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Original Article

On analytic functions and characterization of Schwarz Norms in Banach spaces

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Abstract

In this paper, we characterize Schwarz norms in Banach spaces. We give new results on the s-norms in B (H).

Keywords:
Schwarz Norms, Banach Spaces

1. Introduction

Suppose that f is an analytic function in the open unit disk U ={z ∈ ℂ :|z|<1} and is bounded i.e. $\|f\|_\infty$ = sup{|f(z)| : zℂU } < ∞. If f has the following additional properties, f(0) = 0, $\|f\|_\infty$< 1,then the following (Schwarz Lemma) holds:

If f is analytic in the open unit disk as described above and,

(i.) |f(z)|≤ |z|, z ℂU

(ii.) |f (0)|≤ 1,

and if the equality appears in (i) for one z∈U – {0}, then f(z) = αz ,where $\alpha$is a complex constant with |a/= 1 and also if the equality appears in(ii), f behaves similarly. In case of operators, we have that, if |T|≤ 1, then |f(T)| ≤‖f‖ for each f ∈ R(D) such that f(0) = 0.Here R(D) is the (sup-norm) algebra of the rational functions with no poles in the closed unit diskD and f(T) defined by the usual Cauchy integral around a circle slightly larger than the unit circle.[5]We note here that a contraction (i.e an operator T such that ‖T‖ < 1) T∈B(H) has some relation with the closed unit disk of the complex plane, say for any contraction T and any complex-valued function f(z) defined and analytic on the closed unit disk, then by von Neumann [9],[11] the norm equality holds;‖f(T)‖ ≤ $\|f\|_\infty \equiv$ $\max_{|z|\leq 1}$|f(z)|where the operator f(T) is defined by the usual functional calculus[10].

The above lemma has an interesting application in the theory of operators namely the following assertions hold , if f is analytic in the open unit disk andf(0) = 0 with $\|f\|_\infty$< 1,then for any operatorT∈B(H), ‖T‖< 1,(Berger and Stampfli) [2] we have ‖f(T)‖ <‖T‖.Clearly if we have an equality for some T, then f is of the formf(z) = αz.Where α is a complex constant with |a/= 1. A norm, say,‖T‖< 1 on the algebra B(H) of all bounded operators T , is called a Schwarz norm if it is equivalent to the usual norm ‖.‖and the Schwarz lemma holds for it,i.e for any f analytic in the open unit disc U with f(0) = 0 and $\|f\|_\infty$<1, and for any T∈ B(H), ‖T‖< 1, we have $\|T\|^*$< 1

2. Preliminaries

We will in this section give the definitions that will be essential in our study. In the following **K** = **R** or **C**

Definition 2.1. For a set of points X,the pair (X; K) is called a linear space if for all x,y∈X andα, β ∈**K** then$\alpha x + \beta y \in X$

In case **K** = **R** then the pair is referred to as real linear space but if **K** = **C** then it is a complex linear space.

Definition 2.2. Let (X; **K**) be a linear space as defined above.A mapping ‖.‖: X→ℝ is called a norm on X if it satisfies the following properties (norm axioms);

(i) ‖x‖≥ 0 for all x∈X (non-negativity)

(ii) If x∈X and ‖x‖ = 0, then x = 0 (zero axiom)

(iii) ‖λx‖= |λ|‖x‖ for all x∈X and y∈K (homogenity)

(iv) ‖x + y‖≤ ‖x‖+ ‖y‖$\forall y, z \in X$ (triangular inequality)

The ordered pair (X; ‖:‖) is called a normed linear space (n.l.s) over **K**

Definition 2.3. Suppose property number (ii) (zero axiom) in the above definition fails ,i.e if x∈X and ‖x‖= 0 ; x = 0, then the function , ‖:‖:X→ℝ is referred to as seminorm on X.

Definition 2.4. Let (X,**K**) be a linear space and $\|:\|_1$,$\|:\|_2$ be norms on X we say that $\|:\|_1$ and $\|:\|_2$are equivalent if their exists positive reals ,β , such that

$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\| \forall x \in X$.The two norms generate the same open sets (same topology)

Definition 2.5. A sequence (x_n) is said to converge strongly in a normed linear space (X,‖:‖) if their exists x∈X such that$\lim_{n\to\infty}\|x_n\text{-}x\|$ = 0

Definition 2.6. Let (X,‖:‖) be a normed linear space and ρ be the metric induced by ‖:‖. If (X,ρ) is a complete metric , then we call (X,‖:‖) a Banach space or strongly complete normed linear space.(A normed linear space (X; ‖:‖) is a Banach space if every strong Cauchy sequence of elements of X converges strongly in X)

