Dynamic equivalence and flatness of control systems: some results and open questions

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$\boxed{\text{Everything is } C^\omega.}$

Control systems

$$\dot{x} = f(x,u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \; 0 < m \leq n, \qquad (*)$$

studied locally around some $(\overline{x}, \overline{u}) \in \mathbb{R}^{n+m}$,
or around some germ/jet of solution $t \mapsto (\overline{x}(t), \overline{u}(t))$.

"Geometry": f fiber preserving map $\mathbb{R}^n \times \mathbb{R}^m \to \mathrm{T}\mathbb{R}^n$.
$f(\mathrm{T}\mathbb{R}^n)$ is a
- sub-bundle of $\mathrm{T}\mathbb{R}^n$ if Rank $\frac{\partial f}{\partial u}$ constant,
- affine sub-bundle if $f(x,u) = X_0(x) + \sum_1^m u_k X_k(0)$,
- distribution if $X_0 = 0$...

Underdetermined system of ODEs: the general solution $t \mapsto (x(t), u(t))$ depends on m arbitrary functions of time $u(.)$ and n arbitrary constants $x(0)$. Important object:

"Set of solutions" for under-determined ODEs

$\mathcal{B}$ = set of all (germs of) $t \mapsto (x(t), u(t))$ solution of $(*)$.

(behavior?) **(tribute to J. Willems)**

"Static" equivalence

$$
\begin{array}{lll}
(\Sigma) & \dot{x} = f(x,u)\,, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\
(\Sigma') & \dot{z} = g(z,v)\,, & z \in \mathbb{R}^{n'},\ v \in \mathbb{R}^{m'}
\end{array}
$$

Definition (local static equivalence)

(Σ) and (Σ') are locally static equivalent at $(\overline{x},\overline{u})/(\overline{z},\overline{v})$ iff
there is a diffeo $(\varphi,\psi) : \mathcal{O}_{(\overline{x},\overline{u})} \to \mathcal{O}_{(\overline{z},\overline{v})}$,
of the form $(x,u) \mapsto (\varphi(x), \psi(x,u))$, (bundle isomorphism)
that conjugates f to g.
If control affine & constant control rank, then
- local in state only,
- $\varphi : \mathcal{O}_{\overline{x}} \to \mathcal{O}_{\overline{z}}$ such that φ_* conjugates affine sub-bundles.

Alternative definition:

$(\varphi,\psi) : \mathcal{O}_{(\overline{x},\overline{u})} \to \mathcal{O}_{(\overline{z},\overline{v})}$ maps (by composition), germs of solutions of Σ to germs of solutions of Σ', and vice versa.

Deciding static equivalence

It is feasable. Given Σ and Σ', there is in principle a finite algorithm to write PDEs for φ, ψ and decide (differential elimination) whether there is an obstruction of system formally integrable, Cauchy-Kovalevska, analytic solution.

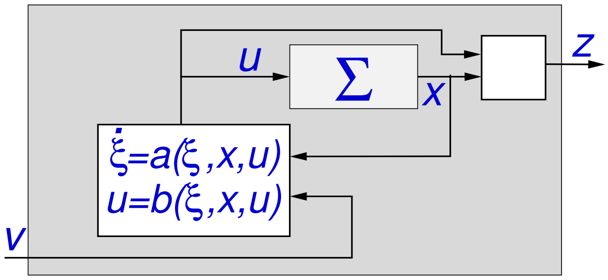
Geometric study. Invariants of distributions, affine sub-bundles.
In non-affine case, affine geometry for submanifolds of $\mathrm{T}\mathbb{R}^n$.
A lot of litterature.
Functionnal moduli, hence equivalence is a "rare" property, or classes are very thin.

Exact linearization: characterization of static equivalence to a linear controllable system [Jakubczyk-Respondek, 80].

Dynamic feedback linearization

Since Static linearization is very restrictive, try dynamic ! (1980's)

Pre-compensator such that $v \to z$ linear controllable:



[Isidori-Moog-de Luca 86]: performing dynamic decoupling, full linearization may occur.
Decide pre-compensator, check for static feedback linearizability [Charlet-Lévine-Marino, 91]
▶ Sufficient conditions.

Linear controllable systems, linearization

Linear controllable system $(\Sigma): \ \dot{x} = Ax + Bu.$

$\mathrm{Rank}\{B, AB, \ldots\} = n.$

Transformation $z = Px, \ v = Kx + Qu$
with $z = (z_{1,1}, \ldots, z_{1,r_1}, \cdots, z_{m,1}, \ldots, z_{m,r_m})$ $\qquad \sum r_k = n$
yields $z_{k,1}^{(r_k)} = v_k, \ 1 \leq k \leq m.$ Brunovsky canonical form.

The general solution is uniquely defined by m arbitrary functions of time $z_{1,1}, z_{2,1}, \ldots, z_{m,1}$ (and no initial conditions).

