Bulletin of Advanced Scientific Research 01 [01] (2015)

www.asdpub.com/index.php/basr

e-ISSN: 2454-3691

Original Article

On reflexivity, denseness and compactness of numerical radius attainable operators

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Abstract

In this paper, we study the properties of normal self-adjoint operators. We concentrate on some of their properties, for example, reflexivity, denseness and compactness. We also give some results on norm-attainability.

Keywords: Reflexivity, Compactness, Denseness, Numerical radius attainability, Normal operators and Self-adjoint operators.

1. Introduction

We consider certain properties of operators. A lot of studies have been done on reflexivity, compactness and numerical radius attainability on Hilbert space operators [1-12] and the reference therein.

2. Preliminaries

2.1 Definition

An operator $A \in B(H)$ attain its numerical radius if there are $x_o \in H$, $f_o \in H^*$ such that $\|x_o\| = \|f_o\| = f_o(x_o) = 1$ and $|f_o(A(x_o))|$=$r(A)$, that is if the supremum defining $r(A)$ is actually a maximum.

2.2 Lemma

Let each operator $S \in M(A)$ be of rank one and attains its numerical radius. Then $M(A)$ is reflexive.

Proof. For proof see [2].

3. Main Results

3.1 Theorem

Let $M(A)$ be reflexive. Then it is Banach and for some y_o in $Q_{M(A)}$ the operator $y^* \otimes y_o^*$ attains its numerical radius for any $y^* \in [M(A)]^*$.

Proof.

Let $M(A)$ be dense and non-reflexive. Suppose that every operator $y^* \otimes y_o^*$ attains its numerical radius. By the Bishop-Phelps Theorem in [4] and the non reflexive of $M(A)$, we find $(y^* \otimes y_o^*) \in \prod(M(A)^*)$ which satisfies $|y_o^{**} - y_o| < 1$ and $y^{**} \notin X$, and since $y_o^{**}(y_o^*) - y_o^*(y_o) < 1$ and since $y_o^{**}(y_o^*) = 1$, then $y_o^*(y_o^*) \neq 1$ and

$$\alpha y_o^{**}(y_o^*) = 1 \tag{1}$$

For some scalar $\beta \neq 0$. By the Hahn-Banach Theorem, there $\xi \in Q_{M(A)^{***}}$ and $t > 0$

Such that $\xi(y) = 0, \forall\ y \in M(A)$ and $Re\ \xi(y_o^{**}) > t. M(A)$ is dense, therefore in $M(A)^{***}$ the topology of strong convergence on $M(A) \cup \{y_o^{**}\}$ is dense. Since $Q_{M(A)^*}$ is w^*-dense in $Q_{M(A)^{***}}$, there exist a sequence $\{y_o^*\}$ in $Q_{M(A)^*}$ converges to φ in $\sigma((A)^{***}, M(A) \cup \{y_o\})$. Then

$$\{y_n^*(y)\} \to 0, \forall \in M(A) \tag{2}$$

And assume

$$Rey_o^{**}(y_n^*) \tag{3}$$

The set $C = \hat{\prod}(M(A))$ and $D = \hat{\prod}(M(A)^*)$ (C) are considered as subsets of D. But the function $f_n: \hat{\prod}(M(A)) \to \mathbb{R}$ given by $f_n(y)^*, y^{**} = y^{**}(y_n^*) y^*(y_o), ((y^*, y^{**})) \in \hat{\prod}(M(A)^*)$. For each sequence $\{g_n\}$ with $0 \leq g_n \leq 1$ and

$$\textstyle\sum_{n=1}^{\infty} g_n\, f_n(y^*, y^{**}) = Rey^{**}(\sum_{n=1}^{\infty} g_n\, y_n^*) y^*(y_o), \forall\ (y^*, y^{**}) \in \hat{\prod}(M(A)^*).$$

We now get $sup_{(y,y^*)\in\hat{\prod}(M(A))} \lim_n supRey_n^*(y) y^*(y_o) \geq inf_{x^* \in CO\{y_n^*\}}\ sup_{(y^*,y^{**})\in\hat{\prod}(M(A)^*)}\ Rey^{**}(x^*) y^*(y_o)$. But, $sup_{(y,y^*)\in\prod(M(A))}$

$$\lim_n supRey_n^*(y)(y_o) = 0 \tag{4}$$

and from (3) and (1), suppose $x^* \in \{y_n^*\}$, then $Rey^{**}(x^*) \frac{\beta}{\beta} y_o^*(y_o) \geq \frac{t}{\beta}$, and

$$inf_{x^*\in CO\{y_n^*\}}\ sup_{(y^*,y^{**})\in\hat{\prod}(M(A)^*)}\ Rey^{**}(x^*) y^*(y_o) \geq \tfrac{t}{\beta}. \tag{5}$$

Finally, from (4), (5) we get $0 \geq \frac{t}{\beta}$, but $t > 0$ which is a contradiction.

