Galois theory in monoidal categories

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Abstract

The Galois theory of Chase and Sweedler [11], for commutative rings, is generalized to encompass commutative monoids in an arbitrary symmetric, closed, monoidal category with finite limits and colimits. The primary tool is the Morita theory of Pareigis [35, 36, 37], which also supplies a suitable definition for the concept of a "finite" object in a monoidal category. The Galois theory is then extended by an examination of "normal" sub-Hopf-monoids, and examples in various algebraic and topological categories are considered. In particular, symmetric, closed, monoidal structures on various categories of topological vector spaces are studied with respect to the existence of "finite" objects.

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Introduction

Essentially, the “fundamental theorem of Galois theory” for finite field extensions refers to the following proposition: The construction of fixed fields generates a bijective mapping between the subgroups of the Galois group and intermediate fields of the field extension. More generally, in 1965, Chase, Harrison, and Rosenberg [10] developed a Galois theory of separable, commutative ring extensions with corresponding Galois group of automorphisms. Following Jacobson [26], instead of a separable field extension, it is possible to use a purely inseparable extension of exponent 1, if instead of the Galois group (of automorphisms), a corresponding p-Lie algebra (of derivations) is used. In [9], Chase describes how higher derivations can be used to include purely inseparable field extensions with a higher exponent. All of these cases can be treated in a uniform way by using the formulation of Hopf algebras. Here, one considers the group algebra generated by the Galois group, the p-universal hull of the p-Lie algebra of derivations, or an algebra generated by higher derivations, along with the canonical Hopf-algebra structure. This general form of Galois theory for commutative ring extensions was treated in 1969 by Chase and Sweedler [11]. The most important resource in this work is the Morita theory, which characterizes all equivalences of module categories. Galois theory was also generalized to skew fields and infinite field extensions, but uses other techniques, which will not be considered here.

All previous Galois theories have one limitation in common, namely a condition of finiteness. For example, in Chase, Harrison, and Rosenberg [10], the Galois group must be finite, and the ring extension must be finitely generated and projective. In the work of Chase and Sweedler [11], we can see that this finiteness condition is a necessary prerequisite for the Morita theorem. The theory of Krull [30] is not free of this restriction either, because in that case it is a matter of injective resp. projective limits of finite objects.

In this work, we want to determine to what extent the fundamental theorem of Galois theory depends on the special properties of the underlying module category, and if the concept of finiteness can be replaced by a more general category-theoretical concept. Among other things, the question of whether Galois theory can be generalized to infinite-dimensional topological algebras and infinite topological groups arises. As an underlying category, we could consider the category of topological vector spaces. It is not abelian: not every monomorphism (= continuous injection) is a kernel (= relatively open, continuous injection). Since the concept “finitely generated and projective” is very important for the above-mentioned Galois theory, we also ask how it might be generalized. In this case, the concept “projective” can be formulated for arbitrary categories, but that is not fruitful in this topological situation, since theorem (10.9) shows, among other things, that $\{0\}$ is the only cokernel-projective object in the category of quasi-complete, barreled spaces. (The concept “cokernel projective” is weaker than “projective”.) As a result, in this work, we want to develop Galois theory without the multiple resources from the theory of modules or abelian categories.

One possible generalization, shared by the theory of k-modules and the theory of topological vector spaces, which does not take the “additive” or “abelian” structure into account, is the theory of monoidal categories. A monoidal category $(\mathcal{C},\otimes,I)$ is a category $\mathcal{C}$ together with a “tensor product”, i.e. a bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a distinguished object I such that some axioms are fulfilled, such as $(A\otimes B)\otimes C\cong A\otimes(B\otimes C)$ and $A\otimes I\cong A$, cf. e.g. MacLane [31]. There are many examples for this, such as $(\boldsymbol{S},\times,e)$, $(k\text{-}\boldsymbol{Mod},\otimes_k,k)$ or $\left(\boldsymbol{Ban},\widehat{\otimes}_\pi,\mathbb{K}\right)$. A monoid in such a category is then nothing else than a classical monoid, a k-algebra, or a Banach algebra. In a monoidal category, all fundamental concepts of Galois theory can be expressed in the Hopf-algebra formulation of Chase and Sweedler [11]. In addition, the theory of monoidal categories has recently been extended by a Morita theory, where in particular the concept of “finitely generated and projective” has a generalization, cf. Pareigis [35, 36, 37]. A k-module is then finitely generated and projective if and only if the canonical morphism $A\otimes_k A^*\to \hom_k(A,A)$ is an isomorphism. In general, we call an object of a monoidal category “finite” if the canonical morphism $A \otimes A^* \to [A, A]$ is an isomorphism, where $[-,-]$ is the "inner hom-functor" and A^* is the corresponding "dual object", cf. Definition (1.4). If the canonical morphism $A^* \otimes_{[A,A]} A \to I$ is also an isomorphism, A is called "faithfully projective". This property is necessary and sufficient for the functor $A \otimes -: \mathcal{C} \to {}_{[A,A]}\mathcal{C}$ to be an equivalence of categories, according to Morita theory, cf. Theorem (1.6).

With help of the Morita theory of Pareigis (cited in Chapter 1), it is possible to generalize the work of Chase and Sweedler [11] to arbitrary symmetrical, closed monoidal categories (Chapters 2 and 4). For this, it is necessary to replace many of the proofs in [11] by new proofs (we refer to that individually in the text), since concepts such as "short, exact sequence", "0-object", "annihilator ideal" etc. do not make sense in an arbitrary monoidal category. However, we can develop a large part of the Galois theory under significantly weaker prerequisites than what was previously required. In particular, this shows that, in large parts, Galois theory is completely independent of any "additive structure" (abelian category) of the underlying category of k-modules.

If S is a commutative monoid in a monoidal category $\mathcal{C}$, and A is a commutative Hopf monoid, whose dual A^* operates in a suitable way on S, then we call S A-Galois over I, if S is faithfully projective over I and the morphism $\gamma: S \otimes S \to S \otimes A$ is an isomorphism, cf. Definition (2.2). Example (2.3) explains this condition for the case that S is a field extension of I. In Theorem (2.11), we show that this condition is equivalent to $\varphi: S\#A^* \to [S, S]$ being an isomorphism. In Theorem (2.17), a third characterization of the concept "Galois" is given, this time by means of a fixed object of A^*-operations and the canonical morphisms of a Morita context.

In Chapter 3, a weakened Frobenius property for finite Hopf-monoids H is proven, namely that an object P exists with $H^* \cong P \otimes H$ in $\mathcal{C}_H$. In contrast with the case of modules, we only know here that P is finite, and not that it has "rank 1".

Chapter 4 is concerned with submonoids of a Galois monoid S. This is where the "fundamental theorem of Galois theory", Theorem (4.11), is proven. It provides an order-reversing injection of the lattice of sub-Hopf-monoids of A^* (that can be decomposed in $\mathcal{C}$) in the lattice of submonoids of S (that can be decomposed in $\mathcal{C}$).

This generalization of Galois theory can also be construed as an axiomatization, in the sense that shaped the mathematics of the twentieth century. For example, the concept of group was discovered on the basis of the transformations of certain sets in geometry, but it was the axiomatic formulation as a set with operations that made it possible to study group structures in general and apply them universally. Another example is the theory of abelian categories. This theory, which investigates the "additive" structure of module categories, is a suitable basis for homological algebra and found its justification in the discovery that sheaf categories are often abelian, even when they are not isomorphic to any module category. In a similar way, the theory of monoidal categories can be seen as a study of the "multiplicative" structure possessed by module categories, among others. With that, this work is a development of Galois theory on the basis of a "multiplicative structure", without help of an "additive structure".

The general Galois theory in monoidal categories is then pursued further in Chapter 5, where we investigate "normal" sub-Hopf-monoids. Here we also find some propositions that were not yet proven in the special case of module theory. Let H' be a sub-Hopf-algebra of a finite, cocommutative Hopf-algebra H and let $H'^+ = \mathrm{Ke}(\varepsilon_{H'})$ (i.e. its augmentation ideal). In Theorem (5.3) (cf. Remark (5.4)), we show that H' is normal in H in the sense of Newman [33] (i.e. $HH'^+ = H'^+H$), if and only if, for all $a \in H, b \in H'$ it follows that $a_1 b \lambda(a_2) \in H'$. In the case of a group ring (resp. of a hull of a Lie-algebra), the last condition is reduced to the definition of a normal subgroup (resp. of a Lie-ideal), cf. Lemma (7.6). In Theorem (5.5), we then show that the fixed-monoid of a normal sub-Hopf-monoid is Galois.

After Galois theory has been generalized to monoidal categories, we investigate it in diverse special cases. The constructions of Chapter 6 can be thought of as modules with additional structures; then, Galois theory is a matter of Galois objects that share this additional structure. In one case, that leads to normal objects, cf. Lemma (6.9) and Corollary (6.10). Chapter 7 investigates the specialization to the classical case of Galois theory of rings and fields. The examples in these chapters demonstrate the scope of the general theory of "purely algebraic" cases.

The other fundamental specialization that we investigate is the specialization to the "topological case", i.e. to the situation where the underlying category $\mathcal{C}$ is a suitable category of topological vector spaces. In Chapters 8 through 11, categories of topological vector spaces are investigated to see how many interesting examples of "finite" objects in the sense of Definition (1.4) exist (i.e. the canonical morphism $E \otimes E^* \to [E, E]$ is an isomorphism). The conclusions are based on well-known properties of the respective spaces but are not found in the literature in the formulation required by our problem. For that, we first need to know which of these categories is symmetric and closed monoidal. In that case, we only want to consider a "finite" object in such a category as interesting if it is infinite-dimensional. The claim that this investigation is meaningful is justified by the following facts:
1) The concept "finite" is a strengthening of the concept "reflexive" (cf. Lemma (3.6)). It is well-known that many reflexive, infinite-dimensional topological vector spaces exist.
2) The category of Banach spaces is known to be symmetric and closed monoidal with the complete, projective tensor product. However, here, the "finite" objects are exactly all finite-dimensional Banach spaces (cf. Example (8.9) and Theorem (8.10).)
3) For almost all of the important nuclear spaces E, in particular all Fréchet-nuclear spaces, the following isomorphism holds: $E \mathbin{\widehat{\otimes}} E'_\beta \to \mathcal{L}_\beta(E, E)$, cf. Köthe [29].

In order to search for "finite" objects, following these clues, we need to describe as many topological tensor products of not-necessarily normed topological vector spaces as possible, cf. Chapter [8]. This is most practical with the concept of hypo-continuity. With that, it is possible to directly construct both many topologies on the algebraic tensor product as well as the corresponding "closure functors" $[E, -]$. In particular, we don't need to resort to the Freyd theorem about the existence of adjoint functors, as was done by U. Seip [47] for the category of compactly determined spaces.

For this investigation, a compatibility theorem of S. Dierolf [15] proves to be very useful. In Chapter 9, we demonstrate a generalized version of this theorem, and some new applications. Let L be an epireflector and R a monocoreflector; then it holds that $LRL = RL \Leftrightarrow RLR = LR.$

In Chapter 10, we consider one of the most important topological vector spaces in analysis, the quasi-complete, barreled spaces. We demonstrate that it is symmetric and closed monoidal, a property that is not treated in the literature for this category. However, it is exactly this property that led to the investigation of the compactly determined (resp. sequence-complete, compactly determined) spaces, cf. Dubuc and Porta [18] and [19]. Following Seip [47], this is the property that makes it possible to develop differential calculus in non-normed spaces.

Chapter 11 treats other categories that are constructed analogously, in particular the above-mentioned compactly-determined spaces. Finally, in Chapters 10 and 11, examples of concrete topological vector spaces are discussed, in order to get information about the property "finite". Here we discover that the most important and known infinite-dimensional spaces are not "finite" in our sense of the word. As a result, this investigation sheds a lot of light on a question that is important for the theory of monoidal categories but doesn't result in any new propositions in the theory of topological algebras.

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1. Notation and Morita theorems in monoidal categories

In this chapter, the general notation and concepts for the following parts are established. Then, a few results from Morita theory that are fundamentally used later are cited.

(1.1.) Notation. *$(\mathcal{C}, \otimes, I)$ refers to a monoidal category $\mathcal{C}$, with bifunctor $\otimes$ and neutral object I, as in MacLane [31] and Pareigis [35]. We also assume that $\mathcal{C}$ is **symmetric**, with functorial isomorphism $\tau: A \otimes B \to B \otimes A$. $\mathcal{C}$ is also assumed to be **closed**, and $[M, -]$ refers to the functor which is right-adjoint to $- \otimes M$, where M* is an abbreviation for $[M, I]$. In any case, we also assume that $\mathcal{C}$ has difference kernels and cokernels, in order to enable the required constructions. A **monoid** A, with multiplication ∇ and unit η is defined exactly as in $(\boldsymbol{S}, \times, \{e\})$ or a unitary algebra in $(k\text{-}\boldsymbol{Mod}, \otimes_k, k)$. The unit $\eta_A \in \mathcal{C}(I, A)$ will also be denoted as 1_A. Dual to (A, ∇, η), one also defines a **comonoid** (C, Δ, ε). **Hopf monoids**, with antipode λ, are defined as Hopf algebras in $(k\text{-}\boldsymbol{Mod}, \otimes_k, k)$. $\boldsymbol{Mon}\mathcal{C}$ (resp. $\boldsymbol{cMon}\mathcal{C}$, $\boldsymbol{coMon}\mathcal{C}$, $\boldsymbol{HopfMon}\mathcal{C}$, $\boldsymbol{cHopfMon}\mathcal{C}$) denotes the category of monoids (resp. commutative monoids, comonoids, Hopf monoids, commutative (not necessarily cocommutative) Hopf monoids) in $\mathcal{C}$. For $A \in \boldsymbol{MonC}$, an **A-left-object** is defined as is an A-set in $(\boldsymbol{S}, \times, \{e\})$ or an A-left-module in $(k\text{-}\boldsymbol{Mod}, \otimes_k, k)$. Dual to that, we define, for $C \in \boldsymbol{coMon}\mathcal{C}$, a C-left-coobject. The category of A-left-objects (resp. A-right-objects, C-left-coobjects, C-right-coobjects) is denoted as ${}_A\mathcal{C}$ (resp. $\mathcal{C}_A$, ${}^C\mathcal{C}$, $\mathcal{C}^C$). We also use the elementwise notation of Pareigis [35], with $A(X) := \mathcal{C}(X, A)$ for $A, X \in \mathcal{C}$. A morphism $h \in \mathcal{C}(A, B)$ is called **rationally surjective** if $h(I): A(I) \to B(I)$ is surjective.*

(1.2) Definition. *A **Morita context** (A, B, P, Q, f, g) consist of:*
(a) $A, B \in \boldsymbol{Mon}\mathcal{C}$,
(b) $P \in {}_A\mathcal{C}_B$, $Q \in {}_B\mathcal{C}_A$,
(c) $f \in {}_A\mathcal{C}_A(P \otimes_B Q, A)$, $g \in {}_B\mathcal{C}_B(Q \otimes_A P, B)$, and
(d) two commutative diagrams:

$$\begin{tikzcd}
P \otimes_B Q \otimes_A P \arrow[r, "f \otimes_A \mathrm{id}_P"] \arrow[d, "\mathrm{id}_P \otimes_B g"] & A \otimes_A P \arrow[d, "\cong"] \\
P \otimes_B B \arrow[r, "\cong"] & P
\end{tikzcd}$$

$$\begin{tikzcd}
Q \otimes_A P \otimes_B Q \arrow[r, "g \otimes_{\boldsymbol{B}} \mathrm{id}_Q"] \arrow[d, "\mathrm{id}_Q \otimes_A f"] & B \otimes_B Q \arrow[d, "\cong"] \\
Q \otimes_A A \arrow[r, "\cong"] & Q
\end{tikzcd}$$

(1.3) Definition. *Let $B \in \boldsymbol{Mon}\mathcal{C}, P \in C_B$.*
*$\left({}_B[P,P], B, P, {}_B[P,B], \tilde{f}, \tilde{g}\right)$ is called a **canonical Morita context**, with $\tilde{f}$ and $\tilde{g}$ defined by:*
${}_B\mathcal{C}_{{}_B[P,P]}({}_B[P,B], {}_B[P,B]) \cong {}_B\mathcal{C}_B\left({}_B[P,B] \otimes_{{}_B[P,P]} P, B\right): \mathrm{id}_{{}_B[P,B]} \longmapsto \tilde{g}$
${}_{{}_B[P,P]}\mathcal{C}_B\left(P \otimes_B {}_B[P,B] \otimes_{{}_B[P,P]} P, P\right) \to {}_{{}_B[P,P]}\mathcal{C}_{{}_B[P,P]}(P \otimes_B {}_B[P,B], {}_B[P,P]): \mathrm{id}_P \otimes_B \tilde{g} \mapsto \tilde{f}$,
i.e. $\tilde{g}(q \otimes p) = q\langle p \rangle$ *and* $\tilde{f}(p' \otimes q')\langle p'' \rangle = p'(q'\langle p'' \rangle)$ *for all* $X, Y, Z \in \mathcal{C}$,
$q \otimes p \in \left({}_B[P,B] \otimes_{{}_B[P,P]}, P\right)(X), p' \otimes q' \in (P \otimes_B {}_B[P,B])(Y), p'' \in P(Z)$.

This notation, along with the next definition, contains a change of sides compared with that of Pareigis [35]. We need to consider $[M, -]$ as right adjoint to $- \otimes M$, and not to $M \otimes -$, which makes it easer to adapt to the customary notation of Galois theory. Now we have a canonical morphism can: $[M, N] \otimes M \to N: f \otimes m \mapsto f\langle m \rangle$ and write both morphisms and elements of

$[M, a](X)$ to the left of the arguments.
$\mathcal{C}([M, N], [M, N]) \cong \mathcal{C}([M, N] \otimes M, N)$: id $\longmapsto$ can.

(1.4) Definition. *Let $B \in \boldsymbol{Mon}\mathcal{C}, P \in \mathcal{C}_B$. P is called*
(a) ***finite*** *over B, if $\tilde{f}$ from (1.3) is an isomorphism,*
(b) ***finitely generated projective*** *over B, if*
$P \otimes {}_B[P, B] \xrightarrow{\text{can}} P \otimes_B {}_B[P, B] \xrightarrow{\tilde{f}} {}_B[P, P]$ *is rationally surjective,*
(c) ***faithfully projective*** *over B, if P is finite and $\tilde{g}$ from (1.3) is an isomorphism,*
(d) a ***progenerator*** *over B, if P is finitely generated projective over B, and*
${}_B[P, B] \otimes P \xrightarrow{\text{can}} {}_B[P, B] \otimes_{{}_B[P,P]} P \xrightarrow{\tilde{g}} B$ *is rationally surjective.*

(1.5) Lemma. *If I is cokernel projective in $\mathcal{C}$, then the concepts "finite over B" and "finitely generated projective over B", as well as "faithfully projective over B" and "progenerator over B" are equivalent to each other.*

Proof. According to Pareigis [37] 5.3, $\tilde{f}$ is an isomorphism if and only if it is rationally surjective. Since can is a cokernel, $\text{can}(I): (P \otimes {}_B[P, B])(I) \longrightarrow (P \otimes_B {}_B[P, B])(I)$ is surjective if I is cokernel projective.

The following theorem was proven in Pareigis [37] 5.1, 5.2, 5.3, and 5.4.

(1.6) Theorem *(Morita-Pareigis) Let $(\mathcal{C}, \otimes, I)$ be a closed monoidal category and (A, B, P, Q, f, g) a Morita context in $\mathcal{C}$. Then the following hold:*
(a) f and g are rationally surjective if and only if P is faithfully projective over A and Q is faithfully projective over B.
(b) If f (resp. g) is rationally surjective, then f (resp. g) is an isomorphism.
(c) If f and g are rationally surjective, then the following hold:
(i) we have canonical isomorphisms:
$P \xrightarrow{\cong} {}_B[Q, B]$ *in* ${}_A\mathcal{C}_B$,
$Q \xrightarrow{\cong} {}_A[P, A]$ *in* ${}_B\mathcal{C}_A$,
$A \xrightarrow{\cong} {}_B[Q, Q]$ *in* ${}_A\mathcal{C}_A$ *and in* $\boldsymbol{Mon}\mathcal{C}$,
$B \xrightarrow{\cong} {}_A[P, P]$ *in* ${}_B\mathcal{C}_B$ *and in* $\boldsymbol{Mon}\mathcal{C}$, *and*
(ii) $P \otimes -: {}_B\mathcal{C} \to {}_A\mathcal{C}$ and $Q \otimes -: {}_A\mathcal{C} \to {}_B\mathcal{C}$ are isomorphisms that are inverse to each other.

(1.7) Remark. Since we assume that $\mathcal{C}$ is closed, $P \otimes -$ preserves cokernels, and tensoring over a monoid is associative, as was used in definition (1.2). When $\mathcal{C}$ is not closed, the situation is much more complicated, cf. Pareigis [37].

2. The concept "Galois"

For a monoid S, on which a Hopf-monoid A^* operates in a suitable fashion, we define the concept "S is A-Galois over I" in such a way that it coincides with the classical concept. A different characterization is given in Theorem (2.11), and Theorem (2.17) describes the concept in terms of a Morita context.

The following definition coincides with the definition of an "A^*-module algebra" in Sweedler [50], page 153, and with an "A-object" in Chase and Sweedler [11], page 55.

(2.1) Definition. *Let $S \in \boldsymbol{Mon}\mathcal{C}, A \in \boldsymbol{Hopfmon}\mathcal{C}, \alpha \in \mathcal{C}(S, S \otimes A)$ and $\beta \in \mathcal{C}(A^* \otimes S, S)$.*

*(a) S is an A^*-**object monoid** if*
  - *(i) $(S, \beta) \in {}_{A^*}\mathcal{C}$, cf. (1.1) and*
  - *(ii) $\beta(x \otimes ss') = \beta(x_1 \otimes s)\beta(x_2 \otimes s')$ and $\beta(x \otimes 1) = \varepsilon(x)1_S$ for all $X, Y, Z \in \mathcal{C}, x \in A^*(X), s \in S(Y), s' \in S(Z)$.*

*(b) S in an A-**coobject monoid** if*
  - *(i) $(S, \alpha) \in \mathcal{C}^A$, cf. (1.1) and*
  - *(ii) $\alpha \in \boldsymbol{Mon}\mathcal{C}(S, S \otimes A)$.*

As in Chase and Sweedler [11] page 138, we can see that definitions (a) and (b) are equivalent to each other if A is finite and α and β correspond to each other via the following isomorphism:
$\mathcal{C}(S, S \otimes A) \cong \mathcal{C}(S, [A^*, S]) \cong \mathcal{C}(A^* \otimes S, S): \alpha \longmapsto \beta$

(2.2) Definition. *Let $S \in \boldsymbol{Mon}\mathcal{C}, A \in \boldsymbol{Hopfmon}\mathcal{C}$ and (S, α) an A-coobject monoid.*

(a) Let $\gamma := (\nabla_S \otimes \mathrm{id}_A)(\mathrm{id}_S \otimes \alpha) \in \boldsymbol{Mon}\mathcal{C}(S \otimes S, S \otimes A)$, i.e. $\gamma(x \otimes y) = xy_1 \otimes y_2$ for all $X, Y \in \mathcal{C}, x \in S(X), y \in S(Y)$, where we write $\alpha(y) = y_1 \otimes y_2$.

*(b) S is called **A-Galois** over $\boldsymbol{I}$ if*
  - *(i) S is faithfully projective over I, and*
  - *(ii) γ is an isomorphism in $\mathcal{C}$.*

(2.3) Example. Let K be a field, $\mathcal{C} = K\text{-}\boldsymbol{Mod}$, and S a finite-dimensional field extension of K.

(a) Let G be a finite group and $A^* := KG$ the group algebra. Then S is an A^*-***object-monoid*** if and only if G operates on S via automorphisms, see Sweedler [50], page 139. Since S is a finite-dimensional K-vector-space, S is a progenerator (faithfully projective) in K-$\boldsymbol{Mod}$. S is A-Galois over K if and only if it is a separable Galois field extension with Galois group G in the classical sense, see Sweedler [50], Theorem (10.2.1) and Bourbaki [2], (V.10).

(b) Let $\mathrm{char}K = p$ and let L be a finite-dimensional p-Lie-algebra over K. Let $A^* = U^{[p]}(L)$ denote the p-universal hull of L. Then S is an A^*-***object-monoid*** if and only if L operates on S via derivations, see Sweedler [50], page 139. S is Galois over K if and only if it is a purely inseparable Galois field extension of K with derivation algebra L in the sense of Jacobson [26], page 186, see Sweedler [50], (10.2.1).

(c) Now let K be simply a commutative ring, S a commutative ring extension of K and G a finite group. Analogous to (a), S is A-Galois over K if and only if it fulfills the definition of Chase, Harrison and Rosenberg [10].

The following lemma, which we need for the next theorem, corresponds to the statement that a module which allows an epimorphism to the base rung is a generator.

(2.4) Lemma. *Let $A \in \boldsymbol{Mon}\mathcal{C}, P \in {}_A\mathcal{C}$ and let $h \in {}_A\mathcal{C}(P, A)$ be rationally surjective. Then $\tilde{g} \in {}_A\mathcal{C}_A({}_A[P, A] \otimes P, A)$ from (1.3) is rationally surjective.*

Proof. Let h' and $\bar{h}$ be defined by
${}_A\mathcal{C}(P, A) \cong {}_A\mathcal{C}(P \otimes I, A) \cong \mathcal{C}(I, {}_A[P, A]) = {}_A[P, A](I): h \longmapsto h'$
$\bar{h}: P(I) \to A(I): \bar{h}(p) = hp$.
By assumption, $\bar{h}$ is surjective, so there exists a $p \in P(I)$ with $hp = \bar{h}(p) = 1_A$. Then we have

$h' \otimes p \in ({}_A[P,A] \otimes P)(I)$ and we have $\tilde{g}(h' \otimes p) = h'\langle p \rangle = hp = 1_A$, and so $\tilde{g}$ is rationally surjective.

(2.5) Theorem. *Let $A \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I and $\alpha \coloneqq \Delta$. Then A is A-Galois over I.*

Proof. From the definition of a Hopf monoid, is follows immediately that A is an A-coobject-monoid. Since the counit $\varepsilon \in \mathcal{C}(A,I)$ is a retraction of the unit $\eta \in \mathcal{C}(I,A)$, it is rationally surjective. Then, from Lemma (2.4), it follows that A is faithfully projective over I. If λ denotes the antipode of A, then
$\vartheta \coloneqq (\nabla \otimes \mathrm{id})(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) \in \mathcal{C}(A \otimes A, A \otimes A)$
is a double-sided inverse to γ, as can be seen here:
$\vartheta\gamma = (\nabla \otimes \mathrm{id})(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(\nabla \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$
$= (\nabla \otimes \mathrm{id})(\nabla \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \lambda \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\mathrm{id} \otimes \Delta)$
$= (\nabla \otimes \mathrm{id})(\mathrm{id} \otimes \nabla \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \lambda \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$
$= (\nabla \otimes \mathrm{id})(\mathrm{id} \otimes \eta\varepsilon \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$
$= (\mathrm{id} \otimes \mathrm{id})$.
$\gamma\vartheta = (\mathrm{id} \otimes \mathrm{id})$ is analogous.

The following definition of a fixed object will be used very frequently in the remaining text.

(2.6) Definition. *Let $A \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I and $(M,\alpha) \in \mathcal{C}^A$. The fixed object M^{A^*} is the difference kernel of the following pair:*

$$M^{A^*} \to M \underset{\mathrm{id}\otimes\eta}{\overset{\alpha}{\rightrightarrows}} M \otimes A.$$

(2.7) Example. Let K be a commutative ring, $\mathcal{C} = K\text{-}\boldsymbol{Mod}$, G a finite group, $A^* = KG$ and β as in (2.1). Then:
$M^{A^*} = \{x \in M \mid \alpha(x) = x \otimes 1_A\}$
$= \{x \in M \mid \beta(y \otimes x) = \varepsilon(y)x \text{ for all } y \in A^*\}$
$= \{x \in M \mid \beta(g \otimes x) = x \text{ for all } g \in G\}$.

(2.8) Theorem. *Let $S \in \boldsymbol{Mon}\mathcal{C}$, $A \in \boldsymbol{Hopfmon}\mathcal{C}$, and S A-Galois over I. Then $S^{A^*} = I$.*

Proof. (a) Claim:

$$I \xrightarrow{\eta} S \underset{\mathrm{id}\otimes\eta}{\overset{\eta\otimes\mathrm{id}}{\rightrightarrows}} S \otimes S$$

is a difference-kernel diagram. After application of $S \otimes -$ we get:

$$S \xrightarrow{\mathrm{id}\otimes\eta} S \otimes S \underset{\mathrm{id}\otimes\mathrm{id}\otimes\eta}{\overset{\mathrm{id}\otimes\eta\otimes\mathrm{id}}{\rightrightarrows}} S \otimes S \otimes S.$$

This is a difference-kernel diagram in ${}_{[S,S]}\mathcal{C}$ because for all $s_1 \otimes s_2 \in (S \otimes S)(X)$ with
$s_1 \otimes 1 \otimes s_2 = (\mathrm{id} \otimes \eta \otimes \mathrm{id})(s_1 \otimes s_2) = (\mathrm{id} \otimes \mathrm{id} \otimes \eta)(s_1 \otimes s_2) = s_1 \otimes s_2 \otimes 1$ holds, and after application of $\nabla \otimes \mathrm{id}$:
$s_1 \otimes s_2 = s_1 s_2 \otimes 1 \in (1 \otimes \eta)S(X)$.
Since S is faithfully projective over I, $S \otimes -: \mathcal{C} \to {}_{[S,S]}\mathcal{C}$ is a category equivalence according to (1.6), so it reflects difference kernels, from which the claim follows.
(b) Since $\gamma \in \mathcal{C}(S \otimes S, S \otimes A)$ is an isomorphism, the theorem follows from (a) and the commutativity of the following diagram:

$$\begin{tikzcd}
S \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\mathrm{id}\otimes\eta"'] & S \otimes A \\
S \arrow[u, "="] \arrow[r, shift left, "\eta\otimes\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\otimes\eta"'] & S \otimes S \arrow[u, "\eta"']
\end{tikzcd}$$

A semidirect product of Hopf algebras that, as a special case, reduces to the group algebra of a semidirect product of groups, was treated by Sweedler [50]. For a more general definition, in arbitrary symmetrical, monoidal categories, the corresponding proofs (unitarity, associativity) were carried out by Wach [53]. Therefore, we will be satisfied here with a definition.

