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On the uniform weak König's lemma

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Abstract

The so-called weak König's lemma WKL asserts the existence of an infinite path b in any infinite binary tree (given by a representing function f). Based on this principle one can formulate subsystems of higher-order arithmetic which allow to carry out very substantial parts of classical mathematics but are Π_2^0 -conservative over primitive recursive arithmetic PRA (and even weaker fragments of arithmetic). In [10] we established such conservation results relative to finite type extensions PRA^ω of PRA (together with a quantifier-free axiom of choice schema). In this setting one can consider also a uniform version UWKL of WKL which asserts the existence of a functional Φ which selects uniformly in a given infinite binary tree f an infinite path Φf of that tree. This uniform version of WKL is of interest in the context of explicit mathematics as developed by S. Feferman. The elimination process in [10] actually can be used to eliminate even this uniform weak König's lemma provided that PRA^ω only has a quantifier-free rule of extensionality QF-ER instead of the full axioms (E) of extensionality for all finite types. In this paper we show that

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in the presence of (E) , UWKL is much stronger than WKL: whereas WKL remains to be Π_2^0 -conservative over PRA, $\text{PRA}^\omega + (E) + \text{UWKL}$ contains (and is conservative over) full Peano arithmetic PA.

1 Introduction

The binary (so-called ‘weak’) König’s lemma WKL plays an important role in the formulation of mathematically strong but proof-theoretically weak subsystems of analysis. In particular the fragment (WKL_0) of second-order arithmetic which is based on recursive comprehension (with set parameters), Σ_1^0 -induction (with set parameters) and WKL occurs prominently in the context of reverse mathematics (see [16]). Although (WKL_0) allows to carry out a great deal of classical mathematics, it is Π_2^0 -conservative over primitive recursive arithmetic PRA as was shown first by H. Friedman using a model-theoretic argument. In [15] a proof-theoretic argument is given for a variant of (WKL_0) which uses function variables instead of set variables. In [10] we established various conservation results for WKL relative to subsystems of arithmetic in all finite types. As a special case these results yield that

(1) $\text{E-PRA}^\omega + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + \text{WKL}$ is Π_2^0 -conservative over PRA,

where $\text{E-PRA}^\omega + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + \text{WKL}$ is a finite type extension of (WKL_0) (see below for a precise definition). The proof of this fact relies on a combination of Gödel’s functional interpretation with elimination of extensionality (see [12]), negative translation and Howard’s [8] majorization technique. The first step of the proof reduces the case with the full axiom of extensionality to a subsystem $\text{WE-PRA}^\omega + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + \text{WKL}$ which is based on a weaker quantifier-free rule of extensionality only (see below) which was introduced in Spector [17]. From this system, WKL is then eliminated. This elimination actually eliminates WKL via a strong uniform version of WKL, called UWKL below, which states the existence of a functional which selects uniformly in a given infinite binary tree an infinite path from that tree. This yields the following conservation result (which isn’t stated explicitly in [10] but which can be obtained from the proofs in section 4 of that paper, see below)

(2) $\text{WE-PRA}^\omega + \text{QF-AC} + \text{UWKL}$ is Π_2^0 -conservative over PRA.¹

For WE-PA^ω we get the following result (with the same convention on $+$ as above)

(3) $\text{WE-PA}^\omega + \text{QF-AC} + \text{UWKL}$ is conservative over PA .

(2) is of interest in the context of so-called explicit mathematics as developed by S. Feferman (starting with [3]) and further investigated also by A. Cantini, G. Jäger and T. Strahm among others, since the uniform weak König's lemma UWKL seems to be a very natural 'explicit' formulation of WKL . We have been asked about the status of UWKL in the presence of full extensionality. In this note we give a surprisingly simple answer to this question showing, in particular, that

(4) $\text{E-PRA}^\omega + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + \text{UWKL}$ contains (and is conservative over) PA

and

(5) $\text{E-PA}^\omega + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + \text{UWKL}$ has the same strength as $(\Pi_1^0\text{-CA})_{<\varepsilon_0}$,

where PA denotes full first-order Peano arithmetic.

