A – contractions is new and also interesting. Works along these lines have recently been initiated by M. Akram, A. A. zafar, A. A. Siddiqui [6] in 2008 and by Gbenga Akinbo, Memudu O. Olatinwo And Alfred O. Bosede [4] in 2010. In this article we introduce a new class of contraction maps, called *A* – contractions in fuzzy metric space. Under different sufficient conditions, existence of common fixed point for a pair of maps, four maps and also for a sequence of maps will be established here. Also it is shown that *A* – contractions is more generalized than TS – Contraction, B – Contraction in FM-space. If two fuzzy metrics are given on a set X, which are related, a pair of self map can have common fixed point though the contractive condition with respect one fuzzy metric is given. Our result extends, generalized and fuzzifies several fixed point theorems with *A* – contractions on metric space.

2. Preliminaries

We quote some definition and statements of a few theorems which will be needed in the sequel.

Definition 1.1 [2] A binary operation*: $[0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t – norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a*1 = a \ \forall \ a \in [0,1]$,

(iv) $a*b \leq c*d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0,1]$.

Result 1.2 [3] (a) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3 \in (0,1)$ such that $r_1 * r_2 > r_2$,

(b) For any $r_5 \in (0,1)$, there exist $r_6 \in (0,1)$ such that $r_6 * r_6 \geq r_5$.

Definition 1.3 [1] The 3-tuple $(X, \mu, *)$ is called a non - Archimedean fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t - norm and μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions :

(i) $\mu(x, y, t) > 0$;

(ii) $\mu(x, y, t) = 1$ if and only if $x = y$;

(iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, \max\{s, t\})$;

(v) $\mu(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous ;

for all $x, y, z \in X$ and $t, s > 0$.

Note that $\mu(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Remark: 1.4 In fuzzy metric space X, $\mu(x, y, \cdot)$ is non – decreasing for all x , y $\in X$ and

$\mu(x, y, t) \geq \mu(x, z, t) * \mu(z, y, t)$ for all $x, y, z \in$ X, $t > 0$.

That is, every non – Archimedean fuzzy metric space is also a fuzzy metric space.

Definition 1.5 [8] A sequence $\{x_n\}_n$ in fuzzy metric space is said to converge to $x \in$ X if and only if $\lim_{n \to \infty} \mu(x_n, x, t) = 1$.

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A sequence $\{x_n\}_n$ in fuzzy metric space is said to be a **Cauchy sequence** if and only if $\lim_{n\to\infty} \mu(x_{n+p}, x_n, t) = 1$.

A fuzzy metric space $(X, \mu, *)$ is said to be **complete** if and only if every Cauchy sequence in X is convergent in X.

Let R_+ denote the set of all non-negative real numbers and A be the set of all functions $\alpha : R_+^3 \to R_+$ satisfying

(i) α is continuous on the set R_+^3.

(ii) $ka \geq b$ for some $k \in (0, 1)$ whenever $a \geq \alpha(a, b, b)$ or $a \geq \alpha(b, a, b)$ or $a \geq \alpha(b, b, a)$ for all $a, b \in R_+$.

Definition 1.6 [9] Let $(X, \mu, *)$ be fuzzy metric space and $T : X \to X$. T is said to be TS – contractive mapping if there exists $k \in (0, 1)$ such that

$$k\,\mu(Tx, Ty, t) \geq \mu(x, y, t)$$
$$\forall\ t > 0.$$

Definition 1.7 Let $(X, \mu, *)$ be fuzzy metric space $T : X \to X$. T is said to be B – contractive mapping if there exists $k \in (0, 1)$ such that

$$k\,\mu(Tx, Ty, t) \geq \min\{\mu(x, Tx, t),\ \mu(y, Ty, t)\}$$
$\forall\ x, y \in X$ and $t > 0$.

Definition 1.8 A self- map T on a non – Archimedean fuzzy metric space X, is said to be A – Contraction if it satisfies the condition:

$$\mu(Tx, Ty, t) \geq \alpha(\mu(x, y, t),\ \mu(x, Tx, t),\ \mu(y, Ty, t))$$
for all $x, y \in X$ and some $\alpha \in A$.

