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Finitisation in Bounded Arithmetic

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Abstract

I prove various results concerning undecidability in weak fragments of Arithmetic. All results are concerned with $S^1_2 \subseteq T^1_2 \subseteq S^2_2 \subseteq T^2_2 \subseteq ....$ a hierarchy of theories which have already been intensively studied in the literature. Ideally one would like to separate these systems. However this is generally expected to be a very deep problem, closely related to some of the most famous open problems in complexity theory.

In order to throw some light on the separation problems, I consider the case where the underlying language is enriched by extra relations and function symbols. The paper introduces a new type of results. These state that the first three levels in the hierarchy (i.e. S^1_2, T^1_2 and S^2_2) are never able to distinguish (in a precise sense) the "finite" from the "infinite". The fourth level (i.e. T^2_2) in some cases can make such a distinction. More precisely, elementary principles from finitistical combinatorics (when expressed solely by the extra relation and function symbols) are only provable on the first three levels if they are valid when considered as principles of general (infinitistical) combinatorics. I show that this does not hold for the fourth level.

All results are proved by forcing.

1 Bounded Arithmetic

The discovery of abstract set theory was like the discovery of the outer space. Set theory provides us with a telescope and has undoubtly affected the general view of the mathematical universe.

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I view Systems of Bounded Arithmetic as a promising framework of studying the mathematical microcosm. I suggest that questions in complexity theory reside outside the macro-world of ordinary mathematics. I belive that most deeper questions in complexity theory in a strong sense require refined "perception". Many principles which reside deep down in most mathematical arguments, appear equivalent from the normal perspective. (Example: Many elementary counting principles, e.g. the different versions of the elementary pigeon-hole principle). Through a microscope things are perceived quite different. Certain powerful extensions of Bounded Arithmetic could provide such a microscope!

Consistency, not truth, is the right starting point when we consider universal problems. What matters must be deductive powerful viewpoints. Certain extensions of subsystems of Bounded Arithmetic seems to provide a very promising basis for this. This paper is the first in series of planed papers. In these my intension is to isolate more and more powerful, (but unsound) systems of Bounded Arithmetic. Notice that if a universal statement (or more generally a Δ_0-statement) ψ is proved from a collection of "false" (in the standard universe $\mathbb{N}$) axioms, which are consistent with Bounded Arithmetic, then we know a priori that ψ must actually be true. Because if ψ ware false this would be witnessed in the standard part of each model $\mathbb{M}$ of Bounded Arithmetic, and so according to Gödels completeness theorem this would contradict the consistency assumption. This observation also seems to apply to the conjecture P $\neq$ NP. To see this recall that P $\neq$ NP is equivalent to the statement that "for all programs $\mathcal{P}$, for all $k \in \mathbb{N}$, there exists an input x, such that (a) $\mathcal{P}$ uses less than $\mid x \mid^k$-steps, and either (b1) accept x but x does not satisfies 3-SAT or (b2) $\mathcal{P}$ does not accept x, but x is an instance of 3-SAT". Now if the existential quantifier in "there exists an input x" is bounded by some term t, we obtain a Δ_0-formula Θ which implies P $\neq$ NP. According to the previous remark, if a (consistent) system of Bounded Arithmetic (however unsound) proves Θ, then actually P $\neq$ NP.

The idea of renouncing central and "obvious" axioms is certainly not new. We recall that there are models in which the self evident parallel postulate of Euclidian Geometry fails, and the wrong principle that each line has many parallel lines-hold true.

2 Making infinite structures finite

We are interested in constructing consistent (but unsound) systems of Bounded Arithmetic. In this paper I show that there are fragments of Bounded Arithmetic which have models $\mathbb{M}$ in which any countable structure S (up to elementarily equivalence) can be elementarily embedded as a "finite" (in the sense of $\mathbb{M}$!) structure.

In this section I will illustrate the basic method of this type of result. Consider the impossible ideal that any consistent theory Σ has always a <u>finite</u> model. I show that there exists a world in which this ideal is realised. In this world the usual induction axioms only hold for purely existentially defined sets.

Construction: According to the completeness theorem, there is a countable structure $\mathbb{M}$ non-isomorphic, but elementarily equivalent to $\mathbb{N}$, so the same set of L-expressible sentences holds in the two structures. There must be an initial segment of $\mathbb{M}$ isomorphic to (and identified with) $\mathbb{N}$. Usually the elements in $\mathbb{N}$ are called standard numbers while the numbers in $\mathbb{M} \setminus \mathbb{N}$ are the so-called "non-standard" numbers.

From an observers perspective outside $\mathbb{M}$ (i.e. from our perspective) there exist "numbers" $n \in \mathbb{M}$ which are infinitely large (i.e $\{1, 2, ..., n\}$) contains infinitely many numbers). However observers inside $\mathbb{M}$ would either not be able to express this, or if we allow them to quantify over second order objects (definable in $\mathbb{M}$) they would disagree. These observers of $\mathbb{M}$ would believe that $\{1, 2, ..., n\}$ was finite simply because it would be finite in their universe $\mathbb{M}$, where they have fewer functions and therefore think more sets are finite! So far all have been folklore. Now the basic part of the argument runs along lines, similar to those in the proof of Theorem 21 in [12].

Suppose ψ states that there exists a bijection from some interval $\{1, 2, .., n\}$ to the universe $\mathbb{M}$. From our outside view $\{1, 2, ..., n\}$ contains infinitely (countable) many numbers, and so our universe must contain a bijection f from $\{1, 2, ..., n\}$ to $\mathbb{M}$. Suppose now that we actually add a such a map f which maps $\{1, 2, ..., n\}$ bijectively to $\mathbb{M}$. Suppose also that we extend the language L with an extra function symbol $\bar{f}$ referring to this f. Also assume that we add names for each of the countable many elements in $\mathbb{M}$. Let us call this new language $L_{\bar{f}}$. Clearly it is not possible for the model $(\mathbb{M}, f)$ to satisfy the principle of induction. However if f is constructed carefully it turns out that we can force the model to satisfy some amount of induction!

In this example I want to show that one can ensure that $(\mathbb{M}, f)$ satisfies induction for sets which can be existentially defined by $L_{\bar{f}}$-formulas. First take an outside view. The model $\mathbb{M}$ is countable so there are only countably many existentially defined sets $S_1, S_2, ..$ defined by formulas $\psi_1(x), \psi_2(x), ...$ of one free-variable. List these formulas such that each formula appears infinitely many times in the list. At the k^{th}-step in the construction consider the formula $\psi_k(x)$, which as an example could be

$$\exists u \ \bar{f}(x + 13) = 2 \cdot u.$$

Suppose that in the previous step f has already been defined on a finite set $A \subseteq \{1, 2, ..., n\}$ with values in $B \subseteq \mathbb{M}$.

We want the least number principle to be true for the formula $\psi_k(x)$. This is done by "forcing" $\psi_k(a)$ to be true for the smallest possible a, i.e. by letting $f(a+13)$ be even for the smallest value a where this is consistent with the fact that f is a 1-1 map. The conditions a has to satisfy can be expressed in the language L without reference to $\bar{f}$. So in $\mathbb{M}$ we are able to search for such an a by a simple search procedure, which only depends on how f has already been defined on A. From an outside view "$<$" does not well order $\mathbb{M}$, so for a moment we take a look at things from inside $\mathbb{M}$. From this perspective "$<$" is a well ordering (this is possible because there are fewer sets in $\mathbb{M}$ than in the real universe). So the search procedure must terminate with some output a. Observers whether inside or outside $\mathbb{M}$, always agree on first order properties, in this case, whether a actually is the smallest such element.

Now go back to the real world outside $\mathbb{M}$ and proceed to the next step where the formula $\psi_{k+1}(x)$ is considered. Again we force $\psi_{k+1}(a)$ to be true for the smallest possible a. Alternatively if we cannot force $\psi_{k+1}(a)$ to be true for any a we know it will never be true (even at doomsday when f is constructed for all formulas).

We must ensure that f eventually defines the required bijection. In the present construction this automatically happens. For instance, for each $a \in \{1, 2, ..., n\}$ the formula $\psi(x) := \exists y \ \bar{f}(x) = y \wedge x = a$ eventually forces a to belong to $\{1, 2, ..., n\}$ (if it does not already do so). The other properties follow for similar reasons.

Now let Σ be any consistent theory. According to Skolem-Löwenheims theorem, Σ has a countable model S. If this model is infinite we assume that $\mathbb{N}$ is the underlying set. If we in the above construction start off by choosing a countable non-standard model $(\mathbb{M}, S_{\mathbb{M}})$ elementarily equivalent to $(\mathbb{N}, S)$, we get a model of existential induction in which Σ has a finite (in the sense of $\mathbb{M}$) model. Thus we have shown:

Proposition 2.0.1 *Any consistent theory Σ has a model S, which is embedded as a finite (=bounded) set in some model $\mathbb{M}$.*

Actually suppose that L is a countable language which extend the language of arithmetic, and suppose that L contains undefined relation and function symbols for the language of Σ. Then the model $\mathbb{M}$ can be chosen such that it satisfies the induction scheme for existential L-formulas.

This shows that any structure S, for example structures of strong systems like set theory, can be embedded as "finite" sets in some super-structure. It also shows that we can always assume that a given mathematical domain is "finite" given that our meta-theory (falsely) believes that all sets (and maps etc) in the universe are purely existentially defined.

As the pigeon-hole principle fails for infinite sets, as a corollary we obtain theorem 21 [12]:

Corollary 2.0.2 (A.Wilkie, J.Paris) *The system $I\exists(f)$ does not prove that f satisfies the pigeon-hole principle.*

The results in this paper resemble the ideas just described. However we need to be more careful.

It follows from the main results that countable structures can always be assumed to be (up to elementarily equivalence) finite in certain fragments of Bounded Arithmetic. As we have already indicated this phenomenon is closely related to the fact that the pigeon-hole principle fails heavily in these fragments. And it illustrates the microscope metaphor. One just has to look through the microscope from the right end!

3 Prelims

First let me recall some basic notations and facts, essentially all from [4]. Let **BASIC** denote a finite set of quantifier free formulas relating constants, functions and relations in the first order language $L = L(0, 1, +, \cdot, | \cdot |, \sharp, \lfloor \frac{x}{2} \rfloor, \leq, =)$. Here $\sharp$ denotes the function given by $a\sharp b = 2^{|a| \cdot |b|}$ where $|a| = \lceil \log_2(a+1) \rceil$. An example of a proper choice of **BASIC** (without coding functions) can be found in [4]. It is convenient to add other functions to the language. We will assume that a function $(w)_x$ which takes the value of the x^{th} element in the sequence coded by w is part of the language. As long as additional functions are polynomially time computable, the results in this section can be stated with no change.

In the first order case atomic formulas are of the form $t = s$ or $t \leq s$ where s, t are terms in L, while in the (monadic) second order case additionally, atomic formulas can be of the form $t \in X$ or $X =_2 Y$. (Where "$=_2$" denotes equality between second order variables.)

A first (second) order formula is *bounded* if all its quantifiers are of the form ....$\forall x \leq t$... or ....$\exists x \leq t$..... Second order quantifiers are not allowed in bounded formulas. Atomic formulas $X =_2 Y$ are not allowed because they smuggle in an unbounded first order quantifier (Extensionality axiom below).

A first (second) order formula is *sharply bounded* if it is bounded and all quantifiers are of the form ....$\forall x \leq | t |$... or ....$\exists x \leq | t |$....

The class of bounded formulas can be stratified as follows: Let $\Sigma_0^b = \Pi_0^b$ be the set of sharply bounded formulas (first or second order formulas depending on the context). Let Σ_{i+1}^b (Π_{i+1}^b) ; $i \geq 0$ be the smallest class of formulas which contains Π_i^b (Σ_i^b) and is closed under $\wedge$, $\vee$, sharply bounded quantification, and bounded existential quantification (bounded universal quantification). Notice that, except for minor syntactical changes, any bounded formula belongs to some Σ_i^b and to some Π_i^b.

