Transition Systems, Event Structures and Unfoldings*

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Introduction

Elementary transition systems were introduced in [NRT2] . They were proved to be, in a strong categorical sense, the transition system version of elementary net systems. The question arises whether the notion of a region and the axioms (mostly based on regions) imposed on ordinary transition systems to obtain elementary transition systems were simply "tuned" to obtain the correspondence with elementary net systems. Stated differently, one could ask whether elementary transition systems could also play a role in characterizing other models of concurrency.

We show here that by smoothly strenghtening the axioms of elementary transition systems one obtains a subclass called occurrence transition systems which turn out to be categorically equivalent to the well-known model of concurrency called prime event structures. Thus there is more to elementary

transition systems than just their (co-reflective) relationship to a basic model of net theory, namely, elementary net systems.

Next we show that occurrence transition systems are to elementary transition systems what occurrence nets are to elementary net systems. We define an "unfold" operation on elementary transition systems which yields occurrence transition systems. We then prove that this operation uniquely extends to a functor which is the right adjoint to the inclusion functor from (the full subcategory of) occurrence transition systems to (the category of) elementary transition systems. Thus the results of this paper also show that the semantic theory of elementary net systems has a nice counterpart in the more abstract world of transition systems.

In the next section a brief review — and a convenient reformulation — of the category of elementary transition systems ETS is provided. Section 2 contains a quick introduction to the category of prime event structures, PES, due to Winskel [W]. In the subsequent section we identify the subcategory of occurrence transition systems, OTS, by a smooth strengthening of the regional axioms for elementary transition systems. We then proceed to establish a few properties of occurrence transition systems. Using these properties, we show in Section 4 that OTS and PES are equivalent categories. Thus, in some sense, occurrence transition systems are *the* transition system model of prime event structures (in the same sense that prime algebraic, coherent domains, are *the* domain model of prime event structures, Winskel [W]). In Section 5 we show that occurrence transition systems can be used to define the unfoldings of elementary transition systems. Exploiting some technical results from the theory of trace languages, we show that the unfold operation, when applied to the objects in ETS, yields objects in OTS. Moreover, we prove that this unfold operation uniquely extends to a functor which is the right adjoint to the inclusion functor from OTS to ETS. This result mirrors the strong result due to Winskel [W] on the side of net theory which established the "correctness" of the unfolding of elementary net systems (and in fact, 1-safe Petri nets) into occurrence nets proposed in [NPW].

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1   Elementary Transition Systems

The purpose of this section is to recall (and rephrase!) the main concepts and results from [NRT2].

Definition 1.1. A *transition system* is a four-tuple $TS = (S, E, T, s^{in})$ where

S is the set of *states*,

E is the set of *events*,

$T \subseteq S \times E \times S$ is the set of *transitions*, and

$s^{in} \in S$ is the *initial* state. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Definition 1.2. A *region* of a transition system $TS = (S, E, T, s^{in})$ is a subset of states, $R \subseteq S$, satisfying:

$$\forall (s_0, e, s_0'), (s_1, e, s_1') \quad \in \quad T.$$
$$(s_0 \in R \wedge s_0' \notin R) \quad \Leftrightarrow \quad (s_1 \in R \wedge s_1' \notin R)$$
$$\text{and } (s_0 \notin R \wedge s_0' \in R) \quad \Leftrightarrow \quad (s_1 \notin R \wedge s_1' \in R).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We shall use the following notation for a given a transition system $TS = (S, E, T, s^{in})$.

R_{TS} — the set of nontrivial (proper, nonempty subsets of S) regions of TS. R_s , where $s \in S$, — the set of nontrivial regions containing s; formally

$$R_s \stackrel{\text{def}}{=} \{R \in R_{TS} \mid s \in R\}.$$

$°R, R°$, where $R \in R_{TS}$, — the set of events entering/leaving R resp.; formally

$$°R \stackrel{\text{def}}{=} \{e \in E \mid \exists (s, e, s') \in T. \ s \notin R \wedge s' \in R\}, \text{ and}$$
$$R° \stackrel{\text{def}}{=} \{e \in E \mid \exists (s, e, s') \in T. \ s \in R \wedge s' \notin R\},$$

${}^\circ e, e^\circ$, where $e \in E$, — the set of pre- and post-regions of e resp., i.e., the set of regions which e is (consistently) leaving/entering; formally

$$
{}^\circ e \overset{\text{def}}{=} \{R \in R_{TS} \mid \exists (s, e, s') \in T.\ s \in R \land s' \notin R\},\ \text{and}
$$
$$
e^\circ \overset{\text{def}}{=} \{R \in R_{TS} \mid \exists (s, e, s') \in T.\ s \notin R \land s' \notin R\},
$$

Proposition 1.3. Let $TS = (S, E, T, s^{in})$ be a transition system. Then

(i) $R \subseteq S$ is a region iff $S \backslash R$ is a region

(ii) $\forall e \in E.\ e^\circ = \{S \backslash R \mid R \in {}^\circ e\}$,

(iii) $\forall (s, e, s') \in T.\ R_s \backslash R_{s'} = {}^\circ\!/!e$ and $R_{s'} \backslash R_s = e^\circ$ and consequently $R_{s'} = (R_s \backslash {}^\circ e) \cup e^\circ$. $\hspace{2cm}\square$

Given a Transition System $TS = (S, E, T, s^{in})$ we s hall use the following notation.