(2.9) Definition. *Let $S \in \boldsymbol{Mon}\mathcal{C}$, $A^* \in \boldsymbol{Hopfmon}\mathcal{C}$, and S an A^*-object-monoid.*

*(a) The **semidirect** or **smash** product $S\#A^*$ is the object $S \otimes A^*$ together with the following structure-morphisms:*
$\eta := \eta_S \otimes \eta_{A^*} \in \mathcal{C}(I, S \otimes A^*)$ *and*
$\nabla \in \mathcal{C}(S \otimes A^* \otimes S \otimes A^*, S \otimes A^*)$ *with*
$\nabla := (\nabla_S \otimes \nabla_{A^*})(\mathrm{id} \otimes \beta \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{A^*} \otimes \mathrm{id} \otimes \mathrm{id})$, *i.e.*
$(x \otimes u)(y \otimes v) = (xu_1 \otimes u_2 v)$ *for all $X, Y \in \mathcal{C}$, $x \otimes u \in (S \otimes A^*)(X)$,*
$y \otimes v \in (S \otimes A^*)(Y)$.

*(b) In this situation, we also define the **canonical morphism** $\varphi \in \boldsymbol{Mon}\mathcal{C}(S\#A^*, [S,S])$ via*
$\mathcal{C}(S \otimes A^* \otimes S, S) \cong \mathcal{C}(S \otimes A^*, [S,S])$: $\nabla_S(\mathrm{id} \otimes \beta) \mapsto \varphi$
i.e. $\big(\varphi(s \otimes u)\big)\langle t\rangle = su(t) = s\beta(u \otimes t)$ *for all $X, Y \in \mathcal{C}$, $s \otimes u \in (S \otimes A^*)(X)$, $t \in S(Y)$.*

For the next theorem, we need a lemma that is known from module theory.

(2.10) Lemma. *Let $S \in \boldsymbol{Mon}\mathcal{C}$, S finite over I, $A, B \in {}_S\mathcal{C}$, A, B finite over S, $h \in {}_S\mathcal{C}(A,B)$ and let ${}_S[h, \mathrm{id}_S]: {}_S[B,S] \to {}_S[A,S]$ be an isomorphism. Then h is also an isomorphism.*

Proof. For all $X \in {}_S\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
({}_S[B,S] \otimes_S X)(I) \arrow[r, "({}_S[h,\mathrm{id}_S] \otimes_S \mathrm{id}_X)(I)"] \arrow[d, "\cong"] & ({}_S[A,S] \otimes_S X)(I) \arrow[d, "\cong"] \\
{}_S[B, S \otimes_S X](I) \arrow[d, "\cong"] & {}_S[A, S \otimes_S X](I) \arrow[d, "\cong"] \\
{}_S[B,X](I) \arrow[d, "\cong"] & ({}_S[A,X])(I) \arrow[d, "\cong"] \\
{}_S\mathcal{C}(B,X) \arrow[r, "{}_S\mathcal{C}(h,\mathrm{id}_X)"] & {}_S\mathcal{C}(A,X)
\end{tikzcd}$$

The upper vertical arrows are isomorphisms due to Pareigis [39], Theorem 1.2b, and the lower ones were specified explicitly in the proof of (2.4). From the commutativity of the diagram it follows that ${}_S\mathcal{C}(h, \mathrm{id}_X)$ is an isomorphism, and as a result also h, because of the Yoneda lemma.

(2.11) Theorem. Let $A \in \boldsymbol{Hopfmon}\mathcal{C}$, *$A$ finite over I, S an A-coobject-monoid, and $S \in \boldsymbol{cMon}\mathcal{C}$. Then the following are equivalent:*

(a) S is A-Galois over I.

(b) S is faithfully projective over I and $\varphi \in \boldsymbol{Mon}\mathcal{C}(S\#A^*, [S,S])$ is an isomorphism.

Proof. "(a) ⇒ (b)": Define an isomorphism h:

$$\begin{tikzcd}
S\#A^* \arrow[r, "h"] \arrow[d, "\xi"', "\cong"] & S \otimes S^* \arrow[r, "\tilde{f}", "\cong"'] \arrow[d, "\eta"', "\cong"] & {[S,S]} \arrow[dl, "\vartheta"', "\cong"] \\
{}_S[S \otimes A, S] \arrow[r, "{}_S[\gamma, \mathrm{id}_S]", "\cong"'] & {}_S[S \otimes S, S] &
\end{tikzcd}$$

Since S is finite, $\tilde{f}$ is an isomorphism, and so is $\varphi = \tilde{f}h$. The equality can be seen as follows:
$\left(\vartheta\tilde{f}(s \otimes v)\right)\langle t \otimes t'\rangle = t\left(\tilde{f}(s \otimes v)\right)\langle t'\rangle = tsv\langle t'\rangle = \big(\eta(s \otimes v)\big)\langle t \otimes t'\rangle$, and so we have $\vartheta\tilde{f} = \eta$.
$\big(\vartheta\varphi(s \otimes u)\big)\langle t \otimes t'\rangle = t\big(\varphi(s \otimes u)\big)\langle t'\rangle = ts\beta(u \otimes t') = ts(\mathrm{id}_S \otimes u)\alpha(t') = ts\alpha_1(t')u\langle\alpha_2(t')\rangle$.

$\big({}_S[\gamma, \mathrm{id}_S]\xi(s \otimes u)\big)(t \otimes t') = (\xi(s \otimes u)\gamma)(t \otimes t') = \xi(s \otimes u)\langle t\alpha_1(t') \otimes \alpha_2(t')\rangle =$
$t\alpha_1(t')su\langle\alpha_2(t')\rangle$.
Since S is commutative, so is $\vartheta\varphi = {}_S[\gamma, \mathrm{id}_S]\xi$. Then we have $\tilde{f}h = \tilde{f}\eta^{-1}{}_S[\gamma, \mathrm{id}_S]\xi = \tilde{f}\eta^{-1}\vartheta\varphi = \varphi$.
"(b) ⇒ (a)" Since $\tilde{f}$ and $\varphi = \tilde{f}h$ are isomorphisms, h is also an isomorphism and thus ${}_S[\gamma, \mathrm{id}_S]$ is an isomorphism. Now is follows from the last lemma (2.10) that γ is an isomorphism.

Now that we can formulate the concept "A-Galois over I" either in terms of γ or of φ, we will head for another characterization, one that will make essential use of the Morita theorems, and that will be very useful for the proof of the fundamental theorem. For that, we will use the following constructions.

(2.12) Definition. *Let $S \in \boldsymbol{Mon}\mathcal{C}$, $A \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I, S an A^*-object-monoid, and $S^{A^*} = I$. Then we define:*
$D := S\#A^*$,
$Q := D^{A^*}$,
$\psi := \nabla_S(\mathrm{id}_S \otimes \beta) \in \mathcal{C}(S \otimes A^* \otimes S, S) = \mathcal{C}(D \otimes S, S)$,
$f := \nabla_D(j_S \otimes j_Q) \in {}_D\mathcal{C}_D(S \otimes Q, D)$, and
$g := \psi(j_Q \otimes \mathrm{id}_S) \in \mathcal{C}(Q \otimes_D S, I)$, where
$j_S: S \to D$ and $j_Q: Q \to D$ are the canonical inclusions. g is well defined because $Q := D^{A^*}$ and $S^{A^*} = I$.

(2.13) Remark. ψ from (2.12) and φ from (2.9) are related in the following way:
$\mathcal{C}(D, [S, S]) \cong \mathcal{C}(D \otimes S, S): \varphi \mapsto \psi$, i.e.
$\big(\varphi(d)\big)\langle s\rangle = \psi(d \otimes s)$ for all $X, Y \in \mathcal{C}, d \in D(X), s \in S(Y)$.

(2.14) Lemma. *(D, I, S, Q, f, g) is a Morita context.*

Proof. We need to demonstrate the commutativity of the following two diagrams:

$$\begin{tikzcd}
S \otimes Q \otimes_D S \arrow[r, "\mathrm{id}_S \otimes g"] \arrow[d, "f \otimes_D \mathrm{id}_S"] & S \otimes I \arrow[d, "\cong"] \\
D \otimes_D S \arrow[r, "\cong"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \otimes_D S \otimes Q \arrow[r, "\mathrm{id}_Q \otimes_{\boldsymbol{D}} f"] \arrow[d, "g \otimes \mathrm{id}_Q"] & Q \otimes_D D \arrow[d, "\cong"] \\
I \otimes Q \arrow[r, "\cong"] & Q
\end{tikzcd}$$

Now let $X, Y \in \mathcal{C}, s \otimes q \otimes s' \in (S \otimes Q \otimes_D S)(X)$, and $p \otimes s \otimes q \in (Q \otimes_D S \otimes Q)(Y)$. Then we have:
$(\mathrm{id}_S \otimes g)(s \otimes q \otimes s') = s \otimes q(s') \longmapsto sq(s')$.
$(f \otimes_D \mathrm{id}_S)(s \otimes q \otimes s') = (s \otimes 1_{A^*})q \otimes s' \longmapsto \big((s \otimes 1)q\big)(s') = (s \otimes 1)\big(q(s')\big) = sq(s')$.
$\big(g \otimes \mathrm{id}_Q\big)(p \otimes s \otimes q) = p(s) \otimes q \longmapsto p(s)q$.
$\big(\mathrm{id}_Q \otimes_{\boldsymbol{D}} f\big)(p \otimes s \otimes q) = p \otimes (s \otimes 1_{A^*})q \longmapsto p(s \otimes 1)q = \big(p(s \otimes 1)\big)q = \big(p(s \otimes 1)\big)(1)q$
since $Q = D^{A^*}$
$= p\big((s \otimes 1)(1)\big)q = p\big(s1(1)\big)q = p(s)q$.

For the proof of the theorem, we will need two more lemmas that are not so customary. They replace 8.3, 8.4, and 8.5 in Chase and Sweedler [10]. The latter are module-theoretic statements that cannot readily be generalized.

(2.15) Lemma. *Let $P \in \mathcal{C}$ and let $\tilde{g} \in \mathcal{C}\big(P^* \otimes_{[P,P]} P, I\big)$ from (1.3) be an isomorphism. Then $P \otimes -: \mathcal{C} \to {}_{[P,P]}\mathcal{C}$ reflects isomorphisms.*

Proof. Let $A, B \in \mathcal{C}, h \in \mathcal{C}(A, B)$ and let $\mathrm{id}_P \otimes h \in {}_{[P,P]}\mathcal{C}(P \otimes A, P \otimes B)$ be an isomorphism. Application of the functor $P^* \otimes_{[P,P]} -: {}_{[P,P]}\mathcal{C} \to \mathcal{C}$ results in the following commutative diagram, from which we can see that h is an isomorphism.

$$\begin{tikzcd}
P^* \otimes_{[P,P]} P \otimes A \arrow[r, "\mathrm{id}_{P^*} \otimes_{[P,P]} \mathrm{id}_P \otimes h"] \arrow[d, "\tilde{g} \otimes \mathrm{id}_A", "\cong"'] & P^* \otimes_{[P,P]} P \otimes B \arrow[d, "\tilde{g} \otimes \mathrm{id}_B", "\cong"'] \\
I \otimes A \cong A \arrow[r, "h"] & B \cong I \otimes B.
\end{tikzcd}$$

(2.16) Lemma. *Let (D, I, P, Q, f, g) be a Morita context, f rationally surjective and let $\tilde{g} \in \mathcal{C}(P^* \otimes_{[P,P]} P, I)$ from (1.3) be an isomorphism. Then f and g are isomorphisms.*

Proof. Because of theorem (1.6), f is an isomorphism and the following diagram commutes:

$$\begin{tikzcd}
P \otimes Q \otimes_D P \arrow[r, "f \otimes_D \mathrm{id}_P"] \arrow[d, "\mathrm{id}_P \otimes g"] & D \otimes_D P \arrow[d, "\cong"] \\
P \otimes I \arrow[r, "\cong"] & P
\end{tikzcd}$$

and so $\mathrm{id}_P \otimes g$ is an isomorphism, and due to the last lemma, so is g.

(2.17) Theorem. *Let $A \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I, S an A-coobject-monoid, and $S \in \boldsymbol{cMon}\mathcal{C}$. Then the following are equivalent:*

(a) S is A-Galois over I.

(b) $S^{A^*} = I$ and f, g from (2.12) are rationally surjective (and as a result isomorphisms).

Proof. "(b) ⇒ (a)": From (b) and the Morita theorems (1.6) it follows that S is faithfully projective over I, and that φ is an isomorphism. So S is A-Galois over I because of theorem (2.11).

"(a) ⇒ (b)": $S^{A^*} = I$ because of theorem (2.8). From (a) and theorem (2.11) it follows that S is faithfully projective over I and that φ is an isomorphism. According to definition (2.6), $I = S^{A^*} \to S \rightrightarrows S \otimes A$ is a difference kernel. Since the functor $[S, -]$ is right adjoint, $[S, I] = [S, S] \to S \rightrightarrows [S, S \otimes A] \cong [S, S] \otimes A$ is also a difference kernel, where the last isomorphism from Pareigis [39] Theorem 1.2 demonstrates that S is finite. As a result: $[S, S]^{A^*} = [S, I]$. Since A^* is a submonoid of D, $D \in {}_{A^*}\mathcal{C}$. Since $S \in {}_{A^*}\mathcal{C}$, $A^* \to [S, S]$ is a monoid morphism, so $[S, S] \in {}_{A^*}\mathcal{C}$. With these structures, we also have $\varphi \in {}_{A^*}\mathcal{C}(D, [S, S])$, or also $\varphi \in \mathcal{C}^A(D, [S, S])$. Therefore, an isomorphism $\bar{\varphi}$ exists such that the following diagram is commutative:

$$\begin{tikzcd}
Q = D^{A^*} \arrow[r] \arrow[d, "\bar{\varphi}"] & D \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\varphi"] & D \otimes A \arrow[d, "\varphi \otimes \mathrm{id}_A"] \\
{[S, I] = [S, S]^{A^*}} \arrow[r] & {[S, S]} \arrow[r, shift left] \arrow[r, shift right] & {[S, S] \otimes A}
\end{tikzcd}$$

Then this diagram is also commutative:

$$\begin{tikzcd}
S \otimes Q \arrow[r, "\mathrm{id}_S \otimes \bar{\varphi}"] \arrow[d, "f"] & S \otimes [S, I] \arrow[d, "\tilde{f}"] \\
D \arrow[r, "\varphi"] & {[S, S]}
\end{tikzcd}$$

$\tilde{f}(\mathrm{id}_S \otimes \bar{\varphi})(s \otimes q) = \tilde{f}(s \otimes \bar{\varphi}(q)) = s\bar{\varphi}(q) = s\varphi(q) = \varphi s(q) = \varphi f(s \otimes q)$ for all $X \in \mathcal{C}$,

$s \otimes q \in (S \otimes Q)(X)$. Since S is finite, $\tilde{f}$ is an isomorphism, and so f is also an isomorphism. Since S is also faithfully projective, g is also an isomorphism according to lemma (2.16).

Remark. Now, after replacing corollary 8.5 of Chase and Sweedler [11] by our lemma (2.16), we can simplify the proof of their theorem 8.6 with the help of our theorem (2.8) and include it in the proof of theorem (2.17).

(2.18) Theorem. *Let $S \in \boldsymbol{cMon}\mathcal{C}$, $A \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I, and S A-Galois over I. Then the following holds:*
(a) $\eta \in \mathcal{C}(I,S)$ has a retraction in $\mathcal{C}$.
(b) S is faithfully projective over D.

Proof. (a) Since $g \in \mathcal{C}(Q \otimes_D S, I)$ is rationally surjective, there is an $s \otimes_D w \in (Q \otimes_D S)(I)$ with $g(s \otimes_D w) = 1_I$. Now let $w' \coloneqq sw = \nabla_D(j_Q \otimes_D j_S)(s \otimes_D w) \in D(I)$. Then $g(w' \otimes_D -) \in \mathcal{C}(S,I)$ is a retraction of $\eta \in \mathcal{C}(I,S)$, because, for all $R \in \mathcal{C}, r \in I(R)$ we have $g(w' \otimes_D r1_S) = rg(w' \otimes_D 1_S) = rg(\nabla_D(s \otimes_D w) \otimes_D 1_S) = rg(s \otimes_D w) = r1_I = r$.
(b) From the last theorem and the Morita theorems (1.6) it follows that S is faithfully projective over D.

(2.19) Corollary. *Let $S \in \boldsymbol{Mon}\mathcal{C}$, $A \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I, and S A-Galois over I. Then the functors*
$\mathcal{C} \to {}_D\mathcal{C}: M \longmapsto S \otimes M$ and ${}_D\mathcal{C} \to \mathcal{C}: N \longmapsto Q \otimes_D N = D^{A^*} \otimes_D N$
are equivalences of categories that are inverse to each other. In particular,
$M \cong D^{A^*} \otimes_D S \otimes M$ and $N \cong S \otimes D^{A^*} \otimes_D N$.

Proof. The statement follows directly from the last theorem (2.18) (b) and the Morita theorems (1.6).

(2.20) Example. Let $I = K$ be a field, $K \subset S$ a finite field extension and $G \subset \mathrm{Aut}_K(S)$, as in example (2.3) (a). Now we will carry out the constructions of chapter 2 for this special case.
Ad (2.9): $D = S\#KG$ is, as an underlying module, $S \otimes_K KG = SG$ and has the following multiplication:
$(sx)(s'x') = sx(s')xx'$ for $x, x' \in G, s, s' \in S$.
$\varphi: SG \to \mathrm{End}_K(S): (\varphi(sx))(s') = sx(s')$.
Ad (2.12): Claim: $Q = D^{A^*} = (SG)^G = NS$, where $N \coloneqq \sum_{x \in G} x \in KG$.
Proof. Obviously, $NS \subset Q$. Now let $\sum_{x \in G} s_x x \in Q$. Then for all $y \in G$ the following holds:
$y(\sum_{x \in G} s_x x) = \sum_{x \in G} y(s_x)yx = \sum_{x \in G} s_x x$. In addition, it holds that: $y(s_e e) = y(s_e)y$.
Since G is an S-basis of SG, it follows that, for all $x \in G$: $s_x = x(s_e)$, and so
$Q = \{\sum_{x \in G} x(s_e)x : s_e \in S\} = \{\sum_{x \in G} xs : s \in S\} = NS$. In this special case, we also have:
$\psi: SG \otimes S \to S: sx \otimes s' \longmapsto sx(s')$,
$f: S \otimes NS \to SG: s \otimes Ns' \longmapsto sNs' = \sum_{x \in G} sx(s')x$,
$g: NS \otimes_{SG} S \to K: Ns \otimes_{SG} s' \longmapsto (Ns)(s') = (\sum_{x \in G} x(s)x)(s') = \sum_{x \in G} x(ss')$.
(2.14) remains basically the same. (2.15) and (2.16) can be disregarded in this special case, since the statements in the proof of (2.17) can be replaced by the argument $\dim_K(S) \geq 1$. Then the proof of (2.17) can be written directly for this special case.

(2.21) Corollary. $\mathrm{Tr}: S \to K: s \longmapsto N(s)$ *is surjective.*

Proof. $g(Ns \otimes_{SG} s') = N(ss')$, and g is surjective according to theorem (2.17).

3. A Frobenius property

The primary result of this chapter is the proof of Theorem (3.8), which states: If H is a finite Hopf-monoid, then a finite object P exists, such that H^* is isomorphic to $P \otimes H$ as a right H-object. Since we cannot prove that P is "finitely generated, projective of rank 1", as in module theory, we have less than the statement that H is a P-Frobenius extension. Nevertheless, this theorem is sufficient for our purposes, and proves to be very useful. In addition to the direct application of this theorem, we also have an important corollary in Theorem (3.12), which will be used in the next chapter.

Our numbers (3.1). (3.2), (3.4), (3.5), (3.7) and (3.8) correspond to 2.9, 2.10, 2.11, 2.5, 2.15 and 2.16 in Pareigis [40].

(3.1) Definition. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$. A triple (M, ρ, χ) is called an H* ***-right Hopf-object*** *if $(M, \rho) \in \mathcal{C}_H$, $(M, \chi) \in \mathcal{C}^H$, see (1.1), and the following diagram commutes:*

$$\begin{array}{ccccc}
M \otimes H & \xrightarrow{\rho} & M & \xrightarrow{\chi} & M \otimes H \\
\downarrow{\scriptstyle \chi \otimes \Delta} & & & & \uparrow{\scriptstyle \rho \otimes \nabla} \\
M \otimes H \otimes H \otimes H & & \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} & & M \otimes H \otimes H \otimes H
\end{array}$$

i.e. if $(mh)_0 \otimes (mh)_1 = m_0 h_1 \otimes m_1 h_2$ for all $X \in \mathcal{C}$, $m \otimes h \in (M \otimes H)(X)$. A ***Hopf-object morphism*** *is a morphism that is in both $\mathcal{C}_H$ and $\mathcal{C}^H$. This category is denoted as H-$\boldsymbol{Hopfobj}\mathcal{C}$.*

(3.2) Theorem. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$. Then the functors*
$H\text{-}\boldsymbol{Hopfobj}\mathcal{C} \to \mathcal{C}: M \mapsto M^{H^*}$ and $\mathcal{C} \to H\text{-}\boldsymbol{Hopfobj}\mathcal{C}: X \mapsto X \otimes H$
are equivalences of categories that are inverse to each other.

Proof. We construct the following functorial isomorphisms:
$M^{H^*} \otimes H \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} M$ and $X \underset{\beta^{-1}}{\overset{\beta}{\rightleftarrows}} (X \otimes H)^{H^*}$
via:
$\alpha(m' \otimes h) = m'h = \rho(m' \otimes h)$,
$\alpha^{-1}(m) = m_0 \lambda(m_1) \otimes m_2$,
$\beta(x) = x \otimes 1_H = x \otimes \eta_H$,
$\beta^{-1}(x' \otimes h') = x' \varepsilon(h')$,
for all $U, V, W, Z \in \mathcal{C}$, $m' \otimes h \in \left(M^{H^*} \otimes H\right)(U)$, $m \in M(V)$, $x \in X(W)$, $x' \otimes h' \in (X \otimes H)^{H^*}(Z)$.
These morphisms are clearly all functorial in M respectively X.
α^{-1} is well defined, because
$\chi\big(m_0 \lambda(m_1)\big)$
$= m_0 \lambda(m_3) \otimes m_1 \lambda(m_2)$ since $M \in H\text{-}\boldsymbol{Hopfobj}\mathcal{C}$
$= m_0 \lambda(m_2) \otimes \eta\varepsilon(m_1)$ since λ is an antipode
$= m_0 \lambda(m_1) \otimes \eta$ since ε is a counit.
χ and $\mathrm{id} \otimes \eta$ thus have the same effect on the first factor of $\alpha^{-1}(m)$, so it is in the difference kernel. ($- \otimes H$ preserves kernels.)
α and α^{-1} are inverse to each other:
$\alpha\alpha^{-1}(m)$
$= \alpha(m_0 \lambda(m_1) \otimes m_2)$
$= m_0 \lambda(m_1) m_2$
$= m_0 \eta \varepsilon(m_1)$ since λ is an antipode
$= m$ since ρ is unitary and a counit.
$\alpha^{-1}\alpha(m' \otimes h)$

$= \alpha^{-1}(m'h)$
$= (m'h)_0\lambda((m'h)_1) \otimes (m'h)_2$
$= m'_0h_1\lambda(m'_1h_2) \otimes m'_2h_3$ since $M \in H\text{-}\boldsymbol{Hopfobj}\mathcal{C}, H \in \boldsymbol{Hopfmon}\mathcal{C}$
$= m'h_1\lambda(h_2) \otimes h_3$ since $m' \otimes h \in \left(M^{H^*} \otimes H\right)(U)$
$= m'\eta\varepsilon(h_1) \otimes h_2$ since λ is an antipode
$= m' \otimes h$ since ε is a counit.
Clearly, $\alpha \in \mathcal{C}_H$. $\alpha^{-1} \in \mathcal{C}^H$, because
$\chi\alpha^{-1}(m)$
$= \chi(m_0\lambda(m_1) \otimes m_2)$
$= m_0\lambda(m_1) \otimes m_2 \otimes m_3$
$= \alpha^{-1}(m_0) \otimes m_1 = (\alpha^{-1} \otimes \mathrm{id})\chi(m)$.
As a result, α and α^{-1} are Hopf-object morphisms and inverse to each other.
β is well defined, because
$\chi(x \otimes \eta)$
$= x \otimes \Delta(\eta)$
$= (x \otimes \eta) \otimes \eta$
$= (\mathrm{id} \otimes \eta)(x \otimes \eta)$.
β and β^{-1} are clearly in $\mathcal{C}$. They are inverse to each other:
$\beta^{-1}\beta(x)$
$= \beta^{-1}(x \otimes \eta)$
$= x\varepsilon(\eta)$
$= x$.
$\beta\beta^{-1}(x' \otimes h')$
$= \beta\big(x'\varepsilon(h')\big)$
$= x'\varepsilon(h') \otimes \eta$
$= x' \otimes \varepsilon(h')\eta$
$=^{(*)} x' \otimes \varepsilon(h'_1)h'_2$
$= x' \otimes h'$.
$=^{(*)}$ holds because $x' \otimes h' \in (X \otimes H)^{H^*}(Z) \Longrightarrow$
$x' \otimes h' \otimes \eta$
$= (\mathrm{id} \otimes \eta)(x' \otimes h')$
$= \chi(x' \otimes h')$
$= x' \otimes h'_1 \otimes h'_2 \Longrightarrow$
$x' \otimes \varepsilon(h') \otimes \eta$
$= x' \otimes \varepsilon(h'_1) \otimes h'_2$
$= x' \otimes h' \otimes \eta \Longrightarrow$
$x' \otimes \varepsilon(h') = x' \otimes h'$.

The next lemma, that we need now, and also later, is again of a general nature. In module theory it is known in the form “Direct sums of finitely generated projective modules are finitely generated projective.” The same proof as below also provides the corresponding statement for our concept of “finitely generated projective”.

(3.3) Lemma. *Let $A \in \boldsymbol{Mon}\mathcal{C}, N, M \in \mathcal{C}_A, j \in \mathcal{C}_A(N,M)$ and let $k \in \mathcal{C}_A(M,N)$ be a retraction of j. If M is finite over A, then N is also finite over A.*

Proof. Since M is finite over A, there is a
$p \otimes_A q \in (M \otimes_A {}_A[M,A])(I)$ with $pq\langle x\rangle = x$ for all $X \in \mathcal{C}, x \in M(X)$.
Now define $p' \otimes_A q' \in (N \otimes_A {}_A[N,A])(I)$ by:
$I \xrightarrow{p\otimes_A q} M \otimes_A {}_A[M,A] \xrightarrow{k\otimes_A[j,\mathrm{id}_A]} N \otimes_A {}_A[N,A]$.
Let $Y \in \mathcal{C}, y \in N(Y)$. Then
$p'q'\langle y\rangle = kp([j,\mathrm{id}_A]q)\langle y\rangle = kp(q\langle jy\rangle) = k(pq\langle jy\rangle) = k(jy) = y$,
so N is also finite over A.

(3.4) Lemma. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$ and $M \in H$-$\boldsymbol{Hopfobj}\mathcal{C}$. Then:*

(a) $M^{H^*} \xrightarrow{\text{can}} M$ has a retraction in $\mathcal{C}$.

(b) If M is also finite over I, then so is M^{H^*}.

Proof. (a) Let $\delta: M \to M^{H^*}$ be defined by: $\delta(m) = m_0\lambda(m_1)$ for all $U \in \mathcal{C}, m \in M(U)$. δ is well defined; the proof is similar to α^{-1} in (3.2). Now let $V \in \mathcal{C}$ and $m' \in M^{H^*}(V)$. Then $m'_0 \otimes m'_1 = m' \otimes \eta$, and so $\delta(m') = m'\lambda(\eta) = m'$.

(b) The statement follows from (a) and the preceding lemma.

(3.5) Lemma. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$ with antipode λ. Then:*

(a) λ is a monoid antimorphism.

(b) λ is a comonoid antimorphism.

Proof. (a) We need to show that:

$\lambda(\eta) = \eta$ and $\lambda(gh) = \lambda(h)\lambda(g)$ for all $X \in \mathcal{C}, g \otimes h \in (H \otimes H)(X)$.

$\lambda(\eta)$

$= \lambda(\eta)\eta$ since η is a unit

$= \lambda(\eta_1)\eta_2$ since $\eta \in \boldsymbol{coMon}\mathcal{C}$

$= \eta\varepsilon(\eta)$ since λ is an antipode

$= \eta.$

$\lambda(gh)$

$= \lambda\big(g_1\varepsilon(g)_2h_1\varepsilon(h_2)\big)$ since ε is a counit

$= \lambda(g_1h_1)g_2\lambda(g_3)\varepsilon(h_2)$ since λ is an antipode

$= \lambda(g_1h_1)g_2\varepsilon(h_2)\lambda(g_3)$

$= \lambda(g_1h_1)g_2h_2\lambda(h_3)\lambda(g_3)$ since λ is an antipode

$= \lambda((gh)_1)(gh)_2\lambda(h_3)\lambda(g_3)$ since $H \in \boldsymbol{Hopfmon}\mathcal{C}$

$= \varepsilon(g_1h_1)\lambda(h_2)\lambda(g_2)$ since λ is an antipode

$= \varepsilon(g_1)\varepsilon(h_1)\lambda(h_2)\lambda(g_2)$ since $\varepsilon \in \boldsymbol{Mon}\mathcal{C}$

$= \lambda(\varepsilon(h_1)h_2)\lambda(\varepsilon(g_1)g_2)$

$\lambda(h)\lambda(g)$ since ε is a counit.

(b) We need to show that:

$\varepsilon(\lambda) = \varepsilon$ and $\big(\lambda(h)\big)_1 \otimes \big(\lambda(h)\big)_2) = \lambda(h_2) \otimes \lambda(h_1)$ for all $Y \in \mathcal{C}, h \in H(Y)$.