Definition 2.7. Let (X,**K**) be a linear space. If M is a subset of X such that x, y∈M and

$\alpha, \beta \in \boldsymbol{K} \rightarrow \alpha x + \beta y \in M$, then M is called a subspace of X

Definition 2.8. Let ~~X be a l~~inear space over **K** and <,> : X↔**K** be a function with,

(i) $<x, x>\geq 0$ for all $x\in X$

(ii) $<x, x> = 0 \rightarrow x = 0$

(iii) $<y, x>=<x,y>^*$ or $<x,y$ if $\mathbf{K} = \mathbb{C}$ or $K = \mathbb{R}$ respectively for all $x, y\in X$. where $<x,y>^*$ denotes the conjugate of the complex number $<x,y>$.

(iv) $<\lambda x, y>= \lambda<x, y>$ for all $x, y \in X$ and all $\lambda\in K$.

(v) $<x + y, z> = <x, z> + <y, z>$ for all $x, y, z \in X$ The function <.> is called inner-product (i.p) function and the real or complex number

$<x, y>$ is called the inner product of x and y (in this order). The ordered pair (X,<.>) is called an inner product space or pre-Hilbert space over K. Let (X,<.>) be an inner-product space. The norm in X is given by $\|x\|= \sqrt{< x,x >}$ for all $x\in X$ and is called the norm determined by (or induced by) the inner-product function of x. The metric ρ determined by this norm $\|.\|$ as defined above is $\rho(x, y) = \|x - y\|$ for all $x, y\in X$ is called the metric induced by the inner-product function<.>. If with respect to this norm $\|x\|$, defined above, $(X,\|.\|)$ is strongly complete i.e $(X,\|.\|)$ is a Banach space,then we refer to $(X,\|.\|)$ as a Hilbert space i.e a Hilbert space is a complete inner product space.

Definition 2.9. Let H be a complex Hilbert space and T be a linear operator from H to H. T is said to be positive if $<Tx, x>\geq 0$, for all x in H. This can be denoted by $T\geq 0$ or $0\leq T$. T is said to be strictly positive or positive definite if $<Tx, x> >0$ for all $x\in H\setminus\{O\}$

Definition 2.10. If $T\in B(H)$, then the operator $T^*: H\rightarrow H$ defined by $<Tx, y> = <x, T^*y>$ for all x,y in H is called the adjoint of T. (T^* is also in B(H)) and $\|T^*\|= \|T\|$

Definition 2.11. An operator $T\in B(H)$ is said to be self – adjoint if $T^* = T$ and if T is linear on a linear subspace M of a Hilbert space H into M, then it is said t be Hermitian addition $\langle Tx, y\rangle = \langle x, Ty\rangle \forall x, y\in M$

Defination 2.12. Let H be a complete Hilbert space and $T\in B(H)$. Then there exist unique self-adjoint operators $A,B\in B(H)$ such that T=A + iB, A and B are given by $A= \frac{1}{2}(T+T^*)$, $B= \frac{1}{2i}(T-T^*)$ so that A is called real part of T denoted by ReT and B the imaginary part of T denoted by ImT. Note that Re $\langle Tx,x\rangle = \langle(\mathrm{Re}\,T)x,x\rangle$ for every $x\in H$. Indeed $\langle Tx,x\rangle = \frac{1}{2}\langle(T+T^*)x,x\rangle + i\frac{1}{2}\left\langle\left(\left(\frac{T-T^*}{2}\right)x,x\right\rangle\right.$, being a complex number, we have $\langle Tx,x\rangle =$ a + ib, where a, b are real numbers given by a= $\langle(\mathrm{Re}\,T)x,x\rangle$, b= $\langle(\mathrm{Im}\,T)x,x\rangle$

Defination 2.13. Let H be a complex Hilbert space and $T\in B(H)$, The numerical range of T is the set $W(T)\subset\mathbb{C}$ defined by $W(T) = \langle Tx,x\rangle : x\in H$ and $\|x\|=1\}$

3. Main Results

It is quite natural to investigate the problem about the existence of Schwarz norms on the algebra B(X) of all bounded operators on a Banach space X. For this we recall that a function [.] on X xX into $\mathbb{C}$ is called a semi-inner product if the following conditions are satisfied:

1. $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
2. $[ax, by] = ab^* [x, y]$
3. $|[x, y]|\leq \|x\|.\|y\|$
4. $[x,x] > 0$ for $x \neq \bar{0}$ for all $x_1, x_2, x, y\in X$ and a, b are complex numbers.