[Fliess-Lévine-Martin-Rouchon 91]: system $\dot{x} = f(x, u)$ is "**flat**"

(at $\overline{x}, \overline{u}, \ldots, \overline{u}^{(J)}$) iff there is a formula giving the general solution a function of arbitrary $y_1, \ldots, y_m$ and time-derivatives and $y_1, \ldots y_m$ may also be recovered from x, u and derivatives.
$(y_1, \ldots, y_m)$ is called a **flat output**.

This yields the dynamic precompensator.

Equivalence

[É. Cartan, "sur l'équivalence absolue...", 1914]:

> La première idée qui vient à l'esprit, et qu'il s'agira de préciser, est la suivante : *deux systèmes seront dits « absolument équivalents » lorsqu'on pourra établir une correspondance univoque* (au moins dans un champ fonctionnel suffisamment petit) *entre les solutions de ces deux systèmes.*

$$
\begin{array}{llll}
(\Sigma) & \dot{x} = f(x,u)\,, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, & \mathcal{B} = \{\text{solutions}\} \\
(\Sigma') & \dot{z} = g(z,v)\,, & z \in \mathbb{R}^{n'},\ v \in \mathbb{R}^{m'}, & \mathcal{C} = \{\text{solutions}\}
\end{array}
$$

Definition

Two systems are "equivalent" iff their (germs of) solutions are in one-to-one correspondence.

$$\boxed{\mathcal{B}} \xrightarrow{\Phi} \boxed{\mathcal{C}}$$

The nature of Φ matters a lot !!

Dynamic Feedback Transformations

$$
\begin{array}{ccccc}
\begin{array}{c}\Sigma \\ \dot{x} = f(x,u)\end{array} &
\begin{pmatrix} x \\ u \\ \dot{u} \\ \ddot{u} \\ \vdots \end{pmatrix} &
\begin{array}{c} (z,v) = \phi(x,u,\dots,u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x,u) = \psi(z,v,\dots,v^{(K')}) \end{array} &
\begin{pmatrix} z \\ v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{pmatrix} &
\begin{array}{c}\Sigma' \\ \dot{z} = g(z,v)\end{array}
\end{array}
$$

(Local) dynamic equivalence: $\exists\, \varphi : \mathcal{O}_{(\overline{x},\overline{u},\bar{\dot{u}},\dots,\overline{u}^{(J)})} \to \mathbb{R}^{n'} \times \mathbb{R}^{m'}$, $\psi : \mathcal{O}_{(\overline{z},\overline{v},\bar{\dot{v}},\dots,\overline{v}^{(J')})} \to \mathbb{R}^{n} \times \mathbb{R}^{m}$ such that the above induces a univocal transformation on solutions.

Flatness

Flatness: Σ is flat if this holds with Σ' trivial:

$$\begin{array}{c} \Sigma \\ \dot{x} = f(x,u) \end{array} \quad \begin{pmatrix} x \\ u \\ \dot{u} \\ \vdots \end{pmatrix} \quad \begin{array}{c} v = \phi(x,u,\ldots,u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x,u) = \psi(v,\ldots,v^{(K')}) \end{array} \quad \begin{pmatrix} v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{pmatrix} \quad \begin{array}{c} \Sigma' \\ \text{no relation} \end{array}$$

Initially, [Fliess & al] stated this in terms of differential fields extensions. Obviously more restrictive (ϕ, ψ should be algebric).

Equivalence can however be translated into
isomorphism of differential algebras or
conjugation of a vector field on a (infinite) jet manifold.

Further characterizations

Differential algebra: $\mathcal{A}_\Sigma = \{$(germs of) C^ω functions of a finite number of variables among $x, u, \dot{u}, \ldots\}$,
dérivation: $F = f(x,u)\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \cdots$

[Jakubczyk 1993]: Σ is equivalent to Σ' if and only $\mathcal{A}_\Sigma$ and $\mathcal{A}_{\Sigma'}$ are isomorphic diff. algebras.

[Fliess & al.], [JBP] 1992: it also translates into a diffeomorphism between "manifolds" where coordinates are $x, u, \dot{u}, \ldots$.. and $z, v, \dot{v}, \ldots$.., that conjugates F to G.

Similar to Lie-Bäcklund transformations.

Checkable conditions ?

How to decide flatness of a system Σ?
or equivalence of Σ to Σ' ? Invariants ?

1. In principle, if K and K' are fixed, one may decide upon existance of ϕ and ψ, that depend on a fixed numer of variables. (see static feedback)
2. ... **but no known a priori bound on** K, K' !!
3. However, many physical systems are flat and this is useful.
4. A lot of current work on all possible choices of flat outputs (they are far from unique), respecting symmetries, etc... see [Respondek, Nicolau et al], [Murray et al.], [Rouchon, Martin et al.] ...

Checkable conditions ?