$\varepsilon\big(\lambda(h)\big) = \varepsilon\Big(\lambda\big(h_1\varepsilon(h_2)\big)\Big)$ since ε is a counit

$= \varepsilon\big(\lambda(h_1)\big)\varepsilon(h_2)$

$= \varepsilon(\lambda(h_1)h_2)$ since $\varepsilon \in \boldsymbol{Mon}\mathcal{C}$

$= \varepsilon\eta\varepsilon$ since λ is an antipode

$= \varepsilon.$

$\big(\lambda(h)\big)_1 \otimes \big(\lambda(h)\big)_2$

$= \big(\lambda(\varepsilon(h_1)h_2)\big)_1 \otimes \big(\lambda(\varepsilon(h_1)h_2)\big)_2$ since ε is a counit

$= \big(\varepsilon(h_1)\lambda(h_2)\big)_1 \otimes \big(\varepsilon(h_1)\lambda(h_2)\big)_2$

$= \big(\lambda(h_2)\big)_1 \otimes \varepsilon(h_1)\big(\lambda(h_2)\big)_2$

$= \big(\lambda(h_3)\big)_1 \otimes \lambda(h_1)h_2\big(\lambda(h_3)\big)_2$ since λ is an antipode

$= \big(\lambda(h_4)\big)_1 \otimes \lambda(h_1)\varepsilon(h_2)h_3\big(\lambda(h_4)\big)_2$ since ε is a counit

$= \varepsilon(h_2)\big(\lambda(h_4)\big)_1 \otimes \lambda(h_1)h_3\big(\lambda(h_4)\big)_2$

$= \lambda(h_2)h_3\big(\lambda(h_5)\big)_1 \otimes \lambda(h_1)h_4\big(\lambda(h_5)\big)_2$ since λ is an antipode

$= \lambda(h_2)\big(h_3\lambda(h_4)\big)_1 \otimes \lambda(h_1)\big(h_3\lambda(h_4)\big)_2$ since $H \in \boldsymbol{Hopfmon}\mathcal{C}$

$= \lambda(h_2)\big(\varepsilon(h_3)\big)_1 \otimes \lambda(h_1)\big(\varepsilon(h_3)\big)_2$ since λ is an antipode

$= \lambda(h_2)\varepsilon(h_3) \otimes \lambda(h_1)$

$= \lambda\big(h_2\varepsilon(h_3)\big) \otimes \lambda(h_1)$

$= \lambda(h_2) \otimes \lambda(h_1)$ since ε is a counit.

Now we need another general lemma that is well known in module theory. The same proof as below also provides the corresponding statement for our concept of "finitely generated projective".

(3.6) Lemma. *Let $A \in$ $\boldsymbol{Mon}\mathcal{C}, M \in \mathcal{C}_A$, and M finite over A. Then:*
(a) ${}_A[M,A]$ is finite over A in ${}_A\mathcal{C}$.
(b) $k: M \to {}_A[{}_A[M,A],A]$ is an isomorphism, whereby
$\big(k(x)\big)\langle y\rangle = y\langle x\rangle$ for all $X, Y \in \mathcal{C}, x \in M(X)$ and $y \in {}_A[M,A](Y)$.

Proof. (a) Since M is finite over A, there is a
$p \otimes_A q \in (M \otimes_A {}_A[M,A])(I)$ with $pq\langle x\rangle = x$ for all $X \in \mathcal{C}, x \in M(X)$.
Now define $p' \otimes q' \in ({}_A[{}_A[M,A],A] \otimes_A {}_A[M,A])(I)$ by:
$I \xrightarrow{p\otimes q} M \otimes_A {}_A[M,A] \xrightarrow{k\otimes \mathrm{id}} {}_A[{}_A[M,A],A] \otimes_A {}_A[M,A]$
i.e. $p' \otimes q' = k(p) \otimes q$. Let $X, Y \in \mathcal{C}, x \in M(X)$ and $y \in {}_A[M,A](Y)$. Then:
$\big((p'\langle y\rangle)q'\big)\langle x\rangle = \big((k(p)\langle y\rangle)q\big)\langle x\rangle = \big((y\langle p\rangle)q\big)\langle x\rangle = y\langle p\rangle q\langle x\rangle = y\langle pq\langle x\rangle\rangle = y\langle x\rangle$, so ${}_A[M,A]$ is finite over A in ${}_A\mathcal{C}$.
(b) Let $l \in {}_A\mathcal{C}({}_A[{}_A[M,A],A], M)$ be defined by:
$l(z) = pz\langle q\rangle$ for all $Z \in \mathcal{C}, z \in {}_A[{}_A[M,A],A](Z)$. Then:
$lk(x) = p\big(k(x)\big)\langle q\rangle = pq\langle x\rangle = x$, and
$\big(kl(z)\big)\langle y\rangle = k(pz\langle q\rangle)\langle y\rangle = y\langle pz\langle q\rangle\rangle = y\langle p\rangle z\langle q\rangle = z\langle y\langle p\rangle q\rangle = z\langle p'\langle y\rangle q'\rangle = z\langle y\rangle$, so l is the inverse of k.

From this lemma we can immediately deduce that the functor ${}_A[-,A]: {}_A\mathcal{C} \longrightarrow \mathcal{C}_A$, restricted to the full subcategory of objects that are finite over A, is a duality, i.e. an anti-equivalence of categories. Since the canonical morphism ${}_A[M,A] \otimes_A {}_A[N,A] \longrightarrow {}_A[M \otimes_A N, A]$, for M and N finite over A and A commutative, is an isomorphism (see Pareigis [39]), this duality is monoidal. Therefore, finite monoids (resp. comonoids, Hopf-monoids) are transferred to comonoids (resp. monoids, Hopf-monoids).

(3.7) Theorem. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$ and H finite over I. Then H^* is an H-Hopf-object.*

Proof. By means of ∇_{H^*}, $H^* \in {}_{H^*}\mathcal{C}$, and so $H^* \in \mathcal{C}^H$ because of
$\xi: \mathcal{C}(H^*, H^* \otimes H) \xrightarrow{\cong} \mathcal{C}(H^* \otimes H^*, H^*)$, where $\xi(f)(g^* \otimes h^*) = (\mathrm{id}_{H^*} \otimes g^*)f(h^*)$ for all $X \in \mathcal{C}$, $g^* \otimes h^* \in (H^* \otimes H^*)(X)$.
The following also holds:
(a) $g^*h^* = \nabla_{H^*}(g^* \otimes h^*) = \xi(\chi)(g^* \otimes h^*) = (\mathrm{id} \otimes g^*)\chi(h^*) = {h^*}_0 g^*\langle {h^*}_1\rangle$
for all $X, Y \in \mathcal{C}, g^* \in H^*(X), h^* \in H^*(Y)$.
Then for all $X, Y, Z \in \mathcal{C}, g^* \in H^*(X), h^* \in H^*(Y), h \in H(Z)$:
(b) $g^*\langle h_1\rangle h^*\langle h_2\rangle = (g^*h^*)\langle h\rangle$ according to the definition of ∇_{H^*}
$= {h^*}_0\langle h\rangle g^*\langle {h^*}_1\rangle$ according to (a).
By means of ∇_H, $H \in \mathcal{C}_H$ and so $H^* \in {}_H\mathcal{C}$ via:
(c) $(hh^*)\langle g\rangle = h^*\langle gh\rangle = h^*\langle \nabla_H(g \otimes h)\rangle$, for all $X, Y, Z \in \mathcal{C}, h \in H(X), h^* \in H^*(Y), g \in H(Z)$.
Since, according to Lemma (3.5), λ is a monoid antimorphism, we can define a structure $H^* \in \mathcal{C}_H$, if we modify the above operation by λ in the following way:
(d) $(h^* \cdot h)\langle a\rangle = h^*\langle a\lambda(h)\rangle$ for all $X, Y, Z \in \mathcal{C}, h^* \in H^*(X), h \in H(Y), a \in H(Z)$. Since λ is also a comonoid antimorphism, see (3.5), the following also holds:
(e) $\big(b\lambda(a_1)\big)_1 \otimes \big(b\lambda(a_1)\big)_2 = b_1\big(\lambda(a_1)\big)_1 \otimes b_2\big(\lambda(a_2)\big)_2 = b_1\lambda(a_2) \otimes b_2\lambda(a_1)$,
for all $X, Y \in \mathcal{C}, a \in H(X), \ b \in H(Y)$.
Now we show that H^* is an H-right Hopf-object, i.e. that χ from (a) is compatible with the structure specified in (d) in the sense of Definition (3.1).
Let $X, Y \in \mathcal{C}, h^* \otimes a \in (H^* \otimes H)(X)$ and $b \otimes g^* \in (H \otimes H^*)(Y)$. Then:
$\big(\chi(h^* \cdot a)\big)\langle b \otimes g^*\rangle$
$= (h^* \cdot a)_0\langle b\rangle (h^* \cdot a)_1\langle g^*\rangle$ (customary orthography)
$= (h^* \cdot a)_0\langle b\rangle g^*\langle (h^* \cdot a)_1\rangle$ via $H \cong H^{**}$, (3.6)

$= (g^*\langle b_1\rangle)(h^* \cdot a)\langle b_2\rangle$ according to (b)
$= g^*\langle b_1\rangle h^*\langle b_2\lambda(a)\rangle$ according to (d)
$= g^*\langle b_1\varepsilon(a_2)\rangle h^*\langle b_2\lambda(a_1)\rangle$ since ε is a counit
$= g^*\langle b_1\lambda(a_2)a_3\rangle h^*\langle b_2\lambda(a_1)\rangle$ since λ is an antipode
$= (a_3 g^*)\langle b_1\lambda(a_2)\rangle h^*\langle b_2\lambda(a_1)\rangle$ according to (c)
$= (a_3 g^*)\left\langle\big(b_1\lambda(a_1)\big)_1\right\rangle h^*\left\langle\big(b_2\lambda(a_2)\big)_2\right\rangle$ according to (e)
$= \big((a_2 g^*)h^*\big)\langle b\lambda(a_1)\rangle$ definition of ∇_{H^*}
$= \Big(\big((a_2 g^*)h^*\big)\cdot a_1\Big)\langle b\rangle$ according to (d)
$= \Big(\Big(h^*{}_0\big((a_2 g^*)\langle h^*{}_1\rangle\big)\Big)\cdot a_1\Big)\langle b\rangle$ according to (a)
$= \Big((h^*{}_0\cdot a_1)\big((a_2 g^*)\langle h^*{}_1\rangle\big)\Big)\langle b\rangle$
$= \big((h^*{}_0\cdot a_1)g^*\langle h^*{}_1\cdot a_2\rangle\big)\langle b\rangle$ according to (c)
$= (h^*{}_0\cdot a_1)\langle b\rangle g^*\langle h^*{}_1\cdot a_2\rangle$
$= (h^*{}_0\cdot a_1)\langle b\rangle (h^*{}_1\cdot a_2)\langle g^*\rangle$ via $H \cong H^{**}$, (3.6)
$= \big((h^*{}_0\cdot a_1)\otimes(h^*{}_1\cdot a_2)\big)\langle b\otimes g^*\rangle$.
Therefore, $\chi(h^*\cdot a) = h^*{}_0\cdot a_1 \otimes h^*{}_1\cdot a_2$, which was to be proven.

(3.8) Theorem. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$ and H finite over I. Then there are $P, P' \in \mathcal{C}$, P, P' finite over I, with $H^* \cong P \otimes H$ in $\mathcal{C}_H$ and $H \cong P' \otimes H^*$ in $\mathcal{C}_{H^*}$.*

Proof. The first statement follows from (3.2), (3.4) (b) and (3.7) with $P := (H^*)^H$. The second is analogous to the first.

Now we will present two more general lemmas that are known from module theory. The proof of the first one supplies an analogous statement for our concept of "finitely generated projective" (resp. "progenerator") in place of "finite" (resp. "faithfully projective"). The proof of the second lemma, which represents a kind of transitivity of the concept "finitely generated projective", cannot be done analogously to the concept "finite", since a B-linearity is always missing. The lemma will only be used for two statements, (3.11) and (4.6), both of which are not essential for this thesis. According to Lemma (1.5), these two concepts coincide if I is cokernel projective in $\mathcal{C}$. This is the case for $\mathcal{C} = k\text{-}\boldsymbol{Mod}$, but not for the category of quasi complete, barreled topological vector spaces, see (10.9).

(3.9) Lemma. *Let $S \in \mathcal{C}$ and $A, B \in \boldsymbol{cMon}\mathcal{C}$. If S is finite (resp. faithfully projective) over I and A finite (resp. faithfully projective) over B, then $S \otimes A$ is finite (resp. faithfully projective) over B.*

Proof. First, we observe: ${}_B[S\otimes A, B] \cong [S, {}_B[A,B]] \cong [S,I]\otimes {}_B[A,B]$, where the last isomorphism, for finite S, follows from Pareigis [39].
(a) Let S (resp. A) be finite over I (resp. B). Then there is
$t\otimes p \in (S\otimes[S,I])(I)$ and $b\otimes q \in (A\otimes_B {}_B[A,B])(I)$ with $tp\langle s\rangle = s$ and $bq\langle a\rangle = a$ for all $X, Y \in \mathcal{C}$, $s\in S(X)$, $a \in A(Y)$. Define $t'\otimes p'$ by:
$(S\otimes[S,I]\otimes A\otimes_B {}_B[A,B])(I) \cong \big((S\otimes A)\otimes_B([S,I]\otimes {}_B[A,B])\big)(I)$
$\cong \big((S\otimes A)\otimes_B [S\otimes A, B]\big)(I)\colon t\otimes p\otimes b\otimes q \mapsto t'\otimes p'$. Then:
$(t'\otimes p')\langle s\otimes a\rangle = (t\otimes b)(p\otimes q)\langle s\otimes a\rangle = tp\langle s\rangle\otimes bq\langle a\rangle = s\otimes a$
for all $X\in\mathcal{C}$, $s\otimes a \in (S\otimes A)(X)$, and so $S\otimes A$ is finite over A.
(b) Let S (resp. A) be faithfully projective over I (resp. B). Then there is
$\tilde{p}\otimes\tilde{t} \in ([S,I]\otimes S)(I)$ and $\tilde{q}\otimes\tilde{b} \in ({}_B[A,B]\otimes_B A)(I)$ with $\tilde{p}\langle\tilde{t}\rangle = \eta_I$ and $\tilde{q}\langle\tilde{b}\rangle = \eta_B$.
Define $\widetilde{p'}\otimes\widetilde{t'}$ by:
$([S,I]\otimes S\otimes {}_B[A,B]\otimes_B A)(I) \cong \big(([S,I]\otimes {}_B[A,B])\otimes_B(S\otimes A)\big)(I)$
$\cong \big({}_B[S\otimes A, B]\otimes_B(S\otimes A)\big)(I)\colon \tilde{p}\otimes\tilde{t}\otimes\tilde{q}\otimes\tilde{b} \mapsto \widetilde{p'}\otimes\widetilde{t'}$. Then:
$\widetilde{p'}\langle\widetilde{t'}\rangle = (\tilde{p}\otimes\tilde{q})\langle\tilde{t}\otimes\tilde{b}\rangle = \tilde{p}\langle\tilde{t}\rangle\otimes\tilde{q}\langle\tilde{b}\rangle = \eta_I\otimes\eta_B = \eta_B$, and so $S\otimes A$ is faithfully projective over B.

(3.10) Lemma. *Let $A, B \in \mathbf{cMon}\mathcal{C}, B \in {}_A\mathcal{C}, C \in {}_B\mathcal{C}$, and C (resp. B) finitely generated projective over B (resp. A). Then C is finitely generated projective over A, where $C \in {}_A\mathcal{C}$ via $A \otimes C \to C: a \otimes c \longmapsto (a \cdot 1_B)c$.*

Proof. Because of the premises, there are $c \otimes p \in (C \otimes {}_B[C, B])(I)$ and $b \otimes q \in (B \otimes {}_A[B, A])(I)$ with $cp\langle c'\rangle = c'$ and $bq\langle b'\rangle = b'$ for all $X, Y \in \mathcal{C}, c' \in C(X), b' \in B(Y)$. Define $cb \otimes qp \in (C \otimes {}_A[C, A])(I)$ by:
$I \xrightarrow{b\otimes q\otimes c\otimes p} B \otimes {}_A[B, A] \otimes C \otimes {}_B[C, B] \xrightarrow{\cong} C \otimes B \otimes {}_A[B, A] \otimes {}_B[C, B] \to C \otimes {}_A[C, A]$. Then: $cb(qp)\langle c'\rangle = cbq\langle p\langle c'\rangle\rangle = cp\langle c'\rangle = c'$ for all $X \in \mathcal{C}, c' \in C(X)$, so C is finitely generated projective over A.

(3.11) Theorem. *Let $S \in \mathbf{Mon}\mathcal{C}, A \in \mathbf{Hopfmon}\mathcal{C}$, A finite over I, S A-Galois over I, and D defined as in (2.12). Then:*
(a) S is finite over A^*.
(b) D is finite over A^*.

Proof. (a) From the definition of "S is A-Galois over I", it follows that S is finite over I, and that $\gamma \in \mathcal{C}(S \otimes S, S \otimes A)$ is an isomorphism in $\mathcal{C}^A$, and so also in ${}_{A^*}\mathcal{C}$, where A^* operates on the right factor. A is finite over A^* because of Theorem (3.8), and so $S \otimes A$ is finite over A^* because of Lemma (3.9), and $S \otimes S$ is finite over A^*, since γ is an isomorphism in ${}_{A^*}\mathcal{C}$. Since, according to Theorem (2.18) (a), $\eta \in \mathcal{C}(I, S)$ has a retraction in $\mathcal{C}$, $\eta \otimes \mathrm{id}_S \in \mathcal{C}(S, S \otimes S)$ has a retraction in ${}_{A^*}\mathcal{C}$ and so S is finite over A^*, due to Lemma (3.3).
(b) From Theorem (2.17) and the Morita Theorems (1.6), it follows that Q is finite over I. Since $D \cong S \otimes Q$ in ${}_D\mathcal{C}$, it is true in particular that $D \cong S \otimes Q$ in ${}_{A^*}\mathcal{C}$. Since S is finite over A^* and Q is finite over I, D is finite over A^*.

The next theorem, a corollary of Theorem (3.8), will be used in the proof of Theorem (4.5), and thus for the fundamental theorem.

(3.12) Theorem. *Let $B \in \mathbf{Hopfmon}\mathcal{C}$, B finite over I, $M \in {}_{B^*}\mathcal{C}$ and $h \in {}_{B^*}\mathcal{C}(B^*, M)$. If h has a retraction in $\mathcal{C}$, then it also has a retraction in ${}_{B^*}\mathcal{C}$.*

Proof. To begin with, we have the following functorial isomorphisms:
${}_{B^*}\mathcal{C}(-, B^*)$
$= {}_{B^*}\mathcal{C}(-, [B, I])$
$\cong \mathcal{C}(B \otimes_{B^*} -, I)$
$\cong \mathcal{C}(P \otimes B^* \otimes_{B^*} -, I)$ because of Theorem (3.18)
$\cong \mathcal{C}(P \otimes -, I)$.
The following diagram commutes:

$$\begin{tikzcd}
{}_{B^*}\mathcal{C}(M, B^*) \arrow[r, "{}_{B^*}\mathcal{C}(h, \mathrm{id}_{B^*})"] \arrow[d, "\cong"] & {}_{B^*}\mathcal{C}(B^*, B^*) \arrow[d, "\cong"] \\
\mathcal{C}(P \otimes M, I) \arrow[r, "\mathcal{C}(\mathrm{id}_P \otimes h, \mathrm{id}_I)"] & \mathcal{C}(P \otimes B^*, I)
\end{tikzcd}$$

If l is a retraction of h in $\mathcal{C}$, then $\mathcal{C}(\mathrm{id}_P \otimes l, \mathrm{id}_I)$ is a section of $\mathcal{C}(\mathrm{id}_P \otimes h, \mathrm{id}_I)$. Therefore, $\mathcal{C}(\mathrm{id}_P \otimes h, \mathrm{id}_I)$ and ${}_{B^*}\mathcal{C}(h, \mathrm{id}_{B^*})$ are surjective, and so there is a $k \in {}_{B^*}\mathcal{C}(M, B^*)$ with $\mathrm{id}_{B^*} = {}_{B^*}\mathcal{C}(h, \mathrm{id}_{B^*})(k) = kh$.

(3.13) Remark. Theorem (3.12) says that B^* is relatively injective with respect to the forgetful functor $V: {}_{B^*}\mathcal{C} \to \mathcal{C}$.

4. Submonoids and fundamental theorem

In this chapter, the fundamental theorem of Galois theory, a statement about the lattice of submonoids of a Galois extension, is proven. An important step in this is Theorem (4.9), which allows us to infer from S^{B^*} back to B^*. This back-inference, which is accomplished in the theory of finite field extensions simply by means of dimension arguments, requires much more effort in the algebraic theory of finite, commutative ring extensions. For this, Morita theory turns out to be well suited.

(4.1) Definition. *Let $S \in \boldsymbol{cMon}\mathcal{C}, A, B \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I and let $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ be a retraction in $\mathcal{C}$. In addition, let S be an A-coobject monoid. We define $T := S^{B^*}$, and $\alpha' := (\mathrm{id}_S \otimes e)\alpha: S \to S \otimes B \cong S \otimes_T (T \otimes B)$.*

(4.2) Lemma. *Under the premises from above, S is a $T \otimes B$*-coobject monoid over T.

Proof. The conditions in Definition (2.1) are trivially preserved by $- \otimes_T -$.

(4.3) Definition. *Let S, A, B, e, T, and α' be as above. Then we define:*
$D' := S\#_T(T \otimes B^*) \in {}_{D'}\mathcal{C}_T$,
$Q' := (D')^{T \otimes B^*} \in {}_T\mathcal{C}_{D'}$,
$f' := \nabla_{D'}\big(j_S \otimes_T j_{Q'}\big) \in {}_{D'}\mathcal{C}_{D'}(S \otimes_T Q', D')$, *and*
$g' := \psi'\big(j_{Q'} \otimes_{D'} \mathrm{id}_S\big) \in {}_T\mathcal{C}_T(Q' \otimes_{D'} S, T)$, *where*
$j_S: S \to D'$,
$j_{Q'}: Q' \to D'$ *and*
$j_{D'}: D' = S \otimes_T (T \otimes B^*) \cong S \otimes B^* \xrightarrow{\mathrm{id}_S \otimes e^*} S \otimes A^* = D$
are the canonical inclusions.

(4.4) Lemma. *(D', T, S, Q', f', g') is a Morita context.*

Proof. The commutativity of the diagrams in Definition (1.2) is trivial; see also Lemma (2.14).

(4.5) Theorem. *Let $S, A, B, e, T, \alpha', D', Q', f'$ and g' be as above and let S be A-Galois over I. Then:*
(a) f' and g' are rationally surjective (and therefore isomorphisms).
(b) S is $T \otimes B$-Galois over T.
(c) S is faithfully projective over T and $\eta: T \to S$ has a retraction in ${}_T\mathcal{C}$.

Proof. (a) (1) Theorem (3.12) provides the existence of a $k \in {}_{B^*}\mathcal{C}(A^*, B^*)$ with $ke^* = \mathrm{id}_{B^*}$. Define $l := \mathrm{id}_S \otimes k \in \mathcal{C}(D, D')$. Since $k \in {}_{B^*}\mathcal{C}$, $l \in {}_{D'}\mathcal{C}$ and $l(Q) \subset Q'$, i.e. there is an $\bar{l} \in \mathcal{C}(Q, Q')$ with $lj_Q = j_{Q'}\bar{l}$. In addition, $lj_{D'} = \mathrm{id}_{D'}$. From (2.17) it follows that $f \in {}_D\mathcal{C}_D(S \otimes Q, D)$ is rationally surjective, so there is an $x \otimes w \in (S \otimes Q)(I)$ with $f(x \otimes w) = \eta_D \in D(I)$. Then:
$\eta_{D'} = lj_{D'}\eta_D = lj_{D'}f(x \otimes w)$
$= lj_{D'}(xw) = x\bar{l}(w)$ since $j_{D'} \in {}_S\mathcal{C}, l \in {}_S\mathcal{C}$ and $l(Q) \subset Q'$
$= f'\Big(x \otimes \bar{l}(w)\Big)$,
so $x \otimes \bar{l}(w) \in (S \otimes_T Q')(I)$ and $f'\Big(x \otimes \bar{l}(w)\Big) = \eta_{D'}$
and therefore $f' \in {}_{D'}\mathcal{C}_{D'}(S \otimes_T Q', D')$ is also rationally surjective.

(2) In order to demonstrate the rational surjectivity of g', we begin by collecting a few concepts:
$\beta(b \otimes t) = (\mathrm{id}_S \otimes b)\alpha(t)$ for all $X \in \mathcal{C}, b \otimes t \in (A^* \otimes S)(X)$,
$\beta'(b \otimes t) = (\mathrm{id}_S \otimes b)\alpha'(t) = (\mathrm{id} \otimes be)\alpha(t) = \big(\mathrm{id} \otimes e^*(b)\big)\alpha(t) = \beta(e^* \otimes \mathrm{id})(b \otimes t)$
for all $X \in \mathcal{C}, b \otimes t \in (B^* \otimes S)(X)$, and so $\beta' = \beta(e^* \otimes \mathrm{id}_S)$,
$\psi' = \nabla_S(\mathrm{id}_S \otimes \beta') \in \mathcal{C}(D' \otimes S, S)$.
Now let $X \in \mathcal{C}, s \otimes b \otimes t \in (D \otimes S)(X) = (S \otimes A^* \otimes S)(X)$. Then:
$\psi'(l \otimes \mathrm{id}_S)(s \otimes b \otimes t)$
$= \psi'(s \otimes k(b) \otimes t)$ according to the definition of l
$= s\beta'(k(b) \otimes t)$ according to the definition of ψ'

$= s\beta\big(e^*(k(b)) \otimes t\big)$ since $\beta' = \beta(e^* \otimes \mathrm{id})$
$= s\beta(b \otimes t)$ since $e^*k = \mathrm{id}_{B^*}$
$= \psi(s \otimes b \otimes t)$ according to the definition of ψ.
So $\psi'(l \otimes \mathrm{id}_S) = \psi$.
(3) From Theorem (2.17), it follows that $g \in (Q \otimes_D S, I)$ is rationally surjective, so there is a $q \otimes_D s \in (Q \otimes_D S)(I)$ with $g(q \otimes_D s) = \eta_I = \mathrm{id}_I \in I(I)$. Then:
$= g(q \otimes_D s)$
$= \psi(j_Q \otimes_D \mathrm{id}_S)(q \otimes_D s)$ according to the definition of g
$= \psi'(lj_Q \otimes_{D'} \mathrm{id}_S)(q \otimes_D s)$ since $\psi'(l \otimes \mathrm{id}_S) = \psi$
$= \psi'(j_{Q'} \otimes_{D'} \mathrm{id}_S)(\bar{l}(q) \otimes_{D'} s)$ since $lj_Q = j_Q\bar{l}$
$= g'(\bar{l}(q) \otimes_{D'} s)$,
and so $\bar{l}(q) \otimes_{D'} s \in (Q' \otimes S)(I)$ and $g'(\bar{l}(q) \otimes_{D'} s) = \eta_I$ and therefore $g' \in \mathcal{C}(Q' \otimes_{D'} S, T)$ is also rationally surjective.
Now (b) and (c) follow from (2.11), (2.17) and (2.18)(a).

Remark. This theorem corresponds to Theorem 10.3 of Chase and Sweedler [11], but the proof was changed. This is because we don't see how to use Theorem 8.4 of Chase and Sweedler without knowing that S is faithful over T, not only over R.

The proof of the next corollary uses Lemma (3.10). It is not used in the sequel.

(4.6) Corollary. *Let $A, B \in \boldsymbol{Hopfmon}\mathcal{C}$, A finite over I and let $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ be a retraction in $\mathcal{C}$. In addition, let I be cokernal-projective in $\mathcal{C}$. Then A^* is finite as a left and right B^*-object.*

Proof. According to Theorem (2.5), A is A-Galois over I. With $T := A^{B^*}$ and Theorem (4.5) (b), A is $T \otimes B$-Galois over T and therefore A is also finite over $T \otimes B^*$, according to Theorem (3.11). Because of Theorem (2.18) (a), $T \to A$ has a retraction in ${}_T\mathcal{C}$, and therefore T is finite over I, according to Lemma (3.3). Then, $T \otimes B^*$ is finite over B^*, according to Lemma (3.9). As a result, A is finite over B^*, according to Lemma (3.10). Then, with Theorem (3.8), we can conclude that A^* is finite over B^*.

(4.7) Lemma. *Let $S \in \boldsymbol{Mon}\mathcal{C}$, $A \in \boldsymbol{Hopfmon}\mathcal{C}$, S A-Galois over I, and $R \in c\boldsymbol{Mon}\mathcal{C}$. Then* $R \otimes S$ is $R \otimes A$-Galois over R.

Proof. The functor $R \otimes -$ trivially preserves all algebraic premises in Definition (2.1) and Definition (2.2). $R \otimes S$ is also faithfully projective (over R) according to Lemma (3.9).

(4.8) Lemma. *Let $A, B \in \boldsymbol{cHopfmon}\mathcal{C}$; A, B finite over I; $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ a retraction in $\mathcal{C}$; $S, R \in c\boldsymbol{Mon}\mathcal{C}$, and S A-Galois over I. Then:* $(R \otimes S)^{R \otimes B^*} \cong R \otimes S^{B^*}$.

Proof. With $T := S^{B^*}$ and Theorem (4.5), we see that S is $T \otimes B$-Galois over T. Then, because of Lemma (4.7), $R \otimes S = R \otimes T \otimes_T S$ is $R \otimes T \otimes B$-Galois over $R \otimes T$. Now we have:
$(R \otimes S)^{R \otimes B^*} \cong \big((R \otimes T) \otimes_T S\big)^{R \otimes T \otimes B^*}$
$\cong \big((R \otimes T) \otimes_T S\big)^{[R \otimes T \otimes B, R \otimes T]}$
$\cong R \otimes T$ according to Theorem (2.8)
$= R \otimes S^{B^*}$.

(4.9) Theorem. *Let $A, B \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ a retraction in $\mathcal{C}$, and $j \in \mathcal{C}(A^{B^*}, A)$ the canonical morphism. Then*

$A^{B^*} \underset{\eta\varepsilon}{\overset{j}{\rightrightarrows}} A \xrightarrow{e} B$ is a difference cokernel in $\boldsymbol{cMon}\mathcal{C}$.