Finally let strict-$\Sigma_{i+1}^b \subseteq \Sigma_{i+1}^b$ denote the set of Σ_{i+1}^b-formulas which are of the form $\exists y_1 \leq t_1 \exists y_2 \leq t_2 ... \exists y_r \leq t_r\ \phi$ where $\phi \in \Pi_i^b$. Similarly, let strict-$\Pi_{i+1}^b \subseteq \Pi_{i+1}^b$ denote the set of Π_{i+1}^b-formulas which are of the form $\forall y_1 \leq t_1 \forall y_2 \leq t_2 ... \forall y_r \leq t_r\ \phi$ where $\phi \in \Sigma_i^b$.

3.1 The first order theories

Let S_2^i denote the first order theory consisting of **BASIC**, together with the following "polynomial time" induction scheme, $\varphi(0) \wedge \forall x(\varphi(\lfloor \frac{x}{2} \rfloor) \Rightarrow \varphi(x)) \Rightarrow \forall x \varphi(x))$, where $\varphi \in \Sigma_i^b$. This scheme is usually denoted by Σ_i^b-PIND.
Let T_2^i denote the first order theory consisting of **BASIC** together with the Σ_i^b induction scheme, $\varphi(0) \wedge \forall x(\varphi(x) \Rightarrow \varphi(x+1)) \Rightarrow \forall x \varphi(x)$, where $\varphi \in \Sigma_i^b$. This scheme is usually denoted by Σ_i^b-IND.

3.2 The second order theories

The (monadic) second order versions of these theories $S_2^i(\alpha)$ ($T_2^i(\alpha)$) consist of **BASIC**, Σ_i^b-PIND (Σ_i^b-IND) together with the extensionality axiom

$$\text{EXT}: \ \forall X, Y(X =_2 Y \Leftrightarrow \forall x(x \in X \Leftrightarrow x \in Y)).$$

We do not allow the full comprehension axiom, but follow [6] and equip $S_2^i(\alpha)$ ($T_2^i(\alpha)$) with the following "NP $\cap$ co-NP"comprehension axiom-scheme: (Δ_1^b-comprehension)

$$\forall x(\varphi(x, \vec{z}, \vec{Z}) \Leftrightarrow \neg \eta(x, \vec{z}, \vec{Z})) \Rightarrow \exists X \forall x\ (x \in X \Leftrightarrow \varphi(x, \vec{z}, \vec{Z}))$$

where $\varphi, \eta \in \Sigma_1^b$.

The underlying logic of these theories is second order predicate logic with second order equality $=_2$. It is easy to prove that no deductive strength is lost if $X =_2 Y$ is taken to be short-hand notation for $\forall z(z \in X \leftrightarrow z \in Y)$, and if EXT and the equality axioms in the underlying logic are dropped.

3.3 Models of second order theories

A model of a second order theory T is a pair $(\mathbb{M}, \hat{R})$, where $\hat{R} \subseteq P(\mathbb{M})$, the power set of $\mathbb{M}$, and where $\mathbb{M}$ is a model for the first order part of T. The satisfaction relation $\models$ is defined inductively such that second order variables are taken to be the subsets of $\mathbb{M}$ which are in $\hat{R}$. The well known main advantage of using this notion of a model, without requiring that $\hat{R} = P(\mathbb{M})$, is that the Compactness Theorem, the Completeness Theorem and Skolem-Löwenheims Theorems hold with minor changes. These facts follow easily (pointed out by A.J.Wilkie in Personal communication) from the natural isomorphism:

Observation 3.3.1 (Transitive collapse) *Let* $\mathbb{M}^{\triangle} = \mathbb{M}^{\triangle}_{num} \cup \mathbb{M}^{\triangle}_{set}$ *be a first order model which contains two kinds of elements. One kind(x,y,z,...) denotes numbers and belongs to* $\mathbb{M}^{\triangle}_{num}$*, the other kind(X,Y,Z,...) denotes "sets" and belongs to* $\mathbb{M}^{\triangle}_{set}$*. If* $\tilde{\in}$ *is a binary relation on* $\mathbb{M}^{\triangle}$*, with domain* $\mathbb{M}^{\triangle}_{num}$ *and range* $\mathbb{M}^{\triangle}_{set}$ *such that*

$$\mathbb{M}^{\triangle} \models (X = Y \Leftrightarrow \forall x(x \tilde{\in} X \Leftrightarrow x \tilde{\in} Y))$$

then the map $\psi_{collapse} : X \longrightarrow \{x \in \mathbb{M}^{\triangle}_{num} : \mathbb{M}^{\triangle} \models "x \tilde{\in} X"\}$ *maps* $\mathbb{M}^{\triangle}_{set}$ *bijectively onto a class* $\hat{S}$ *of subsets of* $\mathbb{M}^{\triangle}_{num}$*.*
Furthermore $(\mathbb{M}, \hat{S}) :\equiv (\mathbb{M}^{\triangle}_{num}, \psi_{collapse}{}''\mathbb{M}^{\triangle}_{set})$ *is isomorphic to* $\mathbb{M}^{\triangle}$*.*

From this we get the following version of the Completeness Theorem.

Proposition 3.3.2 $T \cup \mathrm{EXT}$ *is consistent if and only if* T *has a model.*
$T \cup \mathrm{EXT} \vdash \phi$ *if and only if* $(\mathbb{M}, \hat{S}) \models \phi$ *for all models* $(\mathbb{M}, \hat{S}) \models T$*.*

3.4 Some special results for $S^i_2(\alpha)$

Now I prove that the second order theories $S^i_2(\alpha)$ are conservative over the corresponding first order theories, in the sense that any model $\mathbb{M} \models S^i_2$ has an expansion to a model $(\mathbb{M}, \tilde{S}) \models S^i_2(\alpha)$. A similar result holds for the theories T^i_2. I prove a slightly more general result. Let Ψ be a set of formulas $\psi(x)$, which might contain free second order variables and free first order variables other than x.

Definition 3.4.1 By a Ψ*-substitution scheme* Σ we will understand a first order formula θ_Σ with no second order variables, which contains meta-variables $F_1, F_2, ..., F_l$. To each meta-variable F_j is associated a term t_j. A *substitution instance* $\Sigma(\psi(x))$, $\psi \in \Psi$ is obtained by replacing each F_j in θ_Σ with $\psi(t_j)$. ♣

Example 3.4.2 $\Sigma^b_i(\alpha)$-PIND *and* $\Sigma^b_i(\alpha)$-IND *are both* $\Sigma^b_i(\alpha)$*-substitution schemes.*

Definition 3.4.3 In the following let $T(i) = \Sigma + \mathrm{BASIC} + \mathrm{EXT}$, where Σ is a Σ^b_i-substitution scheme. In $T(i)$, BASIC could be any set of first order formulas. ♣

Proposition 3.4.4 *If* $(\mathbb{M}, \hat{R}) \models T(i)$ *there is an expansion* $\hat{S} \supseteq \hat{R}$ *such that*

$$(\mathbb{M}, \hat{S}) \models T(i) + \Delta^b_1 - \text{comprehension}.$$

Proof: Assume that $(\mathbb{M}, \hat{R}) \models T(i)$ is a given model. Let

$$\hat{S} = \{S \subseteq \mathbb{M} : \exists \psi_i (= \psi_i(x, X_1, X_2, ..., X_k)) \in \Sigma_1^b, \quad i = 1, 2$$

$$\wedge \exists R_1, R_2, ..., R_k \in \hat{R}$$

$$(x \in S \Leftrightarrow \psi_1(x, R_1, R_2, ..., R_k) \Leftrightarrow \neg \psi_2(x, R_1, R_2, ..., R_k))\}$$

I claim that $(\mathbb{M}, \hat{S}) \models T(i) + \Delta_1^b$-comprehension. According to proposition 3.3.2 EXT holds in all second order models, in particular $(\mathbb{M}, \hat{S})$. **BASIC** holds in $(\mathbb{M}, \hat{S})$ because it is a set of first order formulas. It remains to be shown that $(\mathbb{M}, \hat{S}) \models \Sigma$ and to show $(\mathbb{M}, \hat{S}) \models \Delta_1^b$-comprehension.

Sub claim 1: $(\mathbb{M}, \hat{S}) \models \Sigma(\psi(x))$ for all $\psi \in \Sigma_i^b$. Notice that if $\psi(x, S_1, S_2, ..., S_k)$ is a Σ_i^b-formula with set parameters from $\hat{S}$, there is a Σ_i^b-formula $\eta(x, R_1, R_2, ..., R_l)$ with set parameters from $\hat{R}$, such that for all $c \in \mathbb{M}$

$$(\mathbb{M}, \hat{S}) \models \psi(c, S_1, S_2, ..., S_k) \Leftrightarrow \eta(c, R_1, R_2, ..., R_l).$$

Here η is obtained from ψ by the following. First, by replacing each appearance of S_i with either $\psi_1(x, R_1, R_2, ..., R_r)$ or $\neg\psi_2(x, R_1, R_2, ..., R_r)$ according to whether S_i appears positively or negatively. Second, by bringing it in a "prenex like" form if convenient. Now sub claim 1 follows by noticing that

$$(\mathbb{M}, \hat{S}) \models \Sigma(\psi) \quad \Leftrightarrow \quad (\mathbb{M}, \hat{S}) \models \Sigma(\eta) \quad \Leftrightarrow \quad (\mathbb{M}, \hat{R}) \models \Sigma(\eta)$$

and that $(\mathbb{M}, \hat{R}) \models \Sigma(\eta)$ is part of the assumption.

Sub claim 2: $(\mathbb{M}, \hat{S}) \models \Delta_1^b-$comprehension. Let $\psi_1, \psi_2 \in \Sigma_1^b$ be given and assume that for some given $a \in \mathbb{M}$, $(\mathbb{M}, \hat{S}) \models \forall x \leq a(\psi_1(x) \Leftrightarrow \neg\psi_2(x))$. Consider $S = \{x \mid \psi_1(x, S_1, S_2, ..., S_k)\}$ where $S_1, S_2, ..., S_k \in \hat{S}$. As we have already noticed, there are Σ_1^b-formulas $\eta_1(x, \vec{R})$ and $\eta_2(x, \vec{R})$ equivalent to ψ_1 and ψ_2, $S = \{x : \eta_1(x, \vec{R})\} = \{x : \neg\eta_2(x, \vec{R})\} \in \hat{S}$. □

Corollary 3.4.5 *Every model of first order S_2^i, $i \geq 1$ has an expansion to a model of $S_2^i(\alpha)$.*

More generally if $U_1, U_2, ..., U_r$ are unary relation symbols added to L, then every model of $S_2^i(U_1, U_2, ..., U_r)$, has an expansion to a model of $S_2^i(\alpha)$

Proof: The first part of the corollary is just the special case where $r = 0$. If $r > 0$ let $R_i := \{x \in \mathbb{M} : \mathbb{M} \models U_i(x)\}$ and let $\hat{R} := \{R_1, R_2, ..., R_r\}$. Notice $(\mathbb{M}, \hat{R}) \models S_2^i(\alpha) - \Delta_1^b$-comprehension, and use proposition 3.4.4 with Σ as the Σ_i^b-substitution scheme Σ_i^b-PIND. □

Notice that Corollary 3.4.5 remains valid if the theories S_2^i are replaced by T_2^i for $i \geq 1$.

3.5 Some conservation results

In [5] S.Buss gave a precise characterisation of the S_2^i-provable Σ_i^b-definable functions. Actually one of the major justification for dealing with S_2^i lies in this characterisation. Let me make an observation in this connection. According to a general argument by G.Kreisel the class of provable total recursive functions is always insensitive to the addition of extra universal axioms.