Acknowledgement: This paper was prompted by discussions the author has had with Gerhard Jäger and Thomas Strahm who asked him about the status of the uniform weak König's lemma in a fully extensional context.

2 Preliminaries

The set \mathbf{T} of all finite types is defined inductively by

(i) $0 \in \mathbf{T}$ and (ii) $\rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}$.

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ .

The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

(i) $0 \in \mathbf{P}$ and (ii) $\rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}$.

¹In this weakly extensional context based on a quantifier-free rule of extensionality ' $+$ ' must be understood in the sense that the axioms QF-AC and WKL must not be used in the proof of a premise of an application of the extensionality rule. See [10] (where we use a special symbol ' \oplus ' to emphasize this point) for details on this. Actually it is sufficient to impose this restriction on the use of the additional axioms for UWKL only.

The conservation results in [10] are much more general than the one we mentioned. This makes the proofs more involved than is needed for the special $(\Pi_2^0\text{-})$ case relevant here. A corresponding simplification of our argument has been worked out in [1].

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k \dots \rho_1$ instead of $\tau(\rho_k) \dots (\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n + 1$. The types $0, 00, 0(00), 0(0(00)) \dots$ are so represented by $0, 1, 2, 3 \dots$. For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short $\text{deg}(\rho)$) is defined by $\text{deg}(0) := 0$ and $\text{deg}(\tau(\rho)) := \max(\text{deg}(\tau), \text{deg}(\rho) + 1)$. For pure types the degree is just the number which represents this type.

The system E-PRA^ω is formulated in the language of functionals of all finite types and contains $\Pi_{\rho, \tau}, \Sigma_{\delta, \rho, \tau}$ -combinators for all types (which allow to define λ -abstraction) and all primitive recursive functionals in the sense of Kleene (i.e. primitive recursion is available only on the type 0). More formally, E-PRA^ω results from Feferman's system $\widehat{\text{PA}}^\omega \upharpoonright$ in [4] if we add the axioms of extensionality

$$(E) : \forall x^\rho, y^\rho, z^{\tau\rho} (x =_\rho y \rightarrow zx =_\tau zy)$$

for all finite types (where for $\rho = 0\rho_k \dots \rho_1$, $x =_\rho y$ is defined as

$\forall z_1^{\rho_1}, \dots, z_k^{\rho_k} (xz_1 \dots z_k =_0 yz_1 \dots z_k)$).² We only include equality $=_0$ between numbers as a primitive predicate.

E-PA^ω is the extension of E-PRA^ω obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of Gödel [6] and coincides with Troelstra's [18] system $(\text{E-HA}^\omega)^c$.

The 'weakly extensional'³ versions WE-PRA^ω and WE-PA^ω of these systems result if we replace the extensionality axioms (E) by a quantifier-free rule of extensionality (due to Spector [17])

$$\text{QF-ER: } \frac{A_0 \rightarrow s =_\rho t}{A_0 \rightarrow r[s] =_\tau r[t]},$$

where A_0 is quantifier-free, $s^\rho, t^\rho, r[x^\rho]^\tau$ are arbitrary terms of the system and $\rho, \tau \in$ are arbitrary types.

Note that QF-ER allows to derive the extensionality axiom for type 0 but already the extensionality axiom for type-1-arguments, i.e.

$$\forall z^2 \forall x^1, y^1 (x =_1 y \rightarrow zx =_0 zy)$$

²We deviate slightly from our notation in [11]. The system denoted by E-PRA^ω in the present paper results from the corresponding system in [11] if we replace the universal axioms 9) in the definition of the latter by the schema of quantifier-free induction.

³This terminology is due to [18].

is underivable in WE-PA $^\omega$ (see [8]).

The schema of quantifier-free choice is given by

$$\text{QF-AC}^{\rho,\tau} : \forall x^\rho \exists y^\tau A_0(x, y) \rightarrow \exists Y^{\tau(\rho)} \forall x^\rho A_0(x, Yx), \quad \text{QF-AC} := \bigcup_{\rho,\tau \in \mathbf{T}} \{\text{QF-AC}^{\rho,\tau}\},$$

where A_0 is a quantifier-free formula.