3.2 Theorem

Let $Y \in M(A)$ be a rank one operator not attaining its numerical radius. Then $M(A)$ can be renormed if it is infinite dimensional.

Proof.

Let $M(A)$ to be reflexive and for normalized elements $y_o \in B_{M(A), s_o^* \in BM(A)^*}$, the equality $v(s_o^* \otimes y_o) = \|s_o^* \otimes y_o\| = 1$ is true if $s_o^*(y_o) = 1$, since $v(s_o^* \otimes y_o)$ is attained at $y_o, s_o^* \in \hat{\prod}(M(A))$ [1, 2, 3, 4 and 5]. Now if $v(s_o^* \otimes y_o) = 1$ then we have $s_o^*(y_o) = 1 = s_o^*\ (s)$ and commuting the elements s and s^* we obtain in $\hat{\prod}(M(A))$ satisfying

$$s_o^*(y_o) = 1 = s_o^*\ (s) \tag{6}$$

Let y_o^* be unique in the ball of $M(A)^*$ and $y_o^*(y_o) = 1$. From the smoothness of y_o we obtain $s^* = y_o^*$. Since $(s, y_o^*) = (s, s^*) \in \hat{\prod}(M(A))x$ will uniquely be determined by assuming that y_o^* is also smooth and so $s = \lambda y_o$ for some $\lambda = 1$ and $(s, s^*) = (\lambda y_o, y_o^*)$. Using (1) again, $s_o^*(\lambda y_o) = s_o^* = 1$, and the smoothness of y_o gives us $\lambda s_o^* = y_o = s^*$. Finally, the couple (s, s^*) is (y_o, y_o^*). It is sufficient that $s_o^* \otimes y_o$ satisfies

$v(s_o^* \otimes y_o) = \|y_o\| = \|s_o^*\| = 1$, with y_o, s_o^* smooth and hence $s_o \notin \mathbb{K}z_o$, for some $s_o \in B_{M(A)}$ such that $s_o^*(s_o) = 1$. Next if the numerical radius of the operator is 1,then there exist $\{s_n, s_n^*\} \subseteq \prod(M(A))$ so that

$$\{s_n^*(y_o)\} \to 1 \quad (7)$$

By inequality $2 \geq \|s_n + y_o\| \geq s_n^*(s_n + y_o)$ and (8), we have $\{\|s_n + y_o\|\} \to 2$. Similarly, if s_o is a w −cluster point of $\{s_n\}$, (8) will also give us $s_o^*(s_o) = 1$. Conversely, if $\{s_n\}$ converges in the $w-$ topology to an element s_o in the unit ball and $\{\|s_n + y_o\|\} \to 2$, then there is a sequence of norm one functional $\{s_n^*\}$ so that the sequence $\{s_n^*(s_n)\}$ and $\{s_n^*(y_o)\}$ converges to 1. By Bishop-Phelps-Bollobas Theorem [1, 2, 3, 4, 5] we assume that $s_n^*(s_n) = 1$ and so, we fix an element s_n^* in the unit sphere of the dual so that $s_o^*(s_o) = 1$, and we have $\lim_n s_o^*(s_n) = s_o^*(s_o) = 1, \lim_n s_n^*(y_o) = 1$ and therefore

$v(s_o^* \otimes y_o) \geq sup_n s_o^*(s_n) s_n^*(y_o) \geq 1$, implying that the numerical radius of the operator is 1.

3.3 Corollary

Let $M(A)$ be a Banach algebra. Then every operator in $M(A)$ can be perturbed by a normal self-adjoint operator to obtain an operator in $B(H)$.

Proof.