The same argument applies to the class of Σ_i^b-definable functions of a theory. The class is insensitive to addition of Π_i^b-axioms (as long they remain consistent with S_2^i). This is because for a theory T in general:

$$T + \{\forall x \psi(x)\} \vdash \forall x \exists y \theta(x, y) \Rightarrow T \vdash \forall x \exists y \theta(x, y) \vee \neg\psi(y)$$

which does not produce any new provable total function if $\theta(x, y) \vee \neg\psi(y)$ still is provable equivalent to a Σ_i^b-formula. Thus we have

Observation 3.5.1 *The class of provable Σ_i^b-definable functions of a theory T is immune with respect to the underlying Π_i^b-theory.*

The following immediately gives us the inclusions

$$S_2^1(\alpha) \subseteq T_2^1(\alpha) \subseteq S_2^2(\alpha) \subseteq T_2^2(\alpha) \subseteq .....$$

Proposition 3.5.2 (S.Buss) *$S_2^{i+1}(\alpha) \vdash T_2^i(\alpha)$ for all $i \geq 0$.*

Proof: Fix an arbitrary $(\mathbb{M}, \hat{S}) \models S_2^{i+1}(\alpha)$. First notice that if $X < b$ is a Σ_i^b-definable set in $\mathbb{M}$ the convex closure $Y = conv(X) := \{x < b \mid \exists v \in X \quad v \leq x \wedge \exists u \in X \quad x \leq u\}$ is also a Σ_i^b-definable set in $\mathbb{M}$. Let "dist$(Y, Y^c) \leq d$" be the $\Sigma_1^b(\Sigma_i^b)$-formula $\phi_b(d) \equiv \exists x, y < b\ (y - x \leq d \wedge y \in Y \wedge x \notin Y)$. By considering the point $\lfloor \frac{x+y}{2} \rfloor$obviously $S_2^{i+1}(R) \vdash \phi_b(d(\lfloor \frac{1}{2} \rfloor)^k) \Rightarrow \phi_b(d(\lfloor \frac{1}{2} \rfloor)^{k+1})$ so by Σ_{i+1}^b - PIND "$dist(Y, Y^c) \leq b$" $\Rightarrow$ "$dist(Y, Y^c) \leq b(\lfloor \frac{1}{2} \rfloor)^{|b|}$ " and by *modus ponens* and the fact the $S_2^1(\alpha) \vdash b(\lfloor \frac{1}{2} \rfloor)^{|b|} = 1$ "dist$(Y, Y^c) \leq 1$ ". As Y is convex, Y has a smallest element in $\mathbb{M}$, and then by definition X has a smallest element. As Σ_i^b-LNP $\Leftrightarrow$ Σ_i^b-IND we are done. □

In some later examples I will use one of the deeper theorems in the subject:

Theorem 3.5.3 (S.Buss) *For $i \geq 1$, $S_2^{i+1}(\alpha)$ is $\forall\Sigma_{i+1}^b$*-conservative over $T_2^i(\alpha)$.

Proof: Suppose that $S_2^{i+1} \vdash \forall \vec{x} \forall \vec{X} \exists \vec{z} \leq t(\vec{x})\ \psi(\vec{x}, \vec{z}, \vec{X})$ where $\psi \in \Pi_i^b$. It suffices to show that $T_2^i(\vec{R}) \vdash \exists \vec{z} \leq t(\vec{x})\ \psi(\vec{x}, \vec{z}, \vec{R})$, *for* $\quad \psi \in \Pi_i^b$.

This follows again by relativising the proof of S.Buss' theorem stating that for any $i \geq 0 \quad S_2^{i+1}$ is $\forall^{nd}\Sigma_{i+1}^b$-conservative over T_2^i, [7] [5]. □

4 Some finitisation principles

One of our aims is to show that there is a fundamental difference between $S_2^2(\alpha)$ and levels in the hierarchy which are at least as strong as $T_2^2(\alpha)$. There is a general feeling amongst those working in Bounded Arithmetic that a proof of the non-finite axiomatisability of the first order theory S_2 ($= T_2$) would be of great importance. An important step in that direction would be to separate T_2^1, S_2^2 and T_2^2. These theories are only known to be different under an additional hypothesis from complexity theory.

In [9] T_2^i and S_2^{i+1}, $i \geq 1$ were conditionally separated under the conjectural assumption that the polynomial hierarchy (in complexity theory) does not collapse on level $i+2$. In [8] S_2^2 and T_2^2 were conditionally separated under the conjectural assumption $Logspace^{\Sigma_2^p} \neq \Delta_3^p$. However both these assumptions (generally believed to be true) are far beyond current techniques. They both imply P $\neq$ NP.

The relativised cases $S_2^2(\alpha)$ and $T_2^2(\alpha)$ were first separated in [8]. I present new proofs for these relativised cases. This is done by proving the following finitisation principle:

Theorem 4.0.4 *If ψ has an infinite model (in the real world), there are structures of $T_2^1(\alpha)$ in which ψ has a finite (=bounded) model.*

I also get the following principle.

Theorem 4.0.5 *Suppose that $\theta := \exists A\ \psi(A, R)$ is a second order existential statement. Suppose also that θ is expressed solely by unspecified function and relation symbols. Suppose that θ has an infinite model (in the real world) where the existential quantifier is not witnessed by any finite set. Then there are structures of $T_2^1(\alpha)$, in which θ has a finite model $\{1, 2, ..., n\}$ where the existential quantifier is not witnessed by any set A of size $\leq \log(n)$.*

Since an understanding of the relativised cases, seems to precede an understanding of the unrelativised cases these two results could perhaps be useful in separating T_2^1 from T_2^2 unconditionally.

Finally I prove a finitisation principle which can be used to separate the theories $S_2^1(\alpha)$ and $T_2^1(\alpha)$.

Theorem 4.0.6 *Suppose that $\psi(\prec, R)$ is a first order statement which is expressed solely by unspecified function and relation symbols. Suppose that in the real world $\psi(\prec, R)$ holds in an infinite model, where $\prec$ defines a total linear ordering. Then there are structures of $S_2^1(\alpha)$ in which $\psi(\prec, R)$ holds in a finite (=bounded) model and where $\prec$ is the restriction of the order relation $<$.*

These theorems show that when we pass from the real universe to an universe which only contains "feasible" sets, we have the heuristic translations, *countable* $\rightarrow$ *finite, finite* $\rightarrow$ *poly-logarithmic.*

4.1 Further definitions and assumptions

Let $L_2(0, 1, +, \cdot, f_{fast}, f_{slow}, =, \leq)$ be a second order language where through some basic axioms f_{fast} and f_{slow} are ensured to define functions such that f_{fast} is fast-growing, and that f_{slow} is slow growing. Further it is assumed that f_{slow} is slower than f_{fast} is fast (!). More precisely assume:

(1) f_{slow}, f_{fast} are increasing.

(2) $f_{fast} \geq x^2$.

(3) For any fixed k, for all sufficiently large n $f_{slow}(f_{fast}^{(k)}(n)) \leq n$, where

$$f^{(k)} = \underbrace{f \circ f \circ ... \circ f}_{k}.$$

Definition 4.1.1 A formula $\psi \in L_2$ is *sharply bounded* if each quantifier appears in the context $\exists x \leq f_{slow}(t)$ or $\forall x \leq f_{slow}(t)$, where $t \in \mathrm{Term}_L$. A formula $\psi \in L_2$ is *bounded* if each quantifier appears in a context $\exists x \leq t$ or $\forall x \leq t$, where $t \in \mathrm{Term}_L$.

The class of Σ_i^b formulas and the class of Π_i^b formulas are defined similar to the earlier definition. ♣

Definition 4.1.2 $T^1(\alpha)$ denotes the second order theory consisting of a proper base theory **Basic** together with EXT + Δ_1^b-comprehension + Σ_1^b-IND. ♣

Given some additional relations $\vec{R}$ we define the first order theory $T^1(\vec{R})$ in the obvious way. <u>Notice</u> that if $f_{fast}(< x, y >) := x\sharp y$ and $f_{slow}(x) := |x|$, for a proper choice of **BASIC** $T^1(\alpha)$ becomes $T_2^1(\alpha)$. Furthermore notice that in this case

$$f_{fast}(n) \leq f_{fast}(< n, n >) = 2^{(|n|^2)}$$

and therefore $| \; f_{slow}(f_{fast}^{(k)}(n)) \; | \leq | \; f_{slow}(f_{fast}^{(k)}(< n, n >)) \; | = | \; n \; |^{2^k}$. Clearly for any fixed k and non-standard n, $| \; n \; |^{2^k} < n$ so the consideration below applies to the theory $T_2^1(\alpha)$.

Definition 4.1.3 Let $< x_1, x_2, ..., x_k >$ be a natural code of the k-tuple. For each k we introduce quantifiers $\forall^k$ and $\exists^k$ such that $Q^k x \; \psi(x)$ is shorthand notation for $Qx_1 Qx_2 ... Qx_k \; \psi(< x_1, ..., x_k >)$ where $Q \equiv \forall$ or $Q \equiv \exists$. ♣

Definition 4.1.4 Let $Q^k x < t...$ be short-hand notation for $Qx_1 < t...Qx_k < t$, when $< x_1, x_2, ..., x_k >= x$. Let $\theta(\vec{x})$ be a formula in some relational language $L(R_1, R_2, ..., R_l)$. For a relation symbol S, the formula $\theta_S(\vec{x})$ denotes the formula which appears if each quantifier in θ is restricted to S. By $\theta_{<a}(\vec{x})$ we understand the formula which appears by restricting each quantifier in θ to $[0, a)$. ♣

4.2 A version of the completeness theorem

As I have already pointed out, in the real mathematical universe it is not true that any finite consistent set of first-order sentences has a finite model. But there are $T_2^1(\alpha)$-universes where such a strong form of the completeness theorem holds.

Theorem 4.2.1 (Finitisation principle) *Let $\theta(\vec{R})$ be a first order property expressed in some relational language $\tilde{L} = L(\vec{R})$. Suppose that $\eta(n)$ is an arithmetical first order property expressible in the language L of arithmetic and suppose that $\eta(n)$ holds for arbitrary large n. If $\theta(\vec{R})$ has a model then the theory $T^1(\vec{R}) + \exists n\ \eta(n) \wedge \theta_{<n}(\vec{R})$ has a model.*

First we shall make some preparation for the proof. Let $\tilde{L}$ and $\theta(\vec{R})$ be given as in the theorem. We can assume that all relations are r-ary. $\theta(\vec{R})$ is assumed to have an infinite model, so by Skolem-Löwenheims Theorem, we can assume $\theta(\vec{R})$ has an infinite countable model S_{st} on a subset of the natural numbers. Furthermore, we can assume S_{st} is a model with an underlying co-countable set, and if convenient, that an extra unary relation symbol denoting membership of S_{st} is added to the language.

Let $(\mathbb{M}, S)$ be a countable non-standard model for the language $\tilde{L}(R) := L \cup \tilde{L}$ which is elementarily equivalent to the standard model $(\mathbb{N}, S_{st})$. Use overspill to pick a non-standard number n such that $\mathbb{M} \models \eta(n)$.

Fix a non-standard number $b_0 < n$ such that $b_0^k < n$ for each standard number k, and such that $f_{slow}(f_{fast}^{(k)}(n)) < b_0$ for every standard number k. Use overspill to pick $c \in \mathbb{M}$ non-standard such that $f_{slow}(c) < b_0$ and such that $c > f_{fast}^{(k)}(n)$ for all standard numbers k.

Definition 4.2.2 Let $\mathcal{P}_k,\ k \in \omega$, be the set of all partial (1-1)-maps α which have $\mathrm{dom}(\alpha) \subseteq \{1, 2, ..., c\}$ and $\mathrm{ran}(\alpha) \subseteq \mathbb{M}$ such that $\mid \mathrm{dom}(\alpha) \mid \leq b_0^k$, and such that α maps points in $[0, n)$ to S, and maps points in $\{n+1, n+2, ..., c\}$ to $\mathbb{M} \setminus S$.

Let $\mathcal{P} = \cup_{k \in \omega} \mathcal{P}_k$ and let $(\mathcal{P}, \subseteq)$ be $\mathcal{P}$ ordered under inclusion. ♣

4.3 Generic maps

Let us now make a few basic definitions:

Definition 4.3.1 $\mathcal{D} \subseteq \mathcal{P}$ is called *dense* if $\forall \alpha \in \mathcal{P} \exists \beta \in \mathcal{D}$ such that $\beta \supseteq \alpha$.