3. Convergences of Contractive maps

Theorem 3.1 Every TS - Contraction is an A – Contraction on non – Archimedean fuzzy metric space $(X, \mu, *)$, where

$$a * b = \min\{a, b\}\ \ \forall\ a, b \in [0, 1]$$

Proof: Assume that T on the non – Archimedean fuzzy metric space X is TS – Contraction. Define $\alpha : R_+^3 \to R_+$ by

$\alpha(u, v, w) = \frac{1}{k}\min\{v, w\}$ for all $u, v, w \in R_+$ with some fixed $k \in (0, 1)$. Next we show that $\alpha \in A$.

(i) Clearly α is Continuous.

(ii) If $u \geq \alpha(u, v, v)$ then $u \geq \frac{1}{k}\min\{v, v\} = \frac{v}{k}$.

If $u \geq \alpha(v, u, v)$ then $u \geq \frac{1}{k}\min\{u, v\} = \frac{v}{k}$. So that $ku \geq v$.

Similarly, if $u \geq \alpha(v, v, u)$ then $ku \geq v$.

So we deduce that $\alpha \in A$. Further, since T is a TS - Contraction, there exists $k \in (0, 1)$ such that

$$k\,\mu(Tx, Ty, t) \geq \mu(x, y, t) \text{ for all } t > 0.$$

$$\Rightarrow\ k\,\mu(Tx, Ty, t) \geq \mu(x, y, t) \geq \mu(x, Tx, t) * \mu(Tx, Ty, t) * \mu(Ty, y, t)$$

$$\Rightarrow\ k\,\mu(Tx, Ty, t) \geq \min\{\mu(x, Tx, t), \mu(Tx, Ty, t), \mu(Ty, y, t)\}$$

$$\Rightarrow\ \mu(Tx, Ty, t) \geq \frac{1}{k}\min\{\mu(x, Tx, t), \mu(Ty, y, t)\}$$

$$\Rightarrow\ \mu(Tx, Ty, t) \geq \alpha(\mu(x, y, t), \mu(x, Tx, t), \mu(Ty, y, t))$$

This completes the proof.

Theorem 3.2 Every B – Contraction is an A – Contraction on non – Archimedean fuzzy metric space $(X, \mu, *)$, where

$$a * b = \min\{a, b\}\ \ \forall\ a, b \in [0, 1].$$

Proof: Assume that T on the non – Archimedean fuzzy metric space X is B – Contraction. $\alpha : R_+^3 \to R_+$ by $\alpha(u, v, w) = \frac{1}{k}\min\{v, w\}$ for all $u, v, w \in R_+$ with some fixed $k \in (0, 1)$.

From the proof of the above theorem, we see that $\alpha \in A$.

Further, since T is a B – Contractive, we get

$$k\,\mu(Tx, Ty, t) \geq \min\{\mu(x, Tx, t),\ \mu(y, Ty, t)\}\ \forall$$
$x, y \in X$ and $t > 0$

$$\Rightarrow\ \mu(Tx, Ty, t) \geq \alpha(\mu(x, y, t), \mu(x, Tx, t), \mu(Ty, y, t))$$

This completes the proof.

4. Fixed point Theorems

In this section, we give some results on fixed points of A – Contractions.

Let F, G, S and T be self – maps of a fuzzy metric space X satisfying

$$SX \subseteq FX;\ TX \subseteq GX. \qquad (1)$$

Then for any point $x_0 \in X$, we can find points $x_1, x_2, \ldots$, all in X, such that

$$Sx_0 = Fx_1,\ Tx_1 = Gx_2,\ Sx_2 = Fx_3,\ \cdots.$$

Therefore, by induction, we can define a sequence $\{y_n\}_n$ in X as $y_n = Sx_n = Fx_{n+1}$, when n is even and $y_n = Tx_n = Gx_{n+1}$, when n is odd, where $n = 0, 1, 2, \cdots$.

The following theorem establishes existence of coincidence and unique common fixed point of F, G, S and T where the union of the ranges of F and G is required to be complete. The set of coincidence points of T and F is denoted by $C(T, F)$, and the set of natural numbers denoted by N.

Theorem 4.1 Let F, G, S and T be self-maps of a non - Archimedean fuzzy metric space X satisfying (1) and for all $x, y \in X$

$$\mu(Sx, Ty, t) \geq \alpha(\mu(Gx, Fy, t),\ \mu(Gx, Sx, t),\ \mu(Fy, Ty, t)) \quad \cdots \quad (2)$$

where $\alpha \in A$. Suppose $FX \cup GX$ is a complete subspace of X, then the sets $C(T, F)$ and $C(S, G)$ are nonempty. Suppose further that (T, F) and (S, G) are weakly compatible, then F, G, S and T have a unique common fixed point.