In the following we use the formal definition of the binary (‘weak’) König’s lemma as given in [19] (here $*$, $\bar{b}x$, $lth(n)$ refer to the primitive recursive coding of finite sequences from [18]):

Definition 2.1 (Troelstra(74))

WKL: $\equiv \forall f^1 (T(f) \wedge \forall x^0 \exists n^0 (lth(n) = x \wedge fn = 0) \rightarrow \exists b^1 \forall x^0 (f(\bar{b}x) = 0))$, where
 $Tf : \equiv \forall n^0, m^0 (f(n * m) =_0 0 \rightarrow fn =_0 0) \wedge \forall n^0, x^0 (f(n * \langle x \rangle) =_0 0 \rightarrow x \leq_0 1)$
(i.e. $T(f)$ asserts that f represents a 0,1-tree).

Notation 2.2 $T^\infty(f) : \equiv T(f) \wedge \forall x^0 \exists n^0 (lth(n) = x \wedge fn = 0)$, i.e. $T^\infty(f)$ expresses that f represents an infinite binary tree. So

$$\text{WKL} \equiv \forall f^1 (T^\infty(f) \rightarrow \exists b^1 \forall x^0 (f(\bar{b}x) = 0)).$$

Definition 2.3 The uniform weak König’s lemma UWKL is defined as

$$\text{UWKL} : \equiv \exists \Phi^{1(1)} \forall f^1 (T^\infty(f) \rightarrow \forall x^0 (f((\overline{\Phi f})x) = 0)).$$

3 Results

For the weakly extensional systems WE-PRA $^\omega$ and WE-PA $^\omega$ we have the following conservation results for UWKL:

Theorem 3.1 1) WE-PRA $^\omega$ +QF-AC+UWKL is Π_2^0 -conservative over PRA.

2) WE-PA $^\omega$ +QF-AC+UWKL is conservative over PA.

(Here, again, + must be understood in the sense of footnote 1).

Proof: 1) In [10] (4.2-4.7), we constructed a primitive recursive functional

$f^1, g^1 \mapsto \zeta fg := (\widehat{f_g})$ such that

$$(1) \text{ WE-PRA}^\omega \vdash \forall f, g T^\infty(\zeta fg)$$

and

$$(2) \text{ WE-PRA}^\omega \vdash \forall f (T^\infty(f) \rightarrow \exists g (f =_1 \zeta f g)).$$

By the proof of theorem 4.8 in [10] (and the fact that WE-PRA^ω is Π_2^0 -conservative over PRA), it follows that

$$\text{WE-PRA}^\omega + \text{QF-AC} + \text{UWKL}^* \text{ is } \Pi_2^0\text{-conservative over PRA,}$$

where

$$\text{UWKL}^* := \exists B \forall f, g, x ((\zeta f g)((\overline{Bfg})x) =_0 0).$$

It remains to show that

$$\text{WE-PRA}^\omega \vdash \text{UWKL}^* \rightarrow \text{UWKL}.$$

The proof of (2) in [10](4.7) shows that g can be primitive recursively defined in f as

$$\tilde{f}(x) := \begin{cases} \min n \leq \overline{1}x [lth(n) = x \wedge f(n) = 0], & \text{if such an } n \text{ exists} \\ 0^0, & \text{otherwise.} \end{cases}$$

Thus for $\xi f := \zeta(f, \tilde{f})$

$$(2)' \text{ WE-PRA}^\omega \vdash \forall f (T^\infty(f) \rightarrow f =_1 \xi f).$$

Define $\Phi f := B(f, \tilde{f})$ for B satisfying UWKL^* . Then

$$\forall x ((\xi f)((\overline{\Phi f})x) =_0 0)$$

and so for f such that $T^\infty(f)$ (which implies $f =_1 \xi f$)

$$\forall x (f((\overline{\Phi f})x) =_0 0),$$

i.e. Φ satisfies UWKL .