A subset $S \subseteq \mathbb{M}$ is *quasi-definable* in $\mathbb{M}$ if there is $\theta(x) \in L(R_\omega)$ such that $S = \{x \in M : (\mathbb{M}, \omega) \models \theta(x)\}$ where R_w is interpreted by ω. We allow θ to contain parameters from $\mathbb{M}$. ♣

Notice that in [1] the similar notion (just called definable) is $S = \{x \in M \mid \exists n \in R_w \ \theta(x,n)\}$, which would not work in our case because we cannot force formulas in general to be equivalent to existential formulas. It should also be noticed that any extension of the notion of quasi-definability which does not produce uncountably many quasi-definable dense subsets of $\mathcal{P}$, would work.

Example 4.3.2 *The set $\mathcal{P}$ is quasi-definable. The initial segment $[0, b_0^\omega)$ is quasi-definable.*

Definition 4.3.3 $\mathcal{G} \subseteq \mathcal{P}$ is a *generic filter* if

(1) $\forall \alpha, \beta \in \mathcal{G} \ \exists \gamma \in \mathcal{G} \ \gamma \supseteq \alpha \wedge \gamma \supseteq \beta$.

(2) $\forall \alpha \in \mathcal{G} \ \forall \beta \in \mathcal{P} \ \alpha \supseteq \beta \Rightarrow \beta \in \mathcal{G}$.

(3) $\mathcal{G} \cap \mathcal{D} \neq \emptyset$ for each quasi-definable dense set $\mathcal{D} \subseteq \mathcal{P}$. ♣

Lemma 4.3.4 *For every $\alpha_0 \in \mathcal{P}$ $\exists \mathcal{G} \subseteq \mathcal{P}$ generic such that $\alpha_0 \in \mathcal{G}$.*

Proof: $\mathbb{M}$ is assumed to be countable so that there is at most countably many quasi-definable sets $\mathcal{D} \subseteq \mathcal{P}$. List those as $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots$. Pick $\alpha_1 \supseteq \alpha_0$ such that $\alpha_1 \in \mathcal{D}_1$, pick $\alpha_2 \supseteq \alpha_1$ such that $\alpha_2 \in \mathcal{D}_2$, etc. and let $\mathcal{G} := \{\beta \in \mathcal{P} \mid \exists j \in \omega \quad \beta \subseteq \alpha_j\}$. It is straightforward to check that $\mathcal{G}$ is generic. □

Definition 4.3.5 $\tilde{\alpha} \subseteq \mathbb{M}$ is *generic* if there is a generic $\mathcal{G} \subseteq \mathcal{P}$ such that $\tilde{\alpha} = \cup_{\alpha \in \mathcal{G}} \alpha$. ♣

4.4 Sketch of proof

Now let me sketch the proof. We are given a non-standard model $(\mathbb{M}, S) \equiv_e (\mathbb{N}, S_{st})$ in which the "structure" S we want to miniaturise is a part. We can assume that the language contains the language of arithmetic together with extra relations, denoting the relations in S.

First, it is shown that some (actually any) generic $\tilde{\alpha}$ is a bijection from $[0, c)$ to $\mathbb{M}$, mapping $[0, n)$ onto S. At this stage we also have to show certain lemmas about the forcing relation in order to ensure it behaves well.

Second, it is shown that for some (actually any) generic map $\tilde{\alpha}$, $(\mathbb{M}, \tilde{\alpha}) \models \exists^*{-}\mathrm{LNP}$, where $\exists^*$-LNP denotes the least number principle for formulas in which all quantifiers are either existential (positive appearance) or are restricted to $[0, b_0)$. This part of the argument is based on the same idea as the proof of theorem 21 in [12] (see also the introduction).

Third, it is noticed that the constants, relations and functions in the miniaturised structure $S_{mini} := \tilde{\alpha}^{-1}(S)$ are $\exists \cap \forall$-definable in the generic model $(\mathbb{M}, \tilde{\alpha})$.

Fourth, it is shown that each formula expressing a Σ_1^b- property about the miniaturised model and numbers in $[0, c)$, can be translated into an $\exists^*$-formula. This ensures that LNP holds for Σ_1^b-formulas with parameters in $[0, c)$.

Finally a concrete model $\mathbb{M}^*$ is constructed as the smallest initial segment of $[0, c)$ which contains $[0, n)$ and is closed under f_{fast}.

4.5 The forcing relation

First we need to show some basic fact about generic maps. For simplicity we reduce logical constants. So suppose that $\forall :\equiv \neg\exists\neg$ and $\wedge :\equiv \neg \vee \neg$. Extend the language with names for the elements in $\mathbb{M}$. Also extend the language $\tilde{L}(R)$ by an extra binary relation symbol $\bar{\alpha}$. For sentences in this language $\tilde{L}(\vec{R}, \bar{\alpha})$ we define the forcing relation inductively as follows:

$$\alpha \Vdash \psi \text{ if } \psi \text{ does not contain } \bar{\alpha}, \text{ is atomic and true.}$$

$$\alpha \Vdash \bar{\alpha}(a, b) \quad \text{iff} \quad \alpha(a) \text{ is defined and equals } b.$$

$$\alpha \Vdash \psi \vee \eta \quad \text{iff} \quad \alpha \Vdash \psi \textit{ or } \alpha \Vdash \eta.$$

$$\alpha \Vdash \exists x\ \psi(x) \quad \text{iff for } \underline{\text{some}}\ a \in \mathbb{M} \quad \alpha \Vdash \psi(a).$$

The forcing relation for negation satisfies:

$$\alpha \Vdash \neg\psi \text{ iff for no } \beta \supseteq \alpha,\ \beta \in P\ \beta \Vdash \psi.$$

4.6 Soundness of the forcing relation

We have to make sure that the forcing relation satisfies certain key properties. Except for lemma 4.6.4 below, the reader who is familiar with forcing techniques could ignore this section.

Definition 4.6.1 Let $\Sigma(b_0)$ be the set of formulas where all quantifiers are restricted to $[0, b_0)$. ♣

Notice $f_{slow}(c) < b_0$ and therefore in the final model all sharply bounded quantifiers are restricted to $[0, b_0)$.

Lemma 4.6.2 (Forcing lemma) *The forcing relation has the following properties:*

Extension property: If $\alpha \Vdash \psi$ and $\beta \supseteq \alpha$ then $\beta \Vdash \psi$

Consistency: For no $\alpha \in \mathcal{P}$ and for no ψ, does both $\alpha \Vdash \psi$ and $\alpha \Vdash \neg\psi$ hold.

Completeness: For each generic set $\mathcal{G} \subseteq \mathcal{P}$, and for each ψ there is $\alpha \in \mathcal{G}$ such that $\alpha \Vdash \psi$ or $\psi \Vdash \neg\psi$.

Soundness 1: If for a generic map $\tilde{\alpha}$, $(\mathbb{M}, \tilde{\alpha}) \models \psi$, there is $\alpha \in \mathcal{G}$ such that $\alpha \Vdash \psi$.

Soundness 2: If $\alpha \Vdash \psi$ then $(\mathbb{M}, \tilde{\alpha}) \models \psi$ for $\underline{\text{any}}$ generic $\tilde{\alpha} \supseteq \alpha$.

Proof: Extension property: First notice that the claim holds for atomic formulas. Clearly the extension lemma holds if $\psi \equiv \psi_0 \vee \psi_1$, or if $\psi \equiv \exists y \psi_0(y)$. Suppose $\psi \equiv \neg\psi_0$, $\alpha \Vdash \psi$, and $\beta \supseteq \alpha$ is given. By definition for no $\alpha' \supseteq \alpha$, $\alpha' \vdash \psi_0$. The ordering $\supseteq$ of the forcing conditions $\mathcal{P}$ is transitive for no $\alpha' \supseteq \beta$, $\alpha' \Vdash \psi_0$. By definition $\beta \Vdash \psi$.

Consistency: Direct by the inductive definition.

Completeness: Let $\mathcal{D} := \{\alpha : \ \alpha \Vdash \psi \vee \neg\psi\}$. Notice $\mathcal{D}$ is quasi-definable and dense so there is $\alpha_0 \in \mathcal{G} \cap \mathcal{D}$.

By definition either $\alpha_0 \Vdash \psi$ or $\alpha_0 \Vdash \neg\psi$.

Soundness 1 + 2: Both claims are proved simultaneously using induction on the number of logical constants in ψ. The case where ψ is atomic is straightforward, and so is the case where $\psi \equiv \psi_0 \vee \psi_1$ or $\psi \equiv \exists y \ \psi_0(y)$.

If $(\mathbb{M}, \tilde{\alpha}) \models \neg\psi$, by induction there cannot be $\alpha \in \mathcal{G}$ such that $\alpha \Vdash \psi$. By completeness there is $\alpha \in \mathcal{G}$ such that $\alpha \Vdash \neg\psi$.

If $\alpha \Vdash \neg\psi$ but $(\mathbb{M}, \tilde{\alpha}) \models \psi$ for some generic map $\tilde{\alpha} \supseteq \alpha$, there is $\beta \in \mathcal{G}$ such that $\beta \Vdash \psi$. By definition α and β have a common extension in $\mathcal{P}$. By use of the extension and the consistency property we get a contradiction. □

Corollary 4.6.3 *Any generic $\tilde{\alpha}$ is a bijection from $[0, c)$ to $\mathbb{M}$, which maps $[0, n)$ onto S.*

Proof: Let $a \in [0, c]$ be an arbitrary element. Notice that $\mathcal{D}_a := \{\alpha : \ \alpha(a) \text{ is defined}\}$ is both dense and quasi-definable. By definition there is $\alpha_0 \in \mathcal{D} \cap \mathcal{G}$.

As $\alpha_0 \subseteq \tilde{\alpha}$ this shows that $\tilde{\alpha}$ has domain $[0, c)$. The other properties are proved by a similar argument. □

Lemma 4.6.4 *For each $\psi(\vec{x}) \in \Sigma(b_0)$ there is $k \in \omega$ which does not depend on the parameters in $\psi(\vec{x})$ and $\mathbb{M}$-definable maps $\vec{x} \hookrightarrow V^D_{\vec{x}}$ and $\vec{x} \hookrightarrow V^R_{\vec{x}}$ such that*

$$\text{(1)} \qquad \mathrm{Card}(V^D_{\vec{x}}) \leq b_0^k, \quad \mathrm{Card}(V^R_{\vec{x}}) \leq b_0^k.$$

$$\text{(2)} \qquad \text{For all } \alpha \in \mathcal{P} \text{ with } \mathrm{Dom}(\alpha) \supseteq V^D_{\vec{x}} \text{ and } \mathrm{Ran}(\alpha) \supseteq V^R_{\vec{x}}.$$

$$\alpha \Vdash \psi(\vec{x}) \text{ or } \alpha \Vdash \neg\psi(\vec{x}).$$

$$\text{(3)} \qquad \text{For all } \alpha \in \mathcal{P},\ \alpha_{|V^D_{\vec{x}} \cup V^R_{\vec{x}}} \Vdash \psi(\vec{x}) \text{ if } \alpha \Vdash \psi(\vec{x}).$$

Proof: If $\psi(x)$ is atomic and is of the form $\alpha(u,v), \alpha(x,v), \alpha(u,x)$ or $\alpha(x_1,x_2)$ let $V^D_x := \{u\}, V^D_x := \{u\}, V^D_x := \{v\}$ or $V^D_{x_1,x_2} := \{x_1\}$, and similar let $V^R_x := \{v\}$, $V^R_x := \{v\}$, $V^R_x := \{v\}$ and $V^R_{x_1,x_2} := \{x_2\}$.

If $\psi \equiv \psi_0 \vee \psi_1$ let $V^D_{\vec{x}} := V^{D,0}_{\vec{x}} \cup V^{D,1}_{\vec{x}}$, and let $V^R_{\vec{x}} := V^{R,0}_{\vec{x}} \cup V^{R,1}_{\vec{x}}$. If $\psi \equiv \exists u \leq b_0\ \psi_0(u,\vec{x})$ let $V^D_{\vec{x}} := \bigcup_{u \leq b_0} V^{D,0}_{u,\vec{x}}$ and $V^R_{\vec{x}} := \bigcup_{u \leq b_0} V^{R,0}_{u,\vec{x}}$ and notice that $\mathrm{Card}(V^{D/R}_{\vec{x}}) \leq b_0 \cdot b_0^k = b_0^{k+1}$ for some $k \in \omega$. If $\psi \equiv \neg\psi_0$ let $V^{D/R}_{\vec{x}} := V^{D/R,0}_{\vec{x}}$.

