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研究ノート

Recurrence Analysis of Converging and Diverging Trajectories in the Mandelbrot Set

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ABSTRACT

Iterated dynamics of the Mandelbrot set (M-set) were studied at paired points close to, but on either side of seven specific borders regions. Cross recurrence analysis of the real versus imaginary variables of the converging and diverging trajectories were found to be similar until the final divergence was manifested after hundreds of iterations. This study show that recurrence strategies can be used to study the sensitive dependence of M-set dynamics on initial conditions of the Mandelbrot chaotic attractor.

INTRODUCTION

The Mandelbrot set (M-set) is an infinitely deep mathematical structure defined by the iterated equation $Z = Z^2 + K$ where variable Z (Z_{real}, Z_{imag}) and constant K (K_{real}, K_{imag}) are both complex numbers. The M set exists as a geometrical object within the complex plane and features miniaturized or recurrent M sets at all scales. At high magnifications, K points within the M set can lie very close to K points outside the M set, both separated by a single border. In these cases, the associated Z dynamics for converging and diverging trajectories can be very similar over many iterations before divergence separated the two trajectories. The purpose of this study was to characterize these paired Z trajectories using cross recurrence analysis before they diverged.

Cross recurrence analysis is founded upon principles from the recurrence plot (RP) and recurrence quantification analysis (RQA). RPs were originally designed to visualize recurring patterns and non-stationarities within experimental data sets (Eckmann et al., 1987). RQA extended RPs by introducing 5 dynamical variables (Zbilut and Webber, 1992; Webber and Zubilt, 1994): recurrence, determinism, max diagonal line, line entropy, and trend. Later 3 more dynamical variables were added (Marwan et al., 2002): laminarity, max vertical line, trapping time. Cross recurrence analysis compares recurrent patterns in paired trajectories that may or may not run in parallel.

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Keywords : Recurrence plot, Mandelbrot set, Cross recurrence analysis

METHOD

Mandelbrot set calculations

The M-set equation ($Z = Z^2 + K$) was iterated multiple times until the trajectory either converged or diverged. Variable Z and constant K are both complex numbers with real and imaginary coefficients. Z_{real} and Z_{imag} were both initially set to 0.0 whereas C_{real} and C_{imag} were set to specific values corresponding M-set borders in the complex plane.

The complex numbers were squared as follows.

$Z_{real} = (Z_{real})^2 - (Z_{imag})^2$

$Z_{imag} = 2 * Z_{real} * Z_{imag}$

The complex numbers were summed as follows.

$Zr_{eal} = Z_{real} + K_{real}$

$Z_{imag} = Z_{imag} + K_{imag}$

The maximum number of iterations was set to 1,000 before a series was defined as convergent. divergent trajectories never reached this upper-bounds limit.

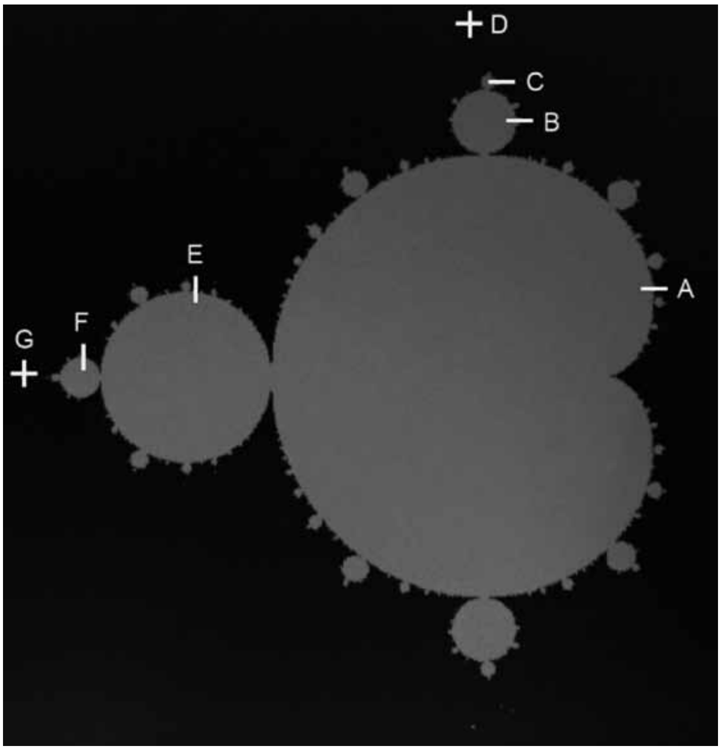

Fig. 1 Seven selected borders of the M-set (A, B, C, D, E, F, G).

