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Original Article

Generalizations on contractive mappings in metric spaces

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Abstract
We present new results on generalizations on metric spaces. New results are given on contractions in metric spaces and their fuzzy sets.

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1 Metric space,
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1. Introduction

In this paper we initiate the study of fixed point theory for fuzzy contraction mappings in the settings of *b*-dislocated metric spaces.

2. Definitions and terminologies

Definition 2.1 [20] Let X be a nonempty set. A mapping $d_l : X \times X \to [0,\infty)$ is called a dislocated metric or d_l-metric if the following conditions hold for any $x, y, z \in X$

(i) If $d_l(x,y)=0$, then $x=y$

(ii) $d_l(x,y)=d_l(y,x)$,

(iii) $d_l(x,z) \le d_l(x,z)+d_l(z,y)$

The pair (X,d_l) is called a dislocated metric space or a d_l-metric space.

Note that when $x=y$, $d_l(x,y)$ may not be 0.

Example: If $X=R^+\cup\{0\}$, then $d_l(x,y)=x+y$ defines a d_l-metric on X.

Definition 2.2 [4] Let X be a nonempty set and let $s\ge 1$ be given real number. A function $d : X\times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x,y,z\in X$ the following conditions are satisfied:

1. $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$;
3. $d(x,z)\le s[d(x,y)+d(y,z)]$

The pair (X,d) is called a *b*-metric space with parameter s.

Definition 2.3 [18] Let X be a nonempty set. A mapping $b_d : X\times X\to[0,\infty)$ is called a *b*-dislocated metric or b_d-metric if the following conditions hold for any $x,y,z\in X$:

(b_{d_1}) If $b_d(x,y)=0$, then $x=y$

(b_{d_2}) $b_d(x,y)=b_d(y,x)$,

(b_{d_3}) $b_d(x,z)\le s(b_d(x,z)+b_d(z,y))$

The pair (X,b_d) is called a *b*-dislocated metric space or a b_d-metric space. It should be noted that the class of b_d-metric spaces is effectively larger than that of d_l-metric spaces, since a b_d-metric is a d_l-metric when $s=1$.

In general a b_d-metric need not be a d_l-metric, as the following Example shows.

Example 1.8 Let $X=[0,+\infty)$. Define the function $b_d : X\times X\to[0,\infty)$ by $b_d(x,y)=(x+y)^2$. Then (X,b_d) is a b_d-metric space with constant $s=2$. It is easy to see that (X,b_d) is not a *b*-metric or d_l-metric space.

In [18] showed that each b_d-metric on X generates a topology τ_{b_d} whose base is the family of open b_d-balls $B_{b_d}(x,\varepsilon)=\{y\in X : b_d(x,y)<\varepsilon\}$.

Also in [18] are presented some topological properties of b_d-metric spaces; (X,b_d,τ_{b_d}) is a Hausdorff space and first countable.

Definition 2.4 Let (X,b_d) a b_d-metric space, and $(x_n)_{n\in N}$ be a sequence of points in X. A point $x\in X$ is said to be the limit of the sequence $(x_n)_{n\in N}$ if $\lim_{n\to\infty} b_d(x_n,x)=0$ and we say that the sequence $(x_n)_{n\in N}$ is b_d-convergent to x and denote it by $x_n\to x$ as $n\to\infty$.

The limit of a b_d-convergent sequence in a b_d-metric space is unique [18, Proposition 1.27].

Definition 2.5 [18] A sequence $(x_n)_{n\in N}$ in a b_d-metric space (X,b_d) is called a b_d-Cauchy sequence iff, given $\varepsilon>0$, there exists $n_0\in N$ such that for all $n,m>n_0$, we have $b_d(x_n,x_m)<\varepsilon$ or $\lim_{n,m\to\infty} b_d(x_n,x_m)=0$.

Every b_d-convergent sequence in a b_d-metric space is a b_d-Cauchy sequence.

Remark 2.6 The sequence $(x_n)_{n\in N}$ in a b_d-metric space (X,b_d) is called a b_d-Cauchy sequence iff $\lim_{n,m\to\infty} b_d(x_n,x_{n+p})=0$ for all $p\in N^*$.

Definition 2.7 A b_d-metric space (X,b_d) is called complete if every b_d-Cauchy sequence in *X* is b_d-convergent.

In general a b_d-metric is not continuous, as in Example 1.31 in [18] showed.

Lemma 2.8 [18, Lemma 1.32] Let (X,b_d) be a b_d-metric space with $s\ge1$, and suppose that $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ are b_d-convergent to x, y respectively. Then, we have

$$\frac{1}{s^2} b_d(x,y) \le \liminf_{n\to\infty} d(x_n,y_n) \le \limsup_{n\to\infty} b_d(x_n,y_n) \le s^2 b_d(x,y)$$

In particular, if $b_d(x,y)=0$, then we have $\lim_{n\to\infty} b_d(x_n,y_n)=0=b_d(x,y)$.