Proof. A^{B^*} is a submonoid of A, so $(A, \nabla') \in \boldsymbol{Mon}_{A^{B^*}}\mathcal{C}$ with

$$\begin{tikzcd}
A \otimes A \arrow[r, "\nabla"] \arrow[d] & A. \\
A \otimes_{A^{B^*}} A \arrow[ur, "\nabla'"'] &
\end{tikzcd}$$

According to Theorem (4.5), A is $A^{B^*} \otimes B$-Galois over A^{B^*}, so
$\gamma' \coloneqq (\nabla' \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \alpha'): A \otimes_{A^{B^*}} A \ \to A \otimes B$ is an isomorphism.
Define $\Lambda \in \mathcal{C}\big(I \otimes_{A^{B^*}} A, B\big)$ by:
$I \otimes_{A^{B^*}} A \cong I \otimes_A A \otimes_{A^{B^*}} A \xrightarrow{\mathrm{id}_I \otimes_A \gamma'} I \otimes_A A \otimes B \cong I \otimes B \cong B.$
Λ is an isomorphism, and $\Lambda(x \otimes a) = xe(a)$ for all $X \in \mathcal{C}, x \otimes a \in \big(I \otimes_{A^{B^*}} A\big)(X)$. By virtue of its construction, the following diagram is a fiber-product diagram in $\boldsymbol{cMon}\mathcal{C}$:

$$\begin{tikzcd}
A^{B^*} \arrow[r, "j"] \arrow[d, "\varepsilon"'] & A \arrow[d, "p(\eta \otimes \mathrm{id})"'] \arrow[ddr, "e"] & \\
I \arrow[r, "p(\mathrm{id} \otimes \eta)"] \arrow[drr, "\eta"'] & I \otimes_{A^{B^*}} A \arrow[dr, "\Lambda"] & \\
 & & B
\end{tikzcd}$$

where $p = \mathrm{can}: I \otimes A \to I \otimes_{A^{B^*}} A$, $\Lambda p(\eta \otimes \mathrm{id}) = e, \Lambda p(\mathrm{id} \otimes \eta) = \eta$. Now the statement follows via a comparison of the universal properties.

Remark. Up to now, the commutativity of A has been used primarily in order to arrive at a <u>symmetric</u> monoidal category ${}_{A^{B^*}}\mathcal{C}$, because that is required for the definition of Hopf monoids. However, even the preceding Theorem (4.9) can be formulated without commutativity of A: If γ' is an isomorphism, then so is Λ, and we get the following difference cokernel diagram in $\mathcal{C}$:
$A^{B^*} \otimes A \cong I \otimes A^{B^*} \otimes A \rightrightarrows I \otimes A \to I \otimes_{A^{B^*}} A \xrightarrow{\Lambda} B.$
This property would also suffice in the proof of the following Theorem.

(4.10) Theorem. *Let $A, B_1, B_2 \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $e_i \in \boldsymbol{Hopfmon}\mathcal{C}(A, B_i)$ a retraction in $\mathcal{C}$, $S \in \boldsymbol{cMon}\mathcal{C}$, and S A-Galois over I. Then:*
$B_1{}^* \subset B_2{}^* \Leftrightarrow S^{B_2{}^*} \subset S^{B_1{}^*}$, and in particular:
$B_1{}^* = B_2{}^* \Leftrightarrow S^{B_2{}^*} = S^{B_1{}^*}$.

Proof. (a) Let $B_1{}^* \subset B_2{}^*$, i.e. there is a monomorphism $B_1{}^* \to B_2{}^*$. Since B_1 and B_2 are finite, the dual is an epimorphism $B_2 \to B_1$. Then, the definition of the fixed object implies the existence of a monomorphism that commutatively completes the following diagram:

$$\begin{tikzcd}
S^{B_1{}^*} \arrow[r] & S \arrow[r, shift left] \arrow[r, shift right] & S \otimes B_1 \\
S^{B_2{}^*} \arrow[r] \arrow[u, dashed] & S \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "="'] & S \otimes B_2, \arrow[u]
\end{tikzcd}$$

i.e. $S^{B_2{}^*} \subset S^{B_1{}^*}$.
(b) Now let $S^{B_2{}^*} \overset{i}{\hookrightarrow} S^{B_1{}^*}$ and look at the large diagram on the next page. k_i, a retraction of j_i, exists because of Theorem (2.18) (a). The squares I, III, V, and VII commute according to Lemma (4.8). γ_1 and γ_2 exist such that II and VI commute, because γ is an $S \otimes A^*$-morphism. IV commutes as a premise. Thus we have a monomorphism $l = (\mathrm{id}_S \otimes k_1 j_1)l = (\mathrm{id}_S \otimes k_1 j_2) \in {}_{[S,S]}\mathcal{C}$. Since S is faithfully projective, $S \otimes -: \mathcal{C} \to {}_{[S,S]}\mathcal{C}$ is an equivalence of categories according to Theorem

(1.6), and thus $k_1 j_2 \in \mathcal{C}\left(A^{B_2^*}, A^{B_1^*}\right)$ is a monomorphism, and the following diagram commutes:

$$
\begin{tikzcd}
A^{B_2^*} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "k_1 j_2"] & A \arrow[r] \arrow[d, "="] & B_2 \arrow[d, dashed] \\
A^{B_1^*} \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r] & B_1,
\end{tikzcd}
$$

where the epimorphism $B_2 \to B_1$ exists because of Theorem (4.9). Therefore, $B_1{}^* \subset B_2{}^*$.

$$
\begin{tikzcd}
S \otimes A^{B_1^*} \arrow[r, shift left, "\mathrm{id}_S \otimes j_1"] \arrow[d, "="] & S \otimes A \arrow[l, shift left, "\mathrm{id}_S \otimes k_1"] \arrow[d, "="] \arrow[dddddddd, bend left, "="] \\
(S \otimes A)^{(S \otimes B_1^*)} \arrow[r] \arrow[d, "\gamma_1", "\cong"'] \arrow[ur, phantom, "\mathrm{I}"] & S \otimes A \arrow[d, "\gamma"', "\cong"] \\
(S \otimes S)^{(S \otimes B_1^*)} \arrow[r] \arrow[d, "="] \arrow[ur, phantom, "\mathrm{II}"] & S \otimes S \arrow[d, "="] \\
S \otimes S^{B_1^*} \arrow[r] \arrow[d, "\mathrm{id}_S \otimes i"] \arrow[ur, phantom, "\mathrm{III}"] & S \otimes S \arrow[d, "="] \\
S \otimes S^{B_2^*} \arrow[r] \arrow[d, "="] \arrow[ur, phantom, "\mathrm{IV}"] & S \otimes S \arrow[d, "="] \\
(S \otimes S)^{(S \otimes B_2^*)} \arrow[r] \arrow[d, "\gamma_2", "\cong"'] \arrow[ur, phantom, "\mathrm{V}"] & S \otimes S \arrow[d, "\gamma"', "\cong"] \\
(S \otimes A)^{(S \otimes B_2^*)} \arrow[r] \arrow[d, "="] \arrow[ur, phantom, "\mathrm{VI}"] & S \otimes A \arrow[d, "="] \\
S \otimes A^{B_2^*} \arrow[r, shift left, "\mathrm{id}_S \otimes j_2"] \arrow[uuuuuuu, bend left, "l"] \arrow[ur, phantom, "\mathrm{VII}"] & S \otimes A. \arrow[l, shift left, "\mathrm{id}_S \otimes k_2"]
\end{tikzcd}
$$

(4.11) Fundamental Theorem. *Let $A, B_1, B_2 \in$ $\boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $e_i \in$ $\boldsymbol{Hopfmon}\mathcal{C}(A, B_i)$ a retraction in $\mathcal{C}$, $S \in$ $\boldsymbol{cMon}\mathcal{C}$, and S A-Galois over I. Then:*

(a) $S^{A^*} = I$.

(b) With $T := S^{B^*}$, S is $T \otimes B$-Galois over T, an thus in particular S is faithfully projective over T, and $T \to S$ is a section in ${}_T\mathcal{C}$.

(c) $B_1{}^* \subset B_2{}^* \Leftrightarrow S^{B_2^*} \subset S^{B_1^*}$, and in particular:
$B_1{}^* = B_2{}^* \Leftrightarrow S^{B_2^*} = S^{B_1^*}$.

Proof. (a) follows from Theorem (2.8), (b) from Theorem (4.5) and (c) from Theorem (4.10).

In the fundamental theorem, we have an injective, order-reversing mapping from the lattice of those sub-Hopf-modules of A^* that are sections in $\mathcal{C}$ to the lattice of submonoids of S. The submonoids of S that are contained in the image of this mapping, i.e. that can be represented as fixed monoids of sub-Hopf-monoids of A^*, must be sections in $\mathcal{C}$, but are otherwise more difficult to characterize. This is already the case in the Galois theory of commutative rings, see Chase, Harrison and Rosenberg [10], Chase and Sweedler [11], Magid [32] and Takeuchi [51].

One step in the direction of characterizing these submonoids of S is the following theorem. It corresponds to proposition (11.4) on page 79, or a part of Theorem 7.6(b) in Chase and Sweedler [11]. However, our proof is simpler than chapter 11 of Chase and Sweedler and requires fewer premises: We only need our Theorem (3.8), and not the full Frobenius property in Lemma 9.5 of Chase and Sweedler. In addition, we do not need to use any null objects or annihilators.

(4.12) Theorem. *Let $A, B \in \mathbf{cHopfmon}\mathcal{C}$, A finite over I, $e \in \mathbf{Hopfmon}\mathcal{C}(A, B)$ a retraction in $\mathcal{C}$, $S \in \mathbf{cMon}\mathcal{C}$, S A-Galois over I, and $i: T \to S$ a submonoid that is a section in $\mathcal{C}$. Then:*

(a) The following statements are equivalent:

(i) $T = S^{B^*}$.

(ii) $\gamma(S \otimes T) = S \otimes A^{B^*}$, i.e. there is an isomorphism $\bar{\gamma}$ that commutatively completes the following diagram:

$$\begin{tikzcd}
S \otimes S \arrow[r, "\gamma"] & S \otimes A \\
S \otimes T \arrow[u, "\mathrm{id}_S \otimes i"] \arrow[r, dashed, "\bar{\gamma}"] & S \otimes A^{B^*} \arrow[u, "\mathrm{id}_S \otimes j"'] \\
& S \otimes \mathrm{Diffker}\left(A \overset{(\mathrm{id}\otimes e)\Delta}{\underset{\mathrm{id}\otimes\eta}{\rightrightarrows}} A \otimes B\right). \arrow[u, "="']
\end{tikzcd}$$

(iii) There is an isomorphism $\bar{\varphi}$ that commutatively completes the following diagram:

$$\begin{tikzcd}
S \otimes A^* \arrow[r, "\varphi"] & {[S, S] \cong S \otimes S^*} \\
S \otimes A^*//B^* \arrow[u, "\mathrm{id}_S \otimes \mathrm{can}"'] \arrow[r, dashed, "\bar{\varphi}"] & {[T, S] \cong S \otimes T^*} \arrow[u, "{[i, \mathrm{id}_S]}"'] \\
S \otimes \mathrm{Diffcoker}\left(A^* \otimes B^* \overset{\nabla(\mathrm{id}\otimes e^*)}{\underset{\mathrm{id}\otimes\varepsilon}{\rightrightarrows}} A^*\right). \arrow[u, "="']
\end{tikzcd}$$

(b) In the case of $\mathcal{C} = k\text{-}\mathbf{Mod}$ with a commutative ring k, the above statements are equivalent to:

(iv) for all $w \in S \otimes A^*[w(T) = 0 \Leftrightarrow w \in S \otimes A^*(B^*)^+]$, where $(B^*)^+$ is the augmentation ideal of B^*.

Proof. (i) $\Leftrightarrow$ (ii): Just as in the proof of Theorem (4.10), we consider the following (abbreviated) diagram:

$$\begin{tikzcd}
S \otimes S^{B^*} \arrow[r, shift left, "\mathrm{id}_S \otimes j"] \arrow[d, "\gamma_B"', "\cong"] & S \otimes S \arrow[l, shift left, "\mathrm{id}_S \otimes k_B"] \arrow[d, "\gamma"', "\cong"] \arrow[dd, bend left=60, "="] \\
S \otimes A^{B^*} \arrow[r] & S \otimes A \\
S \otimes T \arrow[u, "\bar{\gamma}", "\cong"'] \arrow[r, shift left, "\mathrm{id}_S \otimes i"] & S \otimes S. \arrow[l, shift left, "\mathrm{id}_S \otimes k"] \arrow[u, "\gamma", "\cong"']
\end{tikzcd}$$

${\gamma_B}^{-1}\,\bar{\gamma} = (\mathrm{id}_S \otimes k_B j){\gamma_B}^{-1}\,\bar{\gamma} = (\mathrm{id}_S \otimes k_B i) \in {}_{[S,S]}\mathcal{C}$ is an isomorphism and S is faithfully projective, and so $k_B i \in \mathcal{C}\big(T, S^{B^*}\big)$ is also an isomorphism, i.e. the two subobjects are identical.
(ii) $\Leftrightarrow$ (iii): (iii) is a dualization of (ii), with the methods of Theorem (2.11).
(iii) $\Leftrightarrow$ (iv): With help of the homomorphy theorem, (iii) is equivalent to the equality of the kernels of the two mappings, and the following holds:

$\mathrm{Ke}(S \otimes A^* \to [T, S])$
$= \mathrm{Ke}(S \otimes A^* \to S \otimes A^*//B^*)$
$= S \otimes \mathrm{Ke}(A^* \to A^*//B^*)$ since S is faithfully projective
$= S \otimes A^*(\mathrm{Ke}\varepsilon_{B^*}) = S \otimes A^*(B^*)^+,$ see Oberst and Schneider [34] and Lemma (5.1).

5. Normal sub- and factor Hopf monoids

In this chapter, we investigate the concept of normality for sub- and factor Hopf monoids and its meaning in Galois theory. In the special case of a separable field extension, we obtain the concept of a normal subgroup, as can be seen in Example (6.9) and Lemma (7.4). Theorem (5.3) provides a new and interesting characterization of the concept of normality, including the case of $\mathcal{C} = k\text{-}\boldsymbol{Mod}$. Theorem (5.5) has not yet been considered for the case of $\mathcal{C} = k\text{-}\boldsymbol{Mod}$. Our Lemma (5.1), together with Theorem (4.9), contains propositions 2.1 and 2.2 of Oberst and Schneider [34]. We have proven these "purely algebraically", with less effort than in Chase and Sweedler [11] and in particular without use of the theory of algebraic groups.

(5.1) Lemma. *Let $A, B \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ a retraction in $\mathcal{C}$, and $j: A' := A^{B^*} \to A$ the canonical morphism. Let $H, H' \in \boldsymbol{Hopfmon}\mathcal{C}$, H cocommutative, H finite over I, $i \in \boldsymbol{Hopfmon}\mathcal{C}(H', H)$ a retraction in $\mathcal{C}$, and $H \otimes H' \underset{\mathrm{id}\otimes\varepsilon}{\overset{\nabla(\mathrm{id}\otimes i)}{\rightrightarrows}} H \xrightarrow{\pi} H//H'$ a difference cokernel in $\mathcal{C}$. Then:*

(a) $A' \otimes A \underset{\varepsilon\otimes\mathrm{id}}{\overset{\nabla(j\otimes\mathrm{id})}{\rightrightarrows}} A \xrightarrow{e} B$ is a difference cokernel in $\mathcal{C}$ and in $\boldsymbol{cMon}\mathcal{C}$.

(b) $H' \xrightarrow{i} H \underset{\mathrm{id}\otimes\eta}{\overset{(\mathrm{id}\otimes\pi)\Delta}{\rightrightarrows}} H \otimes H//H'$ is a difference kernel in $\mathcal{C}$ and in the category of commutative comonoids in $\mathcal{C}$.

Proof. (a) Consider the commutative diagram:

$$\begin{array}{ccccc}
A' \otimes A & \underset{\varepsilon\otimes\mathrm{id}}{\overset{\nabla(j\otimes\mathrm{id})}{\rightleftarrows}} & A & \xrightarrow{\quad e \quad} & A \\
\uparrow\cong & & \uparrow\cong & & \Lambda\uparrow\cong \\
I \otimes A' \otimes A & \underset{\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}}{\overset{\mathrm{id}\otimes\nabla(j\otimes\mathrm{id})}{\rightleftarrows}} & I \otimes A & \xrightarrow{\quad p \quad} & I \otimes_{A'} A
\end{array}$$

with $p = \mathrm{can}: I \otimes A \to I \otimes_{A'} A$ and $\Lambda: I \otimes_{A'} A \to B$ defined as in the proof of Theorem (4.9). Since the lower diagram is by definition a difference cokernel, and Λ is an isomorphism, the upper diagram is also a difference cokernel. (b) is dual to (a).

(5.2) Definition. *(a) Let $H', H \in \boldsymbol{Hopfmon}\mathcal{C}$, H cocommutative, and $i \in \boldsymbol{Hopfmon}\mathcal{C}(H', H)$ a monomorphism in $\mathcal{C}$. H' is called a **normal sub Hopf monoid** if $H//H'$ has exactly one Hopf monoid structure such that $\pi \in \boldsymbol{Hopfmon}\mathcal{C}(H, H//H')$, where $H//H'$ and π are defined as in Lemma (5.1).*
(b) Let $A, B \in \boldsymbol{cHopfmon}\mathcal{C}$, $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ an epimorphism, and $A' := A^{B^*}$. B is called a normal factor Hopf monoid of A if A' has exactly one Hopf monoid structure such that $j = \mathrm{can} \in \boldsymbol{Hopfmon}\mathcal{C}(A', A)$.

(5.3) Theorem. *Let $H', H \in \boldsymbol{Hopfmon}\mathcal{C}$, H cocommutative, H finite over I, and $i \in \boldsymbol{Hopfmon}\mathcal{C}(H', H)$ a section in $\mathcal{C}$. Then the following statements are equivalent:*

(a) H' is normal in H.

(b) $\mathrm{Diffcoker}\left(H \otimes H' \underset{\mathrm{id}\otimes\varepsilon}{\overset{\nabla(\mathrm{id}\otimes i)}{\rightrightarrows}} H\right) = \mathrm{Diffcoker}\left(H' \otimes H \underset{\varepsilon\otimes\mathrm{id}}{\overset{\nabla(i\otimes\mathrm{id})}{\rightrightarrows}} H\right)$.

(c) There is exactly one $\varrho' \in \mathcal{C}(H \otimes H', H')$ with $i\varrho' = \varrho$, where $\varrho \in \mathcal{C}(H \otimes H', H')$ is defined by $\varrho(a \otimes b) = a_1 i(b)\lambda(a_2)$ for all $X \in \mathcal{C}$, $a \otimes b \in (H \otimes H')(X)$.

Proof. (a) ⇒ (c): Consider the following diagram:

$$\begin{tikzcd}
 & H \otimes H' \arrow[dl, dashed, "\rho'"'] \arrow[d, "\rho"] & \\
H' \arrow[r, "i"] & H \arrow[r, shift left, "(\mathrm{id} \otimes \pi)\Delta"] \arrow[r, shift right, "\mathrm{id} \otimes \eta"'] & H \otimes H//H'.
\end{tikzcd}$$

According to Lemma (5.1), the lower part of the diagram is a difference kernel. Let $\pi \in \boldsymbol{Mon}\mathcal{C}(H, H//H')$. We need to demonstrate the kernel property. Let $X \in \mathcal{C}$ and $a \otimes b \in (H \otimes H')(X)$. Then:

$(\mathrm{id} \otimes \pi)\Delta\rho(a \otimes b)$
$= (\mathrm{id} \otimes \pi)\Delta\big(a_1 i(b)\lambda(a_2)\big)$ according to the definition of ρ
$= a_1 i(b_1)\lambda(a_4) \otimes \pi\big(a_2 i(b_2)\lambda(a_3)\big)$ since $H \in \boldsymbol{Hopfmon}\mathcal{C}$
$= a_1 i(b_1)\lambda(a_4) \otimes \pi\big(a_2 i(b_2)\big)\pi\big(\lambda(a_3)\big)$ since $\pi \in \boldsymbol{Mon}\mathcal{C}$
$= a_1 i(b_1)\lambda(a_4) \otimes \pi\big(a_2 \varepsilon(b_2)\big)\lambda(a_3)$ according to the definition of π
$= a_1 i(b)\lambda(a_4) \otimes \pi\big(a_2 \lambda(a_3)\big)$ since ε is a counit
$= a_1 i(b)\lambda(a_3) \otimes \pi\big(\varepsilon(a_2)\big)$ since λ is an antipode
$= a_1 i(b)\lambda(a_2) \otimes \eta$ since ε is a counit
$= (\mathrm{id} \otimes \eta)\, \rho(a \otimes b)$.

(c) ⇒ (b): Consider the following diagram:

$$\begin{tikzcd}
H \otimes H' \arrow[r, shift left, "\nabla(\mathrm{id} \otimes i)"] \arrow[r, shift right, "\mathrm{id} \otimes \varepsilon"'] \arrow[d, "r"] & H \arrow[r, "\pi_1"] \arrow[d, "="] & H//H' \\
H' \otimes H \arrow[r, shift left, "\nabla(i \otimes \mathrm{id})"] \arrow[r, shift right, "\varepsilon \otimes \mathrm{id}"'] & H \arrow[r, "\pi_2"] & H'\backslash\backslash H
\end{tikzcd}$$

where π_1 and π_2 are defined as difference cokernels, and
$r := (\varrho' \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\Delta \otimes \mathrm{id})\colon H \otimes H' \to H' \otimes H$.
The square on the left commutes:

$(\varepsilon \otimes \mathrm{id})r(a \otimes b)$
$= (\varepsilon \otimes \mathrm{id})(\varrho' \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\Delta \otimes \mathrm{id})(a \otimes b)$
$= (\varepsilon \otimes \mathrm{id})(\varrho' \otimes \mathrm{id})(a_1 \otimes b \otimes a_2)$
$= (\varepsilon \otimes \mathrm{id})(a_1 i(b)\lambda(a_2) \otimes a_3)$
$= \varepsilon\big(a_1 i(b)\lambda(a_2)\big)a_3$
$= \varepsilon(b)a$
$= (\mathrm{id} \otimes \varepsilon)(a \otimes b)$.

$\nabla(i \otimes \mathrm{id})r(a \otimes b)$
$= \nabla(i\varrho' \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\Delta \otimes \mathrm{id})(a \otimes b)$
$= \nabla(\varrho \otimes \mathrm{id})(a_1 \otimes b \otimes a_2)$
$= \nabla(a_1 i(b)\lambda(a_2) \otimes a_3)$
$= a_1 i(b)\lambda(a_2)a_3$
$= a_1 i(b)\varepsilon(a_2)$
$= ai(b)$
$= \nabla(\mathrm{id} \otimes i)(a \otimes b)$.

Therefore, there is exactly one morphism $\bar{r}\colon H//H' \to H'\backslash\backslash H$ with $\bar{r}\pi_1 = \pi_2$. Analogously, there is exactly one $\tilde{r}\colon H'\backslash\backslash H \to H//H'$ with $\tilde{r}\pi_2 = \pi_1$. From that, the equality of the two factor-objects follows.

(b) ⇒ (a): Since the morphisms ∇, i, and ε are all compatible with Δ, ε, and η, $H//H'$ is automatically a coaugmented comonoid. So, we need to demonstrate the existence of suitable morphisms $\overline{\nabla}\colon H//H' \otimes H//H' \to H'//H$ and $\bar{\lambda}\colon H//H' \to H'//H$. The uniqueness and the commutativity of the required diagrams follows from the fact that π is an epimorphism. First, consider the following diagram:

$$\begin{tikzcd}
H \otimes H \otimes H' \arrow[r, shift left, "\mathrm{id} \otimes \nabla(\mathrm{id} \otimes i)"] \arrow[r, shift right, "\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon"'] & H \otimes H \arrow[r, "\mathrm{id} \otimes \pi"] \arrow[d, "\nabla"] & H \otimes H//H' \arrow[d, "\widetilde{\nabla}"] \\
& H \arrow[r, "\pi"] & H//H'
\end{tikzcd}$$

Then:
$\pi\nabla\big(\mathrm{id} \otimes \nabla(\mathrm{id} \otimes i)\big)$
$= \pi\nabla(\nabla(\mathrm{id} \otimes \mathrm{id}) \otimes i)$
$= \pi\nabla(\mathrm{id} \otimes i)(\nabla \otimes \mathrm{id})$
$= \pi(\mathrm{id} \otimes \varepsilon)(\nabla \otimes \mathrm{id})$ since $\pi = \pi_1$
$= \pi\nabla(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)$.
Therefore, there is exactly one $\widetilde{\nabla}\colon H \otimes H//H' \to H//H'$ with $\pi\nabla = \widetilde{\nabla}(\mathrm{id} \otimes \pi)$.
Now consider the following diagram:

$$\begin{tikzcd}
H' \otimes H \otimes H \arrow[r, shift left, "\nabla(\mathrm{id} \otimes i) \otimes \mathrm{id}"] \arrow[r, shift right, "\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}"'] \arrow[d, "\mathrm{id} \otimes \mathrm{id} \otimes \pi"] & H \otimes H \arrow[d, "\mathrm{id} \otimes \pi"] & \\
H' \otimes H \otimes H//H' \arrow[r, shift left, "\nabla(i \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[r, shift right, "\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}"'] & H \otimes H//H' \arrow[r, "\pi \otimes \mathrm{id}"] \arrow[d, "\widetilde{\nabla}"] & H//H' \otimes H//H' \arrow[d, "\overline{\nabla}"] \\
& H//H' \arrow[r, "\mathrm{id}"] & H//H'
\end{tikzcd}$$

Then:
$\widetilde{\nabla}(\mathrm{id} \otimes \pi)\nabla(i \otimes \mathrm{id}) \otimes \mathrm{id}$
$= \pi\nabla(\nabla(i \otimes \mathrm{id}) \otimes \mathrm{id})$ because of the above
$= \pi\nabla\big(i \otimes \nabla(\mathrm{id} \otimes \mathrm{id})\big)$
$= \pi\nabla\big((i \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)\big)$
$= \pi(\varepsilon \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)$ since $\pi = \pi_2$
$= \pi\nabla(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})$
$= \widetilde{\nabla}(\mathrm{id} \otimes \pi)(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})$.
Then the following also holds:
$\widetilde{\nabla}(\nabla(i \otimes \mathrm{id}) \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \pi) = \widetilde{\nabla}(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \pi)$ since the upper square commutes. As a result:
$\widetilde{\nabla}(\nabla(i \otimes \mathrm{id}) \otimes \mathrm{id}) = \widetilde{\nabla}(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})$, since $(\mathrm{id} \otimes \mathrm{id} \otimes \pi)$ is an epimorphism. Therefore, there is exactly one $\overline{\nabla}\colon H//H' \otimes H//H' \to H'//H$ with $\overline{\nabla}(\mathrm{id} \otimes \pi) = \pi\nabla$. Then:
$\overline{\nabla}(\pi \otimes \pi) = \overline{\nabla}(\pi \otimes \mathrm{id})(\mathrm{id} \otimes \pi) = \widetilde{\nabla}(\mathrm{id} \otimes \pi) = \pi\nabla$.
Finally, consider the following diagram:

$$\begin{tikzcd}
H \otimes H' \arrow[r, shift left, "\nabla(\mathrm{id} \otimes i)"] \arrow[r, shift right] & H \arrow[r, "\pi"] \arrow[d, "\lambda"] & H//H' \arrow[d, "\bar{\lambda}"] \\
& H \arrow[r, "\pi"] & H//H'
\end{tikzcd}$$

Then:
$\pi\lambda\nabla(\mathrm{id} \otimes i)$
$= \pi\nabla(\lambda \otimes \lambda)\tau(\mathrm{id} \otimes i)$ because of Lemma (3.5)
$= \pi\nabla(\lambda \otimes \lambda)(i \otimes \mathrm{id})\tau$
$= \pi\nabla(i \otimes \mathrm{id})(\lambda \otimes \lambda)\tau$ since $i \in \boldsymbol{Hopfmon}\mathcal{C}(H', H)$

$= \pi\nabla(\varepsilon \otimes \mathrm{id})(\lambda \otimes \lambda)\tau$ due to cokernel property
$= \pi\lambda(\varepsilon \otimes \mathrm{id})\tau$
$= \pi\lambda(\mathrm{id} \otimes \varepsilon)$.
Therefore, there is exactly one $\bar{\lambda}: H//H' \to H//H'$ with $\bar{\lambda}\pi = \pi\lambda$.

(5.4) Remark. (a) Let k be a commutative ring, $\mathcal{C} = k$-$\boldsymbol{Mod}$, and $H'^{+} = \mathrm{Ke}(\varepsilon: H' \to k)$. With the help of the homomorphy theorem we can see that (5.3) (b) is equivalent to $HH'^{+} = H'^{+}H$. This is the concept of normality of K. Newman [33]. In this case, (5.3)(c) is equivalent to:
$a \in H, b \in H' \Longrightarrow a_1 b \lambda(a_2) \in H'$.
(b) The assertion of Theorem (5.3) can, of course, be dualized.

The following theorem corresponds to the assertion in the classical Galois theory that the fixed field of a normal subgroup of the Galois group is also Galois over the base field. The reciprocal of this theorem is true in the Galois theory of separable field extensions, but not in the theory of separable ring extensions, as can be seen in Example (5.7).