2) As in 1) we obtain from the proof of 4.8 in [10] that

$$\text{WE-PA}^\omega + \text{QF-AC} + \text{UWKL} \text{ is } \forall \underline{x}^\ell \exists \underline{y}^0 A_0(\underline{x}, \underline{y})\text{-conservative over } \text{WE-PA}^\omega,$$

where \underline{x}^ℓ is a tuple of variables of type levels ≤ 1 , A_0 is quantifier-free and contains only $\underline{x}, \underline{y}$ as free variables. Now let A be a sentence in the language of PA which can be assumed to be in prenex normal form and assume that

$$\text{WE-PA}^\omega + \text{QF-AC} + \text{UWKL} \vdash A.$$

Then a fortiori

$$\text{WE-PA}^\omega + \text{QF-AC} + \text{UWKL} \vdash A^H,$$

where A^H is the Herbrand normal form of A . By the conservation result just mentioned we get

$$\text{WE-PA}^\omega \vdash A^H$$

and therefore by [9](theorem 4.1)

$$\text{PA} \vdash A.$$

Remark 3.2 *The passage from the provability of A^H to that of A used in the proof of 2) above does not apply to WE-PRA^ω and PRA (see [9] for a counterexample). Indeed, already $\text{WE-PRA}^\omega + \text{QF-AC}^{0,0}$ is not Π_3^0 -conservative over PRA : the former theory proves the schema of Σ_1^0 -collection $\Sigma_1^0\text{-CP}$, but it is known that there are instances of $\Sigma_1^0\text{-CP}$ (which always can be prenexed as Π_3^0 -sentences)⁴ which are unprovable in PRA (see [14]).*

We now show that the picture changes completely if we consider the systems E-PRA^ω and E-PA^ω with full extensionality instead of WE-PRA^ω , WE-PA^ω . This phenomenon is due to the following

Proposition 3.3

$$\text{E-PRA}^\omega \vdash \text{UWKL} \leftrightarrow \exists \varphi^2 \forall f^1 (\varphi f =_0 0 \leftrightarrow \exists x^0 (fx =_0 0)).$$

Proof: 1) ‘ \rightarrow ’: We first show that any Φ satisfying UWKL is – provably in E-PRA^ω – (effectively) discontinuous⁵, i.e.

$$\text{E-PRA}^\omega \vdash \left\{ \begin{array}{l} \forall \Phi^{1(1)} (\forall f^1 (T^\infty(f) \rightarrow \forall x^0 (f((\overline{\Phi}f)x) =_0 0)) \rightarrow \\ \exists g_{(\cdot)}^{1(0)}, g^1 (T^\infty(g) \wedge \forall i T^\infty(g_i) \wedge \forall i \forall j \geq i (g_j(i) =_0 g(i)) \\ \wedge \forall i, j (\Phi(g_i, 0) = \Phi(g_j, 0) \neq \Phi(g, 0))) \end{array} \right.$$

and, moreover, $g_{(\cdot)}, g$ can be computed uniformly in Φ by closed terms of E-PRA^ω . Define g primitive recursively such that

$$g(k) = \begin{cases} 0, & \text{if } \forall m < \text{lth}(k) ((k)_m = 0) \vee \forall m < \text{lth}(k) ((k)_m = 1) \\ 1, & \text{otherwise.} \end{cases}$$

⁴Here PRA is understood not as a quantifier-free theory but with full first-order predicate logic.

⁵The term ‘effectively discontinuous’ is due to [7] on which we rely in the second part of our proof.

It is clear that (provably in E-PRA^ω) $T^\infty(g)$. Now let $\Phi^{1(1)}$ be such that

$$\forall f^1(T^\infty(f) \rightarrow \forall x(f((\overline{\Phi f})x) =_0 0)).$$

Case 1: $\Phi(g, 0) = 0$. Define a primitive recursive function $\lambda i, k. g_i(k)$ such that

$$g_i(k) = \begin{cases} 0, & \text{if } [lth(k) \leq i \wedge \forall m < lth(k)((k)_m = 0)] \vee [\forall m < lth(k)((k)_m = 1)] \\ 1, & \text{otherwise.} \end{cases}$$