Trajectory Selection

Z dynamics were examined for paired points on either side of seven different borders of the M- set (A, B, C, D, E, F, G in Fig. 1). Region A was the central cardioid itself. Regions B, C and D were smaller and smaller cardioids above the central cardioid. Regions E, F and G were smaller and smaller cardioids to the left of the central cardioid. As shown in Table 1, the number of converging (C) trajectories was consistently 1,000, and the number of corresponding diverging (D) trajectories was close to, but always less than 1,000.

Table 1 K_{real} and K_{imag} values for converging and diverging trajectories across 7 boarders of the M-set.

Region	Converging	Diverging	Num C Iterations	Num D Iterations
A	Kr=+0.374750	Kr=+0.374780	1000	722
	Ki=+0.243573	Ki=+0.243573		
B	Kr=-0.030000	Kr=-0.029900	1000	812
	Ki=+0.748148	Ki=+0.748148		
C	Kr=-0.092000	Kr=-0.091000	1000	887
	Ki=+0.860654	Ki=+0.860654		
D	Kr=-0.152500	Kr=-0.152445	1000	841
	Ki=+1.033791	Ki=+1.033791		
E	Kr=-0.979956	Kr=-0.979956	1000	673
	Ki=+0.249600	Ki=+0.249651		
F	Kr=-1.307190	Kr=-1.307190	1000	508
	Ki=+0.059000	Ki=+0.059300		
G	Kr=-1.476235	Kr=-1.476235	1000	667
	Ki=+0.003000	Ki=+0.004000		

Cross Recurrence Quantification of M-set

Cross recurrence analysis (KRQA) of Zbilut et al. (1998) was applied first to the real and imaginary components of the converging Z trajectories and second to the real and imaginary components of the diverging Z trajectories. KRQA parameters were set as follows: embedding dimension of 3 to 7; delay of 1; Euclidean norm; absolute rescaling of the distance matrix; radius from 0 to 5. The 8 recurrence variables were then studied as functions of increasing radius.

RESULTS

Trajectory Behavior

Phase-space loops (Z real vs. Z image) are illustrated in Figure 2 for converging and diverging trajectories for all 7 borders regions. Converging and diverging loops of region A were pentagon-shaped. Loops of regions B, C and D were triangular shaped. And loops of region E, F and G formed flat bar shapes. The loops from paired points on either side of the borders were very similar, but the diverging trajectories demonstrated more jitter.