Proof: Assuming $n \in N$ is even, then

$$\mu(y_n, y_{n+1}, t) = \mu(Sx_n, Tx_{n+1}, t)$$
$$\geq \alpha(\mu(Gx_n, Fx_{n+1}, t),\ \mu(Gx_n, Sx_n, t),\ \mu(Fx_{n+1}, Tx_{n+1}, t))$$
$$= \alpha(\mu(y_{n-1}, y_n, t),\ \mu(y_{n-1}, y_n, t),\ \mu(y_n, y_{n+1}, t))$$
$$\Rightarrow\ k\,\mu(y_n, y_{n+1}, t) \geq \mu(y_{n-1}, y_n, t)$$

On the other hand, assuming $n \in N$ is odd,

$$\mu(y_n, y_{n+1}, t) = \mu(Tx_n, Sx_{n+1}, t)$$
$$\geq \alpha(\mu(Gx_{n+1}, Fx_n, t),\ \mu(Gx_{n+1}, Sx_{n+1}, t),\ \mu(Fx_n, Tx_n, t))$$

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$= \alpha\big(\mu(y_n, y_{n-1}, t),\ \mu(y_n, y_{n+1}, t),\ \mu(y_{n-1}, y_n, t)\big)$

$\Rightarrow k\mu(y_n, y_{n+1}, t) \geq \mu(y_{n-1}, y_n, t)$

Thus whether n is odd or even, we have

$\Rightarrow \mu(y_n, y_{n+1}, t) \geq \frac{1}{k}\mu(y_{n-1}, y_n, t)$ for some $k \in (0, 1)$.

Inductively, we have

$\Rightarrow \mu(y_n, y_{n+1}, t) \geq \frac{1}{k^n}\mu(y_0, y_1, t)$

$\Rightarrow \lim_{n\to\infty} \mu(y_n, y_{n+1}, t) = 1$

Observe that $\{y_n\}$ is contained in $FX \cup GX$. We now verify that $\{y_n\}$ is Cauchy sequence.

$\mu(y_n, y_{n+p}, t) \geq \mu(y_n, y_{n+1}, t) * \cdots * \mu(y_{n+p-1}, y_{n+p}, t)$

$\Rightarrow \lim_{n\to\infty} \mu(y_n, y_{n+p}, t) \geq 1 * \cdots * 1 = 1$

$\Rightarrow \lim_{n\to\infty} \mu(y_n, y_{n+p}, t) = 1$

Therefore $\{y_n\}$ is Cauchy and $FX \bigcup GX$ is complete, there exists a point $p \in FX \bigcup GX$ such that $\lim_{x\to\infty} y_n = p$. Without loss of generality, let $p \in GX$. It means we can find a point $q \in X$ such that $p = Gq$. Putting $x = q$, $y = x_m$, m is odd, in to (2) yields

$\mu(Sq, Tx_m, t) \geq \alpha\big(\mu(Gq, Fx_m, t),\ \mu(Gq, Sq, t),\ \mu(Fx_m, Tx_m, t)\big)$

$\Rightarrow \mu(Sq, y_m, t) \geq \alpha\big(\mu(p, y_{m-1}, t),\ \mu(p, Sq, t),\ \mu(y_{m-1}, y_m, t)\big)$

Letting $m \to \infty$, recalling that α is continuous on $\mathbb{R}_+^3$, we obtain

$\mu(Sq, p, t) \geq \alpha(\mu(p, p, t),\ \mu(p, Sq, t),\ \mu(p, p, t))$

$\Rightarrow \mu(Sq, p, t) \geq \alpha(1,\ \mu(p, Sq, t),\ 1)$

$\Rightarrow k\,\mu(Sq, p, t) \geq 1$

$\Rightarrow \mu(Sq, p, t) = 1$. Consequently, $p = Sq$.

From $SX \subseteq FX$ we know that there exists a point $u \in X$ such that $Fu = Sq = p = Gq$.