Theorem 3.1. On every Banach space there exist a semi-inner product [,] with the property $[x, x] = \|x\|^2$(i.e it is compatible with the norm).Indeed for any $x\in X$ we define the functional $f_x\in X^2$. (Where X denotes the space of all the bounded functionals on X)with the properties;

(i) $\|f_x\|= \|x\|$

(ii) $f_x(x) = \|x\|^2$

The existence of the functional is guaranteed by Hahn-Banach theorem and we define

$[x, y] = f_y(x)$ and $f_{\lambda x}= \lambda^*f_x$ which satisfy the four conditions above, for each $\lambda\in\mathbb{C}$, $x \in X$. An operator $T \in B(X)$ is called hermitian if $\|e^{iT}\|= 1$ for all real numbers t or equivalently,Bonsall[6] if $W(T) = \{[T x,x] : \|x\|= 1\}$ is a subset of real numbers.

An operator $T \in B(X)$ is called positive if T is hermitian and the spectrum of T is in the subset $\{x \in R : x > 0\}$.

Now the definition of the class S_Q can be as follows.

Definition 3.2. An operator T 2 S_Q if and only if

1. $\delta(T)\subset U$
2. For any $x\in X$, and $|z| <1$ $\mathrm{Re}[(I+\sum Q^{\frac{1}{2}}T^n Q^{\frac{1}{2}}z^n)x,x]\geq 0$ where Q is a hermitian operator such that $Q^{1/2}$ is also a hermitian operator. The following results give indications about the possible existence of Schwarz norms.

Theorem 3.3. There exists a Banach space X and an operator T such that $\mathrm{Re}[T x,x]\geq 0$ does not imply $\mathrm{Re}[T^{-1} x, x]\geq 0$.

As an example to illustrate this,we consider the Banach space ℓ^{p}_2 of all pairs $x = (x_1,x_2)$ with the norm $x\mapsto \|x\|_p = \{|x_1|^p + |x_2|^p\}^{\frac{1}{p}}$ $1 < p < \infty$.

In this case it can be seen that the semi-inner product compatible with the norm $[x, x] = \|x\|^2_p$ is given by $[x, y] = x_1|y_1|^{p-1} + x_2|y_2|^{p-2}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ We consider an operator on this space with the matrix

$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$ where the elements a; b; c are complex numbers. We need to find conditions for the a, b ,c such that $\mathrm{Re}[T x, x]\geq 0$.A straight forward but complicated computation shows that these are :

1. $\mathrm{Re}a\geq 0$, $\mathrm{Re}b\geq 0$
2. $|c|\leq (p\mathrm{Re}a)^{1/p} (q\mathrm{Re}b)^{1/q}$ $(1/p + 1/q= 1)$ and condition for $\mathrm{Re}[T^{-1}x,x]\geq 0$ is

$|\frac{c}{ab}|\geq(p\mathrm{Re}a^{-1})^{1/p}(q\mathrm{Re}b^{-1})^{1/q}$ and thus if $|c|\leq |a|^{1-2/p}(\mathrm{Re}pa)^{1/p}(\mathrm{Re}qb)^{1/q}$ and this gives that $\mathrm{Re}[Tx, x]\geq 0$.

Remark 3.4. In case of Hilbert space (and invertible) operators, the condition , $\mathrm{Re}T\geq 0$ implies the condition $\mathrm{Re}T^{-1}\geq 0$

We now give an example of a Banach space with the property that the induced norm on B(X) is not a Schwarz norm.

Example 3.5 If X = ℓ^1_2 then the induced norm on B(X) is not a Schwarz norm.We consider the operator T with the matrix(triangular) $\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$ and a simple computation shows that ‖T‖=max{|a| +|c|, |b|}. We now take 0<a<1 and in this case the operator with the matrix $\begin{bmatrix} a & 0 \\ 1-a & 1 \end{bmatrix}$ is a contraction operator. An elementary computation shows that |a|<1, the conformal map/function $\vartheta\alpha$ (z) = (z- α)(1- $\overline{\alpha}$ z)$^{-1}$ for all z ∈ ℂ, take contractions; now consider the function fα (T) = (1- $\overline{\alpha}$ T)$^{-1}$ (T- α 1). The computation of the norm of the operator fα (T) shows that this is given by ‖fα (T)‖=a| $\alpha + a$ + (1-a)| $\frac{1+\alpha+1+\overline{\alpha}}{(1+\overline{\alpha a})(1+\alpha)}$ |and thus for ‖fα (T)‖≤1, where α is a real number , we obtain a| $\alpha + a$ | + (1-a)(1 + a) ≤ | $1+\alpha a$ which is not α = -1/2(a+1). In view of the results is of interest.