Again we easily verify within E-PRA^ω that $\forall i T^\infty(g_i)$. From the construction of g_i and g it is clear that

$$\forall k \forall l \geq lth(k)(g_l(k) = g(k)).$$

Since our coding has the property that $lth(k) \leq k$, we get

$$\forall k \forall l \geq k(g_l(k) = g(k)).$$

Since $\lambda x.1$ is the only infinite path of the binary tree represented by g_i , it follows that

$$\forall i(\Phi(g_i, 0) = 1).$$

Case 2: $\Phi(g, 0) = 1$. The proof is analogous to case 1 with

$$g_i(k) := \begin{cases} 0, & \text{if } [lth(k) \leq i \wedge \forall m < lth(k)((k)_m = 1)] \vee [\forall m < lth(k)((k)_m = 0)] \\ 1, & \text{otherwise.} \end{cases}$$

This finishes the proof of the discontinuity of Φ . We now show – using an argument from [7] known as ‘Grilliot’s trick’⁶ – that the functional φ^2 defined by $(+)\forall f^1(\varphi f =_0 0 \leftrightarrow \exists x(fx =_0 0))$ can be defined primitive recursively in Φ such that $(+)$ holds provably in E-PRA^ω :

We can construct a closed term $t^{1(1)}$ of E-PRA^ω such that (provably in E-PRA^ω) we have

$$thi = \begin{cases} g_j(i), & \text{for the least } j < i \text{ such that } h(j) > 0, \text{ if existent} \\ g_i(i), & \text{otherwise.} \end{cases}$$

⁶This argument plays an important role in the context of the Kleene/Kreisel countable functionals, see [13] whose formulation of it we adopt here.

Together with $\forall i \forall j \geq i (g_j(i) = g_i(i))$ this yields

$$\exists j (h(j) > 0) \rightarrow th =_1 g_j \text{ for the least such } j$$

and together with $\forall i (g_i(i) = g(i))$

$$\forall j (h(j) = 0) \rightarrow th =_1 g.$$

Hence using the extensionality axiom for type-2-functionals we get

$$\forall j (h(j) = 0) \leftrightarrow \Phi(th, 0) =_0 \Phi(g, 0).$$

So $\varphi := \lambda h^1. \overline{sg} \circ |\Phi(t(\overline{sg} \circ h), 0) - \Phi(g, 0)|$ where $\overline{sg}(x) := 0$ for $x \neq 0$ and $= 1$ otherwise, does the job.

We now combine the two constructions of φ corresponding to the two cases above into a single functional which defines φ primitive recursively in Φ : Let χ be a closed term such that

$$\text{E-PRA}^\omega \vdash \forall x^0 ((x =_0 0 \rightarrow \chi x =_{1(1)} t) \wedge (x \neq 0 \rightarrow \chi x =_{1(1)} \tilde{t})),$$

where t is defined as above with g_i from case 1 whereas \tilde{t} is defined analogously but with g_i as in case 2. Then define $\varphi := \lambda h^1. \overline{sg} \circ |\Phi((\chi(\Phi(g, 0))(\overline{sg} \circ h), 0) - \Phi(g, 0)|$.
 2) ‘ \leftarrow ’: Primitive recursively in φ one can easily compute a functional Φ which even selects the leftmost infinite branch of an infinite binary tree.

Corollary to the proof of proposition 3.3: One can construct closed terms t_1, t_2 of E-PRA^ω such that

$$\text{E-PRA}^\omega \vdash \begin{cases} \forall \Phi^{1(1)} (\forall f^1 (T^\infty(f) \rightarrow \forall x^0 (f((\overline{\Phi}f)x) = 0)) \rightarrow \\ \forall f^1 ((t_1 \Phi)f =_0 0 \leftrightarrow \exists x (fx = 0))) \end{cases}$$

and

$$\text{WE-PRA}^\omega \vdash \begin{cases} \forall \varphi^2 (\forall f^1 (\varphi f = 0 \leftrightarrow \exists x (fx = 0)) \rightarrow \\ \forall f^1 (T^\infty(f) \rightarrow \forall x^0 (f((\overline{t_2 \varphi}f)x) = 0))). \end{cases}$$

Corollary 3.4

$$\text{E-PRA}^\omega + \text{QF-AC}^{1,0} \vdash \text{UWKL} \leftrightarrow \exists \mu^2 \forall f^1 (\exists x^0 (fx = 0) \rightarrow f(\mu f) = 0).$$