Choosing $x = q$, $y = u$, (2) gives

$\mu(Sq, Tu, t) \geq \alpha(\mu(Gq, Fu, t),\ \mu(Gq, Sq, t),\ \mu(Fu, Tu, t))$

$\Rightarrow \mu(p, Tu, t) \geq \alpha(1, 1, \mu(p, Tu, t))$

$\Rightarrow k\,\mu(p, Tu, t) \geq 1$

$\Rightarrow \mu(p, Tu, t) = 1$. Consequently, $p = Tu$.

Hence, $Fu = Tu = p = Sq = Gq$. This proves the first part of the theorem.

Now suppose (T, F) and (S, G) are weakly compatible pairs, then F and T commute at u, and G and S commute at q so that

$Fp = FFu = FTu = TFu = Tp$ and

$Sp = SSq = SGq = GSq = Gp \quad \cdots \quad (3)$

Now with $x = p$, $y = u$, (2) and (3) yield

$\mu(Sp, Tu, t) \geq \alpha(\mu(Gp, Fu, t),\ \mu(Gp, Sp, t),\ \mu(Fu, Tu, t))$

$\Rightarrow \mu(Sp, p, t) \geq \alpha(\mu(Sp, p, t), 1, 1)$

$\Rightarrow k\,\mu(Sp, p, t) \geq 1 \Rightarrow \mu(Sp, p, t) = 1$

$\Rightarrow p = Sp = Gp$.

In a similar way, letting $x = y = p$, (2) and (3) yield

$p = Tp = Fp$.

Thus, $Sp = Gp = p = Fp = Tp$.

Finally, we show that p is unique in X.

Suppose Z is another common fixed point of the four maps. Then putting $x = z$, $y = p$ in (2), we have

$\mu(Sz, Tp, t) \geq \alpha(\mu(Gz, Fp, t),\ \mu(Gz, Sz, t),\ \mu(Fp, Tp, t))$

$\Rightarrow \mu(z, p, t) \geq \alpha(\mu(z, p, t), 1, 1)$

$\Rightarrow \mu(z, p, t) = 1 \Rightarrow z = p$.

This completes the proof.

Corrolary 4.2 Let S and T be self-maps of a non - Archimedean fuzzy metric space X, satisfying

$\mu(Sx, Ty, t) \geq \alpha(\mu(x, y, t),\ \mu(x, Sx, t),\ \mu(y, Ty, t))$

where $\alpha \in A$ and for all $x, y \in X$. Then S and T have a unique common fixed point.

Theorem 4.3 Let F, G, S and T be self-maps of a non - Archimedean fuzzy metric space X, and let $\{S_n\}$ and $\{T_n\}$ be sequences on S and T satisfying

$S_n X \subseteq FX;\ T_n X \subseteq GX,\ n = 1, 2, \cdots \qquad (4)$

and for all $x, y \in X$,

$\mu(S_i x, T_j y, t) \geq \alpha\big(\mu(Gx, Fy, t),\ \mu(Gx, S_i x, t),\ \mu(Fy, T_j y, t)\big) \quad \cdots \quad (5)$

where $\alpha \in A$. Suppose $FX \cup GX$ is a complete subspace of X, then for each $n \in N$,

(i) the sets $C(F, T_n)$ and $C(G, S_n)$ are nonempty.

Further, if T_n commutes with F and S_n commutes with G at their coincidence points, then

(ii) F, G, S_n and T_n have a unique common fixed point.

Proof: For any arbitrary $x_0 \in X$ and $n = 0, 1, 2, \cdots$, following a similar argument as in the beginning of this section, we can define a sequence $\{y'_n\}_n$ in X as $y'_n = S_n x_n = F x_{n+1}$, when n is even and $y'_n = T_n x_n = G x_{n+1}$, when n is odd, where $n = 0, 1, 2, \cdots$.

Now for each $i = 1, 3, 5, \cdots$ and $n = 2, 4, 6, \cdots$, from (5) we have

$k\mu(y'_i, y'_{i+1}, t) \geq \mu(y'_{i-1}, y'_i, t)$ and

$k\mu(y'_j, y'_{j+1}, t) \geq \mu(y'_{j-1}, y'_j, t)$

That is, $k\mu(y'_n, y'_{n+1}, t) \geq \mu(y'_{n-1}, y'_n, t)$ for all $n = 0, 1, 2, \cdots$.