(5.5) Theorem. *Let $A, B \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $e \in \boldsymbol{Hopfmon}\mathcal{C}(A, B)$ a retraction in $\mathcal{C}$, $S \in \boldsymbol{cMon}\mathcal{C}$, and S A-Galois over I. In addition, let B be a normal factor Hopf monoid of A, and $A' := A^{B^*}$. Then S^{B^*} is A'-Galois over I.*

Proof. (a) First, we show that S^{B^*} is an A' co-object monoid. Consider the following diagram:

$$\begin{tikzcd}
 & S^{B^*} \arrow[d, "j"] \arrow[ddl, dashed, "\tilde{\alpha}"'] & \\
 & S \arrow[d, "\alpha"] & \\
S \otimes A' \arrow[r, "\mathrm{id} \otimes \pi^*"] & S \otimes A \arrow[r, shift left, "\mathrm{id} \otimes (\mathrm{id} \otimes e)\Delta"] \arrow[r, shift right, "\mathrm{id} \otimes \mathrm{id} \otimes \eta"'] & S \otimes A \otimes B
\end{tikzcd}$$

$(\mathrm{id} \otimes (\mathrm{id} \otimes e)\Delta)\alpha j$
$= (\mathrm{id} \otimes \mathrm{id} \otimes e)(\mathrm{id} \otimes \Delta)\alpha j$
$= (\mathrm{id} \otimes \mathrm{id} \otimes e)(\alpha \otimes \mathrm{id})\alpha j$
$= (\alpha \otimes \mathrm{id})(\mathrm{id} \otimes e)\alpha j$
$= (\alpha \otimes \mathrm{id})(\mathrm{id} \otimes \eta)j$ because of the definition of S^{B^*}
$= (\mathrm{id} \otimes \mathrm{id} \otimes \eta)\alpha j$.
Therefore, there is exactly one $\tilde{\alpha}: S^{B^*} \to S \otimes A'$ with $(\mathrm{id} \otimes \pi^*)\tilde{\alpha} = \alpha j$. Now consider the following diagram:

$$\begin{tikzcd}
 & S^{B^*} \arrow[d, "\tilde{\alpha}"] \arrow[dl, dashed, "\bar{\alpha}"'] & S \otimes B \otimes A \\
S^{B^*} \otimes A' \arrow[r, "j \otimes \mathrm{id}"] & S \otimes A' \arrow[r, shift left, "(\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes \eta \otimes \mathrm{id}"'] & S \otimes A \otimes A' \arrow[u, "\mathrm{id} \otimes \mathrm{id} \otimes \pi^*"']
\end{tikzcd}$$

$(\mathrm{id} \otimes \mathrm{id} \otimes \pi^*)\big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)\tilde{\alpha}$
$= \big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)(\mathrm{id} \otimes \pi^*)\tilde{\alpha}$
$= \big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)\alpha j$
$= (\mathrm{id} \otimes e \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\alpha j$ since $(S, \alpha) \in \mathcal{C}^A$
$= (\mathrm{id} \otimes e \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(\mathrm{id} \otimes \pi^*)\tilde{\alpha}$
$= (\mathrm{id} \otimes (e \otimes \mathrm{id})\Delta\pi^*)\tilde{\alpha}$
$= (\mathrm{id} \otimes (\eta \otimes \mathrm{id})\pi^*)\tilde{\alpha}$ since B is normal
$= (\mathrm{id} \otimes \mathrm{id} \otimes \pi^*)(\mathrm{id} \otimes \eta \otimes \mathrm{id})\tilde{\alpha}$.
Since $(\mathrm{id} \otimes \mathrm{id} \otimes \pi^*)$ is a monomorphism, the following also holds:
$\big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)\tilde{\alpha} = (\mathrm{id} \otimes \eta \otimes \mathrm{id})\tilde{\alpha}$. Therefore, there is exactly one $\bar{\alpha}: S^{B^*} \to S^{B^*} \otimes A'$ with

$(j \otimes \mathrm{id})\bar{\alpha} = \tilde{\alpha}$. Then: $(j \otimes \pi^*)\bar{\alpha} = (\mathrm{id} \otimes \pi^*)(j \otimes \mathrm{id})\bar{\alpha} = (\mathrm{id} \otimes \pi^*)\tilde{\alpha} = \alpha j$.
Since j and $(j \otimes \pi^*)$ are monomorphisms, the required commutativity conditions in Definition (2.1) are inherited from α to $\bar{\alpha}$.
(b) Now we demonstrate that S^{B^*} is A'-Galois over I. According to Theorem (4.5), $j: S^{B^*} \to S$ has a retraction in $\mathcal{C}$, so S^{B^*} is finite over I, according to Lemma (3.3). According to Theorem (2.18), $\eta: I \to S$ also has a retraction in $\mathcal{C}$, and so $I \to S^{B^*}$ has a retraction in $\mathcal{C}$ and as a result S^{B^*} is faithfully projective over I, according to Lemma (2.4). According to Definition (2.2), we only need to demonstrate that $\bar{\gamma} \coloneqq (\nabla_{S^{B^*}} \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha}): S^{B^*} \otimes S^{B^*} \to S^{B^*} \otimes A'$ is an isomorphism.
Consider the following diagram:

$$\begin{tikzcd}[column sep=huge]
S \otimes S^{B^*} \arrow[r, "\mathrm{id} \otimes j"] \arrow[d, "\tilde{\gamma}"] \arrow[dr, phantom, "\mathrm{I}"] & S \otimes S \arrow[r, shift left, "\mathrm{id} \otimes (\mathrm{id} \otimes e)\alpha"] \arrow[r, shift right, "\mathrm{id} \otimes \mathrm{id} \otimes \eta"'] \arrow[d, "\gamma"] \arrow[dr, phantom, "\mathrm{II}"] & S \otimes S \otimes B \arrow[d, "\gamma \otimes \mathrm{id}"] \\
S \otimes A' \arrow[r, "\mathrm{id} \otimes \pi^*"] & S \otimes A \arrow[r, shift left, "\mathrm{id} \otimes (\mathrm{id} \otimes e)\Delta"] \arrow[r, shift right, "\mathrm{id} \otimes \mathrm{id} \otimes \eta"'] & S \otimes A \otimes B
\end{tikzcd}$$

with $\tilde{\gamma} = (\nabla_S \otimes \mathrm{id})(\mathrm{id} \otimes j \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha}): S \otimes S^{B^*} \to S \otimes A'$ and so
$\tilde{\gamma}(x \otimes y) = xj\bar{\alpha}_1(y) \otimes \bar{\alpha}_2(y)$.
II commutes:
$(\mathrm{id} \otimes \mathrm{id} \otimes \eta)\gamma(x \otimes y)$
$= (\mathrm{id} \otimes \mathrm{id} \otimes \eta)\big(x\alpha_1(y) \otimes \alpha_2(y)\big)$
$= x\alpha_1(y) \otimes \alpha_2(y) \otimes 1$.
$(\gamma \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta)(x \otimes y)$
$= (\gamma \otimes \mathrm{id})(x \otimes y \otimes 1)$
$= x\alpha_1(y) \otimes \alpha_2(y) \otimes 1$.
$(\mathrm{id} \otimes (\mathrm{id} \otimes e)\Delta)(x \otimes y)$
$= (\mathrm{id} \otimes (\mathrm{id} \otimes e)\Delta)\big(x\alpha_1(y) \otimes \alpha_2(y)\big)$
$= x\alpha_1(y) \otimes \alpha_2(y) \otimes e\alpha_3(y)$.
$(\gamma \otimes \mathrm{id})(\mathrm{id} \otimes (\mathrm{id} \otimes e)\alpha)(x \otimes y)$
$= (\gamma \otimes \mathrm{id})\big(x \otimes \alpha_1(y) \otimes e\alpha_2(y)\big)$
$= x\alpha_1(y) \otimes \alpha_2(y) \otimes e\alpha_3(y)$.
I commutes:
$\gamma(\mathrm{id} \otimes j)$
$= (\nabla_S \otimes \mathrm{id})(\mathrm{id} \otimes \alpha)(\mathrm{id} \otimes j)$
$= (\nabla_S \otimes \mathrm{id})(\mathrm{id} \otimes (j \otimes \pi^*)\bar{\alpha})$
$= (\mathrm{id} \otimes \pi^*)\tilde{\gamma}$.
Therefore, $\tilde{\gamma}$ is an isomorphism, because both rows in the diagram are difference cokernels.
Now consider the following diagram:

$$\begin{tikzcd}[column sep=huge]
S^{B^*} \otimes S^{B^*} \arrow[r, "j \otimes \mathrm{id}"] \arrow[d, "\bar{\gamma}"] \arrow[dr, phantom, "\mathrm{I}"] & S \otimes S^{B^*} \arrow[r, shift left, "(\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes \eta \otimes \mathrm{id}"'] \arrow[d, "\tilde{\gamma}"] \arrow[dr, phantom, "\mathrm{II}"] & S \otimes B \otimes S^{B^*} \arrow[d, "(\mathrm{id} \otimes \tau)(\tilde{\gamma} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)"] \\
S^{B^*} \otimes A' \arrow[r, "j \otimes \mathrm{id}"] & S \otimes A' \arrow[r, shift left, "(\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes \eta \otimes \mathrm{id}"'] & S \otimes B \otimes A'
\end{tikzcd}$$

II commutes:
$(\mathrm{id} \otimes \eta \otimes \mathrm{id})\tilde{\gamma}(x \otimes y)$
$= (\mathrm{id} \otimes \eta \otimes \mathrm{id})\big(xj\bar{\alpha}_1(y) \otimes \bar{\alpha}_2(y)\big)$
$= \alpha j\bar{\alpha}_1(y) \otimes 1 \otimes \bar{\alpha}_2(y)$.
$(\mathrm{id} \otimes \tau)(\tilde{\gamma} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \eta \otimes \mathrm{id})(x \otimes y)$
$= (\mathrm{id} \otimes \tau)(\tilde{\gamma} \otimes \mathrm{id})(x \otimes y \otimes 1)$

$= \alpha j\bar{\alpha}_1(y) \otimes \bar{\alpha}_2(y).$

$\big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)\tilde{\gamma}(x \otimes y)$

$= \big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)(\nabla_S \otimes \mathrm{id})(\mathrm{id} \otimes j \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha})(x \otimes y)$

$= \big((\mathrm{id} \otimes e)(\nabla_S \otimes \nabla_A)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\alpha \otimes \alpha) \otimes \mathrm{id}\big)(\mathrm{id} \otimes j \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha})(x \otimes y)$

$= \big((\mathrm{id} \otimes e)(\nabla_S \otimes \nabla_A)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \otimes \mathrm{id}\big)(\alpha \otimes (j \otimes \pi^*)\bar{\alpha} \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha})(x \otimes y)$

$= \big((\mathrm{id} \otimes e)(\nabla_S \otimes \nabla_A)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \otimes \mathrm{id}\big)(\alpha \otimes (j \otimes \pi^*)\bar{\alpha} \otimes \mathrm{id})\big(x \otimes \bar{\alpha}_1(y) \otimes \bar{\alpha}_2(y)\big)$

$= \big((\mathrm{id} \otimes e)(\nabla_S \otimes \nabla_A)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \otimes \mathrm{id}\big)\big(\alpha_1(x) \otimes \alpha_2(x) \otimes j\bar{\alpha}_1(y) \otimes \pi^*\bar{\alpha}_2(y) \otimes \bar{\alpha}_3(y)\big)$

$= \alpha_1(x)j\bar{\alpha}_1(y) \otimes e\big(\alpha_2(x)\pi^*\bar{\alpha}_2(y)\big) \otimes \bar{\alpha}_3(y)$

$= \alpha_1(x)j\bar{\alpha}_1(y) \otimes e\alpha_2(x)\varepsilon\bar{\alpha}_2(y) \otimes \bar{\alpha}_3(y)$ according to Lemma (5.1)

$= \alpha_1(x)j\bar{\alpha}_1(y) \otimes e\alpha_2(x) \otimes \bar{\alpha}_2(y).$

$(\mathrm{id} \otimes \tau)(\tilde{\gamma} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)\big((\mathrm{id} \otimes e)\alpha \otimes \mathrm{id}\big)(x \otimes y)$

$= (\mathrm{id} \otimes \tau)(\tilde{\gamma} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\alpha_1(x) \otimes e\alpha_2(x) \otimes y)$

$= (\mathrm{id} \otimes \tau)(\tilde{\gamma} \otimes \mathrm{id})\big(\alpha_1(x) \otimes y \otimes e\alpha_2(x)\big)$

$= (\mathrm{id} \otimes \tau)\big(\alpha_1(x)j\bar{\alpha}_1(y) \otimes \bar{\alpha}_2(y) \otimes e\alpha_2(x)\big)$

$= \alpha_1(x)j\bar{\alpha}_1(y) \otimes e\alpha_2(x) \otimes \bar{\alpha}_2(y).$

I commutes:

$\tilde{\gamma}(j \otimes \mathrm{id})$

$= (\nabla_S \otimes \mathrm{id})(j \otimes j \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha})$

$= (j \otimes \mathrm{id})\big(\nabla_{S^{B^*}} \otimes \mathrm{id}\big)(\mathrm{id} \otimes \bar{\alpha})$

$= (j \otimes \mathrm{id})\bar{\gamma}.$

Therefore, $\bar{\gamma}$ is an isomorphism, because both rows in the diagram are difference cokernels.

(5.6) Remark. With the notation of above, the following assertions are equivalent:

(a) $\bar{\alpha}: S^{B^*} \to S^{B^*} \otimes A'$ exists such that diagram I (below) commutes.

(b) $\bar{\beta}: H//H' \otimes S^{B^*} \to S^{B^*}$ exists such that diagram II (below) commutes.

$$\begin{tikzcd}
S^{B^*} \arrow[r, "\bar{\alpha}"] \arrow[d, "j"'] \arrow[dr, phantom, "\mathrm{I}"] & S^{B^*} \otimes A' \arrow[d, "j \otimes \pi^*"] \\
S \arrow[r, "\alpha"] & S \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
H//H' \otimes S^{B^*} \arrow[r, "\bar{\beta}"] & S^{B^*} \arrow[dd, "j"] \\
H \otimes S^B \arrow[u, "\pi \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes j"] \arrow[r, phantom, "\mathrm{II}"] & \\
H \otimes S \arrow[r, "\beta"] & S
\end{tikzcd}$$

Proof. (a) ⇒ (b):

$j\bar{\beta}(\pi \otimes \mathrm{id})(h \otimes y)$

$= j\bar{\beta}(\pi(h) \otimes y)$

$= j\big(\mathrm{id} \otimes \pi(h)\big)\bar{\alpha}(y)$

$= (\mathrm{id} \otimes h)(j \otimes \pi^*)\bar{\alpha}(y)$

$= (\mathrm{id} \otimes h)\alpha(jy)$

$= \beta(h \otimes jy)$

$= \beta(\mathrm{id} \otimes j)(h \otimes y).$

(b) ⇒ (a):

$(\mathrm{id} \otimes h)(j \otimes \pi^*)\bar{\alpha}(y)$

$= j\big(\mathrm{id} \otimes \pi(h)\big)\bar{\alpha}(y)$

$= j\bar{\beta}(\pi(h) \otimes y)$

$= j\bar{\beta}(\pi \otimes \mathrm{id})(h \otimes y)$

$= \beta(\mathrm{id} \otimes j)(h \otimes y)$

$= \beta(h \otimes jy)$

$= (\mathrm{id} \otimes h)\alpha(jy),$
$\Longrightarrow (j \otimes \pi^*)\bar{\alpha} = \alpha j.$

(5.7) Example. Let L be the splitting field of $X^3 - 2$ over $\mathbb{Q}$. $\mathrm{Aut}(L, \mathbb{Q}) = S_3$. Let $k = L^{A_3}$ and $S = L \times L$. Then $\mathrm{Aut}_k(L) \cong \mathbb{Z}/3\mathbb{Z}$ and $\mathrm{Aut}_k(S) \cong \mathbb{Z}/3\mathbb{Z} \wr S_2$ (wreath product), see Villamayor and Zelinski [52]. Now let
$G_1 = \{((\bar{0},\bar{0}),(1)),((\bar{1},\bar{1}),(1)),((\bar{2},\bar{2}),(1)),((\bar{0},\bar{0}),(12)),((\bar{1},\bar{1}),(12)),((\bar{2},\bar{2}),(12))\} = (\Delta\mathrm{Aut}_k L) \times S_n \cong \mathbb{Z}/6\mathbb{Z}$. Clearly, $|G_1| = 6$ and $S^{G_1} = k$, and so S is Galois over k with group G_1.Now let
$G_2 = \{((\bar{0},\bar{0}),(1)),((\bar{2},\bar{1}),(1)),((\bar{1},\bar{2}),(1)),((\bar{0},\bar{0}),(12)),((\bar{2},\bar{1}),(12)),((\bar{1},\bar{2}),(12))\}$.
Let $(a,b) \in S^{G_2}$.Then:
$(a,b) = ((\bar{0},\bar{0}),(12))(a,b) = (b,a) \Rightarrow a = b$, and
$(a,a) = ((\bar{2},\bar{1}),(1))(a,a) = (\bar{2}a,\bar{1}a) \Rightarrow a = \bar{1}a = \bar{2}a.$
Therefore, $S^{G_2} = k$ and $|G_2| = 6$. As a result, S is Galois over k with group G_2. Now let
$U = \{((\bar{0},\bar{0}),(1)),((\bar{0},\bar{0}),(12))\}$. Then U is a subgroup of G_1 and of G_2. $S^U = \Delta L$ is Galois over k with group $G_1/U \cong \mathbb{Z}/3\mathbb{Z}$. ($U$ is normal in G_1). Then:
$((\bar{2},\bar{1}),(1))^{-1} = ((\bar{1},\bar{2}),(1))$ and
$((\bar{2},\bar{1}),(1))((\bar{0},\bar{0}),(12))((\bar{1},\bar{2}),(1)) = ((\bar{2},\bar{1}),(1))((\bar{2},\bar{1}),(12)) = ((\bar{1},\bar{2}),(12)) \notin U.$
As a result, U is not a normal subgroup of G_2. Now we define $A = (kG_2)^* \xrightarrow{e} (k\bar{0})^* = B$. Then S is A-Galois over $I = k$ and S^{B^*} is Galois over I, but $A \xrightarrow{e} B$ is not normal.

(5.8) Corollary. *Let $A \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $S \in \boldsymbol{cMon}\mathcal{C}$, and S A-Galois over I. Then the mapping $(e: A \to B) \mapsto (j: S^{B^*} \to S)$ is an injective, order-reversing mapping from the lattice of normal sub-Hopf-monoids of A^* that decompose in $\mathcal{C}$ to the lattice of sub-monoids of S that decompose in $\mathcal{C}$ and are Galois over I.*

Proof. The assertion follows directly from Theorem (4.11) and Theorem (5.5).

6. The category of *H* objects

As was mentioned in Example (2.3), the theory that was developed in Chapters 1 and 2 can be reduced to the Galois theory of commutative rings by defining the category $(\mathcal{C},\otimes,I)$ to be the category of k modules $(k\text{-}\boldsymbol{Mod},\otimes_k,k)$. However, this is not the only "algebraic" category in which interesting examples for the above theory exist. One possibility is to give the category $k\text{-}\boldsymbol{Mod}$ additional structure and obtain an interesting symmetric, closed monoidal category, and that is investigated in this chapter. As "additional structure", an additional module structure or a grading can be chosen, see F. Long [55]. The Galois theory for these categories is closely related to the classical theory, and the correspondence applies especially to the sub-lattice of the original one. This results in new view of classical Galois theory.

(6.1) Theorem. *Let* $(\mathcal{C},\otimes,I,[-,-])$ *be a symmetric, closed, monoidal category and* $H \in \boldsymbol{biMon}\mathcal{C}$, *$H$ cocommutative. Then* $({}_H\mathcal{C},{}_\Delta\otimes,{}_\varepsilon I,{}_H[H\otimes -,-])$ *is symmetric, closed, and monoidal, where the H-structure of the new objects is defined as follows:*
$H\otimes A\otimes B \to A\otimes B\colon h\otimes a\otimes b \mapsto h(a\otimes b) = h_1a\otimes h_2b$,
$H\otimes I \to I\colon h\otimes x \mapsto \varepsilon(h)x$,
$H\otimes {}_H[H\otimes A,B] \to {}_H[H\otimes A,B]\colon h'\otimes f \mapsto h'f$,
with $(h'f)\langle h\otimes a\rangle = f\langle hh'\otimes a\rangle$.
The closedness is defined as:
${}_H\mathcal{C}(A\otimes B,C) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} {}_H\mathcal{C}(A,{}_H[H\otimes B,C])$ *with*
$\big(\Phi(f)(a)\big)\langle h\otimes b\rangle = f(ha\otimes b)$ *and*
$\big(\Psi(g)\big)(a\otimes b) = \big(g(a)\big)\langle \eta_H\otimes b\rangle$.

Proof. ${}_H\mathcal{C}$ is clearly symmetric and monoidal. We only need to prove that Φ and Ψ are inverse to each other.
$\big(\Psi\Phi(f)\big)(a\otimes b)$
$= \big(\Phi(f)\big)(a)\langle\eta_H\otimes b\rangle$
$= f\langle\eta_H a\otimes b\rangle$
$= f\langle a\otimes b\rangle$.
$\Big(\big(\Phi\Psi(g)\big)(a)\Big)\langle h\otimes b\rangle$
$= \big(\Psi(g)\big)\langle ha\otimes b\rangle$
$= \big(g(ha)\big)\langle\eta_H\otimes b\rangle$
$= \Big(h\big(g(a)\big)\Big)\langle\eta_H\otimes b\rangle$ since $g\in {}_H\mathcal{C}(\dots)$
$= \big(g(a)\big)\langle\eta_H h\otimes b\rangle$ because of the *H*-structure of $g(a)$
$= \big(g(a)\big)\langle h\otimes b\rangle$.

(6.2) Remark. The forgetful functor $V\colon {}_H\mathcal{C}\to\mathcal{C}$ is monoidal. It preserves finite and faithfully projective objects.

Proof. The first assertion is clear, and the second follows from Pareigis [38], Theorem 17 and Corollary 19.

(6.3) Lemma. *Let the premises be the same as in Theorem (6.1), and* $H\in\boldsymbol{Hopfmon}\mathcal{C}$. *Then*
${}_H[H\otimes A,B] \underset{\Sigma}{\overset{\Gamma}{\rightleftarrows}} [A,B]$ *are functorial isomorphisms, where*
$\Gamma(f)\langle a\rangle = f\langle\eta_H\otimes a\rangle$ *and* $\big(\Sigma(g)\big)\langle h\otimes a\rangle = h_1(g\langle\lambda(h_2)a\rangle)$.

Proof. $\big(\Gamma\Sigma(g)\big)\langle a\rangle$
$= \big(\Sigma(g)\big)\langle\eta_H\otimes a\rangle$
$= \eta_H(g\langle\lambda(\eta_H)a\rangle)$
$= g\langle a\rangle$.

$\big(\Sigma\Gamma(f)\big)\langle h \otimes a\rangle$
$= h_1\big(\Gamma(f)\big)\langle \lambda(h_2)a\rangle$
$= h_1(f\langle \eta_H \otimes \lambda(h_2)a\rangle)$
$= f\langle h_1(\eta_H \otimes \lambda(h_2)a)\rangle$ since $f \in {}_H[\dots]$
$= f\langle h_1\eta_H \otimes h_2\lambda(h_3)a\rangle$ because of the H-structure of $H\otimes H$
$= f\langle h_1 \otimes \varepsilon(h_2)a\rangle$ since λ is an antipode
$= f\langle h_1\varepsilon(h_2) \otimes a\rangle$ since $\varepsilon(h_2) \in I$
$= f\langle h \otimes a\rangle$ since ε is a counit.

(6.4) Corollary. *In this case, P is finite (resp. faithfully projective) in ${}_H\mathcal{C}$ if and only if it is finite (resp. faithfully projective) in $\mathcal{C}$. Let $S \in \mathbf{cMon}_H\mathcal{C}$, $A \in \mathbf{cHopfmon}_H\mathcal{C}$, and S an A-coobject monoid in ${}_H\mathcal{C}$. Then S is A-Galois over I in ${}_H\mathcal{C}$ if and only if it is A-Galois over I in $\mathcal{C}$.*

Proof. The assertions follow directly from Remark (6.2) and Lemma (6.3).

(6.5) Description of the inner composition. *In $\mathcal{C}$, along with the inner hom-functor $[-,-]$, there is also an **inner evaluation***
$\alpha\colon \mathcal{C}([A,B],[A,B]) \cong \mathcal{C}([A,B]\otimes A, B)\colon \mathrm{id} \mapsto (\alpha\colon f \otimes a \mapsto f\langle a\rangle)$.
*This leads to an **inner composition***
$\kappa\colon \mathcal{C}([B,C]\otimes[A,B]\otimes A, C) \cong \mathcal{C}([B,C]\otimes[A,B],[A,C])\colon (f' \otimes f \otimes a \mapsto f'\langle f\langle a\rangle\rangle) \mapsto \kappa$.
In ${}_H\mathcal{C}$, we denote the inner evaluation (resp. composition) as α' (resp. κ').
${}_H\mathcal{C}({}_H[H\otimes A,B], {}_H[H\otimes A,B],) \cong {}_H\mathcal{C}({}_H[H\otimes A,B]\otimes A, B)\colon \mathrm{id} \mapsto (\alpha'\colon g\otimes a \mapsto g\langle a\rangle)$
and therefore $\alpha' = \Psi(\mathrm{id})$. Then:
$g\langle a\rangle = \alpha'(g\otimes a) = \big(\Psi(\mathrm{id})\big)(g\otimes a) = \big(\mathrm{id}(g)\big)\langle \eta_H \otimes a\rangle = g\langle \eta_H\otimes a\rangle$.
${}_H\mathcal{C}({}_H[H\otimes B,C]\otimes {}_H[H\otimes A,B]\otimes A, C) \cong$
${}_H\mathcal{C}({}_H[H\otimes B,C]\otimes_H [H\otimes A,B], {}_H[H\otimes A,C])\colon (\beta\colon g'\otimes g\otimes a \mapsto g'\langle g\langle a\rangle\rangle) \mapsto \kappa'$
and therefore $\kappa' = \Phi(\beta)$. Then:
$\big(\kappa'(g'\otimes g)\big)\langle h\otimes a\rangle$
$= \big(\Phi(\beta)(g'\otimes g)\big)\langle h\otimes a\rangle$
$= \beta(h(g'\otimes g)\otimes a)$
$= \beta(h_1g'\otimes h_2g\otimes a)$
$= h_1g'\langle h_2g\langle a\rangle\rangle$
$= h_1g'\langle h_2g\langle \eta_H\otimes a\rangle\rangle$
$= h_1g'\langle \eta_H \otimes h_2g\langle \eta_H\otimes a\rangle\rangle$
$= g'\langle h_1\otimes g\langle h_2\otimes a\rangle\rangle$.
The following diagram commutes:

$$\begin{tikzcd}
{}_H[H\otimes B,C]\otimes {}_H[H\otimes A,B] \arrow[r, "\kappa'"] \arrow[d, shift left, "\Gamma\otimes\Gamma"] & {}_H[H\otimes A,C] \arrow[d, shift left, "\Gamma"] \\
{[B,C]\otimes[A,B]} \arrow[r, "\kappa"] \arrow[u, shift left, "\Sigma\otimes\Sigma"] & {[A,C]} \arrow[u, shift left, "\Sigma"]
\end{tikzcd}$$

$\big(\Sigma\kappa(\Gamma\otimes\Gamma)\big)(g'\otimes g)\langle h\otimes a\rangle$
$= h_1\Big(\big(\kappa(\Gamma\otimes\Gamma)(g'\otimes g)\big)\langle\lambda(h_2)a\rangle\Big)$
$= h_1(\Gamma g'\langle \Gamma g\langle \lambda(h_2)a\rangle\rangle)$
$= h_1(g'\langle \eta_H\otimes \Gamma g\langle\lambda(h_2)a\rangle\rangle)$
$= h_1(g'\langle \eta_H \otimes g\langle \eta_H\otimes\lambda(h_2)a\rangle\rangle)$
$= g'\langle h_1\eta_H\otimes h_2g\langle\eta_H\otimes\lambda(h_3)a\rangle\rangle$ *since $g' \in {}_H[\dots]$ and (6.1)*
$= g'\langle h_1\otimes g\langle h_2\eta_H\otimes h_3\lambda(h_4)a\rangle\rangle$ *since $g \in {}_H[\dots]$ and (6.1)*
$= g'\langle h_1\otimes g\langle h_2\otimes\varepsilon(h_3)a\rangle\rangle$ *since λ is an antipode*
$= g'\langle h_1\otimes g\langle h_2\otimes a\rangle\rangle$ *since ε is a counit*
$= \big(\kappa'(g'\otimes g)\big)\langle h\otimes a\rangle$ *see above.*

(6.6) Remark. Let $A, H \in \boldsymbol{Hopfmon}\mathcal{C}$ and A cocommutative. Then $H \in \boldsymbol{Hopfmon}_A\mathcal{C}$ by means of $\rho: A \otimes H \to H$ if the following 7 diagrams commute:

$$\begin{tikzcd}
H \arrow[r, "\cong"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\mathrm{I}"] & I \otimes H \arrow[d, "\eta \otimes \mathrm{id}"] \\
H \arrow[r, "\rho"] & A \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes H \arrow[r, "\mathrm{id} \otimes \rho"] \arrow[d, "\nabla \otimes \mathrm{id}"'] \arrow[dr, phantom, "\mathrm{II}"] & A \otimes H \arrow[d, "\rho"] \\
A \otimes H \arrow[r, "\rho"] & H
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \otimes H \arrow[r, "\Delta \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \nabla"'] & A \otimes A \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] \arrow[d, phantom, "\mathrm{III}"] & A \otimes H \otimes A \otimes H \arrow[r, "\rho \otimes \rho"] & H \otimes H \arrow[d, "\nabla"] \\
A \otimes H \arrow[rrr, "\rho"] & & & H
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes I \arrow[r, "\varepsilon \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \eta"'] \arrow[dr, phantom, "\mathrm{IV}"] & I \arrow[d, "\eta"] \\
A \otimes H \arrow[r, "\rho"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes H \arrow[r, "\rho"] \arrow[d, "\mathrm{id} \otimes \varepsilon"'] \arrow[dr, phantom, "\mathrm{V}"] & H \arrow[d, "\varepsilon"] \\
A \otimes I \arrow[r, "\varepsilon \otimes \mathrm{id}"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes H \arrow[r, "\rho"] \arrow[d, "\mathrm{id} \otimes \lambda"'] \arrow[dr, phantom, "\mathrm{VI}"] & \mathrm{H} \arrow[d, "\lambda"] \\
A \otimes H \arrow[r, "\rho"] & H
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes H \arrow[rrr, "\rho"] \arrow[d, "\mathrm{id} \otimes \Delta"'] & & & H \arrow[d, "\Delta"] \\
A \otimes H \otimes H \arrow[r, "\Delta \otimes \mathrm{id} \otimes \mathrm{id}"] & A \otimes A \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] \arrow[u, phantom, "\mathrm{VII}"] & A \otimes H \otimes A \otimes H \arrow[r, "\rho \otimes \rho"] & H \otimes H
\end{tikzcd}$$

I.e.:
I: $\rho(1 \otimes h) = h$
II: $\rho(aa' \otimes h) = \rho(a \otimes \rho(a' \otimes h))$
III: $\rho(a \otimes hh') = \rho(a_1 \otimes h)\rho(a_2 \otimes h)$
IV: $\rho(a \otimes 1) = \varepsilon_A(a)$
V: $\varepsilon_H\, \rho(a \otimes h) = \varepsilon_A(a)\varepsilon_H(h)$
VI: $\lambda\rho(a \otimes h) = \rho(a \otimes \lambda(h))$
VII: $\Delta\rho(a \otimes h) = \rho(a_1 \otimes h_1)\rho(a_2 \otimes h_2)$
for all $X, Y, Z \in \mathcal{C}, h \in H(X), h' \in H(Y), a \in A(Z), a' \in A(W)$.
If A is commutative, then the conditions are valid for
$H \in \boldsymbol{Hopfmon}^A\mathcal{C}$ by means of $\psi: H \to A \otimes H$.

The next lemma, for which we need the proof of Theorem (6.8), is known for the case of $\mathcal{C} = k$-***Mod***, see Pareigis [40], Theorem 2.5.