Moreover, for each $z\in X$ we have

$$\frac{1}{s} b_d(x,z) \le \liminf_{n\to\infty} b_d(x_n,z) \le \limsup_{n\to\infty} b_d(x_n,z) \le s b_d(x,z)$$

In particular, if $b_d(x,z)=0$, then we have $\lim_{n\to\infty} b_d(x_n,z)=0=b_d(x,z)$.

Lemma 2.9 Let (X,b_d) be a b_d-metric space with parameter s and $(x_n)_{n\in N}$ a sequence in X such that:

$b_d(x_{n+1},x_{n+2}) \le q b_d(x_n,x_{n+1})$ for all $n\in N$ where $0\le q<1$. Then the sequence $(x_n)_{n\in N}$ is b_d-Cauchy sequence in X provided that $sq<1$.

Proof. For any $n\in N$ we have that:

$$b_d(x_{n+1},x_{n+2}) \le q b_d(x_n,x_{n+1}) \le q^2 b_d(x_{n-1},x_n) \le \ldots \le q^{n+1} b_d(x_0,x_1)$$

For all $p\in N$

$$\begin{aligned}
&b_d(x_n,x_{n+p}) \le s b_d(x_n,x_{n+1}) + s^2 b_d(x_{n+1},x_{n+2}) + \ldots \\
&+s^{p-1} b_d(x_{n+p-2},x_{n+p-1}) + s^{p-1} b_d(x_{n+p-1},x_{n+p}) \\
&\le sq^n b_d(x_0,x_1) + s^2 q^{n+1} b_d(x_0,x_1) + \ldots + s^{p-1} q^{n+p-2} b_d(x_0,x_1) + \\
&+s^{p-1} q^{n+p-1} b_d(x_0,x_1) \\
&= sq^n (1+sq+s^2q^2+\ldots+(sq)^{p-2}+s^{p-2}q^{p-1}) b_d(x_0,x_1) \\
&\le sq^n [1+sq+s^2q^2+\ldots+(sq)^{p-2}+s^{p-1}q^{p-1}] b_d(x_0,x_1) \\
&= \frac{sq^n}{1-sq}[1-(sq)^p] b_d(x_0,x_1)
\end{aligned}$$

Since $\frac{sq^n}{1-sq}[1-(sq)^p] d(x_0,x_1) \to 0$, as $n\to\infty$, the sequence $(x_n)_{n\in N}$ is b_d-Cauchy sequence in X.

We need the following Definitions and Lemmas according to [1].

Definition 2.10 Let (X,b_d) be a b_d-metric space with parameter s. The Hausdorff b_d-metric H on $C(X)$; the collection of all nonempty compact subsets of (X,b_d) is defined as follows:

$$H(A,B)=\begin{cases} \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\} \\ \qquad\qquad\qquad \text{if the maximum exists} \\ \infty, \quad \text{otherwise} \end{cases}$$

It is known [1] that $(C(X),H)$ is a complete b-metric space provided (X,b_d) is a complete b-metric space.

We cite the following Lemma:

Lemma 2.11 Let (X,b_d) be a b_d-metric space with parameter s and $A,B,C\in C(X)$.

Then:

(i) $d(x,B)\le d(x,y)$ for any $y\in B$,

(ii) $d(A,B)\le H(A,B)$

(iii) $d(x,B)\le H(A,B)$, $x\in A$,

(iv) $H(A,C)\le s[H(A,B)+H(B,C)]$

(v) $d(x,A)\le s[d(x,y)+d(y,A)],\ x,y\in X$.

Let (X,b_d) be a b_d-metric space with parameter s and $A,B\in C(X)$. Then for all $b\in B$ there exists $a\in A$ such that:

$d(a,b)\le sH(A,B)$

Let (X,b_d) be a b_d-metric space with parameter s. A fuzzy set in X is a function with domain in X and values in [0,1]. If A is a fuzzy set and $x\in X$, then the function-value $A(x)$ is called the *grade of membership* of x in A. The collection of all fuzzy sets in X denoted by $\mathscr{F}(X)$.

Let $A\in\mathscr{F}(X)$ and $\alpha\in[0,1]$. The α-level set of A, denoted by A_α, is defined by

$A_\alpha=\{x\in X : A(x)\ge\alpha\}$ if $\alpha\in(0,1]$, $A_0=\overline{\{x\in X : A(x)>0\}}$

whenever $\overline{B}$ is the closure of set (non-fuzzy) B with respect to topology induced by b-metric.