Proof: The existence of μ obviously implies the existence of φ in proposition 3.3 and hence of Φ . For the other direction we only have to observe that the existence of φ implies the existence of μ by applying QF-AC^{1,0} to

$$\forall f \exists x (\varphi(f) = 0 \rightarrow fx = 0).$$

Remark 3.5 In contrast to the corollary to the proof of proposition 3.3 above there exists no closed term t in E-PRA ^{ω} which computes μ in Φ , i.e.

$$\mathcal{S}^\omega \not\models \forall \Phi^{1(1)} (\forall f^1 (T^\infty(f) \rightarrow \forall n^0 (f(\overline{\Phi}f)n = 0)) \rightarrow \forall f^1 (\exists x (fx = 0) \rightarrow f(t\Phi f) = 0))$$

for every closed term t (of appropriate type) of E-PRA ^{ω} , since – by [8] – every such term has a majorant t^* , Φ is majorized by $\lambda f^1, x^0.1$ and so μ would have a majorant $\lambda f^M.t^*(1^{1(1)}, f^M)$ (where $f^M(x) := \max(f0, \dots, fx)$), which contradicts the easy observation that μ has not even a majorant in \mathcal{S}^ω (here \mathcal{S}^ω denotes the full set-theoretic type structure).

Theorem 3.6 1) E-PRA ^{ω} +UWKL contains Peano arithmetic PA.

2) E-PRA ^{ω} +QF-AC^{1,0}+QF-AC^{0,1}+UWKL is conservative over PA.

3) E-PA ^{ω} +QF-AC^{1,0}+QF-AC^{0,1}+UWKL proves the consistency of PA and has the same proof-theoretic strength as (and is Π_2^1 -conservative over) the second order system $(\Pi_1^0\text{-CA})_{<\varepsilon_0}$.

Proof: 1) Using φ from proposition 3.3 one easily gets characteristic functions for all arithmetical formulas $A(\underline{x})$. By applying the quantifier-free induction axiom of E-PRA ^{ω} to them, one obtains every arithmetical instance of induction.

2) This follows from corollary 3.4 and the conservation of E-PRA ^{ω} +QF-AC^{1,0}+QF-AC^{0,1} + μ over PA which is due [4] (note that the usual elimination of extensionality procedure – which applies to the existence of μ but not to UWKL – yields a reduction of E-PRA ^{ω} +QF-AC^{1,0}+QF-AC^{0,1} + μ to its variant where the extensionality axioms for types > 0 are dropped, see [12] for details on this).

3) follows from [4],[5] using again corollary 3.4 above and elimination of extensionality.

Remark 3.7 1) The functionals φ and μ from proposition 3.3 and corollary 3.4 provide uniform versions (in the same sense in which UWKL is a uniform version of WKL) of

$$(1) \Pi_1^0\text{-CA} : \forall f \exists g \forall x^0 (g(x) =_0 0 \leftrightarrow \exists y^0 (f(x, y) =_0 0))$$

respectively of

$$(2) \Pi_1^0\text{-}\widehat{\text{CA}} : \forall f \exists g \forall x^0, z^0 (f(x, gx) =_0 0 \vee f(x, z) \neq 0),$$

but yet φ, μ are not stronger than (1), (2) relative to E-PRA^ω (but only relative to E-PA^ω) as Feferman's results cited in the proof above show. The reason for this is, that E-PRA^ω is too weak to iterate φ or μ uniformly since this would require a primitive recursion of type level 1. In contrast to this fact, UWKL is stronger than WKL already relative to E-PRA^ω .

- 2) One might ask whether UWKL gets weaker if we allow $\Phi^{1(1)}$ to be a partial functional which is required to be defined only on those functions f which represent an infinite binary tree. However the construction ξ (used in the proof of theorem 3.1) such that

$$(1) \forall f^1 T^\infty(\xi f)$$

and

$$(2) \forall f^1 (T^\infty(f) \rightarrow \xi f =_1 f)$$

shows that any such partial Φ could be easily extended to a total one.

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