(6.7) Lemma. *Let* $H \in \boldsymbol{Hopfmon}\mathcal{C}$ *and* H *cocommutative. Then* $\lambda^2 = \mathrm{id}$.

Proof. Let $X \in \mathcal{C}, h \in H(X)$. Then:
$\lambda^2(h)$
$= \lambda^2(\varepsilon(h_1)h_2)$ since ε is a counit
$= \varepsilon(h_1)\lambda^2(h_2)$
$= h_1\lambda(h_2)\lambda^2(h_3)$ since λ is an antipode
$= h_1\lambda(\lambda(h_3)h_2)$ because of Lemma (3.5)
$= h_1\lambda(\lambda(h_2)h_3)$ since H is cocommutative
$= h_1\lambda(\eta\varepsilon(h_2))$ since λ is an antipode

$= h_1\varepsilon(h_2)$ since $\lambda\eta = \eta$
$= h$ since ε is a counit.

(6.8) Theorem. *Let $H \in \boldsymbol{Hopfmon}\mathcal{C}$.*
(a) If H is cocommutative, then $H \in \boldsymbol{Hopfmon}_H\mathcal{C}$ by means of $\rho: H \otimes H \to H: a \otimes h \mapsto a_1 h\lambda(a_2)$.
(b) If H is commutative, then $H \in \boldsymbol{Hopfmon}^H\mathcal{C}$ by means of $\psi: H \to H \otimes H: h \mapsto h_1\lambda(h_3) \otimes h_2$.

Proof. (a) The 7 conditions in Remark (6.6) need to be demonstrated:
I: $\rho(1 \otimes h) = 1h\lambda(1) = h$.
II: $\rho\big(a \otimes \rho(a' \otimes h)\big)$
$= \rho\big(a \otimes a'_1\lambda(a'_2)\big)$
$= a_1a'_1h\lambda(a'_2)\lambda(a_2)$
$= a_1a'_1h\lambda(a_2a'_2)$ because of Lemma (3.5)
$= (aa')_1h\lambda((aa')_2)$
$= \rho(aa' \otimes h)$.
III: $\rho(a \otimes hh')$
$= a_1hh'\lambda(a_2)$
$= a_1h\varepsilon(a_2)h'\lambda(a_3)$ since ε is a counit
$= a_1h\lambda(a_2)a_3h'\lambda(a_4)$ since λ is an antipode
$= \rho(a_1 \otimes h)\rho(a_2 \otimes h')$.
IV: $\rho(a \otimes 1)$
$= a_1 1\lambda(a_2)$
$= \varepsilon(a)$ since λ is an antipode.
V: $\varepsilon\rho(a \otimes h)$
$= \varepsilon\big(a_1h\lambda(a_2)\big)$
$= \varepsilon\big(a_1\lambda(a_2)\big)\varepsilon(h)$
$\varepsilon(a)\varepsilon(h)$.
VI: $\lambda\rho(a \otimes h)$
$= \lambda\big(a_1h\lambda(a_2)\big)$
$= \lambda^2(a_1)\lambda(h)\lambda(a_2)$ because of Lemma (3.5) and H cocommutative
$= a_1\lambda(h)\lambda(a_2)$ because of Lemma (6.7)
$\rho\big(a \otimes \lambda(h)\big)$.
VII: $\Delta\rho(a \otimes h)$
$= \big(\rho(a \otimes h)\big)_1 \otimes \big(\rho(a \otimes h)\big)_2$
$= \big(a_1h\lambda(a_2)\big)_1 \otimes \big(a_1h\lambda(a_2)\big)_2$
$= a_1h_1\big(\lambda(a_3)\big)_1 \otimes a_2h_2\big(\lambda(a_3)\big)_2$
$= a_1h_1\lambda((a_3)_2) \otimes a_2h_2\lambda((a_3)_1)$ because of Lemma (3.5)
$= a_1h_1\lambda(a_4) \otimes a_2h_2\lambda(a_3)$
$= a_1h_1\lambda(a_2) \otimes a_3h_2\lambda(a_4)$ since H is cocommutative
$= \rho(a_1 \otimes h_1) \otimes \rho(a_2 \otimes h_2)$.
(b) is the dual of (a).

(6.9) Lemma. *Let $H \in \boldsymbol{cHopfmon}\mathcal{C}$, H cocommutative, H finite over I, $H \in \boldsymbol{Hopfmon}_H\mathcal{C}$ by means of $\rho: H \otimes H \to H: a \otimes h \mapsto a_1h\lambda(a_2)$, $H' \in {}_H\mathcal{C}$, and $i \in {}_H\mathcal{C}(H', H)$ a section in $\mathcal{C}$. Then H' is a sub-Hopf-monoid of H in ${}_H\mathcal{C}$ if and only if it is a normal sub-Hopf-monoid of H in $\mathcal{C}$.*

Proof. The assertion follows immediately from Theorem (5.3) and Theorem (6.8).

(6.10) Corollary. *Let $A \in \boldsymbol{cHopfmon}\mathcal{C}$, A finite over I, $H := A^*$, $S \in \boldsymbol{cMon}\mathcal{C}$, and S A-Galois over I. Then S is A-Galois over I in ${}_H\mathcal{C}$, where $S \in {}_H\mathcal{C}$ (resp. $H \in {}_H\mathcal{C}$) by means of β (resp. ρ). The Galois correspondence is the same as in Corollary (5.8).*

Proof. For the first assertion, we only need to demonstrate that the operation β (or α) is compatible with the H-structure, i.e. that the following diagram commutes:

$$\begin{tikzcd} H \otimes (H \otimes S) \arrow[r, "\mathrm{id} \otimes \beta"] \arrow[d, "\beta'"] & H \otimes S \arrow[d, "\beta"] \\ H \otimes S \arrow[r, "\beta"] & S \end{tikzcd}$$

where $\beta' \colon H \otimes (H \otimes S) \to H \otimes S \colon h \otimes (h' \otimes s) \mapsto \rho(h_1 \otimes h') \otimes \beta(h_2 \otimes s)$.
Then:
$\beta\big(\rho(h_1 \otimes h') \otimes \beta(h_2 \otimes s)\big)$
$= \beta\big(h_1 h' \lambda(h_2)\big) \otimes \beta(h_3 \otimes s)$
$= \beta(h_1 h' \lambda(h_2) h_3 \otimes s)$
$= \beta(h_1 h' \varepsilon(h_2) \otimes s)$
$= \beta(hh' \otimes s)$
$= \beta\big(h \otimes \beta(h' \otimes s)\big)$.
The remainder now follows from Corollary (6.4) and Lemma (6.9).

From Corollary (6.10) we see that the constructions of this chapter fit in a very natural way in Galois theory, and Theorem (5.3) shows how much the morphism ρ from Theorem (6.8) has to do with the concept of normality. In a special case, namely $H = kG'$ and $k \subset S$ a separable Galois field extension, the mapping ρ is the only one that fulfills the stated conditions. That is illustrated by Example (6.11).

(6.11) Example. Let $k \subset S$ be a separable field extension, G and G' groups and $\mathcal{D} = kG'$-$\boldsymbol{Mod}$ with the structure of Theorem (6.1), with $\mathcal{C} = k$-$\boldsymbol{Mod}$. If S is kG^*-Galois over k in the category $\mathcal{D}$, then G' can be construed as a subgroup of G, and G' operates on G by means of $g' \otimes g \mapsto g' g g'^{-1} = g'_1 g \lambda(g'_2)$, and thus as in Theorem (6.8).

Proof. According to Corollary (6.4), S is also kG^*-Galois over k in the category $\mathcal{C} = k$-$\boldsymbol{Mod}$, and so $G = \mathrm{Aut}_k(S)$, see Example (2.3). Since $S \in \boldsymbol{cMon}\mathcal{D}$, the multiplication and unit of S are kG'-module morphisms, and as a result, G' operates on S via automorphisms. Without loss of generality (i.e. up to the ineffectivity kernel of the operation of G' on S), we can assume that G' is a subgroup of G. Now let $g' \in G', g \in G$, and $x \in S$. Then:
$\big(g'(g)\big)g'(x)$
$= g'\big(g(x)\big)$ since $\beta \in kG'$-$\boldsymbol{Mod}(kG \otimes S, S)$
$= (g'g)(x)$ since $G' \subset G = \mathrm{Aut}_k(S)$
$= \big(g' g g'^{-1}\big)\big(g'(x)\big)$.
By choosing a k-base of S, we can conclude that
$g'(g) = g' g g'^{-1}$.

7. Special case: Galois theory of rings and fields

This chapter serves mainly to explain the connection between Galois theory in monoidal categories, which we have developed above, and classical Galois theory. First, that means that we consider the special case of $\mathcal{C} = k\text{-}\boldsymbol{Mod}$, as in Examples (2.3) and (2.7), and then that we ask to what extent Galois theory with groups or Lie algebras is equivalent to Galois theory with Hopf algebras. Theorem (7.1) and Lemma (7.4) demonstrate the connection between subgroups (resp. sub-Lie-algebras) and sub-Hopf-algebras. In Examples (7.7) and (7.8), the meaning of the construction in Chapter 6 for a separable field extension appears more clearly. Finally, more is said about the structure of field extensions that are Galois with respect to a pointed Hopf algebra.

(7.1) Theorem. *Let k be a commutative ring, G a set, and $H \subset kG$ a sub-coalgebra that is a direct summand as a k-modul. In addition, assume that one of the following conditions is met:*
(a) k is a field.
(b) G is finite and k is connected (i.e. $e = e^2 \in k \Rightarrow e \in \{0,1\}$.
Then there is a subset $G' \subset G$ with $H = kG'$. If G is a group, and H is a sub-Hopf-algebra of kG, then G' is a subgroup of G.

Proof. Let B be a pointed coalgebra and $A \subset B$ a sub-coalgebra. Then A is pointed, see Sweedler [50] page 157. Now let $g \in G(A)$ (i.e. similar to a group) and A^g be the irreducible component of g in A. Then $A^g \subset B^g$, see Sweedler [50] page 163. As a result, $(kG)^g = kg$, because $\bigoplus_{g \in G}(kG)^g = kG = \bigoplus_{g \in G} kg$ follows from Sweedler [50] (8.1.2). From that, it also follows that $H = \bigoplus_{g \in G(H)} H^g$. Since $G(H) \subset G$, it follows that $H^g \subset (kG)^g = kg$. Therefore $H = kG'$, with $G' \coloneqq G(H)$. If G is a group, then G' is closed under multiplication and the antipode and is thus a subgroup of G.

(b) By assumption, $kG \cong \bigoplus_{g \in G} kg$ as a coalgebra, so $kG^* \cong \prod_{g \in G} k_g$ as an algebra, and there is a k-module morphism $\sigma: H^* \to kG^*$ with $p\sigma = \mathrm{id}$, where p is the dual of the inclusion, and thus a k-algebra morphism. Now let $\mathfrak{A} = \mathrm{Ke}(p)$ and $\mathfrak{A}_g = \mathrm{pr}_g(\mathfrak{A})$. Then the following diagram commutes:

$$\begin{tikzcd}
kG^* \arrow[r, "\cong"] \arrow[d, shift right, "p"'] & \prod_{g \in G} k_g \arrow[r, shift left, "\mathrm{pr}_g"] \arrow[r, shift right, "\mathrm{inj}_g"'] \arrow[d, shift right, "\prod p_g"'] & k_g \arrow[d, shift right, "p_g"'] \\
H^* \arrow[r, "\cong"] \arrow[u, dashed, shift right, "\sigma"'] & \prod_{g \in G} k_g/\mathfrak{A}_g \arrow[r, shift left, "\overline{\mathrm{pr}}_g"] \arrow[r, shift right, "\overline{\mathrm{inj}}_g"'] \arrow[u, dashed, shift right, "\tilde{\sigma}"'] & k_g/\mathfrak{A}_g \arrow[u, dashed, shift right, "\sigma_g"']
\end{tikzcd}$$

see e.g. Bourbaki [2] (I.8.10). σ_g is a k-module section of p_g, and so $k_g = \mathfrak{A}_g \oplus \sigma\left(k_g/\mathfrak{A}_g\right)$. Since k is connected, either $\mathfrak{A}_g = 0$ or $\mathfrak{A}_g = k_g$. As a result, there is a subset $G' \subset G$ with $H^* \cong \prod_{g \in G} k_g$, and so $H = kG'$. The remainder is as in part (a).

(7.2) Example. Let k be a non-connected, commutative ring, $k \ni e = e^2 \notin \{0,1\}$, $G = \{1, g\}$ a subgroup, and $H = k \cdot 1 \oplus keg$. Then H is a sub-Hopf-algebra of kG, as a k-module a direct summand, but not of the form $kG', G' \subset G$.

Proof. H is clearly a sub-Hopf-algebra of kG, and is a k-module direct summand by means of $kG \to H: x \cdot 1 + yg \mapsto x \cdot 1 + yeg$. The last assertion is also clear, since $\{1\}$ is the only genuine subgroup of G, and $k\{1\} \subsetneq H \subsetneq kG$, where $\subsetneq$ denotes subset but not equal to.

(7.3) Corollary. *Let k be a connected, commutative ring and $k \subset S$ a separable ring extension. Then the Galois theory of Chase, Harrison and Rosenberg [10] with a group G corresponds exactly to the Galois theory with the Hopf algebra kG.*

Proof. The "equality" of the subobject lattices of G and kG follows from Theorem (7.1), of the fixed objects from Example (2.7), and of the concept "Galois" from Example (2.3).

(7.4) Lemma. *Let k be a field of characteristic $p > 0$, L a p-Lie-algebra, $U^p(L)$ the universal p-hull of L, and $H \subset U^p(L)$ a sub-Hopf-algebra. Then there is a sub-Lie-algebra $L' \subset L$ with $H = U^p(L')$.*

Proof. By assumption, $U^p(L)$ is cocommutative and irreducible, so H is as well. According to Sweedler [50] Proposition 13.2.3, for all $x \in \left(U^p(L)\right)^*$, $x^p \in k$. Now let $i \in \boldsymbol{Hopfalg}\left(H, U^p(L)\right)$ be the inclusion and $r \in k\text{-}\boldsymbol{Mod}(U^p(L), H)$ a retraction of i. Then, for every $y \in H^*$, $y^p = (yri)^p = (yr)^p i \in k$. The condition "height" or "exponent" = 1 in Sweedler [50] Proposition 13.2.3 is thus also fulfilled for H, and so $H = U^p(L')$. If $P(H)$ denotes the set of primitive elements of H, then: $L' = P(H) \subset P\left(U^p(L)\right) = L$, so L' is a sub-Lie-algebra.

(7.5) Corollary. *Let k be a field and $k \subset S$ a purely inseparable field extension. Then the Galois theory of Jacobson [26] with a p-Lie-algebra L corresponds exactly to the Galois theory with the Hopf algebra $U^p(L)$.*

Proof. The assertion is analogous to (7.3), where it is necessary to be aware that the "Lie algebra" in Jacobsen [26] is actually the set $S \otimes_k L$.

(7.6) Lemma. *Let H be a finite group algebra over a connected, commutative ring k (resp. the p-universal hull of a finite-dimensional p-Lie-algebra over a field k). Let $H' \subset H$ be a sub-Hopf-algebra that is a direct summand as a k-module. Then H' is normal in H if and only if it stems from a normal subgroup (resp. from a p-Lie-ideal).*

Proof. We make use of Theorem (5.3), Theorem (7.1), Lemma (7.4), and the consideration that, for $g \in G, h \in G', a \in L$ and $b \in L'$ the following holds: $g_1 h\lambda(g_2) = ghg^{-1}$ and $a_1 b\lambda(a_2) = ab - ba = [a, b]$.

(7.7) Example. Let $k \subset S$ be a finite, separable, Galois field extension and $G = \mathrm{Aut}_k(S)$. We define the symmetric, closed monoidal categories $\mathcal{C} = k\text{-}\boldsymbol{Mod}$ and $\mathcal{D} = kG\text{-}\boldsymbol{Mod}$, where $\mathcal{D}$ has the structure specified in Theorem (6.1). The Galois correspondence, considered in the category $\mathcal{D}$, is then the customary bijection between all normal subgroups of G and all intermediate fields $k \subset S' \subset S$ that are Galois over k.

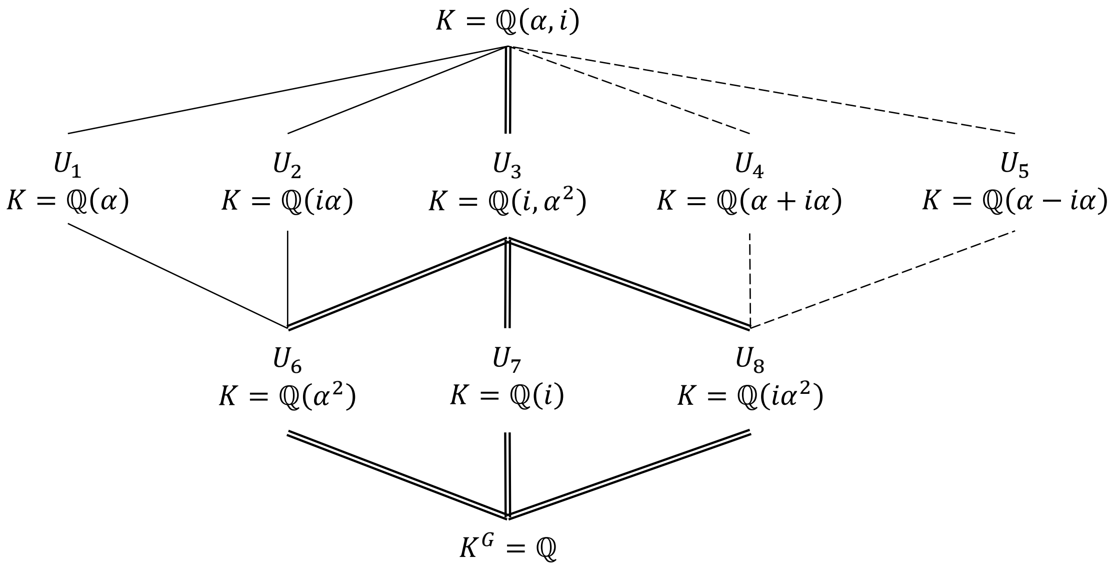
Proof. See Corollary (5.8), Corollary (6.10), Corollary (7.3), and Lemma (7.6).

(7.8) Example. In this example, we investigate a Galois field extension in the category $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]\text{-}\boldsymbol{Mod}$. According to Corollary (6.4), this is a matter of taking a Galois field extension $\mathbb{Q} \subset K$ in the category of $\mathbb{Q}\text{-}\boldsymbol{Mod}$ and then specifying a $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$-structure.
The field extension: As K, we take the splitting field of $X^4 - 2$, see Lang [54] page 200. α denotes a real root of the polynomial, $i = \sqrt{-1}$, and σ and τ automorphisms, defined by $\tau(\alpha) = \alpha$, $\tau(i) = -i, \sigma(i) = i, \sigma(\alpha) = i\alpha$. Then the Galois group is generated by σ and τ, and is isomorphic to the dihedral group D_4. The subgroups of G are $U_0 := \{1\}, U_1 := \{1, \tau\}, U_2 := \{1, \sigma^2\tau\}$, $U_3 := \{1, \sigma^2\}, U_4 := \{1, \sigma\tau\}, U_5 := \{1, \sigma^3\tau\}, U_6 := \{1, \sigma^2, \tau, \sigma^2\tau\}, U_7 := \{1, \sigma, \sigma^2, \sigma^3\}, U_8 := \{1, \sigma^2, \sigma\tau, \sigma^3\tau\}$ and G.
The $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$-structure is determined by the assertion in Example (6.11). That means that $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$ either operates trivially or as one of the subgroups U_1 through U_5. U_i operates on K via automorphisms and on $\mathbb{Q}G$ via conjugation.
The Galois correspondence in the category $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]\text{-}\boldsymbol{Mod}$ is described similarly to Example (7.7). A sub-Hopf-algebra $\mathbb{Q}G'$ of $\mathbb{Q}G$ is a sub-Hopf-monoid of $\mathbb{Q}G$ in the category $\mathbb{Q}U_i\text{-}\boldsymbol{Mod}$ if and only if $\mathbb{Q}G'$ is globally invariant under the operation of $\mathbb{Q}U_i$, i.e. if G' is transformed into itself under conjugation by elements of U_i. An intermediate field K' of $\mathbb{Q} \subset K$ is an intermediate

monoid of $\mathbb{Q} \subset K$ in the category $\mathbb{Q}U_i$-$\boldsymbol{Mod}$ if and only if K' is globally invariant under the operation of U_i, i.e. if K' is transformed into itself under conjugation by elements of U_i. Which subgroups (resp. intermediate fields) are selected during the transition from the Galois correspondence in $\mathbb{Q}$-$\boldsymbol{Mod}$ to that in $\mathbb{Q}U_i$-$\boldsymbol{Mod}$ can be seen with help of the following tables. In the first table, $\varepsilon = +$ if U_i is globally invariant under conjugation by all elements of U_j, and otherwise $\varepsilon = 0$.

	0	1	2	3	4	5	6	7	8	G
0	+	+	+	+	+	+	+	+	+	+
1	+	+	+	+	0	0	+	0	0	0
2	+	+	+	+	0	0	+	0	0	0
3	+	+	+	+	+	+	+	+	+	+
4	+	0	0	+	+	+	0	0	+	0
5	+	0	0	+	+	+	0	0	+	0
6	+	+	+	+	+	+	+	+	+	+
7	+	+	+	+	+	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+
G	+	+	+	+	+	+	+	+	+	+

	j
i	ε



If we take for example U_1 as $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$-structure, then exactly those objects that are globally invariant under complex conjugation are selected.

(7.9) Lemma. *Let k be a field and H a Hopf algebra of the form $H = H^1 \otimes kG(H)$, where H^1 is the irreducible component of 1 and $G(H)$ denotes the set of group-like elements of H.*
Let $H' \subset H$ be a sub-Hopf-algebra. Then $H' = (H')^1 \otimes kG(H')$, where $(H')^1 \subset H^1$ (resp. $G(H') \subset G(H)$) is a sub-Hopf-algebra (resp. a subgroup).

Proof. According to the remark of Sweedler [50] page 177: $H' = (H')^1 \# kG(H')$ and $G(H') \subset G(H)$ is a subgroup. According to Sweedler [50] Theorem 8.0.5 and Corollary 8.0.8, $(H')^1 \subset H$ is pointed irreducible, and therefore $(H')^1 \subset H^1$. As a result, the component-wise structure is inherited, i.e. $H' = (H')^1 \# kG(H') = (H')^1 \otimes kG(H')$.

(7.10) Theorem. *Let H be a finite, cocommutative, pointed k-Hopf-algebra, $k \subset K$ a field extension, and let K be H^*-Galois over k. Then $K = L \otimes_k M$, where L is purely inseparable and M separable Galois over k. Every intermediate field $k \subset K' \subset K$ in the Galois correspondence is of the form $K' = L' \otimes_k M'$, with $k \subset L' \subset L$ and $k \subset M' \subset M$.*

Proof. According to Sweedler [50] Theorem 10.2.3, $H = H^1 \# kG(H)$ and $K = L \otimes_k M$, *where* $L \coloneqq K^{kG}$ is purely inseparable and $M \coloneqq (K^H)^1$ is separable Galois. Since L (resp. M) is $(H^1)^*$-Galois (resp. kG^*-Galois) over k, it is easy to check that $K = L \otimes_k M$ is $(H^1 \otimes kG)^*$-Galois over k. Therefore, we can assume, without loss of generality, that $H = H^1 \# kG = H^1 \otimes kG$. The assertion over K' follows from Lemma (7.9).

The last theorem says that a field extension that is Galois with a pointed Hopf algebra splits into a tensor product of a separable with a purely inseparable component. There seems to be very little known about the non-pointed case and thus the question of whether all A-Galois field extensions split in a similar way. In general, the "mixed case" of an A-Galois ring or field extension that is not necessarily separable or purely inseparable, has not been considered very much.

8. Topological tensor products

In this chapter, a few topologies are defined on the tensor product of two Hausdorff, locally convex topological vector spaces, and some general properties stated. Then we ask if a category of topological vector spaces exists in which infinite-dimensional spaces can be "finite" in the sense of Definition (1.4). Based on some examples of non-existence we then see which constraints must be fulfilled.

(8.1) Notation. *$\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.*
$\boldsymbol{HfLc}$ *:= the category of Hausdorff, locally convex, topological vector spaces over $\mathbb{K}$, with linear, continuous mappings as morphisms.*
Now let $E, F, G \in \boldsymbol{HfLc}$.
$\mathcal{P}(E) :=$*power set of E.*
$\mathcal{B}(E) := \{X \subset E\colon X \text{ is bounded}\}$.
$\mathcal{E}(E) := \{X \subset E\colon X \text{ is bounded and finite dimensional}\}$.
$\mathcal{K}(E) := \{X \subset E\colon \text{there is a } K \subset E \text{ with } K \text{ compact and } X \subset K\}$.
$\mathrm{Bilin}(E \times F, G) := \{f \in \mathrm{Map}(E \times F, G)\colon f \text{ is bilinear}\}$.
$\mathfrak{U}_0(E) := \{X \subset E\colon X \text{ is a null neigborhood}\}$.
$\mathfrak{L}(E, F) := \{f \in \mathrm{Map}(E, F)\colon f \text{ is linear and continuous}\}$.
$\mathfrak{L}_{\mathfrak{S}}(E, F)$ *as in Bourbaki* [4] $(III,\ 3, 1)$.
$\mathfrak{L}_\beta(E, F)$ $\big(resp.\, \mathfrak{L}_\sigma(E, F), \mathfrak{L}_\kappa(E, F)\big)$ *if* $\mathfrak{S} = \mathcal{B}(E)$ $\big(resp.\ \mathfrak{S} = \mathcal{E}(E), \mathfrak{S} = \mathcal{K}(E)\big)$.

The following definitions were taken from Schwartz [45] page 9. If $\mathfrak{S}$ and $\mathfrak{T}$ are bounded, they agree with the customary definitions, Bourbaki [4] (III, 4, 2), as can be seen from the next Lemma (8.3).

(8.2) Definition. *Let $E, F, G \in \boldsymbol{HfLc}, \mathfrak{S} \subset \mathcal{P}(E), \mathfrak{T} \subset \mathcal{P}(F)$. $f \in \mathrm{Bilin}(E \times F, G)$ is called $(\mathfrak{S}, \mathfrak{T})$-**hypo-continuous** if, for all $S \in \mathfrak{S}, T \in \mathfrak{T}$: $f|_{S\times F}$ and $f|_{E\times T}$ are continuous.*

(8.3) Lemma. *Let $E, F, G \in \boldsymbol{HfLc}, \mathfrak{S} \subset \mathcal{B}(E), \mathfrak{T} \subset \mathcal{B}(F)$ $\bigcup_{S\in\mathfrak{S}} S = E, \bigcup_{T\in\mathfrak{T}} T = F$,*
$f \in \mathrm{Bilin}(E \times F, G)$, and f separately continuous. Then the following are equivalent:
(a) f is $(\mathfrak{S}, \mathfrak{T})$-hypo-continuous.
(b) (i) for all $W \in \mathfrak{U}_0(G)$ there is $S \in \mathfrak{S}$ such that for all $V \in \mathfrak{U}_0(F)$ $[f(S \times V) \subset W]$ and
(ii) for all $W \in \mathfrak{U}_0(G)$ there is $T \in \mathfrak{T}$ such that for all $V \in \mathfrak{U}_0(F)$ $[f(V \times T) \subset W]$.
(c) (i) for all $S \in \mathfrak{S}$ $f(S \times -) \subset \mathfrak{L}(F, G)$ is equicontinuous and
(ii) $f \in \mathrm{Map}\big(E, \mathfrak{L}_{\mathfrak{T}}(F, G)\big)\colon f'(x)(y) = f(x, y)$ is continuous.

Proof. The assertion (a)⇒(b) (resp. (b) ⇒ (a), (b) ⇔ (c)) follows directly from Bourbaki [4] (III, 4, ex. 2), (resp. (III, 4, 2, prop. 4), (III, 4, 2, prop. 3)).

(8.4) Notation. *Let $E, F, G \in \boldsymbol{HfLc}, \mathfrak{S} \subset \mathcal{P}(E), \mathfrak{T} \subset \mathcal{P}(F)$.*
$\mathcal{H}^{(\mathfrak{S},\mathfrak{T})}(E \times F, G) := \{f \in \mathrm{Bilin}(E \times F, G)\colon f \text{ is } (\mathfrak{S}, \mathfrak{T})\text{-hypo-continuous}\}$.
$\mathcal{L}^{(\mathfrak{S})}\big(E, \mathfrak{L}_{\mathfrak{T}}(F, G)\big) := \big\{f \in \mathfrak{L}\big(E, \mathfrak{L}_{\mathfrak{T}}(F, G)\big)\colon \text{for all } S \in \mathfrak{S}, f(S \times -) \subset \mathfrak{L}(F, G) \text{ is equicontinuous}\big\}$.

(8.5) Theorem. *Let $E, F \in \boldsymbol{HfLc}, \mathfrak{S} \subset \mathcal{P}(E), \mathfrak{T} \subset \mathcal{P}(F)$, and $\mathfrak{S}, \mathfrak{T}$ saturated, see Bourbaki [4] (III, 3, ex. 2).*
(a) There is exactly one topology on $E \otimes_{(\mathfrak{S},\mathfrak{T})} F$ such that:
$\mathfrak{L}\big(E \otimes_{(\mathfrak{S},\mathfrak{T})} F, G\big) \cong \mathcal{H}^{(\mathfrak{S},\mathfrak{T})}(E \times F, G)$ for all $G \in \boldsymbol{HfLc}$.
It is the finest $\boldsymbol{HfLc}$-topology for which the canonical mapping $E \times F \to E \otimes F$ is $(\mathfrak{S}, \mathfrak{T})$-hypo-continuous.
Let $\mathfrak{S} \subset \mathcal{B}(E), \mathfrak{T} \subset \mathcal{B}(F)$. Then:
(b) $\mathfrak{L}\big(E \otimes_{(\mathfrak{S},\mathfrak{T})} F, G\big) \cong \mathcal{L}^{(\mathfrak{S})}\big(E, \mathfrak{L}_{\mathfrak{T}}(F, G)\big)$.

Proof. (a) is a known assertion, see Schwartz [45], page 9 and Grothendieck [22], page 74. (b) follows from (a) and Lemma (8.3).