We consider a sub-collection of $\mathscr{F}(X)$ denoted by $\mathscr{W}(X)$; for any $A\in\mathscr{W}(X)$, its α-level set is a nonempty compact subset (non-fuzzy) of b_d-metric space (X,b_d) for each $\alpha\in[0,1]$ and $\sup_{x\in X} A(x)=1$; named the collection of all *approximate quantities*.

Definition 2.12 Let (X,b_d) be a b_d-metric space with parameter s and let $A,B\in\mathscr{W}(X)$ and $\alpha\in[0,1]$. Then we define,

$$\begin{aligned}
&p_\alpha(A,B)=\inf\{d(x,y): x\in A_\alpha, y\in B_\alpha\}=d(A_\alpha,B_\alpha)\\
&D_\alpha(A,B)=H(A_\alpha,B_\alpha)\\
&p(A,B)=\sup\{p_\alpha(A,B):\alpha\in[0,1]\}\\
&D(A,B)=\sup\{D_\alpha(A,B):\alpha\in[0,1]\}.
\end{aligned}$$

It is easy to see that D is a b_d-metric with parameter s in $\mathscr{W}(X)$, provided that (X,b_d) is a b_d-metric space with parameter s.

Definition 2.13 Let (X,b_d) be a b_d-metric space with parameter s. The mapping T is said to be a *fuzzy mapping* if and only if is mapping from the set X into $\mathscr{W}(X)$, i.e., $T(x)\in\mathscr{W}(X)$ for each $x\in X$.

Definition 2.14 The point $z\in X$ is called fixed point for the fuzzy mapping T if and only if $\{z\}\subset T(z)$.

The analogues lemmas of Heilpern in settings of b_d-metric spaces are:

Lemma 2.15 Let $x\in X$, $A\in\mathscr{W}(X)$ and $\{x\}$ be a fuzzy set with membership function equal to characteristic function of the set $\{x\}$. Then $\{x\}\subset A$ if and only if $p_\alpha(x,A)=0$ for each $\alpha\in[0,1]$.

Lemma 2.16 $p_\alpha(x,A)\le s[d(x,y)+p_\alpha(y,A)]$ for all $x,y\in X$ and $A\in\mathscr{W}(X)$.

Lemma 2.17 If $\{x_0\}\subset A$, then $p_\alpha(x_0,B)\le D_\alpha(A,B)$ for each $B\in\mathscr{W}(X)$.

Lemma 2.18 Let (X,b_d) be a complete b_d-metric space with parameter s, $T:X\to\mathscr{W}(X)$ be a fuzzy map and $x_0\in X$. Then there exists $x_1\in X$ such that $\{x_1\}\subset T(x_0)$.

Let (X,b_d) be a b-metric space with parameter s.

We consider a sub-collection of $\mathscr{F}(X)$ denoted by $\mathscr{W}^*(X)$ such that for any $A\in\mathscr{W}^*(X)$, its $\alpha-$level set is a nonempty compact subset (non-fuzzy) of X for each $\alpha\in[0,1]$. It is obvious that each element $A\in\mathscr{W}(X)$ leads to $A\in\mathscr{W}^*(X)$ but the converse is not true.

3. Main results

Next, we introduce the improvements of the lemmas in Heilpern [6] as follows.

Lemma 3.1 Let (X,b_d) be a b_d-metric space with parameter s and $x\in X$, $A\in\mathscr{W}^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to characteristic function of the set $\{x\}$.

Then $\{x\}\subset A$ if and only if $p_\alpha(x,A)=0$ for each $\alpha\in[0,1]$.

Lemma 3.2 $p_\alpha(x,A)\le s[d(x,y)+p_\alpha(y,A)]$ for all $x,y\in X$ and $A\in\mathscr{W}^*(X)$.

The proof of this lemma is particular case of Lemma 2.4 (v)

Lemma 3.3 If $\{x_0\}\subset A$, then $p_\alpha(x_0,B)\le D_\alpha(A,B)$ for each $B\in\mathscr{W}^*(X)$.

Lemma 3.4 Let (X, b_d) be a complete b_d -metric space with parameter s, $T: X \to \mathscr{W}^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$.

Theorem 3.5. Let (X, b_d) be a complete b_d -metric space with parameter s and with continuous b_d -metric in each coordinate, $T: X \to \mathscr{W}^*(X)$ be a fuzzy mapping such that:

$$D(Tx, Ty) \leq q[d(x, y)]$$

where q is real constant and $0 \leq q < \frac{1}{s}$. Then T has a fixed point.