- For every $e \in E$, $\overset{e}{\to} \subseteq S \times S$, where $(s, s') \in \overset{e}{\to} \Leftrightarrow (s, e, s') \in T$.

- Let $\rho \in E^*$, $\rho = e_1 e_2 \ldots e_n, n \geq 1$. Then $\Rightarrow^\rho \subseteq S \times S$ where $(s, s') \in \Rightarrow^\rho$ iff $\exists s_0, s_1, \ldots, s_n$ such that $s = s_0 \overset{e_1}{\to} s_1 \cdots s_{n-1} \overset{e_n}{\to} s_n = s'$. By convention, $\Rightarrow^\Lambda = \{(s, s) \mid s \in S\}$,

where Λ denotes the null string.

- The *computations* of TS is defined as
  $C_{TS} \overset{\text{def}}{=} \{\rho \in E^* \mid \Rightarrow^\rho \cap (\{s^{in}\} \times S) \neq \emptyset\}$, and nonempty computations of TS is defined as
  $C_{TS+} \overset{\text{def}}{=} C_{TS} \cap E^+$.

- $\to \subseteq S \times S$, where $\to \overset{\text{def}}{=} \bigcup_{e \in E} \overset{e}{\to}$, and

- $\overset{*}{\to}$ is the transitive and reflexive closure of $\to$.

- For every $s \in S, \uparrow s \overset{\text{def}}{=} \{s' \in S \mid (s, s') \in \overset{*}{\to}\}$.

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So $\uparrow s$ denotes the set of states reachable from s via the transitions of TS.

The results of [NRT2] show that the category of elementary transition systems, ETS, introduced below is the category of the (sequential) case graphs of elementary net systems. We recall that elementary net systems is the basic system model of net theory in which fundamental behavioural aspects of distribute systems such as causality, concurrency, conflict and confusion can be made transparent [Th]. We also recall that there is a natural way of associating a transition system with an elementary net system using the notion of a sequential case graph which explicates the operational behaviour of elementary net system [Ro].

We present the definition of ETS as it was stated in [NRT2].

Definition 1.4. (ETS-objects)

A Transition System $TS = (S, E, T, s^{in})$ is said to be *elementary* iff it satisfies the following axioms:

(S1) $\uparrow s^{in} = S$ (every state reachable from s^{in}).

(S2) $\forall s, s' \in S . \ R_s = R_{s'} \Rightarrow s = s'$ (regional separability of states).

(T1) $\forall s \in S, e \in E. \ [^\circ e \subseteq R_s \Rightarrow \exists s' \in S. \ (s, e, s') \in T]$ (enabling of transitions).

(T2) $\forall (s, e, s') \in T . \ s \neq s'$ (i.e., $\xrightarrow{e}$ irreflexive for every $e \in E$).

(T3) $\forall (s, e_1, s_1), (s, e_2, s_2) \in T. \ [s_1 = s_2 \Rightarrow e_1 = e_2)$

$$(\text{i.e.,} e_1 \neq e_2 \Rightarrow \xrightarrow{e_1} \cap \xrightarrow{e_2} = \emptyset).$$

(E) $\forall e \in E . \ \exists (s, e, s') \in T . \ $ (i.e., $\xrightarrow{e}$ nonempty). $\qquad\qquad \square$

Definition 1.5. (ETS-morphisms)

Let $TS_i = (S_i, E_i, T_i, s_i^{in})$ for $i = 0, 1$ be a pair of transition systems. A *morphism* from TS_0 to TS_1, is a pair(f, η) where

$f : S_0 \longrightarrow S_1$ is a total function from S_0 t o S_1, and

$\eta : E_0 \longrightarrow E_1$ is a partial function from E_0 to E_1 such that

1. $f(s_0^{in}) = s_1^{in}$,

2. $\forall (s_0, e_0, s_0') \in T_0.$ $\begin{cases} f(s_0) = f(s_0'), & \text{if } \eta(e_0) \text{ undefined.} \\ (f(s_0), \eta(e_0), f(s_0')) \in T_1, & \text{if } \eta(e_0) \text{ defined.} \end{cases}$

$\square$

Composition of morphisms is componentwise composition of the total/partial functions and identity is the pair of identity functions.