Now we prove (2) and (3) by induction on the number of logical constants in $\psi(\vec{x})$. If $\psi(\vec{x})$ is atomic it is easy to check (2) and (3). Suppose $\alpha \not\Vdash \exists x \leq b_0\ \psi(x)$ where $\psi \in \Sigma(b_0)$.

We need to show that $\alpha_{|V} \Vdash \neg\exists x \leq b_0\ \psi(x)$. Conversely suppose that for some $\beta \supseteq \alpha_{|V}$, $\beta \Vdash \exists x \leq b_0\ \psi(x)$. By definition for some $a \leq b_0$, $\beta \Vdash \psi(a)$, and by induction $\beta_{|V^D_a \cup V^R_a} \Vdash \psi(a)$. Now as $\alpha_{|V^D_a \cup V^R_a} = \beta_{|V^D_a \cup V^R_a}$ $\alpha_{|V^D_a \cup V^R_a} \Vdash \exists x \leq b_0\ \psi(x)$ and then by the extension lemma $\alpha_{|V} \Vdash \exists x \leq b_0\ \psi(x)$, which is in contradiction to the assumption $\alpha \not\Vdash \exists x \leq b_0\ \psi(x)$. □

4.7 Some properties of the generic objects

We have already defined $\exists^*$ to be the set of formulas in which all quantifiers are either existential which appear positively or are restricted to $[0, b_0)$ (Sharply bounded quantifiers). According to our plan in order to prove the main theorem we have to prove that some (any) generic $\tilde{\alpha}$ satisfies the $\exists^*$ -LNP scheme. Let $\exists^*_{Strict}$ be the set of formulas $\exists\vec{x}\psi(\vec{x})$, where $\psi \in \Sigma(b_0)$ and where there are no restrictions on the parameters in ψ. First we prove that:

Lemma 4.7.1 *For any generic map $\tilde{\alpha}$, $(\mathbb{M}, \tilde{\alpha})$ satisfies the $\exists^*_{Strict}$-LNP scheme.*

Proof: Let $\psi(z) \equiv \exists\vec{x}\psi_1(\vec{x}, z)$ be given. ($\psi_1 \in \Sigma(b_0)$). Let $\alpha_0 \in \mathcal{P}_k$ be given such that $\alpha_0 \Vdash \psi(a)$ for some $a \in \mathbb{M}$. Let $a_0 \in \mathbb{M}$ be the smallest element such that for some $\beta \supseteq \alpha_0$ with $\beta \in \mathcal{P}_{k+r}$ (for suitable r), and some $\vec{x}_0$, $\beta \Vdash \psi_1(\vec{x}_0, a_0)$. This definition makes sense because the forcing relation is definable for $\Sigma(b_0)$-formulas. (By lemma 4.6.4). Let $\mathcal{D}$ be the set of $\beta' \in \mathcal{P}$ which are either incompatible to α_0 or extensions of a β with the property just mentioned above. $\mathcal{D}$ is quasi-definable and dense so there is $\alpha \in \mathcal{D} \cap \mathcal{G}$. Clearly $\alpha \Vdash \psi(a_0)$. All that remains is to check that if r is chosen properly (lemma 4.6.4) there is no $a_1 < a_0$ and $\alpha' \supset \alpha$ with $\alpha' \Vdash \psi(a_1)$. $\Box$

4.8 A minor problem

Now we want to prove that for some (any) generic $\tilde{\alpha}$, $(\mathbb{M}, \tilde{\alpha})$ actually satisfies the $\exists^*$-LNP scheme. It should be noticed that because of the presence of the generic object $\tilde{\alpha}$ it is not entirely clear why any $\exists^*$-formula should be equivalent to an $\exists^*_{Strict}$ formula.

Let $\psi(z)$ be a given $\exists^*$-formula. Let us try to follow the same strategy as in the proof of lemma 4.7.1 above. Without loss of generality we can assume that

$$\psi(z) \equiv \exists x_0 \forall u_0 \leq b_0 \exists x_1 \forall u_1 \leq b_0 ... \exists x_k \theta(\vec{x}, \vec{u}, z)$$

where $\theta \in \Sigma(b_0)$. Find smallest z_0 such that there are a_0,$a_{1,0}, a_{1,1}, ..., a_{1,b}$, and $a_{2,0,0}, a_{2,0,1}, ..., a_{2,0,b_0}$ and $a_{2,1,0}, a_{2,1,1}, a_{2,1,2}, ..., a_{2,1,b_0}$ $..., a_{2,b_0,b_0}, ...$

$$... a_{\underbrace{k, b_0, b_0, ... b_0}_{k \ b_0's}}$$

and $\beta \supseteq \alpha_0$ in $\mathcal{P}_{k+r}$ such that for any choice $u_0 \leq b_0$, $u_1 \leq b_0$, ... $u_k \leq b_0$

$$\beta \Vdash \theta(a_0, a_{1,u_0}, a_{2,u_0,u_1}, ..., a_{k,u_0,...,u_{k-1}}, u_0, u_1, ... u_{k-1}, z_0)$$

Again if there is any z_0 satisfying the condition, there is also a smallest such z_0. This holds for each choice of $r \in \omega$. As a minor technical problem we need to show (what might be obvious to the reader) that if β is chosen as above, for no $\beta' \supseteq \beta$ we can have $\beta' \Vdash \psi(z_1)$ for some $z_1 < z_0$. Now in general $(\mathbb{M}, \tilde{\alpha})$ has <u>more</u> definable functions on $[0, b_0)$ than $\mathbb{M}$. This is because for each formula $\theta(x, \alpha)$ and each generic map $\tilde{\alpha}$, α could be eliminated (i.e. there was a formula $\psi(x)$ such that $\forall x \in [0, b_0)$ $\theta(x, \alpha) \Leftrightarrow \psi(x)$ in the case where $f_{slow}(x) := \log(x)$), then induction would hold up to $b_0 \geq log(c)$ and therefore up to c, which would be a contradiction. Essentially we have to check that $\psi(z_1)$ can not have a sequence of witnesses which was not definable in $\mathbb{M}$. In the case where $\psi(x)$ was a $\exists^*_{Strict}$-formula there was no such problem because the search in $\mathbb{M}$ was only a search for standard finitely many witnesses. We have to show that there cannot be such a z_1. This is essentially done by showing that in the case of $\exists^*$-formulas for any generic $\tilde{\alpha}$ there is an $\mathbb{M}$ definable sequence of witnesses.

For $\theta \in \exists^*$ we make the following definition:

Definition 4.8.1 $\alpha \Vdash^D \ \forall x \leq b_0 \ \theta(x)$ *iff* $\forall x \leq b_0 \ \alpha \Vdash^D \ \theta(x)$.
$\alpha \Vdash^D \ \exists x \theta(x)$ *iff* $\exists x \ \alpha \Vdash^D \ \theta(x)$
$\alpha \Vdash^D \ \neg\neg\theta$ *iff* $\alpha \Vdash^D \ \theta$. ♣

Notice that for $\theta \in \exists^*$ if $\alpha \Vdash^D \theta$ for each extension (not necessarily a generic extension!) $\tilde{\alpha}$ of α then $(\mathbb{M}, \tilde{\alpha}) \models \theta$.

Observation 4.8.2 *For each $\alpha_0 \in \mathcal{P}$, each $k \in \omega$ and each $\psi \in \exists^*$ the set $\{<\alpha, x>: \alpha \supseteq \alpha_0 \wedge \alpha \in \mathcal{P}_k \wedge \alpha \Vdash^D \ \psi(x)\}$ is definable in $\mathbb{M}$.*

The problem we are concerned with at this stage is whether it is possible for given α_0 and $\theta \in \Sigma(b_0)$ to have $\alpha \supseteq \alpha_0$ with $\alpha \Vdash \forall x \leq b \exists y \ \theta(x, y)$ but for <u>no</u> $\beta \supseteq \alpha_0, \ \beta \in P_r : \ \beta \Vdash^D \ \forall x \leq b \exists y \ \theta(x, y)$?

<u>Assume</u> that there is $\alpha \supseteq \alpha_0$ such that $\alpha \Vdash \forall x \leq b_0 \exists y \ \theta(x, y)$.
Claim: There is an extension $\alpha_1 \supseteq \alpha$ such that $\forall x \leq b_0 \exists y \ \alpha_1 \Vdash \theta(x, y)$. Pick for each $j \leq b_0, \ \alpha^j$ such that $\alpha \subseteq \alpha^0 \subseteq \alpha^1 \subseteq ...\alpha^{b_0} = \alpha_1$ and such that $\alpha_j \Vdash \exists y \theta(j, y)$. Furthermore by lemma 4.6.4 each extension can be chosen to be of size $\leq b_0^l$ for some <u>fixed</u> $l \in \omega$ (l can be chosen to be the number of "$\forall x \leq b_0$" quantifiers in θ). So α_1 is a $\leq b_0^{l+1}$ extension of α. By lemma 4.6.4 for each $x \leq b_0$ and witness $y(x)$ there is a set $V_{x,y(x)}$ with cardinality $\leq b_0^l$ for some <u>fixed</u> standard l. Let $V := \bigcup_{x \leq b_0} \ V_{x,y(x)}$. Notice that Card$(V) \leq b_0^{l+1}$ and that $\forall x \leq b_0 \exists y \ \alpha_{1,|V} \Vdash \psi(x, y)$. Let $\beta = (\alpha_1)_{|V}$ and notice that $\beta \Vdash^D \ \forall x \leq b_0 \exists y \ \theta(x, y)$. We have just proved the first part of the next lemma:

Lemma 4.8.3 *Let $\psi(z) = \forall x \leq b_0 \exists y \ \theta(x, y, z)$ where $\theta \in \Sigma(b_0)$. There is $r \in \omega$ such that if $\beta \supseteq \alpha_0$ is the b_0^r-extension of α_0 with the smallest z_0 such that*

$$\beta \Vdash^D \ \forall x \leq b_0 \exists y \ \theta(x, y, z_0)$$

then for <u>any</u> generic model $(\mathbb{M}, \tilde{\alpha})$ with $\tilde{\alpha} \supseteq \alpha_0$ $\ (\mathbb{M}, \tilde{\alpha}) \models \psi(z_0) \wedge \forall z < z_0 \ \neg\psi(z)$.