By induction (as in the proof of Theorem 4.1), we have

$\mu(y'_n, y'_{n+1}, t) \geq \frac{1}{k^n}\mu(y'_0, y'_1, t)$ for some $k \in (0, 1)$.

Consequently, $\{y'_n\}_n$ is Cauchy in $FX \bigcup GX$, a complete subspace of X.

The rest of the proof is similar to the corresponding part of the proof of Theorem 4.1.

Theorem 4.5 Let T be an *A*-Contraction on a complete non - Archimedean fuzzy metric space X. Then T has a unique fixed point in

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X such that the sequence $\{T^n x_0\}$ converges to the fixed point, for any $x_0 \in X$.

Proof: Fix $x_0 \in X$ and define the iterative sequence $\{x_n\}$ by $x_n = T^n x_0$ (equivalently, $x_{n+1} = T x_n$) where T^n stands for the map obtained by n - time composition of T with T. Since T is an A-Contraction, $\exists\ \alpha \in A$ such that the definition 1.8 holds, i.e.,

$$\mu(Tx,Ty,t) \geq \alpha(\mu(x,y,t),\ \mu(x,Tx,t),\ \mu(y,Ty,t)) \quad \cdots \quad (6)$$

for all $x, y \in X$. Now,

$$\mu(x_n, x_{n+1}, t) = \mu(Tx_{n-1}, Tx_n, t)$$

$$\geq \alpha(\mu(x_{n-1}, x_n, t),\ \mu(x_{n-1}, Tx_{n-1}, t),\ \mu(x_n, Tx_n, t))$$

$$= \alpha(\mu(x_{n-1}, x_n, t),\ \mu(x_{n-1}, x_n, t),\ \mu(x_n, x_{n+1}, t))$$

$$\Rightarrow\ k\,\mu(x_n, x_{n+1}, t) \geq \mu(x_{n-1}, x_n, t)$$

Continuing this way, we get

$$\mu(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\mu(x_0, x_1, t)$$

$$\Rightarrow\ \lim_{n\to\infty} \mu(x_n, x_{n+1}, t) = 1$$

We now verify that $\{x_n\}$ is Cauchy sequence.

$$\mu(x_n, x_{n+p}, t) \geq \mu(x_n, x_{n+1}, t) * \cdots * \mu(x_{n+p-1}, x_{n+p}, t)$$

$$\Rightarrow\ \lim_{n\to\infty} \mu(x_n, x_{n+p}, t) \geq 1 * \cdots * 1 = 1$$

$$\Rightarrow\ \lim_{n\to\infty} \mu(x_n, x_{n+p}, t) = 1$$

Thus $\{x_n\}$ is Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that

$x_n \to x$ as $n \to \infty$.

Again, with $x = x'$ and $y = x_n$, the inequality (6) gives

$$\mu(Tx, x_{n+1}, t) = \mu(Tx', Tx_n, t)$$

$$\geq \alpha(\mu(x', x_n, t),\ \mu(x', Tx', t),\ \mu(x_n, Tx_n, t)) \quad \forall\ n \in N.$$

By allowing n $\to \infty$ and using the continuity of α, we get

$$\mu(Tx', x', t) \geq \alpha(\mu(x', x', t),\ \mu(x', Tx', t),\ \mu(x', x', t))$$

i.e., $\mu(Tx', x', t) \geq \alpha(1,\ \mu(Tx', x', t),\ 1)$

$k\,\mu(Tx', x', t) \geq 1 \qquad \Rightarrow \mu(Tx', x', t) = 1$

$\Rightarrow\ Tx' = x'$.

Now, if $w \in X$ satisfies, $Tw = w$, then by taking $x = w$ and $y = x'$ in (6) we get

$$\mu(w, x', t) = \mu(Tw, x', t)$$

$$\geq \alpha(\mu(w, x', t),\ \mu(Tw, w, t),\ \mu(Tx', x', t))$$

$$\geq \alpha(\mu(w, x', t),\ 1,\ 1)$$

$$\Rightarrow\ k\,\mu(w, x', t) \geq 1\ \Rightarrow\ \mu(w, x', t) = 1$$

$$\Rightarrow\ w = x'.$$

This completes the proof.