We let ETS denote the category of objects and morphisms as defined in Definitions 1.4 and 1.5. In [NRT2] a category ENS of elementary net systems as objects and suitably defined behaviour preserving net-morphisms is introduced. We recall the main result from [NRT2].

Theorem 1.6. There exists a coreflection between ETS and ENS, where the rightadjoint is the well-known case-graph construction from Net Systems, and the left adjoint constructs an elementary net system from an ETS-object, in which the regions play the role of local states (conditions i n net theory). $\square$

As stated earlier, the importance of this result is that the axioms from Definition 1.4 identify a transition system based model of "true concurrency" — *not* by adding structure, but by imposing the six axioms of Definition 1.4. The reader will have noticed that the notion of regions play a central role in the axiomatization $(S2, T1)$, but that the axiomatization also contains structural/syntactical axioms like $T2$, TS and E. For the purpose of the following sections we provide here an almost purely regional axiomatization of elementary transition systems.

Theorem 1.7. A transition system $TS = (S, E, T, s^{in})$ is elementary iff it satisfies axioms **S1**, **S2**, **T1** from Definition 1.4, and

(E1) $\forall e \in E.\ {}^\circ e \neq \emptyset$.

(E2) $\forall e, e' \in E.\ {}^\circ e = {}^\circ e' \Rightarrow e = e'$ (regional separability of events). $\square$

Proof.

$\Rightarrow$ The fact that $E1$ and $E2$ follow from the original ETS-axioms is immediate from the proof of Proposition 4.2 in [NRT2].

$\Leftarrow$ Assume TS satisfies $E1$. Let $R \in \ ^\circ e$. From Definition this implies that we must have $(s, e, s') \in T$ such that $s \in R$ and $s' \in R$. Hence axiom E follows from $E1$. Further assume $(s, e, s') \in T$ for some $s, s' \in S$. This implies $s \in R$, $s' \notin R$, i.e. $s \neq s'$, and hence $T2$ also follows from $E1$. Assume TS satisfies $E2$, $(s, e_1, s'), (s, e_2, s') \in T$. Clearly this means that $\forall R \in R_{TS}$. $[R \in \ ^\circ e_1 \Leftrightarrow s \in R$ and $s' \notin R \Leftrightarrow R \in ^\circ e_2]$.
I.e., assuming $E2$ we get $e_1 = e_2$, and hence $T3$ follows from $E2$. $\square$

It is maybe worth noticing that the "if part" of the proof above shows that $T2, T3$ and E (the old structural axioms) follow from $E1$ and $E2$ (the new regional axioms). The other direction of this implication does *not* hold (the proof of the "only if part" from [NRT2] makes use of axioms $S2$ and $T1$!).

$\square$

2 Prime Event Structures

In this section we briefly introduce one of the fundamental models of concurrency, prime event structures, originally introduced in [NPW], and since then studied extensively by primarily Winskel [W]. It is important to realize, that event structures is basically a model of concurrency on the behavioural level, i.e., events represent unique temporal occurrences of actions, as opposed to the models mentioned in the previous section, ETS and ENS, both of which are basically models on the system level, in which events may have repeated occurrences at different times in different contexts. We now introduce the category of prime event structures, PES.

Definition 2.1. (PES-objects)

A prime event structure is a triple $ES = (E, \leq, \#)$ where

E is a set of events,

$\leq \ \subseteq E \times E$ is a partial order (*causaltiy*) ,

$\# \subseteq E \times E$ is a symmetric relation (*conflict*) , where

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(A1) $\forall e_0, e_1, e_2 \in E$. $e_0 \# e \leq e \Rightarrow e_0 \# e_2$ (conflict inheritance) ,

(A2) $\forall e \in E$. $[e] = \{e \in E \mid e' \leq e\})$ is finite and #-free $\qquad\qquad$ □

Given ES as above - the *configurations* of ES are defined as

$$C(ES) \stackrel{\text{def}}{=} \{c \subseteq E \mid (\forall e, e' \in c. \text{ not } (e \# e')) \text{ and } \forall e, e' \in E. e' \leq e \in c \Rightarrow e' \in c\}$$

So, configurations of ES are the downwards (w.r.t. $\leq$) closed and conflict-free subsets of E. We use the notation $FC(ES)$ for the set of finite configurations of ES. $\qquad\qquad$ □

Definition 2.2 (*PES*-morphisms)

Let $ES_i = (E_i, \leq_i, \#_i)$, for $i = 0, 1$ be two Prime Event Structures. A morphism from ES_0 to ES_1 is a partial function from E_0 to E_1 satisfying $\forall c \in C(ES_0)$.