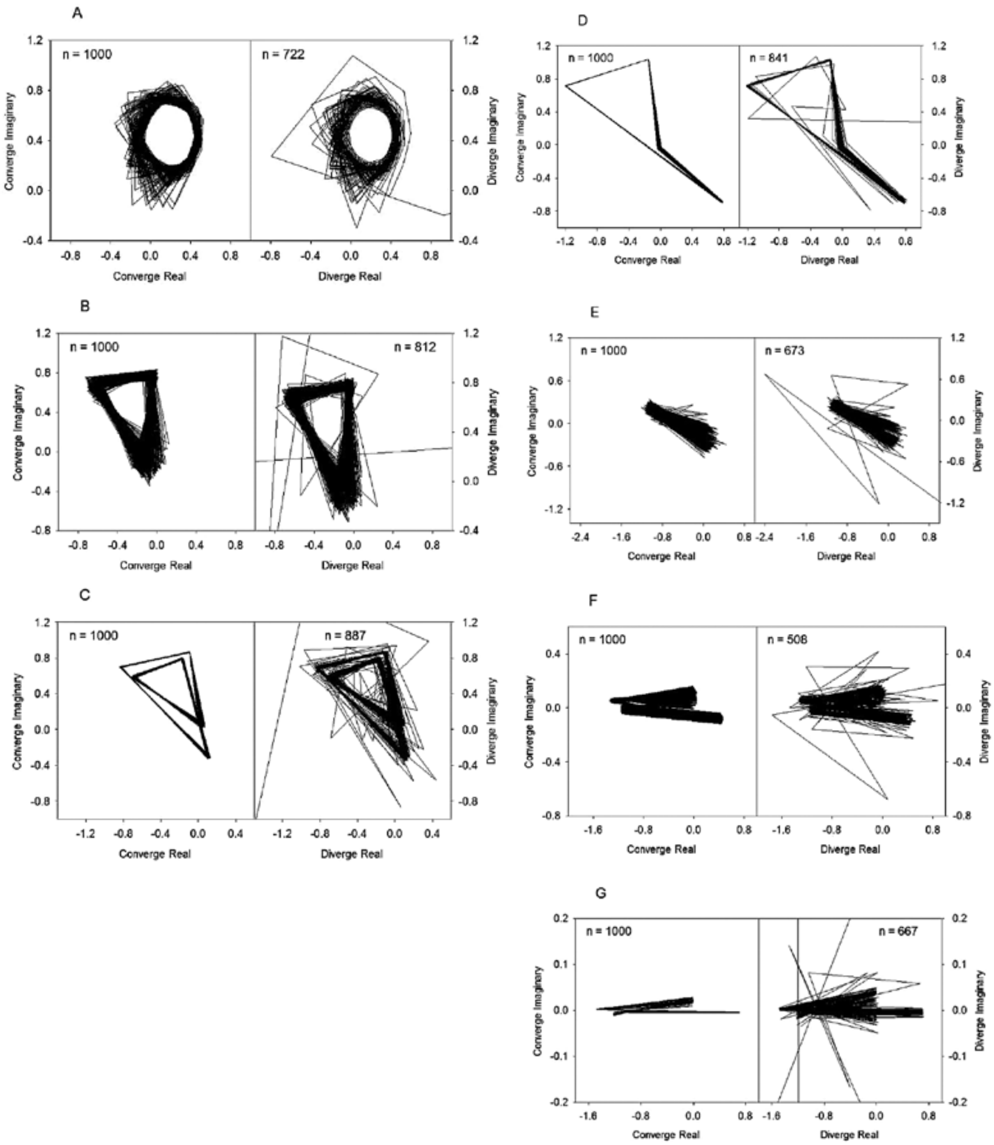

Fig. 2 Phase-space loops of Z_{real} vs. Z_{imag} for converging and diverging trajectories.

Cross-Recurrence Analysis

Cross-recurrence plots are illustrated for M-set region F in Figures 3 and 4 for C_{real} vs. C_{imag} and D_{real} vs. D_{imag} respectively. Quantitative values are also shown to the left of the plots. In Fig. 5 the behavior of the 10 RQA variables in region F are plotted as functions of radius for converging trajectories (blue) and diverging trajectories (red). Again, the individual variables are almost superimposable except for trend, trapping time and entropy. Normalized areas between the two curves (summed differences between red and blue traces) are quantified in Figure 6. Here it is shown that these three RQA variables (trend > trapping time > entropy) have the highest sensitivity to subtle differences in the two dynamics occurring just before the diverging trajectories finally fly far away from the converging trajectories.