Theorem 4.6 Let $\alpha \in A$ and $\{T_n\}$ be a sequence of self-maps on the complete non - Archimedean fuzzy metric space $(X, \mu, *)$ such that

$$\mu(T_i x, T_j y, t) \geq \alpha(\mu(x,y,t),\ \mu(x, T_i x, t),\ \mu(y, T_j y, t)) \quad \cdots \quad (7)$$

for all $x, y \in X$ and $k \in (0, 1)$. Then $\{T_n\}$ has a unique common fixed point in X.

PROOF. Taking any $x_0 \in X$, we define $x_n = T_n x_{n-1}$ for each $n \in N$. Now from (7), we have

$$\mu(x_1, x_2, t) = \mu(T_1 x_0, T_2 x_1, t)$$

$$\geq \alpha(\mu(x_0, x_1, t),\ \mu(x_0, T_1 x_0, t),\ \mu(x_1, T_2 x_1, t))$$

$$\geq \alpha(\mu(x_0, x_1, t),\ \mu(x_0, x_1, t),\ \mu(x_1, x_2, t))$$

$$\Rightarrow\ k\,\mu(x_1, x_2, t) \geq \mu(x_0, x_1, t)$$

Similarly, we have

$$k\,\mu(x_2, x_3, t) \geq \mu(x_1, x_2, t)$$

$$\Rightarrow\ \mu(x_2, x_3, t) \geq \frac{1}{k}\mu(x_1, x_2, t) \geq \frac{1}{k^2}\mu(x_0, x_1, t)$$

Inductively, we have

$$\mu(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\mu(x_0, x_1, t)$$

$$\Rightarrow\ \lim_{n\to\infty} \mu(x_n, x_{n+1}, t) = 1$$

We now verify that $\{x_n\}$ is Cauchy sequence.

$$\mu(x_n, x_{n+p}, t) \geq \mu(x_n, x_{n+1}, t) * \cdots * \mu(x_{n+p-1}, x_{n+p}, t)$$

$$\Rightarrow\ \lim_{n\to\infty} \mu(x_n, x_{n+p}, t) \geq 1 * \cdots * 1 = 1$$

$$\Rightarrow\ \lim_{n\to\infty} \mu(x_n, x_{n+p}, t) = 1$$

Therefore $\{x_n\}$ is Cauchy sequence in the complete fuzzy metric space X, so it converges to $x' \in X$. Next,

$$\mu(x', T_n x', t) \geq \mu(x', x_{m+1}, t) * \mu(x_{m+1}, T_n x', t)$$

$$= \mu(x', x_{m+1}, t) * \mu(T_{m+1} x_m, T_n x', t)$$

$$\geq \mu(x', x_{m+1}, t) * \alpha(\mu(x_m, x', t),\ \mu(T_{m+1} x_m, x_m, t),\ \mu(T_n x', x', t))$$

$$\geq \mu(x', x_{m+1}, t) * \alpha(\mu(x_m, x', t),\ \mu(x_{m+1}, x_m, t),\ \mu(T_n x', x', t))$$

Letting $m \to \infty$, recalling that α is continuous on R_+^3, we obtain

$$\mu(T_n x', x', t) \geq \mu(x', x', t) * \alpha(\mu(x', x', t),\ \mu(x', x', t),\ \mu(T_n x', x', t))$$

$$\Rightarrow\ \mu(T_n x', x', t) \geq \alpha(1,\ 1,\ \mu(T_n x', x', t))$$

$$\Rightarrow\ \mu(T_n x', x', t) = 1 \qquad \Rightarrow T_n x' = x' \quad \forall\ n \in N.$$

For uniqueness of the fixed point x', we suppose $T_n y = y$ for some y $\in$ X and for all $n \in N$.

Then by (7), we have

$$\mu(x', y, t) = \mu(T_i x', T_j y, t)$$

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$$\geq \alpha\left(\mu(x', y, t), \mu(x', T_i x', t), \mu(y, T_j y, t)\right)$$
$$= \alpha\left(\mu(x', y, t), 1, 1\right) \Rightarrow x' = y.$$

Theorem 4.7 Let X be a set with two non - Archimedean fuzzy metrics μ and ∂ satisfying the following conditions:

(i) $\mu(x, y, t) \geq \partial(x, y, t)$ for all $x, y \in X$.

(ii) X is complete with respect to μ.

(iii) S, T are self maps on X, such that T is continuous with respect to μ and

$$\partial(Tx, Sy, t) \geq \alpha\left(\partial(x, y, t), \partial(x, Tx, t), \partial(y, Sy, t)\right)$$

for all $x, y \in X$ and for some $\alpha \in A$.