Proof. For a arbitrary point $x_0 \in X$ there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$ and $x_2 \in X$ such that $\{x_2\} \subset T(x_1)$ and

$$d(x_1, x_2) \leq D(Tx_0, Tx_1) \leq qd(x_0, x_1)$$

Similarly, there exists $x_3 \in X$ such that

$$d(x_2, x_3) \leq D(Tx_1, Tx_2) \leq qd(x_1, x_2) \leq q^2(d(x_0, x_1))$$

So we construct a sequence $(x_n)_{n \in N}$ such that:

$$d(x_n, x_{n+1}) \leq q(d(x_{n-1}, x_n))$$

By Lemma 2.9 the above sequence is Cauchy in complete b-dislocated metric space (X, b_d), so there exists a $z \in X$ such that $\lim_{n \to \infty} x_n = z$. By Lemmas 2.16 and 2.17 we get:

$$\begin{aligned} p_\alpha(z, Tz) &\leq s[d(z, x_n) + p_\alpha(x_n, Tz)] \\ &\leq s[d(z, x_n) + D_\alpha(Tx_{n-1}, Tz)] \\ &\leq s[d(z, x_n) + D(Tx_{n-1}, Tz)] \\ &\leq s[d(z, x_n) + d(x_{n-1}, z)] \end{aligned}$$

Since d is continuous, letting $n \to \infty$, we have $p_\alpha(z, Tz) = 0$ and By Lemma 2.15, $\{z\} \subset T(z)$.

References

[1] Czerwik, S, Nonlinear set valued contraction mappings in b-metric spaces, Atti Sem Math. Univ. Modena, 46(1998), 263-276.

[2] S L Singh, B. Prasad, Some coincidence theorems and stability of iterative procedures, Comput. Math. Appl. 55 (2008), 2512-2520.

[3] Berinde, V, Generalized contractions in quasimetric spaces, Seminar on fixed point theory, Preprint no. 3(1993), 3-9.

[4] Czerwik, S, Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostraviensis 1 (1993), 5–11.

[5] S. L. Singh, S. Czerwik, Krzysztof Krol and Abha Singh, *Coincidences and Fixed points of hybrid contractions*, Tamsui Oxford Journal of Mathematical Sciences 24 (4) (2008) 401-416.

[6] S. Heilpern, Fuzzy mappings and fixed point theorems, J. Math. Anal. Appl. 83 (1981) 566-569.

[7] I. Beg, M. A. Ahmed, Common fixed point theorem for generalized fuzzy contraction mappings satisfying an implicit relation, Applied Mathematics Letters, In Press, Available online 17 December 2011.

[8] M. Akkouchi, A common fixed point Theorem for Expansive Mappings under Strict Implicit Conditions on b-Metric Spaces, Acta Univ. Palacki. Olomuc., Mathematica 50, 1 (2011) 5-15.

[9] M. Akkouchi, Common Fixed Point Theorems for Two Self-mappings of a b-metric Space under an Implicit Relations, Hacettepe Journal of Mathematics and Statistics, Vol. 40 (6) (2011), 805-810.

[10] S. L. Singh, B. Prasad, Some coincidence theorems and stability of iterative procedures, Computers and Mathematics with Applications 55 (2008) 2512-2520.

[11] S. Czerwik, Contraction mappings in *b*-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, Vol. 1 (1993), No. 1, 5—11.

[12] M. Boriceanu, M. Bota, A. Petrusel, Multivalued fractals in b-metric spaces, Central European Journal of Mathematics, 8(2), 2010, 367-377.

[13] S. B. Nadler, Multivalued contraction mappings, Pacific J. of Math., 1969, 30, 475-488.

[14] V. Berinde, Sequences of operators and fixed points in quasimetric spaces, stud. Univ. "Babes-Bolyai", Math., 16(4)(1996), 23-27.

[15] S. C. Arora, V. Sharma, Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems 110 (2000) 127-130.

[16] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal. 30(1989), 26-37.

[17] M. Bota, A. Molnar, C. Varga, On Ekeland`s variational principle in b-metric spaces, Fixed Point Theory, 12(2011), No. 2, 21-28.

[18] Hussain, N, Roshan, JR, Parnaveh, V, Abbas, M,:Common fixed point results for weak contractive mappings in ordered *b*-dislocated metric spaces with applications, Journal of Inequalities and Applications 2013, 486(2013).

[19] I. Beg, M. A. Ahmed, Common fixed point theorem for generalized fuzzy contraction mappings satisfying an implicit relation, *Mat. Vesnik*, Available online 20.04.2013.

[20] Hitzler, P, Seda, AK: Dislocated topologies. J. Electr. Eng. 51(12), 3-7 (2000).

[21] A. Isufati, I. Vardhami, K. Gjino, Fuzzy mappings in b-metric spaces, Proceddings of ICBWM 2013.