More generally Let $\psi(z) = \exists x_0 \forall u_0 \leq b_0 \exists x_1 \forall u_1 \leq b_0 .... \exists x_k \ \theta(\vec{u}, \vec{x}, z)$ *where* $\theta \in \Sigma(b_0)$. *Then there is a standard number r which does not depend on the parameters in ψ such that if $\beta \supseteq \alpha_0$ is the b_0^r-extension of α_0 with the smallest z_0 satisfying $\beta \Vdash^D \ \psi(z_0)$ then for <u>any</u> generic model $(\mathbb{M}, \tilde{\alpha})$ with $\tilde{\alpha} \supseteq \alpha_0$*

$$(\mathbb{M}, \tilde{\alpha}) \models \psi(z_0) \wedge \forall z < z_0 \ \neg\psi(z).$$

Proof: Let α_0 be given. Assume that for some $\alpha \supseteq \alpha_0$

$$\alpha \Vdash \forall x_1 \leq b_0 \exists y_1 \forall x_2 \leq b_0 ... \exists x_k \; \theta(\vec{x}, \vec{y})$$

but that for no $\beta \supseteq \alpha_0$, $\beta \in \mathcal{P}_r$ $\beta \Vdash^D \forall x_1 \leq b_0 \exists y_1 ... \theta(\vec{x}, \vec{y})$. There is $\alpha' \supseteq \alpha$ such that $\forall x_1 \leq b_0 \exists y_1$ $\alpha' \Vdash \forall x_2 \leq b_0 \exists y_2 .... \theta(\vec{x}, \vec{y})$. By induction on k, we can assume that we have already proved that if for some fixed x_1 and y_1, $\beta \supseteq \alpha$, $\beta \Vdash \forall x_2 \leq b_0 \exists y_2 ... \theta(\vec{x}, \vec{y}), \beta$ can be chosen to be an b_0^{l+k-1}-extension, where l is the number of $\forall$-quantifiers in θ. There is $\alpha'' \supseteq \alpha'$ (making b_0^2 b_0^{k+l-2}-extensions of α'), such that

$$\forall x_1 \leq b_0 \exists y_1 \forall x_2 \leq b_0 \exists y_2 \; \alpha'' \Vdash \forall x_3 \leq b_0 ... \theta(\vec{x}, \vec{y})$$

Continuing like this, after k steps we obtain $\alpha^* \supseteq \alpha$ such that $\alpha^* \Vdash^D \psi$. We can not use α^* as β in the lemma, because α (and therefore α^*) could be too large. But by lemma 4.6.4, there are b_0^k sets $V_{x_1, y(x_1), x_2, y(x_2), ..., x_k, y(x_k)}$ each with $\leq b_0^l$ elements where l can be taken to be the number of $\forall$-quantifiers in θ. Let

$$V := \bigcup_{x_1 \leq b_0} \bigcup_{x_2 \leq b_0} ... \bigcup_{x_k \leq b_0} V_{x_1, y(x_1), ..., x_k, y(x_k)}.$$

Notice that $\mathrm{Card}(V) \leq b_0^{k+l}$. Let $\beta := \alpha^*_{|V}$ and notice that $\beta \Vdash^D \psi$. This contradicts the assumption. $\square$

Corollary 4.8.4 *For any generic map* $\tilde{\alpha}$ $(\mathbb{M}, \tilde{\alpha}) \models \exists^* {-}\mathrm{LNP}$

Proof: Let $\tilde{\alpha}$ be an arbitrary generic map. We need to show that $(\mathbb{M}, \tilde{\alpha}) \models \exists x \psi(x) \Rightarrow \exists x_0 \leq x \forall z < x_0 \; \psi(x_0) \wedge \neg\psi(z)$. Suppose $(\mathbb{M}, \tilde{\alpha}) \models \psi(a)$ for some $a \in \mathbb{M}$ (otherwise there is nothing to prove). According to the completeness property there is $\alpha_0 \subseteq \tilde{\alpha}$ such that $\alpha_0 \in P$ and $\alpha_0 \Vdash \psi(a)$. Assume $\alpha_0 \in \mathcal{P}_k$. Consider $\mathcal{D} \subseteq \mathcal{P}$ defined by $\mathcal{D} := \{\alpha \mid \alpha \not\supseteq \alpha_0 \; \vee \; (\alpha \supseteq \alpha_0 \wedge \exists x \; \alpha \Vdash^D \psi(x) \wedge \forall y < x (\forall \alpha' \supseteq \alpha \; \alpha' \not\Vdash^D \psi(y)))\}$. From what has already been proved it follows that $\mathcal{D}$ is well defined, dense and quasi-definable. For any generic $\mathcal{G} \subseteq \mathcal{P}$ there are $\alpha_1 \in \mathcal{D} \cap \mathcal{G}$. By lemma 4.8.3 for any $\tilde{\beta} \supseteq \alpha$ (in particular $\tilde{\alpha}$) $(\mathbb{M}, \tilde{\beta}) \models \exists x \psi(x) \Rightarrow \exists x_0 \leq x \forall z < x_0 \; \psi(x_0) \wedge \neg\psi(z)$. As $\alpha_1 \subseteq \tilde{\alpha}$ we are done. $\square$

4.9 Proof of the first finitisation principle

The previous section has given us a "generic model" $(\mathbb{M}, \tilde{\alpha})$ which satisfies the $\exists^*$-LNP scheme. Clearly $\tilde{\alpha}$ induces a miniaturised version of S on $[0, n)$. Constants, relations and functions on S correspond to constants, relations and functions on $[0, n)$. Now I prove the important fact that all the miniaturised relations etc. are $\exists \cap \forall$-definable in $(\mathbb{M}, \tilde{\alpha})$.

For each relation $R \subseteq S^r$ (with quantifier-free definition in $\mathbb{M}$), we define the corresponding miniaturised relation R_{mini} "existentially" by:

$$\{(x_1, x_2, ..., x_r) \mid \exists z_1, z_2, ..., z_r \ \wedge_{i=0}^{i=r} \ \tilde{\alpha}(x_i, z_i) \wedge R(z_1, z_2, ..., z_r)\}.$$

Notice that R_{mini} also has a "universal" definition:

$$\{(x_1, x_2, ..., x_r) \mid \forall z_1, z_2, ..., z_r \ \wedge_{i=0}^{i=r} \ \tilde{\alpha}(x_i, z_i) \Rightarrow R(z_1, z_2, ..., z_r)\}.$$

Add the miniaturised relations to the language. Consider the sub-language L_{mini} which contains L and names for the miniaturised constants, relations and functions.

Lemma 4.9.1 *If $\psi(x) \in L_{mini}$ is a Σ_1^b-formula with all quantifiers restricted to $[0, c)$, there is a $\exists^*$-formula $\psi^*(x)$ such that:*

$$\{x \leq c : \ (\mathbb{M}, \tilde{\alpha}) \models \psi(x)\} = \{x \leq c : \ (\mathbb{M}, \tilde{\alpha}) \models \psi^*(x)\}.$$

Proof: Let $\psi(x) \in L_{mini}$ be a given Σ_1^b property. Replace each appearance of a "miniaturised" relation R_{mini} by either the $\exists$-definition or the $\forall$-definition according to whether the R_{mini} appears positively or negatively in $\psi(x)$. Notice that this gives an $\exists^*$-formula $\psi^*(x)$ which satisfies the lemma. $\square$

Now we are ready to construct a model of $T_2^1(\alpha)$ in which there is a bounded sub-structure S_{mini} on $[0, n)$ where $\eta(n)$ such that $S_{mini} \models \theta_{<n}(\vec{R})$. We have already got a countable model $\mathbb{M}$ of true arithmetic in which a possibly unbounded model of $\theta(\vec{R})$ is coded. We have $n, b_0, c \in \mathbb{M}$ as above. Let $\tilde{\alpha}$ be an arbitrary generic map, as described earlier. Let

$$\mathbb{M}^*_{\tilde{\alpha}} = \{x \leq c : \ \exists k \in \omega \quad x < f^{(k)}_{slow}(n)\}.$$

Let $\hat{R}$ be the set of all L_{mini}-definable relations on $[0, n)$. It follows from what has already been proved that $(\mathbb{M}, \hat{R}) \models \Sigma_1^b$-IND scheme, hence because of corollary 3.4.5 there is an expansion of $\hat{R}$ to $\hat{S} \supseteq \hat{R}$ such that $(\mathbb{M}, \hat{S}) \models T^1(\alpha)$. As second order existence statements are absolute with respect to expansions, we have proved the main theorem in the $T^1(\alpha)$-case.

Corollary 4.9.2 *Let S be a countable mathematical structure which can be encoded in a non-standard model $\mathbb{M}$ which satisfies true arithmetic. There is a model of $T_2^1(\alpha)$ in which S appears as a bounded set.*

Notice that our results hold in the special case where $T^1(\alpha)$ is the second order theory which consists of

1) Induction for existential formulas.

2) The $\forall \cap \exists$-comprehension axiom scheme.

5 Separating $T_2^1(\alpha)$ and $S_2^2(\alpha)$

In the process as a by-product we have obtained a new proof of the separation of the theories $T_2^1(\alpha)$ and $S_2^2(\alpha)$.

Theorem 5.0.3 *For <u>any</u> generic map* $\tilde{\alpha}$ $(\mathbb{M}^*_{\tilde{\alpha}}, \tilde{S}_{\tilde{\alpha}}) \models T_2^1(\alpha)$.

If S defines two disjoint infinite sets, then for <u>any</u> generic map $\tilde{\alpha}$ $(\mathbb{M}^*_{\tilde{\alpha}}, \tilde{S}_{\tilde{\alpha}}) \not\models S_2^2(\alpha)$.

Proof: The first part of the theorem has already been proven. It follows from the examples below that the second part holds at least for some constructions (i.e. for some S). To prove the second part let $U(x)$ be a new unary predicate symbol which holds exactly in one component of S. Let U_{mini} denote the corresponding predicate in L_{mini}. Put $f_{slow} :=\mid x \mid$ and $f_{fast} := x^{|x|}$. Consider the formula

$$A(z) := \exists u_1 < u_2 \leq n \ u_2 - u_1 \geq b^z \ \forall x \in [u_1, u_2) \ U_{mini}(x).$$

Clearly $A(k)$ is valid for all standard k and hence, by overspill, $A(z)$ holds for some non-standard z (and $z \leq\mid n \mid$). But this is a contradiction as for any given interval of length $< b^\omega$ the set of $\alpha \in \mathcal{P}$ such that α maps an element of $S \setminus U$ into it, is dense and quasi-definable.

According to [5] Σ_2^b-PIND is sufficient to ensure the validity of Σ_2^b-LMAX principle which in turn implies the validity of the Σ_2^b-overspill just used. $\Box$

Notice that if S is an infinite (co-countable) set with no additional structure then $(\mathbb{M}^*_{\tilde{\alpha}}, \tilde{S}_{\tilde{\alpha}}) \models S_2(\alpha)$. This shows that the second part of the theorem becomes false if there is no condition on S.

Corollary 5.0.4 $T_2^1(\alpha) \neq S_2^2(\alpha)$.

This is a new proof of the result which was first proved in [8].

6 Some examples

Example 6.0.5 *Fix $p \geq 2$. There is an infinite model A where "R defines a partition of A into disjoint p-subsets". Let $\eta(n) \equiv$ "n is not divisible by p'.*

According to the first principle $T^1(\alpha) \not\vdash$ Count(p).

In [2] and [16] it is shown that this holds for much stronger theories.

Example 6.0.6 *According to the first principle:*

(1) $T^1(\alpha) \not\vdash$ *"every linear ordering R (of a finite set) has an isolated point".*

(2) $T^1(\alpha) \not\vdash$ *"every linear ordering R (of a finite set) is discrete".*

(3) $T^1(\alpha) \not\vdash$ *"every linear ordering R (of a finite set) is a well ordering".*

By the results below in all cases $T^1(\alpha)$ can be replaced by $S^2_2(\alpha)$.

Notice that example 6.0.6 shows (using $T^1_2(\alpha) \equiv$ WOA, Proposition 8.0.9) that the well ordering axiom for arbitrary linear orderings (WOA*) R does not follow from WOA.

7 Another principle

The second finitisation principle says that for any given $r \in \mathbb{N}$ if some second order existential relational property $\tilde{P} \equiv P(\vec{R}, X)$ is only witnessed by infinite sets (in the real universe), there are models of $T^1_2(\alpha)$ in which there is an n and relations $\vec{R} \subseteq [0, n)$ such that no subset $X \subseteq [0, n)$ with size $\leq \log^r_2(n)$ witness $P(\vec{R}, X)$.

Theorem 7.0.7 (Finitisation principle) *Let $\theta \equiv \exists X \psi(\vec{R}, X)$ be a second order existential formula where ψ is a first order formula in the language $L(\vec{R}, X, =)$. Let $k \in \mathbb{N}$ be a given natural number. In general (1) implies (2):*

(1) *There is a countable model S of the language $L(\vec{R}, =)$, such that for no finite set $X \subseteq S$, $S \models \psi(\vec{R}, X)$.*

(2) $T^1(\alpha) +$ "$\exists n \forall X[0, n)\ \mathrm{Card}(X) \leq (\log(n))^k \Rightarrow \neg\psi_{<n}(\vec{R}, X)$" *has a model.*

This principle states that if a second order existential property has (in the real universe) a countable model where the existential quantifier is not finitely witnessed then it is consistent with $T^1_2(\alpha)$ (and by the results below $S^2_2(\alpha)$) that there is a finite model where the existential quantifiers is not "polylog"-witnessed.