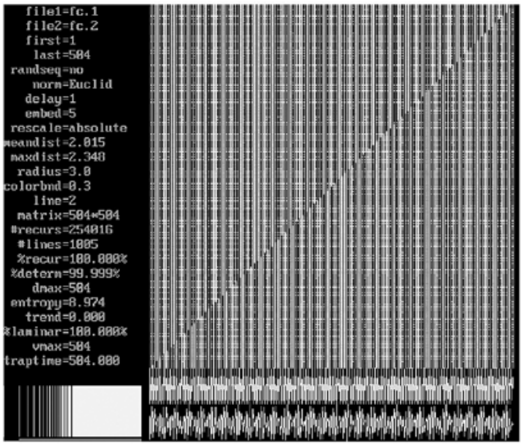

Fig. 3

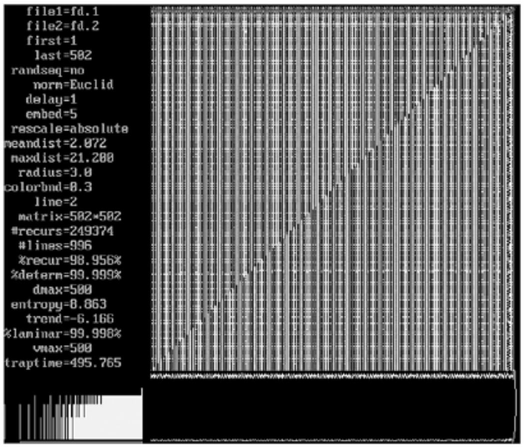

Fig. 4

Cross recurrence plots were computed for both C_{real} vs. C_{imag} (Fig. 3) and D_{real} vs. D_{imag} (Fig. 4) pairs in M-set region F.