Difficult point is: necessary conditions.
If no ϕ, ψ for some K, why not for $K+1$?
Invariants:

1. m (number of inputs) is an invariant of dynamic equivalence.
2. If $m=1$, dynamic equivalence is static equivalence. [Charlet et al., 1991], [JBP 1993].
3. Singular curves are an obstruction to flatness: if $t \mapsto (x(t), u(t)))$ is singular, flatness fails at any truncated jet $(x(0), u(0), \dot{u}(0), \dots, u^{(J)}(0))$ [common knowledge ?]
4. Ruled manifold criterium

Ruled systems

Regularity assumption

Σ and Σ' define smooth (C^ω) sub-bundles of $T\mathbb{R}^n$ and $T\mathbb{R}^{n'}$.
E.g. $\text{Rank}\dfrac{\partial f}{\partial u} = m$, $\text{Rank}\dfrac{\partial g}{\partial v} = m'$

- $\Sigma \to \mathbb{R}^n$ sub-bundle of $T\mathbb{R}^n \to \mathbb{R}^n$.
  $\Sigma_x = \{f(x,u), u \in \mathbb{R}^m\}$ sub-manifold of the linear space $T_x\mathbb{R}^n$.
- A submanifold of an affine space is called (locally) **ruled** if it is a union of straight lines (locally of open segments).

Definition (**ruled system**)

Σ is ruled iff each Σ_x is a ruled submanifold of $T_x\mathbb{R}^n$ $(\approx \mathbb{R}^n)$.

Note: This property is preserved by static equivalence.

Necessary condition for flatness

$$(\Sigma)\ \ \dot{x} = f(x,u)\,,\quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m\,, \qquad (\Sigma')\ \text{“trivial”}.$$

Theorem ([Rouchon], [Sluis], 1992)

A flat system must be ruled

What if

$$\begin{array}{lll} (\Sigma) & \dot{x} = f(x,u)\,, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ (\Sigma') & \dot{z} = g(z,v)\,, & z \in \mathbb{R}^{n'},\ v \in \mathbb{R}^m \end{array} \quad \text{dynamic equivalent ?}$$

Necessary condition for equivalence

$$\begin{array}{lll}(\Sigma) & \dot{x} = f(x,u)\,, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ (\Sigma') & \dot{z} = g(z,v)\,, & z \in \mathbb{R}^{n'},\ v \in \mathbb{R}^{m'}\end{array}$$

Theorem (JBP, 2009)

If Σ and Σ' are dynamic equivalent and $n = n'$, then, locally,

1. **either** *they are* **static equivalent**,
2. **or** *they are* **both ruled**.

If $n < n'$, Σ' must be ruled.

In the C^∞ case, 1 and 2 may both occur for the same Σ,Σ'.

Note: the condition for flatness is a consequence
(trivial system is ruled).

Flatness with 2 controls and 3 states

$$\dot{z} = f(x,y,z,\dot{x},\dot{y})$$

[Avanessof,JBP 2007].
We look for a parameterization x,y,z function of $(u,\dot{u},\ldots,u^{(k)},v,\ldots,u^{(\ell)})$, $k \leq \ell$, such that all solutions are covered. one has

- Either no parameterization because of ruled criterion of non-controllability,
- or parameterization with $(k,\ell) = (1,2)$,
- or there could be a parameterization, but k and ℓ have to be no smaller than 3 and the largest one no smaller than 4.

Examples

Example 1: $\dot{z} = y + (\dot{y} - z\dot{x})\dot{x}$. Parameterization of order $(1, 2)$ around jets such that $\ddot{x} + \dot{x}^3 \neq 1$, given by :

$$x = v\,, \quad y = \frac{\dot{v}^2 u + \dot{u}}{\ddot{v} + \dot{v}^3 - 1}\,, \quad z = \frac{(1 - \ddot{v})u + \dot{v}\dot{u}}{\ddot{v} + \dot{v}^3 - 1}\,. \qquad (*)$$

Formulas can be "inverted" by $u = -z + y\dot{x}$, $v = x$, hence flatness.
Note on singularity: $\ddot{x} + \dot{x}^3 = 1$ is the eqution of the singular curves. There is a parameterization of higher order at jets of order 3 such that $x^{(3)} + 3\dot{x}^2\ddot{x} \neq 0$.

Example 2: $\dot{z} = y + (\dot{y} - z\dot{x})^2\dot{x}$: if there is a parameterization, it has order at least $(3, 4)$. Conjecture: no parameterization.

Open questions

1. Given Σ, give an a priori bound on the number K of derivatives.
2. Prove that at least one system for which above mentionned necessary conditions do not work is not flat, or does not admit a parameterization.
3. Does existance of a parameter imply flatness ? (exogenous implies exogenous, according to [Fliess et al.])

Thank you all for attention, and
thank you for your action in the community, Bronek !