Example 7.0.8 *In the real world there is a binary tree with no finite branch. By the second finitisation principle T^1_2 is consistent with the existence of a tree $T \subseteq [0, n)$, which has no branch (coded by a number) of length $\leq \log^k_2(n)$".*

In the real word there is a vector space over Z_2 with no finite basis, so by the second principle there exists a model of $T^1_2(\alpha)$ in which there exists a vector space $V \subseteq [0, n)$ with no basic (coded by a number) of size $\leq \log^k_2(n)$".

Proof: Proved by a construction very similar to the proof of the first finitisation principle. Pick S according to (1). Choose S as a countable model in the standard model on say the even numbers. Extend the standard model to an elementary equivalent countable non-standard model. Pick non-standard numbers $b_0 < n < c$ as above. Consider the same set of forcing conditions $(\mathcal{P}, \subseteq)$ as above. The forcing conditions ensure that each generic $\tilde{\alpha}$ maps small (i.e. $\leq b_0^k$ points) $\mathbb{M}$-definable sets to "small" $\mathbb{M}$-definable sets. The map $\tilde{\alpha}$ induces a miniaturised structure S_{mini} with underlying set $[0, n)$. For each $\mathbb{M}$-definable subset $B \subseteq [0, n)$, $A := \tilde{\alpha}^{-1}(B)$ can not witness $\psi(\vec{R}, A)$ because the set A has cardinality $\leq b_0^k$ which is finite in the model $\mathbb{M}$. As in the proof of the first principle without any complications we construct a model $T_2^1(\alpha)$ where $\psi_{<n}(\vec{R}, X)$ is not witnessed by any set A of size $\leq b_0^\omega$. $\square$

8 The well ordering axiom in S_2^1.

The well ordering axiom (WOA) is the principle:

WOA $$\forall X (X \neq \emptyset \Rightarrow \exists y (y \in X \wedge \forall z < y \quad z \notin X))$$

WOA says that "<" well orders any set X, and should not be confused with the stronger principle WOA* stating that any linear ordering of a bounded set is a well ordering.

Proposition 8.0.9 (Suggested by A.J.Wilkie) WOA *is equivalent to* $T_2^1(\alpha)$ *in models of* $S_2^1(\alpha)$

Proof: $T_2^1(\alpha) \Rightarrow$ **WOA:** To reach a contradiction let $(\mathbb{M}, \hat{S}) \models T_2^1(\alpha) + \neg$WOA. There is $R \in \hat{S}$ such that $R \neq \emptyset$ and such that $(\mathbb{M}, \hat{S}) \models \forall y (y \in R \Rightarrow \exists z < y \; z \in R)$. As $R \neq \emptyset$ there is $u_0 \in R$ such that:

$(*)$ $$(\mathbb{M}, \hat{S}) \models \forall y \leq u_0 (y \in R \Rightarrow \exists z < y \; \in R).$$

Let $Y = \{x \in \mathbb{M} : \; \exists z \in R \quad z < x \wedge x \leq u_0\} \subseteq \mathbb{M}$. Notice that Y is Σ_1^b-definable in $\mathbb{M}$ with set parameter R (Y is not required to belong to $\tilde{S}$). As Σ_1^b-LNP $\Leftrightarrow \Sigma_1^b$- IND, and as Y is non-empty, there is a minimal $x_0 \in Y$. By definition of Y there is $z < x_0$ such that $z \in R$. By $(*)$ this contradicts the minimality of x_0.

WOA $\Rightarrow T_2^1(\alpha)$: Let $(\mathbb{M}, \hat{S}) \models S_2^1(\alpha)$ + WOA be an arbitrary model. Let $\psi \in \Sigma_0^b \quad (= \Pi_0^b)$ be an arbitrary formula with possible set variables $\vec{Z} \in \hat{S}$. It is enough to prove that the Σ_1^b-definable set $Y := \{x \in \mathbb{M} : \; \exists z \leq t(x) \; \psi(x, z, \vec{Z})\} \subseteq \mathbb{M}$ has a smallest element if it is not empty. Pick $u_0 \in Y$. To this end consider

$$\phi(w) :\equiv \exists w_1 \leq u_0 \exists w_2 \leq u_0 \; w = w_1 \cdot u_0 + w_2 \wedge \psi(w_1, w_2; \vec{Z}).$$

As $\phi \in \Delta^b_1$ $R_{b,u_0} :\equiv \{w \leq b \mid \phi(w)\} \in \hat{S}$ for any $b, u \in M$. Choose b such that for each pair (x, z) with $z \leq t(x)$ and $x \leq u_0$, $(x, z) \leq b$. According to WOA R_{b,u_0} has a smallest element w_0. Let y_0 be the unique element with $y_0 \cdot u_0 \leq w_0 < (y_0 + 1) \cdot u_0$, and notice that $M \models y_0 = \min(Y)$. We used the fact that strict-Σ^b_1-LNP is equivalent to Σ^b_1-IND over $S^1_2(\alpha)$. So to prove Proposition 8.0.9 all that remains is to prove this, and the equivalence of Σ^b_1-IND and strict-Σ^b_1-LNP over $S^1_2(\alpha)$. Both these facts follow from [4],[7]. $\square$

8.1 A finitisation principle for $S^1_2(\alpha)$

As already proved $T^1_2(\alpha)$ is equivalent to the well ordering axiom WOA over the theory $S^1_2(\alpha)$. In this section I show that $S^1_2(\alpha)$ does not prove this axiom, and thus as a corollary we get a new proof of the separation result $S^1_2(\alpha) \neq T^1_2(\alpha)$. Parts of the argument are very similar to the proof of the two previous principles so I only emphasise the new ideas in the construction.

Theorem 8.1.1 (Finitisation principle for $S^1_2(\alpha)$) *Let $\psi(\prec, R)$ be a sentence in the first order language $L(\prec, R, =)$. Suppose that $\psi(\prec, R)$ has an infinite model S where $\prec$ defines a total linear ordering. Then $\exists n \exists U \subseteq [1, n]$ $\psi_U(<, R)$ holds in some models of $S^1_2(\alpha)$.*

Proof: I put the emphasis on the new ideas. Let $(\mathbb{N}, S_{\mathbb{N}}, R_{\mathbb{N}})$ be an expansion of the standard model $\mathbb{N}$ to an infinite model of S, where $R \subseteq S^r$, and where $\psi(\prec, R)$ holds. Let $(\mathbb{M}, S_{\text{large}}, R_{\text{large}})$ be an elementary equivalent (in a language expanded by relation symbols for R_{large} and S_{large}) non-standard model. Let $n \in \mathbb{M}$ be a non-standard number, and let $I^* \subseteq \mathbb{M}$ be the initial sequent defined by

$$I^* := \{m \in \mathbb{M} : \ \exists t \text{ term } m < t(n)\}.$$

A $\Sigma^b_1(I^*)$-formula θ, is a Σ^b_1-formula where all parameters belong to I^*. Notice that all quantifiers in such a θ also are restricted to I^*. Choose b_0 such that $I^* < 2^{b_0}$, and such that $b_0^\omega < n$. Notice that sharply bounded quantifiers in a $\Sigma^b_1(I^*)$-formula are bound by b_0.

We want to construct $U \subseteq [1, n]$ and $R \subseteq U^r$ such that $\psi_U(<, R)$ is forced true, and such that the polynomial induction scheme

$$(\theta(0) \wedge \forall x(\theta(\lfloor \frac{x}{2} \rfloor) \Rightarrow \theta(x))) \Rightarrow \forall x \theta(x)$$

is forced true for all $\Sigma^b_1(I^*)$-formulas θ. Let $< \theta_1(x), a_1 >, < \theta_2, a_2 >, \ldots.$ be an enumeration of the countably many pairs $< \theta_j(x), a_j >$ where $\theta_j(x)$ is a $\Sigma^b_1(I^*)$-formula with one free variable (namely x), and where $a_j \in I^*$.

Let $U_0 = W_0 = R_0 = \emptyset$. Choose $d_0 < e_0 \in \mathbb{M}$ with $e_0 - d_0 > b_0^\omega$, and $e_0 < \frac{n}{b_0}$. Let $f_0 = \emptyset$.

In general **after stage** j we have constructed $U_j, W_j \subseteq [1, n]$ and $R_j \subseteq U_j^r$, such that $U_j \cap W_j = \emptyset$. The map f_j maps U_j bijectively onto a subset of S_{large}, such that $< u_1, .., u_r > \in R_j$ precisely when $R_{\text{large}}(f(u_1), ..., f(u_r))$. Furthermore, we have $| U_j | \leq j \cdot b_0$, $| W_j | \leq j \cdot b_0$, and $[d_j, e_j]$ such that $e_j - d_j > b_0^\omega$. The points in U_j are very sparsely distributed in the sense that $\forall u_1 \neq u_2 \in U_j \ \ | u_2 - u_1 | \geq e_j$.

Now consider $< \theta_{j+1}(x), a_{j+1} >$. Our aim is to force

$$\theta_{j+1}(0) \wedge \forall x(\theta_{j+1}(\lfloor \frac{x}{2} \rfloor) \Rightarrow \theta_{j+1}(x)) \Rightarrow \theta_{j+1}(a_{j+1})$$

true. Let $a_{j+1}^1 := a_{j+1}, ..., a_{j+1}^{r+1} := \lfloor \frac{a_{j+1}^r}{2} \rfloor, ...$ for $r \leq \lceil \log(a_{j+1}) \rceil$. Now for each $u \in [d_j, e_j]$ define the number $l(u) \in \mathbb{M}$ as the largest number ($\leq \log(a_{j+1})$) such that there are extensions $U'_{j+1} \supseteq U_j$, $W'_{j+1} \supseteq W_j$, and $R'_{j+1} \supseteq R_j$, such that

(1) (U'_{j+1}, R'_{j+1}) are isomorphic by an extension $f'_{j+1} \supseteq f_j$ to a subset of S_{large}.

(2) The elements in U'_{j+1} has pairwise distance $\geq u$

(3) $| U'_{j+1} | \leq (j + 1) \cdot b_0$

(4) $| W'_{j+1} | \leq (j + 1) \cdot b_0$.

(5) At least $l(u)$ of the $\Sigma_1^b(I^*)$-formulas $\theta(a_{j+1}^1), ..., \theta(a_{j+1}^{\lceil \log(a_{j+1}) \rceil})$ are forced true.

Now because l takes less than b_0 values, there must be an interval $[d_{j+1}, e_{j+1}] \subseteq [d_j, e_j]$ such that l is constant on this interval, and such that $e_{j+1} - d_{j+1} > b_0^\omega$. Choose such an interval, and extend U_j, W_j and R_j by letting $U_{j+1} := U'_{j+1}$, $W_{j+1} := W'_{j+1}$, $R_{j+1} := R'_{j+1}$, and $f_{j+1} := f'_{j+1}$ where these extensions correspond to the case where $u = e_{j+1}$. Proceed with the construction in this way:

Let $U := \cup_{j \in \omega} U_j$ and let $R = \cup_{j \in \omega} R_j$ and notice $R \subseteq U^r$. It is not hard to show that $f = \cup_{j \in \omega} f_j$ defines a bijection from the miniaturised model U on $[1, n]$ to the large model S_{large}. This ensures that $\psi_U(<, R)$ holds in the generic model.

The $\Sigma_1^b(I^*)$ polynomial induction schema is forced true. To see this suppose contrarily that for some $k \in \omega$ in the generic model we have

$$\theta_k(0) \wedge \forall x(\theta_k(\lfloor \frac{x}{2} \rfloor) \Rightarrow \theta_k(x)) \wedge \neg\theta_k(a_k).$$

In stage k in the construction we have forced a maximum number of the $\Sigma_1^b(I^*)$-formulas $\theta_k(a_k^1), \theta_k(a_k^2), .., \theta_k(a_k^j), .., \theta_k(0)$ true. As obviously $\theta_k(a_k)$ cannot have been forced true, there must be $j < \lceil \log(a_k) \rceil$ such that $\theta_k(a_k^{j+1})$ is already forced true (after stage k) while $\theta_k(a_k^j)$ is not forced true at this stage (here we are using that the search procedure, for <u>fixed</u> $u \in [d_k, e_k]$, is $\mathbb{M}$-definable). Could it be that $\theta_k(a_k^j)$ get forced true at a later stage $k' > k$? No, because if this were the case for some $u \in [d_k, e_k]$, l would take a larger value than $l(e_k)$, which would be a contradiction.