Remark. Part (a) of the theorem is also valid for $\mathfrak{T} \subset \mathcal{P}(F)$ with $\mathfrak{T} \not\subset \mathcal{B}(F)$, even though the topology on $\mathfrak{L}_{\mathfrak{T}}(F,G)$ is no longer compatible with the vector-space structure.

(8.6) Examples.
$E \otimes_\pi F$, the **projective tensor product**, is defined by $\mathfrak{S} \subset \mathcal{P}(E), \mathfrak{T} \subset \mathcal{P}(F)$. The following holds:
$\mathfrak{L}(E \otimes_\pi F, G) = \{f \in \mathrm{Bilin}(E \times F, G) : f \text{ is continuous}\}$
$E \otimes_\iota F$, the **inductive tensor product**, is defined by $\mathfrak{S} \subset \mathcal{E}(E), \mathfrak{T} \subset \mathcal{E}(F)$. The following holds:
$\mathfrak{L}(E \otimes_\iota F, G) = \{f \in \mathrm{Bilin}(E \times F, G) : f \text{ is separately continuous}\}$
$E \otimes_\beta F$ is defined by $\mathfrak{S} \subset \mathcal{B}(E), \mathfrak{T} \subset \mathcal{B}(F)$.
$E \otimes_\kappa F$ is defined by $\mathfrak{S} \subset \mathcal{K}(E), \mathfrak{T} \subset \mathcal{K}(F)$.

(8.7) Theorem. *$\boldsymbol{HfLc}$ is a symmetric, closed, monoidal category. In particular, for all $E, F, G \in \boldsymbol{HfLc}$:*
(a) $E \otimes_\iota \mathbb{K} \cong E$.
(b) $(E \otimes_\iota F) \otimes_\iota G \cong E \otimes_\iota (F \otimes_\iota G)$.
(c) $E \otimes_\iota F \cong F \otimes_\iota E$.
(d) $\mathfrak{L}(E \otimes_\iota F, G) \cong \big(E, \mathfrak{L}_\sigma(F,G)\big)$.

Proof. (a) Claim: $E \otimes_\iota \mathbb{K} \cong E \otimes_\pi \mathbb{K} \cong E$. Let $u \in \mathrm{Bilin}(E \times \mathbb{K}, G)$ be separately continuous and $W \in \mathfrak{U}_0(G)$. Then there is a $U \in \mathfrak{U}_0(E)$ with $u(U,1) \subset W$. Let $y \in \mathbb{K}$ with $|y| < 1$. Then $u(U,y) = u(yU,1) \subset u(U,1)$, since u is bilinear, and U is circular. Therefore, $u\big(U, K_1(\mathbb{K})\big) \subset W$, where $K_1(\mathbb{K})$ denotes the open unit sphere of $\mathbb{K}$. As a result, u is in fact continuous and therefore $E \otimes_\iota \mathbb{K} \cong E \otimes_\pi \mathbb{K}$. The canonical vector-space morphism $l : E \otimes_\pi \mathbb{K} \to E$ with $l(x \otimes y) = xy$ is also a homomorphism, because the topology on $E \otimes_\pi \mathbb{K}$ can be defined by the tensor products of the continuous semi-norms p on E with the norm on $\mathbb{K}$, and $(p \otimes |{-}|)(x \otimes y) = p(x)|y| = p(xy) = pl(x \otimes y)$.
(b) Let $H \in \boldsymbol{HfLc}$. The continuity of a $u \in \mathrm{Lin}\big((E \otimes_\iota F) \otimes_\iota G, H\big)$ is equivalent to the separate continuity of the corresponding $u_1 \in \mathrm{Bilin}\big((E \otimes_\iota F) \times G, H\big)$, i.e. to the continuity of $u_1(x,-) : G \to H$ and $u_1(-,g) : E \otimes_\iota F \to H$ for all $x \in E \otimes F$ and $g \in G$. This is in turn equivalent to the separate continuity of $u_2 \in \mathrm{Trilin}(E \times F \times G, H)$, i.e. to the continuity of $u_2(e,f,-) : G \to H$, $u_2(e,-,g) : F \to H$, and $u_2(-,f,g) : E \to H$ for all $e \in E, f \in F, and\ g \in G$, as can be seen in the following way: The continuity of $u_1(-,g) : E \otimes_\iota F \to H$ is equivalent to the separate continuity of $u_2(-,-,g) : E \times F \to H$ and $u_1(x,-) = u_1(\sum_{i=1}^n e_i \otimes f_i, -) = \sum_{i=1}^n u_1(e_i \otimes f_i, -) = \sum_{i=1}^n u_2(e_i, f_i, -)$, and finite sums of continuous mappings are continuous. By analogous argumentation, this is equivalent to the continuity of $u \in \mathrm{Lin}(E \otimes_\iota (F \otimes_\iota G), H)$.
(c) follows immediately from the symmetry of the corresponding definition.
(d) follows from (8.5)(b) and the fact that $\mathfrak{L}_\sigma(F,G)$ is from $\boldsymbol{HfLc}$.
(e) All "coherence" conditions in the definition of a monoidal category hold as in the category of $\mathbb{K}$-vector-spaces, since we are concerned with the same canonical isomorphisms.

(8.8) Example. Let $E \in \boldsymbol{HfLc}$ be finite in the sense of Definition (1.4). Then E is finite dimensional.

Proof. Here, "finite" means that the canonical morphism $E \otimes_\iota E'_\sigma \to \mathfrak{L}_\sigma(E,E) : x \otimes y \mapsto \big(z \mapsto xy(z)\big)$ is an isomorphism. Since the image of this mapping consists only of operators of finite rank, id_E must also have finite rank, and so E is finite dimensional.

If the concept "finite" is to cover more than just finite-dimensional spaces, the monoidal structure must be more than just a topology on the tensor product. A likely possibility is the complete tensor product of Banach spaces, which we consider in the next example.

(8.9) Example. Let $\boldsymbol{Ban}$ (resp. $\boldsymbol{Ban}_1$) be the category of Banach spaces with continuous, linear mappings (resp. continuous contractions) as morphism and $E \mathbin{\widehat{\otimes}_\pi} F$ the completion of the projective tensor product. Then:
(a) $\boldsymbol{Ban}$ and $\boldsymbol{Ban}_1$ are symmetric, closed, monoidal categories. In particular:
$\mathfrak{L}\big(E \mathbin{\widehat{\otimes}_\pi} F, G\big) \cong \mathfrak{L}\Big(E, \mathfrak{L}_\beta(F,G)\Big)$ and

$\mathfrak{L}_1\big(E \mathbin{\widehat{\otimes}_\pi} F, G\big) \cong \mathfrak{L}_1\Big(E, \mathfrak{L}_\beta(F,G)\Big)$ for all $E, F, G \in \boldsymbol{Ban}$.

(b) If E is finite in $\boldsymbol{Ban}$ or $\boldsymbol{Ban}_1$ in the sense of Definition (1.4), then E is finite dimensional.

Proof. (a) is known, see for example Semadeni [48], and can also be demonstrated directly with help of the corresponding norms.
(b) If E is finite, then $\hat{\xi}$, in the following commutative diagram, is surjective.

$$\begin{tikzcd} E \otimes_\pi E'_\beta \arrow[drr, "\hat{\xi}"] & & \\ E \otimes_\pi E'_\beta \arrow[u, "="] \arrow[rr, "\xi"] & & \mathfrak{L}_\beta(E,E) \end{tikzcd}$$

$\mathrm{Bi}(\xi)$, the set of operators of finite rank, is contained in the set of compact operators, which is closed, see Schäfer [44] page 98 and 110. Therefore, $\mathrm{id}_E \in \mathrm{Bi}\big(\hat{\xi}\big)$ is also compact, and so E is finite dimensional.

The lack of function of the last example has two reasons: 1) $E \otimes E'_\beta$ is not dense in $\mathfrak{L}_\beta(E,E)$, and 2) the topology on $E \otimes_\pi E'_\beta$ is in general genuinely finer than the subspace topology $E \otimes_\varepsilon E'_\beta$ (see Köthe [29]). The fact that an investigation of the "weak tensor product" $E \otimes_\varepsilon F$ with regards to finding non-trivial finite objects in the category $\boldsymbol{Ban}_1$ is futile can be seen from the following theorem.

(8.10) Theorem. *There is no monoidal structure on $\boldsymbol{Ban}_1$ for which infinite-dimensional, finite objects exist.*

Proof. Semadeni and Wiweger [49] have proven a theorem of Eilenberg-Watts for Banach spaces, which states that every functor that preserves limits (resp. colimits) is already isomorphic to an $\mathfrak{L}_\beta(E,-)$ (resp. to $- \mathbin{\widehat{\otimes}_\pi} E$). This puts us in the same situation as in the last example.

Now that we have seen that "finite" Banach spaces must be finite dimensional, we want to turn to the question of to what extent the projective tensor product is useful for us at all. The following theorem serves that purpose.

(8.11) Theorem. *Let $\mathcal{C}$ be a full subcategory of $\boldsymbol{HfLc}$, $E \in \mathcal{C}$, $\mathfrak{S} \subset \mathcal{B}(E)$, and $\bigcup_{S\in\mathfrak{S}} S = E$. In addition, let $- \otimes_\pi E\colon \mathcal{C} \to \mathcal{C}$ be left adjoint to $\mathfrak{L}_\mathfrak{S}(E,-)$. Then E is normable and $E'_\mathfrak{S} \cong E'_\beta$.*

Proof. Since the two functors are adjoint, it follows that the following canonical mapping is continuous: $\mathcal{C}(E'_\mathfrak{S}, E'_\mathfrak{S}) \overset{\cong}{\to} \mathcal{C}(E'_\mathfrak{S} \otimes_\pi E, \mathbb{K})\colon \mathrm{id} \mapsto k$ with $k(y \otimes x) = y(x)$.
From the universal property of the projective tensor product, it follows that the evaluation $E'_\mathfrak{S} \times E \to \mathbb{K}$ is continuous. The assertion now follows from Bourbaki [4] (IV, 3, ex. 2).

At this point, we want to ask if the "inner hom-functor" $[E,-]$ must always have the form $\mathfrak{L}_\mathfrak{S}(E,-)$. From the next theorem, we can see that that is not quite always the case: The topology can be a bit more general.

(8.12) Theorem. *Let $\mathcal{C}$ be a closed category of topological vector spaces. Then $[E,-]$ can be represented by $\mathfrak{L}(E,-)$ with a suitable topology.*

Proof. Since $\mathbb{K}$ has the finest topology, we have the following isomorphisms of (set-valued) functors: $[E,F] \cong \mathfrak{L}(\mathbb{K},[E,F]) \cong \mathfrak{L}(\mathbb{K} \otimes E, F) \cong \mathfrak{L}(E,F)$.

Remark. That the "suitable topology" on $\mathfrak{L}(E,F)$ is not always an $\mathfrak{S}$-topology can be seen in the example of barreled spaces in Chapter 10. They are a refinement (associated barreled topology) of the finest $\mathfrak{S}$-topology (topology of bounded convergence).

In our efforts to find a monoidal category of topological vector spaces in which infinite-dimensional "finite" spaces exist, we have recognized the following: 1) The monoidal structure should be more than just a topological tensor product. 2) We cannot restrict ourselves to the projective tensor product of Banach spaces. In order to get a closed monoidal category, we will restrict ourselves to a class of spaces for which $\mathfrak{L}^{(\mathfrak{S})}(E,-)=\mathfrak{L}(E,-)$ holds. As a result, the spaces will have two properties, a completeness property and the one just mentioned. However, they cannot be arbitrary, e.g. the completeness property must be inherited from G to $\mathfrak{L}_{\mathfrak{T}}(F,G)$. As we will also see, the two properties must be compatible in a certain sense with each other. This concept of compatibility will be treated in the next chapter, and then the category of quasi-complete, barreled spaces, which fulfills the above conditions in an excellent fashion, will be treated in the subsequent chapter.

9. A concept of compatibility for functors

In the previous chapter, we have seen that our category of locally convex spaces should have two properties, a completeness property, and another, yet to be mentioned. These two properties must fulfill a compatibility condition which will be treated in this chapter. A compatibility topic like this has already appeared in the literature, for example in the following situation: Frölicher and Jarchow [20] posed two questions, namely: 1) Is the completion of a Kelley space also a Kelley space, and 2) Is the Kelleyfication of a complete space also complete? A counterexample was found by Haydon [24], and S. Dierolf [15] demonstrated that the two questions are equivalent to each other in a general situation in the category of (locally convex) topological vector spaces. Motivated by this theorem, we have formulated and proven Theorem (9.3). It is of general (category-theoretical) nature and is very useful in the following chapters.

The following theorem characterizes a certain type of subcategory that appears frequently in the examples. A subcategory $\mathcal{D} \subset \mathcal{C}$ is called **epireflective** if $\mathcal{D}$ is reflective with an epimorphic reflector. $\mathcal{D}$ is called **strongly closed with respect to formation of limits** if products and difference kernels of diagrams $A \rightrightarrows B$ are in $\mathcal{D}$, where $A \in \mathcal{D}, B \in \mathcal{C}$.

(9.1) Theorem. *Let $\mathcal{C}$ be complete, locally small and cosmall and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ that is closed with respect to isomorphisms. Then the following are equivalent:*
(a) $\mathcal{D}$ is epireflective in $\mathcal{C}$.
(b) $\mathcal{D}$ is strongly closed with respect to formation of limits in $\mathcal{C}$.

Proof. See Herrlich [25] (10.2.1).

The next Lemma establishes a connection between the above concept and that of S. Dierolf [15]. We call a subcategory **bimorreflectiv** (resp. **bimorcoreflective**) if the reflector (resp. coreflector) is a bimorphism (epimorphism and monomorphism). This way, we avoid the ambiguity of the word "bireflective" in Herrlich [25].

(9.2) Lemma. *Let $\mathcal{D}$ be a full subcatgory of $\mathcal{C}$, the category of (locally convex) topological vector spaces. Then the following are equivalent:*
(a) $\mathcal{D}$ is bimorreflective (resp. bimorcoreflectiv) in $\mathcal{C}$.
(b) $\mathcal{D}$ is defined by a property that is inherited by products and linear subspaces and has the coarsest topology (resp. is inherited by linear locally convex direct sums and quotients and has the finest topology).

Proof. (a)⇒(b): The bimorphisms in $\mathcal{C}$ are the bijective morphisms. The inheritance by products and linear subspaces follows from the last theorem (strongly closed). Since the reflector is defined globally, a space with the coarsest topology must be contained in $\mathcal{D}$, because the reflector is just a coarsening of the topology.
(b)⇒(a): follows from the above Theorem (9.1) and the fact that the corresponding coarsening of the topology is a bimorphism.

(9.3) Theorem. *Let $\mathcal{C}$ be a category and $\mathcal{L}$ (resp. $\mathcal{R}$) a full epireflective (resp. monocoreflective) subcategory of $\mathcal{C}$ that is closed with respect to isomorphisms.*

$$\mathcal{L} \underset{L,\,l.a.}{\overset{V_L,\,r.a.}{\leftrightarrows}} \mathcal{C} \qquad \mathcal{R} \underset{R,\,r.a.}{\overset{V_R,\,l.a.}{\leftrightarrows}} \mathcal{C}$$

Then the following are equivalent:
(1) $R(\mathcal{L}) \subset \mathcal{L}$.
(1') $X \xrightarrow{\eta} LX$ is an isomorphism $\Rightarrow$ $RX \xrightarrow{\eta} LRX$ is an isomorphism.
(1'') $RL \xrightarrow{\eta} LRL$ is an isomorphism.
(2) $L(\mathcal{R}) \subset \mathcal{R}$.
(2') $RX \xrightarrow{\varepsilon} X$ is an isomorphism $\Rightarrow$ $RLX \xrightarrow{\varepsilon} LX$ is an isomorphism.
(2'') $RLR \xrightarrow{\varepsilon} LR$ is an isomorphism.

Remark. By assumption, the forgetful functors V_L and V_R are fully faithful, and so $\varepsilon \in \mathcal{L}(LV_LX, X)$ and $\eta \in \mathcal{R}(Y, RV_RY)$ are isomorphisms for all $X \in \mathcal{L}$ and $Y \in \mathcal{R}$. Then the following also holds: $V_LLV_LL \cong V_LL$ and $V_RRV_RR \cong V_RR$. Therefore, we can now leave V_L and V_R off and write $L^2 \cong L, R^2 \cong R$. Then we also have: $L|_{\mathcal{L}} = \mathrm{id}_{\mathcal{L}}, R|_{\mathcal{R}} = \mathrm{id}_{\mathcal{R}}$ and $\mathcal{L} = L(\mathcal{C})$ (resp. $\mathcal{R} = R(\mathcal{C})$). By assumption $\eta \in \mathcal{C}(X, LX)$ (resp. $\varepsilon \in \mathcal{C}(RX, X)$) is an epimorphism (resp. monomorphism) for all $X \in \mathcal{C}$.

Proof of Theorem (9.3). To begin with, it is easy to see that (1), (1'), and (1'') (resp. (2), (2'), and (2'')) are equivalent to each other. Then we consider the following commutative diagram:

$$\begin{tikzcd}
RX \arrow[r, "\varepsilon_1"] \arrow[d, "\eta_2"'] \arrow[drr, "\alpha"] & X \arrow[r, "\eta_1"] & LX \\
LRX \arrow[rr, dashed, "\gamma"] \arrow[urr, "\beta"] & & RLX \arrow[u, "\varepsilon_2"'] \arrow[d, "\eta_3"] \\
RLRX \arrow[u, "\varepsilon_3"] & & LRLX
\end{tikzcd}$$

Here, $\varepsilon_1, \varepsilon_2$, and ε_3 (resp. η_1, η_2, and η_3) are monomorphisms (resp. epimorphisms). Because of the universal property of R (resp. L), there is exactly one α (resp. β) with $\varepsilon_2\alpha = \eta_1\varepsilon_1$ (resp. $\beta\eta_2 = \eta_1\varepsilon_1$). We can also write: $\alpha = R(\eta_1)$ (resp. $\beta = L(\varepsilon_1)$). Now we only need to show that (1'') $\Rightarrow$ (2'), because (2'') $\Rightarrow$ (1') is analogous (dual). So now assume that (1'') is true, i.e. η_3 is an isomorphism and let ε_1 also be an isomorphism. Since η_3 is an isomorphism, there is exactly one γ with $\gamma\eta_2 = \alpha$, and so $\varepsilon_2\gamma\eta_2 = \varepsilon_2\alpha = \eta_1\varepsilon_1 = \beta\eta_2$ and, since η_2 is an epimorphism, so is $\varepsilon_2\gamma = \beta$. Since ε_1 is an isomorphism, so is $\beta = L(\varepsilon_1)$. Therefore, $\varepsilon_2\gamma\beta^{-1} = \mathrm{id}_{LX}$ and as a result ε_2 is a monomorphic retraction, so it is also an isomorphism, which was to be proven.

(9.4) Definition. *If the conditions of Theorem (9.3) (1) are met, we call L and R* ***compatible*** *with each other.*

(9.5) Corollary. *Let R and L be compatible with each other as in (9.3) and (9.4). Then:*
(a) $(RL)(RL) = RL, (LR)(LR) = LR$.
(b) $RL(\mathcal{C}) = LR(\mathcal{C})$.
(c) $\mathcal{RL} := RL(\mathcal{C})$ *is a full, epireflexive subcategory of* $\mathcal{R}$ *and a full, monocoreflective subcategory of* $\mathcal{L}$.

Proof. (a) $R(LRL) = RRL = RL$. LR is analogous.
(b) Let $X \in \mathcal{C}$ with $X \cong RLX$. Then: $LRX \cong LRRLX \cong LRLX \cong RLX \cong X$. The reverse is analogous.
(c) For all $X \in \mathcal{RL}, Y \in \mathcal{R}$ we have: $\mathcal{R}(Y, X) = \mathcal{C}(Y, X) = \mathcal{C}(Y, V_LX) \cong \mathcal{L}(LY, X) = \mathcal{RL}(LY, X)$.

As we can easily see, the commutation of the two functors $\mathcal{R}$ and $\mathcal{L}$ is stronger than their compatibility. The fact that it is genuinely stronger can be seen from the following example. In particular, we cannot conclude from Corollary (9.5) (b) that $RL = LR$. However, from the proof of Theorem (9.3), we know that the two canonical functorial morphisms from LR to RL are equal.

(9.6) Example. (a) In the category of abelian groups, we define $L(X) = X/2X$ and $R(X) = T(X) =$ the torsion subgroup of X. Then the conditions of Theorem (9.3) are fulfilled, R and L are compatible, but $RL(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \neq 0 = LR(\mathbb{Z})$.
(b) Let $(X, \mathcal{T}) = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\}$ with the relative product topology and Ba (resp. Qc) the associated barreled topology (resp. quasi-completion) as in (10.1) (resp. (10.4)). $(X, \mathcal{T})$ has a countable dimension and is metrizable, Ba$\mathcal{T}$ is the finest locally convex topology on X and is complete. $\mathrm{Qc}(X, \mathcal{T}) = (\mathbb{K}^{\mathbb{N}}, \text{product topology})$ is barreled. Therefore, $\mathrm{BaQc}(X, \mathcal{T}) = \mathrm{Qc}(X, \mathcal{T}) = \omega \neq \varphi = \mathrm{Ba}(X, \mathcal{T}) = \mathrm{QcBa}(X, \mathcal{T})$.

(9.7) Corollary. *In the following, (i) (a) and (i) (b) are equivalent to each other:*

(1) In $\boldsymbol{HfLc}$*:*
   (a) The completion of a compactly determined space is compactly determined.
   (b) The associated compactly determined space to a complete space is complete.

(2) In the category of locally convex spaces:
   (a) The associated barreled (resp. bornological) space to a nuclear (resp. Schwartz) space is nuclear (resp. Schartz).
   (b) The associated nuclear (resp. Schwartz) space to a barreled (resp. bornological) space is barreled (resp. bornological).

(3) In the category of Hausdorff topological spaces:
   (a) The Stone-Čech compactification of the quotient of a metrizable space is quotient of a metrizable space.
   (b) Let K *be a compact topological space. On* K*, we define a new topology by means of:* $A \subset K$ *is closed if and only if* A *fulfills the following condition: a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* A *converges in* $K \Rightarrow \lim(x_n) \in A$*. Then,* K *with this topology is compact.*

Remark. (1) was the first application of the theorem, see the introduction of this chapter and Chapter 11, in particular (11.6) through (11.10). The assertions are false according to an example of Haydon [24]. The assertions (2) are applications of our theorem in the category of locally convex spaces and was not yet covered by the theorem of S. Dierolf [15]. Example (9.8) shows that Ba and Nu are not compatible. The assertions in (3) are a case of compatibility between the functors "Stone-Čech compactification" and "associated sequence-determined topology", see Herrlich [25]. In order to see that the assertions are false, it suffices to find a compact, but not sequence-determined space, because, in that case, the associated sequence-determined topology is genuinely finer, and therefore not compact. That will be shown in Example (9.9), which was kindly communicated by S. Dierolf.

(9.8) Example. Let $(X,\mathcal{T})$ be an infinite-dimensional Banach space and Ba (resp. Nu) the associated barreled (resp. nuclear) topology. $(X,\mathcal{T})$ is barreled. $\big(X,\sigma(X,X')\big)$ is nuclear, since it is a weak topology. Therefore, $\sigma(X,X') \subset \mathrm{Nu}\mathcal{T} \subsetneq \mathcal{T} = \tau(X,X)$. $\mathrm{Nu}\mathcal{T}$, the associated nuclear space, is not a Mackey space, and therefore neither barreled, bornological, nor quasi-barreled. $\mathrm{Ba}\big(X,\sigma(X,X')\big) = (X,\mathcal{T})$ is not nuclear, and therefore $\mathrm{BaNu} \neq \mathrm{NuBaNu}$.

(9.9) Example. Let ω be the smallest non-countable ordinal number and let $X \coloneqq \{x : x \leq \omega\}$ with the order topology, see e.g. Kelley [27]. Then X is Hausforff and compact, but not sequence determined.

Proof. Let $f: X \to \mathbb{R}$ be defined by $f(\omega) = 1$ and $f(x) = 0$ for all $x < \omega$. Clearly, f is not continuous. Now we will show that f is sequence continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X, and $a = \lim(x_n)$.
Case 1: $a < \omega$. Then $x_n < \omega$ for almost all $n \in \mathbb{N}$ and as a result $f(x_n) = 0$ for almost all $n \in \mathbb{N}$ and therefore $f(x_n)$ converges towards $0 = f(a)$.
Case 2: $a = \omega$. Then $x_n = \omega$ for almost all $n \in \mathbb{N}$ (because, for every countable set of ordinal numbers, there is a countable upper bound). Therefore, $f(x_n) = 1$ for almost all $n \in \mathbb{N}$ and, as a result, $f(x_n)$ converges towards $1 = f(a)$.

The next lemma shows that our compatibility concept cannot be interesting in the customary "algebraic" categories.

(9.10) Lemma. *Let* $\mathcal{C}$ *be a category in which either every monomorphism is a kernel, or every epimorphism is a cokernel. Then the conditions of Theorem (9.3) are always fulfilled.*

Proof. If every monomorphism is a kernel, then $RLX \to LX$ is a kernel, and so $RLX \in \mathcal{L}$ according to Theorem (9.1), i.e. $RLX = LRLX$. The other case is analogous.

10. The category of quasi-complete, barreled spaces

The most important category-theoretical properties of the category of quasi-complete, Hausdorff, barreled, topological vector spaces are covered in this chapter. The closed monoidal structure, which is given by the functors $\mathrm{Qc}(- \otimes_\iota -)$ and $\mathrm{Ba}\mathcal{L}_\beta(-,-)$, are investigated. In this category, we have many examples of spaces for which $E \otimes E'_\beta$ is dense (and in fact sequence dense) in $\mathcal{L}_\beta(E,E)$. With that, we have found a closed, monoidal category in which the first obstacle of Example (8.9) has been overcome. However, the topology on $E \otimes E'$ is still finer than the subspace topology, so that all examples are negative, see Corollary (10.16) and Lemma (10.19).

(10.1) Notation. *With $\boldsymbol{HfBa} \subset \boldsymbol{HfLc}$ we denote the full subcategory of Hausdorff, barreled spaces, see Bourbaki [4] (III, 1, 1).* Ba: $\boldsymbol{HfLc} \to \boldsymbol{HfBa}$ *is "barrelize" (associated barreled space), as in Robert [43].* $\boldsymbol{HfBa}$ *is a full, monocoreflective subcategory of* $\boldsymbol{HfLc}$.

(10.2) Theorem. *Let $E, F, G \in \boldsymbol{HfLc}$. Then:*

(a) $F \in \boldsymbol{HfBa} \Rightarrow \mathcal{L}(E \otimes_\beta F, G) \cong \mathcal{L}\left(E, \mathcal{L}_\beta(F,G)\right)$.

(b) $E, F \in \boldsymbol{HfBa} \Rightarrow E \otimes_\iota F = E \otimes_\beta F \in \boldsymbol{HfBa}$.

Proof. (a) Let $B \in \mathcal{B}(E)$ and $f \in \mathcal{L}\left(E, \mathcal{L}_\beta(F,G)\right)$. Then $f(B)$ is bounded in $\mathcal{L}_\beta(F,G)$, see Bourbaki [4] (III, 2, 3, Cor. 1), and so it is also bounded in $\mathcal{L}_\sigma(F,G)$ and, as a result, uniformly continuous in $\mathcal{L}(F,G)$, see Bourbaki [4] (III, 3, 6, Thm. 2). Therefore, we have in fact $f \in \mathcal{L}^{\mathcal{B}(E)}\left(E, \mathcal{L}_\beta(F,G)\right)$ and (a) follows from Theorem (8.5).
(b) Let $k = \mathrm{can}\colon E \times F \to E \otimes F$. By definition, $E \otimes_\iota F$ has the finest $\boldsymbol{HfLc}$-topology such that k is $\left(\mathcal{E}(E), \mathcal{E}(E)\right)$-hypo-continuous, i.e. separately continuous. In other words, $E \otimes_\iota F$ has the $\boldsymbol{HfLc}$-final-topology with respect to all mappings $k(x,-), k(-,y), x \in E, y \in F$. So, $E \otimes_\iota F$ is a colimit (in $\boldsymbol{HfLc}$) of barreled spaces, and so it is itself a barreled space, according to Theorem (9.1) or Bourbaki [4] (III, 1, 2, Prop. 2). The proof of part (a) shows that, for $E, F \in \boldsymbol{HfBa}$, $\mathcal{K}^{(\mathcal{E}(E),\mathcal{E}(F))}(E \times F, G) = \mathcal{K}^{(\mathcal{B}(E),\mathcal{B}(F))}(E \times F, G)$ holds, see Bourbaki [4] (III, 4, 2). $E \otimes_\iota F = E \otimes_\beta F$ then follows from the universal property in Theorem (8.5).

Remark. We could have also used the concept "quasi-barreled" here, because of Bourbaki [4] (III, 3, ex. 17). But, since we later only consider quasi-complete spaces, that would not be a true generalization, see e.g. Schäfer [44] page 142.

(10.3) Corollary. $\boldsymbol{HfBa}$ *is a symmetric, closed, monoidal category. In particular,* $\mathcal{L}(E \otimes_\iota F, G) \cong \mathcal{L}\left(E, \mathrm{Ba}\mathcal{L}_\beta(F,G)\right)$.

Proof. The assertion follows immediately from Theorem (8.7), Theorem (10.2) and the universal property of Ba.

(10.4) Notation. *With $\boldsymbol{QcHfLc} \subset \boldsymbol{HfLc}$ (resp. $\boldsymbol{QcHfBa} \subset \boldsymbol{HfBa}$) we denote the full subcategory of quasi-complete spaces, see Bourbaki [4] (III, 2, 5, Def. 3).* Qc: $\boldsymbol{HfLc} \to \boldsymbol{QcHfLc}$ *denotes the quasi-completion, see Robert [43] or Schwartz [46].* $\boldsymbol{QcHfLc}$ *is a full, epireflective subcategory of* $\boldsymbol{HfLc}$.

(10.5) Lemma.

(a) $\mathrm{Qc}(\boldsymbol{HfBa}) \subset \boldsymbol{HfBa}$.

(b) $\mathrm{Ba}(\boldsymbol{QcHfLc}) \subset \boldsymbol{QcHfLc}$.

(c) $\boldsymbol{QcHfBa}$ *is a full, epireflective subcategory of* $\boldsymbol{HfBa}$ *and a full, monocoreflective subcategory of* $\boldsymbol{QcHfLc}$.

Proof. (a) follows directly from Robert [43] (1.1.5) (a), (b) follows from (a) and Theorem (9.3), and (c) follows from (a), (b), and Theorem (9.5).