Proposition 3.6 If X is a complex Banach space and for any contraction T, f(T) is also a contraction for all |f|≤1, then X is a Hilbert space .

Proof:

Let $x_0 \in X$ be arbitrary $x_0 \in X$ such that ‖x_0‖ ‖x^*_0‖≤ 1and define the operator on X by the relation $T_x = x_0^*(x)x_0$. Its clear that T is a contraction . From the hypothesis it follows that ‖$x_0^*(x)x_0$ + x≤‖x+ α^* $x^*(x)x_0$‖. Now if x,y ∈ X and ‖x‖ ≥ ‖y‖ ≥ 0 , we obtain from the H-Banach theorem that there exists $x_0^* \in X^*$ such that ‖x_0^*‖= ‖x‖$^{-1}$, $x_0^*(x)$=1.

We take x_0 =y and remark that the operator T constructed with these element gives us ‖y+ α x‖≤‖x + α^* y‖< 1 and from the continuity argument, it follows that this relation holds for | α^* |= 1. Now if ‖x‖= ‖y, changing the role of x with y and α with α^*, we obtain‖≤‖x + α^* y‖ ≥ ‖y + αx ‖, thus we have the equality‖x + α^* y‖= ‖y + α x‖. Now if | α |>1, then for $\beta = \frac{1}{\alpha}$ we have by the above result‖x + α^* y‖=| α || β x +y|= | α |‖x + β y‖= ‖ α x + y‖ and thus the relation is true for any α Now for α =p/q, p and q being real numbers we obtain thatp/q, p and q being real numbers we obtain that ‖px +qy‖= |q|‖p/qt+x‖=|q|‖p/qt+x‖ = |q|‖y + p/qx‖= ‖qy+px‖ and thus for any x and y , ‖x‖=‖y‖ and any p,q real numbers we obtain that ‖px + qy‖=‖qx + py‖ and by a famous result of F.A.Ficken,this relation is characteristic for a norm to be inner product norm, i.e, there exists an inner product $\langle . \rangle$ on X such that for all x ∈ X , ‖X‖^2 = $\langle x, x \rangle$

4. Conclusion

A Schwarz norm can be constructed from the sum of a norm and a seminorm and that Schwarz norms are easily realizable in the Hilbert space context.

References

[1] J.P Williams,(1968),Schwarz norms for operators, *Paci c Journal of Mathematics*. 24, No.1

[2] C.A Berger and Stamp i,(1967), Mapping theorems for the numerical range, to appear in *American J. Math.* 26, 247-250.

[3] B.S.Z Nagy and C.Foias,(1983), On certain classes of power-bounded operators, *Acta.Sci. Math. Ser.* III 18(38) 317-320.

[4] C. Foias,(1957), Sur certain stheoremes de von Neumann concernant les ensemplesspectraux , *Acta. Math. Sci*.(Szeged) 85 15-20.

[5] J.G Stamp i,(1966),Normality and the numerical range of an opera-tor, *Bull Amer. Math. Soc.*. 72 1021-23.

[6] F.F. Bonsall, J. Duncan, Numerical Ranges of operators on Normed spaces and elements of Normed algebras, London Math. Soc. Lecture Notes series 2, Cambridge University Press, London-New York, 1971.

[7] F.F. Bonsall, J. Duncan, Numerical Ranges II, London Math. Soc. Lec-ture notes Series 10, Cambridge University Press, London-New York, 1973.

[8] E. Kreyszig, Introduction to functional analysis with applications, Uni-versity of Windsor, 1978.

[9] J.A.R Holbrook, Inequalities of von Neumann type for small matrices, Function Spaces(ed.K.Jarosz), Marcel Dekker,1992,273-280

[10] T. Ando Construction of Schwarz norms, Operator Theory. *Advances and Application*.,127(2001) 29-39

[11] J.VonNeumann,EineSpektraltheorie fur allgemeineOperatoreneinesUnitaren Raumes, *Math. Nach.,* 4 (1951),258-281

[12] C.A Berger, A strange dilation theorem (Ab-stract), *Amer. Math. Soc. Notice*, 12 (1965)590