$\square$

Corollary 8.1.2 *There are models of $S_2^1(\alpha)$ in which* WOA *fails.*

Proof: Let $\psi(\prec) := \forall x \exists y\ y \prec x$. This sentence holds in an infinite model (in the real world), so by theorem 8.1.1 the sentence $\forall x \in U \exists y \in U\ y < x$ holds in some models of $S_2^1(\alpha)$. $\square$

This gives a new proof of the separation first proved in [10]

Corollary 8.1.3 $S_2^1(\alpha) \neq T_2^1(\alpha)$.

9 Lifting the finitisation principles

Now I show that both finitisation principles for $T_2^1(\alpha)$ hold for the theory $S_2^2(\alpha)$ but both fail to hold for $T_2^2(\alpha)$. This, in particular separates the theories $S_2^2(\alpha)$ and $T_2^2(\alpha)$. In this I rely heavily on a proof sketch ed by J.Krajicek (personal communication). The proof relies again strongly on S.Buss' result [5], that $S_2^{i+1}(\alpha)$ is $\forall\Sigma_{i+1}^b$-conservative over $T_2^i(\alpha)$.

Theorem 9.0.4 *The first finitisation principle holds if $T^1(\alpha)$ is replaced by $S_2^2(\alpha)$*

Proof: To get a contradiction assume that $\theta(R) \equiv \forall x_1 \exists y_1 .... \forall x_k \exists y_k\ \tilde{\theta}(\vec{x}, \vec{y}; R)$ ($\tilde{\theta}$ is quantifier free) does have an infinite model, but $S_2^2(\alpha) \vdash \forall n\ (\eta(n) \Rightarrow \forall R < n\ \neg\theta_{<n}(R))$. Consider the skolemisation

$$\tilde{\theta}(x_1, F_1(x_1), x_2, F_2(x_1, x_2), ..., x_k, F_k(x_1, x_2, ..., x_k); R)$$

of $\theta(R)$. By the first finitisation principle $T_2^1(\alpha) + \exists n\ (\eta(n) \wedge \exists R < n \exists F_1, ..\exists F_k < n$ $\forall \vec{x} \leq n\ \tilde{\theta}(x_1, F_1(x_1)....; R))$ is consistent.

Add an extra constant $\bar{n}$, extra function symbols $\bar{F}_1, \bar{F}_2, .., \bar{F}_k$ and an extra relational symbol $\bar{R}$ to the language. Call this language $\tilde{L}_2$. The first order theory $T_2^1(\bar{n}, \bar{F}_1, ..., \bar{F}_k, \bar{R}) + \forall x \leq \bar{n}\ \tilde{\theta}(x_1, \bar{F}_1(x_1), ..., \bar{F}_k(\vec{x}); \bar{R})$ is consistent.

Now by theorem 3.5.3 $S_2^2(\bar{n}, \bar{F}_1, ..., \bar{F}_k, \bar{R}) + \forall x \leq \bar{n}\ \tilde{\theta}(x_1, \bar{F}_1(x_1), ..., \bar{F}_k(\vec{x}); \bar{R})$ is consistent. As a skolemisation of θ implies θ (even in pure predicate logic) this contradicts our initial assumption about $S_2^2(\alpha)$. $\square$

This give us the following:

Corollary 9.0.5 *Let $\theta(\vec{R})$ be any first order property in some relational language $\tilde{L} := L(\vec{R})$. Then the following are equivalent:*

(a) Predicate logic $\vdash \theta(\vec{R})$

(b) $$T^1(\alpha) \vdash \forall n\ \theta_{<n}(\vec{R})$$

(c) $$S^2_2(\alpha) \vdash\ \forall n\ \theta_{<n}(\vec{R}).$$

Proof: Let T be $T^1(\alpha)$ or $S^2_2(\alpha)$. Assume that $\theta(\vec{R})$. Assume that T does <u>not</u> prove $\forall n\ \theta_{<n}(\vec{R})$. There exists a non-standard model $\mathbb{M}^*$ of T in which $\neg\theta_{<n}(\vec{R})$ holds in some initial segment $[0, n)$, so $([0, n), \vec{R}) \models \neg\theta(\vec{R})$. But then by the completeness theorem $\theta(\vec{R})$ is not provable in predicate logic.

Contrarily if predicate logic does not prove $\theta(\vec{R})$ there is a model S of $\neg\theta(\vec{R})$. If S is finite certainly $\forall n\ \theta_{<n}(\vec{R})$ does not have a T-proof (because T is consistent). If S is infinite, by the finitisation principle $\exists n\ \neg\theta_{<n}(\vec{R})$ is consistent with T. $\Box$

10 Separating $S^2_2(\alpha)$ and $T^2_2(\alpha)$

Theorem 10.0.6 *The statement "Every linear ordering of an interval has a smallest element" is provable in $T^2_2(\alpha)$, but <u>not</u> provable in $S^2_2(\alpha)$.*

The statement "There is no dense linear ordering of an interval of length ≥ 2" is provable in $T^2_2(\alpha)$ but not provable in $S^2_2(\alpha)$.

Proof: The second part of the claim follows from the finitisation principle for $S^2_2(\alpha)$. To prove the first part let $(\mathbb{M}, \hat{S}) \models T^2_2(\alpha)$ be an arbitrary model. Assume that R is an $\mathbb{M}$-definable linear ordering of an interval $[0, n)$ for some $n \in M$. Let

$$Y = \{b \in [0, n) :\ \exists a_0 \leq b \forall x \leq b\ xRa_0 \Rightarrow x = a_0\}.$$

Notice that Y is Σ^b_2-definable and that $0 \in Y$ and $b \in Y \Rightarrow b + 1 \in Y$ for $b < n$. As $(\mathbb{M}, \hat{S}) \models \Sigma^b_2 - \mathrm{IND}$, $(n - 1) \in Y$. By definition of Y, R has a smallest element. The second part follows from the first part. $\Box$

This gives us a new proof of

Corollary 10.0.7 $S^2_2(\alpha) \neq T^2_2(\alpha)$.

The following were proved in [10]. Above we have obtained a new proof of the second part of the theorem.

Theorem 10.0.8 *The weak pigeon-hole principle (*WPHP*) is provable in $T^2_2(\alpha)$, but <u>not</u> provable in $S^2_2(\alpha)$.*

Proof: The second part of the theorem has already been proved. The first part follows from the fact that $S^3_2(\alpha)$ is $\forall\Sigma^b_3$-conservative over $T^2_2(\alpha)$, that WPHP is a $\forall\Sigma^b_2$-formula and from the following sub claim:

Lemma 10.0.9 WPHP *is provable in* $S_2^3(\alpha)$.

Proof: A careful analysis of the S_2-proof in [12] (see also [8]) of WPHP shows that the proof is actually a $S_2^3(R)$-proof. $\square$

11 Lifting the second principle

For the sake of completeness let me sketch how the second finitisation principle can be lifted to $S_2^2(\alpha)$.

Theorem 11.0.10 *Let* $\theta \equiv \exists X\ \psi(\vec{R}, X)$ *be a second order existential formula where* ψ *can be any first order formula in the relational language* $L(\vec{R}, =)$. *Let* $k \in \mathbb{N}$ *be a given natural number. In general (1) implies (2):*

(1) *There is a* <u>*countable*</u> *model* S *of the language* $L(\vec{R}, =)$, *such that for* <u>*no finite*</u> *set* $X \subseteq S$, $S \models \theta(\vec{R}, X)$.

(2) $S_2^2(\alpha) +$ "$\exists n \forall X \subseteq [0, n)\ \mathrm{Card}(X) \leq (log(n))^k \rightarrow \neg\theta_{<n}(\vec{R}, X)$" *has a model.*

Proof:(sketch) Essentially we lift the result as we did for the first finitisation principle. Suppose that $S_2^2(\alpha) \vdash \forall n \forall \vec{R} \subseteq [0, n) \exists b\ \psi_{<n}(\vec{R}, \mathrm{Seq}(b))$ where $\psi_{<n} \equiv \forall x_1 < n \exists x_2 < n ... \exists x_{2r} < n\ \tilde{\theta}(\vec{x}, n, \vec{R}, \mathrm{Seq}(b))$ with $\tilde{\theta}$ quantifier free.

We have already proved the theorem in the $T_2^1(\alpha)$-case, so

$$T_2^1(\alpha) + \forall \vec{x} < n \forall b \neg\tilde{\theta}(x_1, \bar{F}_1(x_1), x_3, ..., \bar{F}_r, n, \vec{R}, \mathrm{Seq}(b))$$

is consistent, where the language is extended by extra relation symbols $\bar{n}, \vec{R}, \bar{F}_1, ..., \bar{F}_r$. By S.Buss' conservation result, (theorem 3.5.3), which also works in this case where we have added extra function symbols to the language

$$S_2^2(\alpha) + \forall \vec{x} < \bar{n} \forall b\ \neg\tilde{\theta}(x_1, \bar{F}_1(x_1), ..., \bar{n}, \vec{R}, \mathrm{Seq}(b))$$

is consistent in this extended language. Clearly this contradicts our initial assumption. $\square$

Notice the following phenomenon:

Corollary 11.0.11 *If*

$$\exists X \subseteq [0, n)\ \mathrm{Card}(X) \leq (\log(n))^{k_1} \wedge \theta_{<n}(\vec{R}, X)$$

holds in the real world for some constant k_1 *but does not hold for some other constant* $k_2 < k_1$, $S_2^2(\alpha)$ *does not prove the formula for any constant* k.

12 Some applications

By use of the second finitisation principle we get the following theorem. The last case (3) shows that the theorem is not entirely a curiosity.

Theorem 12.0.12 *If k is a natural number, then $S_2^2(\alpha)$ does not prove any of the following statements:*

(1) *"Every vector space $V \subseteq [0, n)$ over Z_2 has a basis of size $\leq \log^k(n)$".*

(2) *"Every binary tree $T \subseteq [0, n)$ has a branch of length $\leq \log^k(n)$".*

(3) *"Every total irreflexively oriented graph on $[0, n)$ has a dominating set of size $\leq \log^k(n)$".*

On the other hand $T_2^2(\alpha)$ proves (1)-(2).

At present, it is not known whether $T_2^2(\alpha) \vdash (3)$. A proof of even $S_2(\alpha) \vdash (3)$ would have interesting consequences. (See [13] and [10] for more details).

Proof: The first part of the theorem follows from the second finitisation principle (in case (1) from a slightly modified version of the principle) and the facts:

There is an (infinite) vector space with no finite basic.

There is an (infinite) binary tree with no finite branch.

There is an (infinite) irreflexively oriented graph with no finite dominating set.

To prove the last part of the theorem, assume V is a vector space on $[0, n)$ with no coded basic. Pick independent vectors $v_1, v_2, ..., v_{\lfloor \log(2n) \rfloor + 1}$ and get a Δ_1^b-definable injective map from $[0, 2n)$ to $[0, n)$. This violates WPHP which are provable in $T_2^2(\alpha)$ by [4] (or [8]).

Finally $T_2^2(\alpha)$ proves that every tree $T \subseteq [0, n)$ has a coded branch. Let $T^* := \{x \in T : \exists b\ b$ codes a path of length $\leq \lfloor \log(n) \rfloor + 1$ from the root to $x\}$. Notice that T^* also have the Π_1^b-definition: $T^* := \{x \in T : \forall b\ b$ codes a path from the root to x and the length of the path is $\leq \lfloor \log(n) \rfloor + 1\}$. Define a Δ_1^b-relation by $F(x) = y$ iff $\forall b\ b$ codes a path $x_1, x_2, ..., x_r$ from the root to x, and $\mathrm{bit}(y, j) = 1 \leftrightarrow x_j$ is a right son. Again we get a contradiction to the WPHP in [11]. □

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