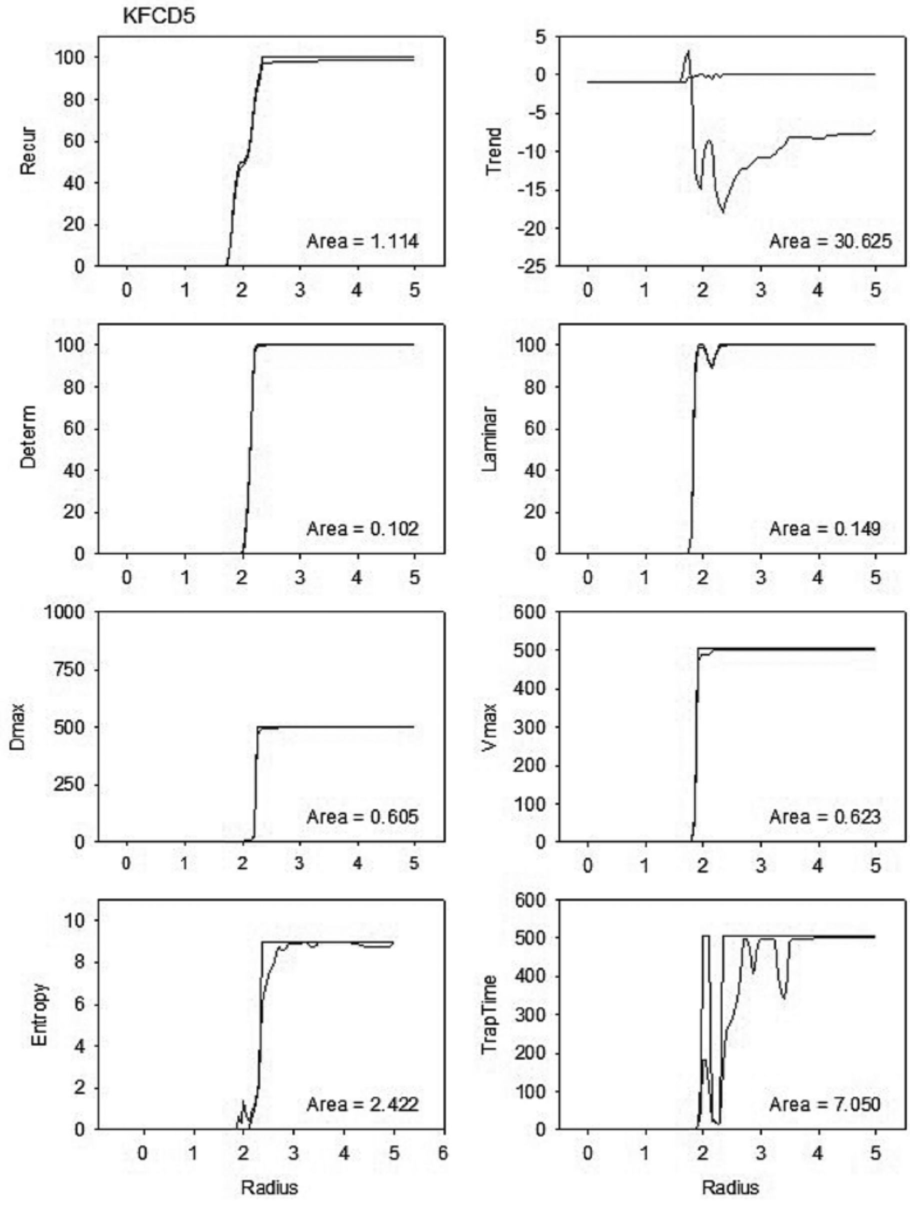

Fig. 5 KRQA variables as functions of radius for converging (blue) and diverging (red) trajectories of the M-set in region F (embed dim = 5).

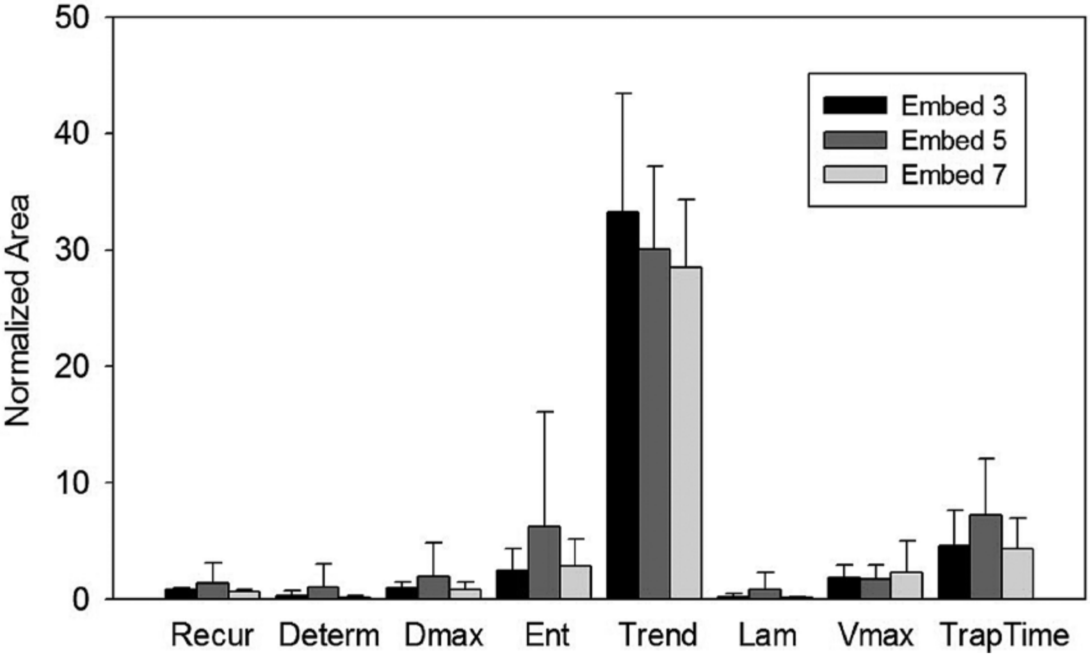

Fig. 6 Normalized areas between convergent and divergent trajectories from recurrence variable profiles in Figure 5 (red vs. blue differences).

CONCLUSIONS

Phase-space loops and cross recurrence quantification analysis were successfully applied to reveal subtle dynamical differences between converging and diverging trajectories at 7 different border regions of the Mandelbrot set. The fact that diverging trajectories were not seen to be unstable until after hundreds of iterations underscores a fundamental property of chaotic dynamics. That is, the evolution of Z dynamics as either converging or diverging is sensitively dependent upon the initial values the K constant. As the magnification scale of the M-set increases, so does the number of iterations required increase to determine whether or not the Z dynamic will converge or diverge.

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