Then S and T have a unique common fixed point.

Proof: Take any $x_0 \in X$. For each $n \in N$, we define $x_n = S x_{n-1}$, when n is even and $x_n = T x_{n-1}$, when n is odd. Then, by inequality in the above condition (iii) we get

$$\partial(x_1, x_2, t) = \partial(T x_0, S x_1, t)$$

$$\geq \alpha\left(\partial(x_0, x_1, t), \partial(x_0, T x_0, t), \partial(x_1, S x_1, t)\right)$$

$$\geq \alpha\left(\partial(x_0, x_1, t), \partial(x_0, x_1, t), \partial(x_1, x_2, t)\right)$$

$$\Rightarrow k\partial(x_1, x_2, t) \geq \partial(x_0, x_1, t)$$

In general, for any $n \in N$ we get (as in the proof of the of the previous theorem) that

$$\partial(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\partial(x_0, x_1, t)$$ for some $k \in (0, 1)$.

$$\Rightarrow \mu(x_n, x_{n+1}, t) \geq \partial(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\partial(x_0, x_1, t)$$

(By the condition (i))

$$\Rightarrow \lim_{n\to\infty} \mu(x_n, x_{n+1}, t) = 1$$

This implies that $\{x_n\}$ is a Cauchy sequence in X with respect to μ and hence by condition (ii), we have

$$\lim_{n\to\infty} \mu(x_n, x', t) = 1$$ for some $x' \in X$.

Since T is given to be continuous with the respect to μ we have

$$1 = \lim_{n\to\infty} \mu(x_{2n+1}, x', t) = \lim_{n\to\infty} \mu(T x_{2n}, x', t) = \mu(T x', x', t)$$

$$\Rightarrow T x' = x'.$$

Now , by condition (iii)

$$\partial(x', S x', t) = \partial(T x', S x', t)$$

$$\geq \alpha\left(\partial(x', x', t), \partial(x', T x', t), \partial(x', S x', t)\right)$$
$$= \alpha\left(1, 1, \partial(x', S x', t)\right)$$

$$\Rightarrow S x' = x'.$$

Thus x' is a common fixed point of S and T.

For the uniqueness, let y be any common fixed point of S and T in X.

Then by condition (iii) ,

$$\partial(x, y, t) = \partial(T x, S y, t)$$

$$\partial(x', y, t) = \partial(T X', Sy, t)$$

$$\geq \alpha\left(\partial(x, y, t), \partial(x, T x, t), \partial(y, S y, t)\right)$$
$$= \alpha\left(\partial(x, y, t), 1, 1\right)$$

$$\Rightarrow \partial(x, y, t) = 1 \quad \Rightarrow x = y.$$

This completes the proof.

Reference

[1] George and P. Veeramani. *On Some result in fuzzy metric spaces*, Fuzzy Sets and Systems Vl. 64 (1994) p.395 - 399.

[2] B. Schweizer , A. Sklar, *Statistical metric space*, Pacific journal of mathematics Vol.10 (1960) p.314 - 334.

[3] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer , Dordrecht, 2000.

[4] Gbenga Akinbo, Memudu O. Olatinwo and Alfred O. Bosede. *A Note on A - Contractions and Common Fixed Points*, Acta Universitatis Apulensis, Vol. 23 (2010), p.91-98.

[5] L. A. Zadeh Fuzzy sets, *Information and control* Vol.8 (1965), p.338-353.

[6] M. Akram, A.A. zafar, A.A.Siddiqui. *A General Class of Contractions: A – Contractions*, Novi Sad J. Math. Vol. 38 (2008) p.25-33.

[7] O. Kramosil, J. Michalek , *Fuzzy metric and statisticalmetric spaces*, Kybernetica Vol.11 (1975) p.326 - 334.

[8] T. K. Samanta and Iqbal H. Jebril , *Finite dimentional intuitionistic fuzzy normed linear space*, Int. J. Open Problems Compt. Math., Vol 2, No. 4 (2009) p.574 - 591.

[9] T. K. Samanta, Sumit Mohinta, Iqbal, H. Jebril, *On Fixed-point theorems in Intuitionistic Fuzzy metric Space* , Int. J. Open Problems Compt. Math., Vol. 5, No. 2,(2012) p.15 – 27.

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