Suppose $X \in M(A)$ with $\|X\| = 1$ and $0 < \varepsilon < \frac{1}{2}$ given. From [2, 3 and 4] two decreasing sequences of positive numbers, $\{\alpha_n\}$ and $\{\delta_n\}$ are chosen with the following conditions satisfied

$$\sum_{i=1}^{\infty}(\alpha_i + 2\alpha_i^2) < \varepsilon; \lim_{n=\infty\alpha_n} \frac{1}{2}\sum_{i=n+1}^{\infty}(\alpha_i + 2\alpha_i^2) =; \left\{\frac{\delta_n}{\alpha_n^2}\right\} \to 0 \quad (8)$$

(We choose $\alpha_n = \frac{\varepsilon}{3.2^{n!}}$, for example, and $\delta = \alpha_n^3$). The sequence X_n in $M(A)$ and $\{a_n, f_n\}$ in $\prod(A)$ are constructed satisfying

$$X_1 = X, \quad (9)$$

$$|f_n(X_n(a_n))| > v(X_n) - \delta_n$$

(10)

$$X_{n+1}(a) = X_n(a) + \alpha_n \lambda_n f_n(a) a_n + \alpha_n^2 f_n(X_n(a)) a_n (a \in A) \quad (11)$$

Where $|\lambda_n| = 1$ and $f_n(X_n(a_n)) = \lambda_n |f_n(X_n(a_n))|$. It can be verified by induction that

$$\|X_{n+1}\| \leq 1 + \sum_{i=1}^{\infty}(\alpha_i + 2\alpha_i^2) \leq 2, \forall\, n \quad (12)$$

It follows that

$$\|X_{n+1} - X_n\| \leq 1 + \sum_{i=1}^{n+k-1}(\alpha_i + 2\alpha_i^2), \forall\, n, k \quad (13)$$

By (12) and (7), the norm of the sequence $\{X_n\}$ converges to an operator G in $M(A)$ satisfying

$$\|G - X_n\| \leq \sum_{i=1}^{n+k-1}(\alpha_i + 2\alpha_i^2), \forall\, n, k. \quad (14)$$

For all n, and particularly $\|G - X\| < \varepsilon$. With X_n playing the role of $X, \delta = \delta_n, \alpha = \alpha_n, \rho = \alpha_{n+k+2\sum_{i=n+1}^{\infty}(\alpha_i + 2\alpha_i^2)}$, $(a, f) = a_n, f_n$ and $(y, h) = (a_{n+k}, f_{n+k})$, so that the operator X' agrees with X_{n+1} and we have

$1 + \alpha_n v(X_n) \leq |f_n(a_{n+k})| + \alpha_n |f_n(X_n(a_{n+k}))| + \frac{1}{\alpha_n}[\delta_{n+k+}2\sum_{i=n+1}^{\infty}(\alpha_i +$ *2αi2+δn1+αn2*

$\leq |f_n(a_{n+k})| + \alpha_n |f_n(X_n(a_{n+k}))| + \frac{1}{\alpha_n}[\delta_{n+k+}2\sum_{i=n+1}^{\infty}(\alpha_i + 2\alpha_i^2) +$ *δn1+αn2*

Here, the fact that δ_n is a decreasing sequence is used for the last inequality. We now replacing X_n by G in the inequality above and use the estimate of $G - X_n$ given by (13) (to neutralize the errors) and we get $1 + \alpha_n v(G) \leq |f_n(a_{n+k})| + \alpha_n |f_n(G(a_{n+k}))| + \varepsilon_n$ where $\varepsilon_n = \frac{1}{\alpha_n}[\delta_{n+k+}2\sum_{i=n+1}^{\infty}(\alpha_i + 2\alpha_i^2) + \delta_n(1 + \alpha_n^2)] + 2\alpha_n \sum_{i=n+1}^{\infty}(\alpha_i + 2\alpha_i^2)$.

Hence by (7) and due to the fact that the sequence $\alpha_n \to 0$ and $\delta_n \to 0$, then $G \in B(H)$.

3.4 Theorem

Let $A \in B(H)$ be normal and $M(A)$ be compact and dense in $B(H)$. Then A is compact.

Proof.

Let $A \in B(H)$ and $M(A) \subseteq B(H)$. Suppose that x_n is a strongly convergent sequence in H then Ax_n is also a strongly convergent sequence in $M(A)$. As A is normal then $M(Ax_n) \to 0$ hence $M(A)$ is normal. But $M(A)$ is compact and dense. Then $Ax_n \to 0$ for every strongly convergent sequence (x_n) from H. Then we also have $Ax_n \to 0$. Since A is normal [4,7] then the operator A^* is also normal. Since x_n is a strongly convergent sequence in H then $A^* Ax_n \to 0$ and A is closed. This implies that A is compact.

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