(10.6) Theorem. *$\boldsymbol{QcHfBa}$ is a symmetric, closed monoidal category. In particular:*
(a) $\mathrm{Qc}(E \otimes_\iota \mathbb{K}) \cong E$.
(b) $\mathrm{Qc}(\mathrm{Qc}(E \otimes_\iota F) \otimes_\iota G) \cong \mathrm{Qc}\big(E \otimes_\iota \mathrm{Qc}(F \otimes_\iota G)\big)$.
(c) $\mathrm{Qc}(E \otimes_\iota F) \cong \mathrm{Qc}(F \otimes_\iota E)$.
(d) $\mathcal{L}(\mathrm{Qc}(E \otimes_\iota F) \otimes_\iota G) \cong \mathcal{L}\left(E, \mathrm{Ba}\mathcal{L}_\beta(F,G)\right)$ $for\ all\ G \in \boldsymbol{QcHfBa}$.

Proof. (a), (b), and (c) follow immediately from Corollary (10.3) and Lemma (10.5) and, for (b), the following consideration: For all $H \in \boldsymbol{QcHfBa}$:
$\mathcal{L}(\mathrm{Qc}(\mathrm{Qc}(E \otimes_\iota F) \otimes_\iota G), H)$
$\cong \mathcal{L}(\mathrm{Qc}(E \otimes_\iota F) \otimes_\iota G, H)$
$\cong \mathcal{L}\left(\mathrm{Qc}(E \otimes_\iota F), \mathcal{L}_\beta(G,H)\right)$
$\cong \mathcal{L}\left(E \otimes_\iota F, \mathcal{L}_\beta(G,H)\right)$
$\cong \mathcal{L}\big((E \otimes_\iota F) \otimes_\iota G, H\big)$
$\cong \mathcal{L}(E \otimes_\iota (F \otimes_\iota G), H)$
$\cong \mathcal{L}\big(\mathrm{Qc}\big(E \otimes_\iota \mathrm{Qc}(F \otimes_\iota G)\big), H\big)$.
(d) follows from (10.3), (10.5), and the fact that $\mathcal{L}_\beta(F,G)$ is quasi-complete, see Bourbaki [4] (III, 3, 7, Cor. 2).

(10.7) Theorem. *$\boldsymbol{QcHfBa}$ is complete and cocomplete. Limits are the* Ba*-images of limits formed in* $\boldsymbol{HfLc}$ *and colimits are the* Qc*-images of colimits formed in* $\boldsymbol{HfLc}$*.*

Proof. The assertion follows from the fact that $\boldsymbol{HfLc}$ is complete and cocomplete, Lemma (10.5) (c), Theorem (9.1) and Pareigis [41], 2.12, Prop. 4 or also Herrlich [25].

(10.8) Lemma. *Let $A, B \in \boldsymbol{QcHfBa}$ and $h \in \mathcal{L}(A,B)$. Then the following equivalences hold in $\boldsymbol{QcHfBa}$:*
(a) (i) h is a monomorphism $\Leftrightarrow$
(ii) h is injective.
(b) (i) h is an epimorphism $\Leftrightarrow$
(ii) $\mathrm{Bi}(h)$ *is dense in B.*
(c) (i) h is a kernel $\Leftrightarrow$
(ii) A is the barrelization of a closed subspace of B $\Leftrightarrow$
(iii) h is injective, $\mathrm{Bi}(h) = \overline{\mathrm{Bi}(h)}$*, and A has the coarsest barreled topology for which h is continuous.*
(d) (i) h is a cokernel $\Leftrightarrow$
(ii) B is the quasi-completion of a quotient with respect to a closed subspace $\Leftrightarrow$
(iii) $\mathrm{Bi}(h)$ *is strictly dense in B and h is open.*

Proof. (a) (ii) $\Rightarrow$ (i) is clear.
(i) $\Rightarrow$ (ii): Assume that h is not injective. Then $\mathrm{BaKe}(h) \underset{0}{\overset{i}{\rightrightarrows}} A \xrightarrow{h} B$ is a contradiction to h being a monomorphism.
(b) (ii) $\Rightarrow$ (i) is clear because A and B are Hausdorff.
(i) $\Rightarrow$ (ii): Assume that $\mathrm{Bi}(h)$ is not dense in B. Then $A \xrightarrow{h} B \underset{0}{\overset{p}{\rightrightarrows}} B/\overline{\mathrm{Bi}(h)} \to \mathrm{Qc}(B/(\mathrm{Bi}(h)))$ is a contradiction to h being an epimorphism.
(c) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear because of Theorem (10.7).

(iii) ⇒ (i): Claim: h is a kernel of jp in the following diagram:

$$
\begin{tikzcd}
A \arrow[r] & \mathrm{Bi}(h) \arrow[r, "i"] & B \arrow[r, shift left, "p"] \arrow[r, shift right, "0"'] & B/\mathrm{Bi}(h) \arrow[r, "j"] & \mathrm{Qc}\big(B/\mathrm{Bi}(h)\big) \\
A' \arrow[u, dashed, "g'"] \arrow[ur, dashed, "g"'] \arrow[urr, "h'"']
\end{tikzcd}
$$

Let $A' \in \boldsymbol{QcHfBa}$ and $h' \in \mathcal{L}(A', B)$ with $jph' = j0h' = 0$. j is a monomorphism $\Rightarrow ph' = 0 \Rightarrow$ $\mathrm{Bi}(h') \subset \mathrm{Ke}(h') = \mathrm{Bi}(h)$, so there is exactly one g with $h' = ig$. g is continuous because of the subspace topology on $\mathrm{Bi}(h)$. Since A' is barreled, there is exactly one g' with $h' = g'h$.
(d) (i) ⇒ (ii) ⇒ (iii) is clear because of Theorem (10.7).
(iii) ⇒ (i): Claim: h is a cokernel of i in the following diagram:

$$
\begin{tikzcd}
\mathrm{BaKe}(h) \arrow[r] & \mathrm{Ke}(h) \arrow[r, shift left, "i"] \arrow[r, shift right, "0"'] & A \arrow[r, "h"] \arrow[d, "h_1"] & B \arrow[dr, dashed, "g'"] & \\
 & & A/\mathrm{Ke}(h) \arrow[r, "h_2"] & \mathrm{Bi}(h) \arrow[u, "h_3"] \arrow[r, dashed, "g"] & B'
\end{tikzcd}
$$

Let $B' \in \boldsymbol{QcHfBa}$ and $h' \in \mathcal{L}(A, B')$ with $h'i = h'0 = 0$. $\Rightarrow \mathrm{Ke}(h) \subset \mathrm{Ke}(h') \Rightarrow$ there is exactly one g with $gh_2h_1 = h'$, since h_2 is an isomorphism and h_1 is open. Since h_3 is injective, continuous, relatively open, and strictly dense, there is exactly one g' with $h' = gh_2h_1 = g'h_3h_2h_1 = g'h$.

(10.9) Theorem. *In* $\boldsymbol{QcHfBa}$ *:*
(a) (i) $\mathbb{K}$ is a generator.
(ii) $\mathbb{K}$ is a cogenerator.
(b) (i) $\mathbb{K}$ is injective.
(ii) $\mathbb{K}$ is not cokernel projective, and thus not projective.
(c) 0 is the only cokernel-projective object.

Proof. (a) Let $A, B \in \boldsymbol{QcHfBa}$ and $f, g \in \mathcal{L}(A, B)$ with $f \neq g$. Then there is an $x \in A$ with $f(x) \neq g(x)$.
(i) Let $h \in \mathcal{L}(\mathbb{K}, A)$ be defined by $1 \mapsto x$. h exists and is uniquely determined by the preceding, since $\mathbb{K}$ has the finest topology. Now we have $fh \neq gh$, so $\mathbb{K}$ is a generator.
(ii) Define $y \coloneqq f(x) - g(x) \in B$. Since $y \neq 0$ and B is locally convex, there is an $h \in \mathcal{L}(B, \mathbb{K})$ with $h(y) \neq 0$, according to the theorem of Hahn-Banach. Therefore, $hf \neq hg$ and as a result $\mathbb{K}$ is a cogenerator.
(b) (i) Let $h: A \to B$ be a monomorphism and $g \in \mathcal{L}(A, \mathbb{K})$. According to the last Lemma (10.8) (a), h is injective, and so there is a $g' \in \mathcal{L}(B, \mathbb{K})$ with $g'h = g$. g' is continuous, because $\mathbb{K}$ has the coarsest Hausdorff linear topology.
(c) According to Bourbaki [4] (IV, 4, ex. 10) there is an $A \in \boldsymbol{QcHfBa}$ with a closed subspace U such that the quotient A/U is not quasi-complete. Therefore, there is an $h: A \xrightarrow{p} A/U \xrightarrow{i} \mathrm{Qc}(A/U) \eqqcolon B$ cokernel in $\boldsymbol{QcHfBa}$ according to Lemma (10.8) (d), and h is not surjective. Let $x \in B \backslash \mathrm{Bi}(h)$ and define $g' \in \mathcal{L}(\mathbb{K}, B)$ by $g'(1) = x$. Now let $0 \neq P \in \boldsymbol{QcHfBa}$. Since P is locally convex, there is a $g'' \in \mathcal{L}(P, \mathbb{K})$ with $g'' \neq 0$, so there is a $y \in P$ with $g''(y) = 1$. With $g \coloneqq g'g''$ we then have: $g(y) = x$, and so g cannot be factored over h, and as a result P is not cokernel projective. With that, (c) and also (b) (ii) have been demonstrated.

Remark. This theorem, (b) (ii), has repercussions in the Morita theory of monoidal categories because, according to Lemma (1.5), the concepts “finite” and “finitely generated projective” must coincide in this case.

The assertion (c) also shows how little use the concept "projective" has in this category. Similar questions were also considered by Dostal [17] and Geiler [21].

(10.10) Corollary. *Let* $E \in \boldsymbol{QcHfBa}$*. Then:*

(a) $\tilde{g}\colon \mathrm{Qc}\big(\mathrm{Ba}E'_\beta \otimes_\iota E\big) \to K\colon l \otimes x \mapsto l(x)$ *and*
$\tilde{f}\colon \mathrm{Qc}\big(E \otimes_\iota \mathrm{Ba}E'_\beta\big) \to \mathrm{Ba}\mathcal{L}_\beta(E,E)\colon x \otimes l \mapsto \big(y \mapsto xl(y)\big)$ *are continuous and linear.*

(b) $\mathrm{Qc}(E \otimes_\iota -)\colon \boldsymbol{QcHfBa} \to \boldsymbol{QcHfBa}$ *preserves colimits, and*
$\mathrm{Ba}\mathcal{L}_\beta(E,-)\colon \boldsymbol{QcHfBa} \to \boldsymbol{QcHfBa}$ *preserves limits.*

(c) $\mathrm{Ba}\mathcal{L}_\sigma(E,-)$ *and* $\mathrm{Ba}\mathcal{L}_\beta(E,-)\colon \boldsymbol{HfBa} \to \boldsymbol{HfBa}$ *are functorially isomorphic.*

(d) $E, F \in \boldsymbol{Ban} \Rightarrow \mathrm{Ba}\mathcal{L}_\sigma(E,F) \cong \mathcal{L}_\beta(E,F)$ *and* $\mathrm{Ba}E'_\sigma \cong E'_\beta$*.*

Proof. (a) follows from the construction in Definition (1.3) and (b) is a general property of adjoint functors, see e.g. Pareigis [41] 2.7 Thm. 3.
(c) With Theorem (8.7) and Corollary (10.3) one can see that $\mathrm{Ba}\mathcal{L}_\sigma(E,-)$ and $\mathrm{Ba}\mathcal{L}_\beta(E,-)$, as endofunctors on $\boldsymbol{HfBa}$, are both right adjoint to $(E \otimes_\iota -)$. Then the assertion follows from the uniqueness of adjoint functors, see Pareigis [41]. 2.1 Prop. 1.
(d) is a special case of (c) since, in this case, strong topologies are barreled.

(10.11) Lemma. *Let* $E \in \boldsymbol{QcHfLc}$ *and* E *normable. Then* E *is finite in* $\boldsymbol{QcHfLc}$ *if and only if* E *is finite dimensional.*

Proof. Since normable spaces are metrizable, the quasi completion and the completion are the same in this case. In addition, Banach spaces are barreled. Therefore, we have $\mathrm{Qc}\big(E \otimes_\beta \mathrm{Ba}E'\big) = E \mathbin{\widehat{\otimes}_\pi} E'$ and $\mathrm{Ba}\mathcal{L}_\beta(E,E) = \mathcal{L}_\beta(E,E)$. This puts us into exactly the same situation as in Example (8.9).

For the question of which spaces are "finite" in $\boldsymbol{QcHfBa}$, we will take care of the "trivial" or "uninteresting" case first.

(10.12) Lemma. *Let* $E \in \boldsymbol{QcHfBa}$ *and* E *finite dimensional. Then* E *is finite in the sense of Definition (1.4).*

Proof. Because of the finite dimension, the canonical morphism $f\colon \mathrm{Qc}(E \otimes_\iota E') = E \otimes_\iota E' \to \mathcal{L}_\beta(E,E)$ is bijective, and the condition "finite dimensional and Hausdorff" assures that, everywhere, only one topology is possible.

(10.13) Lemma. *Let* E *be a Fréchet space and let the canonical bilinear form (evaluation)* $k\colon E'_\beta \times E \to \mathbb{K}$ *(resp.* $E'_\kappa \times E \to \mathbb{K}$*) be continuous. Then* E *is normable (resp. finite dimensional).*

Proof. Because of the continuity of k, there is a null neighborhood $U \subset E$ and a bounded (resp. compact) set $B \subset E$ (resp. $K \subset E$) with $k(B^\circ, U) \subset [-1,1]$ (resp. $k(K^\circ, U) \subset [-1,1]$), where A° denotes the polar of A. Then, $U \subset B^{\circ\circ}$ (resp. $U \subset K^{\circ\circ}$), i.e. there is a bounded (resp. compact) null neighborhood, from which the assertion follows.

(10.14) Corollary. *Let* E *be a non-normable (resp. an infinite dimensional) Fréchet space. Then the topology* $E \otimes_\iota E'_\beta$ *(resp.* $E \otimes_\iota E'_\kappa$*) is genuinely finer than* $E \otimes_\pi E'_\beta$ *(resp.* $E \otimes_\pi E'_\kappa$*).*

Proof. According to Lemma (10.13), the canonical bilinear form is separably continuous, but not continuous.

(10.15) Theorem. *Let* $E \in \boldsymbol{QcHfBa}$ *such that* E'_β *and* $\mathcal{L}_\beta(E,E)$ *are also barreled. If E is finite in the sense of Definition (1.4), then E is finite dimensional.*

Proof. By assumption, $\mathrm{Qc}\big(E \otimes_\iota E'_\beta\big) \cong \mathcal{L}_\beta(E,E)$, and therefore the canonical morphism $E \otimes_\iota E'_\beta \to \mathcal{L}_\beta(E,E)$ is relatively open. However, since the subspace topology is $E \otimes_\varepsilon E'_\beta$, see Köthe [29], the following holds: $E \otimes_\iota E'_\beta = E \otimes_\pi E'_\beta = E \otimes_\varepsilon E'_\beta$. From Corollary (10.14), we conclude that E is normable and that E is quasi complete and is in fact a Banach space. Then the assertion follows from Lemma (10.11).

(10.16) Corollary. *The following spaces are not finite in $\boldsymbol{QcHfBa}$ in the sense of Definition (1.4):*
(a) $\omega = \mathbb{K}^{\mathbb{N}} \cong \mathcal{C}^{\infty}(\mathbb{N}, \mathbb{K}) = \mathcal{E}(\mathbb{N})$
(b) $\varphi = \mathbb{K}^{(\mathbb{N})} \cong \mathcal{C}_{\mathcal{C}}^{\infty}(\mathbb{N}, \mathbb{K}) = \mathcal{D}(\mathbb{N})$
(c) $\mathcal{E}(M) = \mathcal{C}^{\infty}(M, \mathbb{K})$*, M a compact $\mathcal{C}^{\infty}$-manifold,* $|M| \geq |\mathbb{N}|$
(d) $s \cong \mathcal{S}(R^n) \cong \mathcal{E}([0,1]) \cong \mathcal{E}(S^1)$
(e) $\mathcal{E}'(M), s', \mathcal{S}'$.
Here, ω has the product topology, φ the Lk-⊕-topology, $\mathcal{E}(M)$ the topology of compact convergence of functions and all derivatives, and $\mathcal{E}'$ the strong topology.

Proof. Let $E \in \{\omega, \mathcal{E}(M), s\}$. Then E is Fréchet, nuclear, reflective and E'_β is barreled. According to Bourbaki [4] (III, 3, ex. 9), the following holds: $\mathcal{L}_\beta(\omega, \omega) = \mathcal{L}_\beta(\omega, \mathbb{K}^{\mathbb{N}}) \cong \left(\mathcal{L}_\beta(\omega, \mathbb{K})\right)^{\mathbb{N}} \cong \varphi^{\mathbb{N}} = \omega\varphi$ and $\varphi^{\mathbb{N}}$ is, as a product of barreled spaces, itself barreled, see Bourbaki [4] (IV, 2, ex. 9b). $\mathcal{L}_\beta(s, s)$ and $\mathcal{L}_\beta(\mathcal{E}(M), \mathcal{E}(M))$ are complete and bornological, so they are also barreled, according to Grothendieck [23], page 128 and 129. Therefore, $\mathrm{Ba}\mathcal{L}_\beta(E, E) = \mathcal{L}_\beta(E, E)$. The assertion for E follows from the last Theorem (10.15). The assertion for E' follows from Lemma (3.6), since all spaces are reflective, and $\varphi = \omega'$.

If M is a non-compact $\mathcal{C}^{\infty}$-manifold, we can no longer apply the above argument, since then $\mathcal{L}_\beta(\mathcal{E}(R^n), \mathcal{E}(R^n))$ is not barreled, according to Grothendieck [23] page 98. In order to clarify the question of finiteness of $\mathcal{E}(M)$, we need a few more lemmas.

(10.17) Lemma. *Let M be a $\mathcal{C}^{\infty}$-manifold and $A \subset M$ a closed submanifold. Then $\mathcal{E}(M) \to \mathcal{E}(A)$ has a continuous, linear section.*

Proof. According to Bröcker and Jänich [7] 12.11, there is a tube neighborhood U of A in M. Then, there is a $\varphi \in \mathcal{E}(M)$ with $\mathrm{supp}\varphi \subset U$ and $\varphi(x) = 1$ for all $x \in A$ (e.g. because $U, M\backslash A$ is a locally finite covering of M, see Dieudonné [16] (16.4.1)). Every $f \in \mathcal{E}(M)$ can be trivially extended to all of U (i.e. constant in the transverse direction), because of the bundle property of the tube neighborhood. From the definition of the topology on $\mathcal{E}(-)$, we can see that this extension process gives us a continuous, linear mapping $\mathcal{E}(A) \to \mathcal{E}(U)$. The multiplication by φ is then a continuous, linear mapping $\mathcal{E}(U) \to \mathcal{E}(M)$, according to Dieudonné [16] (17.1.4). The composition $\mathcal{E}(A) \to \mathcal{E}(U) \to \mathcal{E}(M)$ provides us with the desired section.

(10.18) Lemma. *Let M be a $\mathcal{C}^{\infty}$-manifold. Then $\mathbb{N}$ can be embedded in M as a closed subset if and only if M is not compact.*

Proof. Since $\mathbb{N}$, with the discrete topology, must also have a trivial $\mathcal{C}^{\infty}$-structure, is a topological embedding also a $\mathcal{C}^{\infty}$-embedding. That is equivalent to the existence of a sequence of points in M that have no limit point in M. This is possible if and only if M is not compact, see Bourbaki [3] (IX, 2, 9, Prop. 15).

(10.19) Lemma. *Let M be a non-compact $\mathcal{C}^{\infty}$-manifold. Then $\mathcal{E}(M)$ is not finite in $\boldsymbol{QcHfBa}$ in the sense of Definition (1.4).*

Proof. According to Lemma (10.18), $\mathbb{N}$ can be embedded in M as a closed subset, and, according to Lemma (10.17), we have $\mathcal{L}(\mathbb{N}) \cong \omega$ is a direct factor of $\mathcal{E}(M)$. If $\mathcal{E}(M)$ were finite, then $\mathcal{E}(\mathbb{N})$ would also be finite, according to Lemma (3.3), in contradiction to Corollary (10.16) (a).

(10.20) Corollary. *Let M be a $\mathcal{C}^{\infty}$-manifold. Then the following assertions are equivalent:*
(a) $\mathcal{E}(M)$ *is finite in* $\boldsymbol{QcHfBa}$.
(b) M *is a finite set.*
(c) $\mathcal{E}(M)$ *is finite dimensional.*

Proof. The assertion follows directly from Corollary (10.16) (c) and Lemma (10.19).

11. Other categories of topological vector spaces

In this chapter, we examine a few categories that are to a large part analogous to the category of quasi-complete, barreled spaces, in particular, the category of Mackey-sequence-complete, bornological spaces, and the category of sequence-complete, compactly-determined spaces.

(11.1) Notation. *With $\boldsymbol{HfBo} \subset \boldsymbol{HfLc}$, we denote the full subcategory of bornological spaces, see Bourbaki [4] (III, 2, ex. 12), and with* $\mathrm{Bo}\colon \boldsymbol{HfLc} \to \boldsymbol{HfBo}$ *the bornologicalization (associated bornological space), see Bourbaki [4] (III, 2, ex. 13). $\boldsymbol{HfBo}$ is a full, monocoreflective subcategory of $\boldsymbol{HfLc}$.*

(11.2) Theorem. *Let $E, F, G \in \boldsymbol{HfLc}$. Then:*
(a) $F \in \boldsymbol{HfBo} \Rightarrow \mathcal{L}(E \otimes_\beta F, G) \cong \mathcal{L}\left(E, \mathcal{L}_\beta(F, G)\right)$.
(b) $E, F \in \boldsymbol{HfBo} \Rightarrow E \otimes_\iota F = E \otimes_\beta F \in \boldsymbol{HfBo}$.

Proof. The proof is analogous to (10.2) when we consider the fact that bounded subsets of $\mathcal{L}_\beta(F, G)$ are uniformly continuous, see Bourbaki [5] Prop. 6 and that colimits of bornological spaces are bornological because of Theorem (9.1) or Bourbaki [4] (III, 2, ex. 17a).

(11.3) Corollary. *$\boldsymbol{HfBo}$ is a symmetric, closed, monoidal category. In particular:*
$\mathcal{L}(E \otimes_\iota F, G) \cong \mathcal{L}\left(E, \mathrm{Bo}\mathcal{L}_\beta(F, G)\right)$.

Proof. The assertion follows from (11.2), analogous to (10.3).

(11.4) Notation. *With $\boldsymbol{McHfLc} \subset \boldsymbol{HfLc}$ (resp. $\boldsymbol{McHfBo} \subset \boldsymbol{HfBo}$) we denote the full subcategory of* ***Mackey-sequence-complete*** *spaces, as in P. Dierolf [13] page 23.* $\mathrm{Mc}\colon \boldsymbol{HfLc} \to \boldsymbol{McHfLc}$ *denotes the Mackey-sequence completion. $\boldsymbol{McHfLc}$ is a full, epireflective subcategory of $\boldsymbol{HfLc}$.*

(11.5) Lemma.
(a) $\mathrm{Mc}(\boldsymbol{HfBo}) \subset \boldsymbol{HfBo}$.
(b) $\mathrm{Bo}(\boldsymbol{McHfLc}) \subset \boldsymbol{McHfLc}$.
(c) $\boldsymbol{McHfBo}$ is a full, epireflective subcategory of $\boldsymbol{HfBo}$ and a full, monocoreflective subcategory of $\boldsymbol{McHfLc}$.
(d) $E \in \boldsymbol{McHfBo} \Rightarrow E$ is ultrabornological $\Rightarrow E \in \boldsymbol{HfBo}$.
(e) $\boldsymbol{QcHfLc} \subset \boldsymbol{McHfLc}$.

Proof. (a) follows from P. Dierolf [14] Prop. 1, (b) from (a) and Theorem (9.3), and (c) from (a), (b) and Theorem (9.5). (d) follows from P. Dierolf [13] page 27 or Köthe [28] page 384(2) and Bourbaki [4] (III, 3, ex. 11a). (e) follows from P. Dierolf [13] page 24.

So, we can see that Mc and Bo are compatible with each other, and also that Mackey-sequence-complete, bornological spaces are automatically barreled. Since we were looking for a completeness property that is as strong as possible, it appears that the category of quasi-complete, barreled spaces is more suitable for our purposes than the category of Mackey-sequence-complete, bornological spaces.

(11.6) Notation. *With $\boldsymbol{HfCd} \subset \boldsymbol{HfLc}$ we denote the full subcategory of* ***compactly determined*** *spaces in the sense of "compactly determined" in Porta [42], "ck-Räume" in Frölicher and Jarchow [20] and "lokon*hVl" in Seip [47].* $\mathrm{Cd}\colon \boldsymbol{HfLc} \to \boldsymbol{HfCd}$ *denotes the "compact determinization", i.e. the functor ℓ in Porta, ck in Fröhlicher and Jarchow and $LK \circ KE$ in Seip. $\boldsymbol{HfCd}$ is a full, monocoreflective subcategory of $\boldsymbol{HfLc}$.*

(11.7) Theorem. *Let $E, F, G \in \boldsymbol{HfLc}$. Then:*
(a) $F \in \boldsymbol{HfCd} \Rightarrow \mathcal{L}(E \otimes_\kappa F, G) \cong \mathcal{L}\left(E, \mathcal{L}_\kappa(F, G)\right)$.
(b) $E, F \in \boldsymbol{HfCd} \Rightarrow E \otimes_\iota F = E \otimes_\kappa F \in \boldsymbol{HfCd}$.

Proof. The proof is analogous to (10.2.), when we consider the fact that the pre-compact subsets of $\mathcal{L}_\kappa(E,F)$ are uniformly continuous, according to the theorem of Ascoli, see Bourbaki [3] (X, 2, 5, Thm. 2). Colimits of compactly determined subspaces are compactly determined according to Theorem (9.1) or Porta [42].

(11.8) Corollary. *$\boldsymbol{HfCd}$ is a symmetric, closed, monoidal category. In particular:* $\mathcal{L}(E \otimes_\iota F, G) \cong \mathcal{L}\big(E, \mathrm{Cd}\mathcal{L}_\kappa(F,G)\big)$.

Proof. The assertion follows from (11.7), analogous to (10.3).

(11.9) Notation. *With $\boldsymbol{ScHfLc} \subset \boldsymbol{HfLc}$ (resp. $\boldsymbol{ScHfCd} \subset \boldsymbol{HfCd}$) we denote the full subcategory of **sequence-complete** spaces in the sense of "semi-complet" in Bourbaki [4] (III, 3, ex, 10) or Köthe [28] (18.4).* Sc: $\boldsymbol{HfLc} \to \boldsymbol{ScHfLc}$ *denotes the sequence completion.* $\boldsymbol{ScHfCd}$ *is a full, epireflective subcategory of* $\boldsymbol{HfLc}$.

(11.10) Lemma.
(a) $\mathrm{Sc}(\boldsymbol{HfCd}) \subset \boldsymbol{HfCd}$.
(b) $\mathrm{Cd}(\boldsymbol{ScHfLc}) \subset \boldsymbol{ScHfLc}$.
(c) $\boldsymbol{ScHfLc}$ *is a full, epireflective subcategory of* $\boldsymbol{HfCd}$ *and a full, monocoreflective subcategory of* $\boldsymbol{ScHfLc}$.

Proof. (a) follows from Seip [47], (b) from (a) and Theorem (9.3), and (c) from (a), (b) and Corollary (9.5).

(11.11) Theorem. *$\boldsymbol{ScHfCd}$ is a symmetric, closed monoidal category. In particular:*
(a) $\mathrm{Sc}(E \otimes_\iota \mathbb{K}) \cong E$.
(b) $\mathrm{Sc}(\mathrm{Sc}(E \otimes_\iota F) \otimes_\iota G) \cong \mathrm{Sc}\big(E \otimes_\iota \mathrm{Sc}(F \otimes_\iota G)\big)$.
(c) $\mathrm{Sc}(E \otimes_\iota F) \cong \mathrm{Sc}(F \otimes_\iota E)$.
(d) $\mathcal{L}(\mathrm{Sc}(E \otimes_\iota F) \otimes_\iota G) \cong \mathcal{L}\big(E, \mathrm{Cd}\mathcal{L}_\kappa(F,G)\big)$ *for all* $G \in \boldsymbol{ScHfCd}$.

Proof. The assertion follows from (11.8) and (11.10), analogously to (10.6), when we consider that $\mathcal{L}_\kappa(F,G)$ is sequence complete.

(11.12) Theorem.
(a) $\boldsymbol{ScHfCd}$ *is complete and cocomplete. Limits are the* Cd*-images of limits formed in* $\boldsymbol{HfLc}$ *and colimits are the* Sc*-images of colimits formed in* $\boldsymbol{HfLc}$.
(b) $\mathbb{K}$ *is a generator and a cogenerator in* $\boldsymbol{ScHfCd}$.
(c) $\mathbb{K}$ *is injective in* $\boldsymbol{ScHfCd}$.

Proof. (a) follows from Lemma (11.10) (c) and Theorem (9.1) as in Theorem (10.7). (b) and (c) are analogous to Theorem (10.9).

(11.13) Remark. If we replace "compact" (resp. "compactly determined", "sequence complete") by "precompact" (resp. "p-determined", "p-complete"), we can carry out the last example completely analogously, see Brauner [6], where different methods are used.

(11.14) Remark. If we work with the concept "absolutely convex, compact" instead of "compact", we arrive at the concept of "espace de Kelley" in Buchwalter [8].

(11.15) Example. The following spaces are not finite in $\boldsymbol{ScHfCd}$: $\omega = \mathcal{E}(\mathbb{N})$, $\varphi = \mathcal{E}(\mathbb{N})$, $\mathcal{E}(M)$ with M a C^∞-manifold, $|M| = |\mathbb{N}|$, $s \cong \mathcal{S}(R^n)$, $\mathcal{E}'(M)$, s', $\mathcal{S}'$.

Proof. The proof is completely analogous to Corollary (10.16) and Lemma (10.19), when we consider that all spaces mentioned are Montel, and thus $\mathcal{L}_\beta(E,-) = \mathcal{L}_\kappa(E,-)$, that bornological spaces are compactly determined, and that the topology of $E \otimes_\kappa F$ is at most finer than $E \otimes_\beta F$.

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