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Søren Riis

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BRICS
Department of Computer Science
University of Aarhus
Ny Munkegade, building 540
DK - 8000 Aarhus C
Denmark

Telephone: +45 8942 3360
Telefax: +45 8942 3255
Internet: BRICS@daimi.aau.dk

A Fractal which violates the Axiom of Determinacy

Søren Riis*
BRICS†

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Abstract

By use of the axiom of choice I construct a symmetrical and self-similar subset $A \subseteq [0,1] \subseteq \mathbb{R}$. Then by an elementary strategy stealing argument it is shown that A is not determined. The (possible) existence of fractals like A clarifies the status of the controversial Axiom of Determinacy. [1]

In this note, I present an argument against the unrestricted axiom of determinacy (=**AD**).

Fix $A \subseteq [0,1] \subseteq \mathbb{R}$. We define an infinite game $\mathcal{G}_A$ as follows. The initial position is $I := [0,1]$. Each position will be an interval $[a,b] \subseteq [0,1]$, $a, b \in \{\frac{p}{2^k} : p, k \in \mathbb{N}\}$. In each position $[a,b]$ there are always two legal moves. The player who has the turn can move "left" or can move "right". If the player moves "left" the new position is $[a, \frac{a+b}{2}]$. If the player moves "right" the new position is $[\frac{a+b}{2}, b]$. Player $\mathcal{A}$ makes the first move. In each actual game, successively the players construct a sequence $I = [0,1] \supseteq I_1 \supseteq I_2 \supseteq ....$ of closed intervals. The interval I_{j+1} is either the left or the right closed interval of I_j. Each actual game produces a point $p := \cap_{j \in \mathbb{N}} I_j \in [0,1]$.

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†Basic Research in Computer Science, Centre of the Danish National Research Foundation.

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According to the rules of $\mathcal{G}_A$ player $\mathcal{A}$ wins if $p \in A$. Otherwise (i.e. when $p \notin A$) player $\mathcal{B}$ wins.

A player has a winning strategy if there is a protocol (i.e. a map from the set of positions to { 'left','right'}) which guarantees victory. The set of positions can be divided into 3 classes. The positions which are won for player $\mathcal{A}$, the positions which are won for player $\mathcal{B}$, and the controversial class of the positions which are undetermined. According to the axiom of determinacy the last class is always empty. Each actual game has a winner. So if the players are clever enough it must be determined who will win the game. The intuition behind **AD** is that if $\mathcal{A}$ and $\mathcal{B}$ have infinite powers the same player will win each game. If for instance $\mathcal{A}$ wins the first game, $\mathcal{A}$ ought also to win the second game. The argument is that if player $\mathcal{B}$ wins this new game, sometimes in the first game $\mathcal{B}$ could not have possibly played optimally [2] .

It is well-known that **AC** (the axiom of choice) and **AD** are contradictory. The usual proof uses a diagonal argument combined with the fact that the number of strategies is $2^{\aleph_0}$ [1], [2]. The status of **AD** has been examined in great depth [4],[5]. There seems to be two approaches. One can accept **AC** and ask which sets are determined. This leads to questions which are independent of the usual axiomatization of set theory [2],[4], [5]. The other and more radical approach is to discard the axiom of choice [3]. The main argument is that **AD** is deductively strong and has many nice consequences, [2],[4]. Still there is no doubt that most of us prefer **AC**.

Let $\mathbb{Q}$ denote the rational numbers.

Theorem (AC) *There exists a set $A \subseteq [0,1]$ such that A and $A \cap [0, \frac{1}{2}]$ are isomorphic under the map $x \to \frac{x}{2}$, and $A \setminus \mathbb{Q}$ and $A^c \setminus \mathbb{Q}$ are isomorphic under the map $x \to\ 1-x$.*

Proof: Consider the collection **J** of pairs (A, B) where $A, B \subseteq [0,1] \setminus \mathbb{Q}$, where $A \cap B = \emptyset$ and where $A = A \cdot \mathbb{Q} \cap [0,1] \setminus \mathbb{Q}$ and $B = B \cdot \mathbb{Q} \cap [0,1] \setminus \mathbb{Q}$. The set **J** of such pairs are ordered inductively under inclusion. According to Zorn's lemma (which is equivalent to **AC**) there must be a pair $(A, B) \in \mathbf{J}$ which is maximal with respect to inclusion. We claim that $A \cup B = [0,1] \setminus \mathbb{Q}$. Otherwise there would be $x \in ([0,1] \setminus \mathbb{Q}) \setminus (A \cup B)$. Notice that, $(1-x) \cdot \mathbb{Q} \cap A = \emptyset$, $x \cdot \mathbb{Q} \cap (1-x) \cdot \mathbb{Q} = \emptyset$ and $x \cdot \mathbb{Q} \cap B = \emptyset$. Thus $(A \cup x \cdot \mathbb{Q}, B \cup (1-x) \cdot \mathbb{Q})$

[2] Unless of course $\mathcal{A}$ first deviate from the line of play in the first play. But to deviate and lose does not seem wise.

is well-defined and belongs to **J**. This violates the maximality of (A, B). $\Box$

Theorem *All the positions $(I', A \cap I')$ are isomorphic, when the points in $\mathbb{Q}$ are ignored. No move can make any difference to the outcome because all positions are isomorphic. Each game produces a winner.*

Proof: We ignore the points in $\mathbb{Q}$. Notice that the two positions which can be reached from I both are isomorphic to I (when the role of A is changed with that of A^c). This is because $f_1 : x \to 1 - 2x$ maps $A^c \cap [0, \frac{1}{2}]$ isomorphic onto A, and because $f_2 : x \to 2 - 2x$ maps $A^c \cap [\frac{1}{2}, 1]$ isomorphic onto A. $\Box$

Corollary (AC) $\mathcal{G}_A$ *is not determined.*

Proof: If player $\mathcal{A}$ has a winning strategy, player $\mathcal{B}$ can steal it. It is not difficult to show that the points in $\mathbb{Q}$ do not affect this argument. $\Box$

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