(∗) $[\eta(c) \in C(ES_1)$ and $\forall e, e' \in c$.

$[\eta(e) = \eta(e')]$ (and both defined) $\Rightarrow e = e']]$. $\qquad\qquad$ □

Composition of morphisms is normal composition of partial functions, and the identity is the identity function. $\qquad\qquad$ □

We refer the reader to [W] for detailed intuition, explanation and results for the category *PES* of prime event structures with objects and morphisms defined in Definitions 2.1 and 2.2. We only mention that the configurations of a prime event structure may be thought of as the states of a distributed system, where the state is identified with the "events having occurred" at the given state. The fundamental notions of causality (or rather dependence of) and conflict (exclusion/choice among events) are captured directly by the relations $\leq$ and $\#$ in the definition of a prime event independence) between events structure. The notion of concurrency (or may be derived as follows:

$$e \text{ co } e' \stackrel{\text{def}}{\Leftrightarrow} \text{not } (e \leq e' \text{ or } e' \leq e \text{ or } e \# e').$$

We shall use the notation $c_0 \stackrel{e}{\relbar\mkern-10mu\prec} c_1$ for a structure evolving from c_0 to c_1 through the occurrence of event e, i.e., for a prime event structure, ES, as in

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Definition 2.1. Actually it is sufficient to consider just finite configurations. $-\!\!\!\prec \,\subseteq FC(ES) \times E \times FC(ES)$ is given by:

$$(c_0, e, c_1) \in \,-\!\!\!\prec \quad \textit{iff} \;\; c_0 \subseteq c_1 = c_0 \cup \{e\}.$$

As usual, we will often write $c_0 \xrightarrow{\;e\;}\!\!\!\prec$ instead of $(c_0, e, c_1) \in \,-\!\!\!\prec$.
We shall use the following facts about prime event structures.

Proposition 2.3. Let $ES = (E, <, \#)$ be a prime event structure. Then for every $c \in FC(ES)$, and for every linearization $e_0, e_1, \dots, e_n$ of the events there exist configurations $c, c_1, \dots, c_n$ such that

$$\emptyset \xrightarrow{\;e_0\;}\!\!\!\prec c_0 \xrightarrow{\;e_1\;}\!\!\!\prec c_1 \xrightarrow{\;e_2\;}\!\!\!\prec \cdots c_{n-1} \xrightarrow{\;e_n\;}\!\!\!\prec c_n = c.$$

Proof. See [W].

Lemma 2.4. Let ES_i be two Prime Event Structures as in Definition 2.2, and let η be a partial function from E_0 to E_1 . Then η is a morphism from ES_0 to ES_1 iff the condition $(*)$ of Definition 2.2 is satisfied for all *finite* configurations c of ES_0 .

Proof.

The "only if" part of the Lemma is trivial, so we concentrate on the nontrivial "if part". Let η satisfy $(*)$ for all finite configurations and let c be a (infinite) configuration of ES_0 .

We first prove that $\eta(c) \in C(ES_1)$. Assume $e_1 \in \eta(c)$ and $e_1' \leq_1 e_1$. $e_1 \in \eta(c)$ implies that we must have e_0 such that $\eta(e_0) = e_1$, and since from definition $[e_0] \in FC(ES_0)$, we have from our assumption $\eta([e_0]) \in FC(ES_1)$. Now, from this we have $e_1' \in \eta([e_0])$, and hence there must exist $e_0' \in [e_0]$ such that $\eta(e_0') = e_1'$. Since c is downwards closed, $e_0' \in c$, and hence $e_1' \in \eta(c)$, i.e., $\eta(c)$ is downwards closed.

Assume $\eta(e_0), \eta(e_0') \in \eta(c)$, $e_0, e_0' \in c$. Then it follows as above that $[e_0] \cup [e_0'] \in FC(ES_0)$, hence $\eta([e_0] \cup [e_0']) \in FC(ES_1)$ (from the assumption of Lemma), and hence not $(\eta(e_0) \,\#\, \eta(e_0'))$, i.e., $\eta(c)$ is conflict free.

Finally, let $e_0, e_0' \in c$ and $\eta(e_0) = \eta(e_0')$ and both defined. Then again, since $[e_0] \cup [e_0'] \in FC(ES_0)$, we get from assumption of Lemma, that not only is $\eta([e_0] \cup [e_0'])$ a configuration of ES_1 , but also $e_0 = e_0'$.   □

3 Occurrence Transition Systems

In this section we introduce a (full) subcategory of ETS, called the category of occurrence transition systems, OTS, and prove some properties of this subcategory. The main point is that OTS is defined as a simple strengthening of the axiomatization of ETS-objects, and it will be proved in the next section that OTS is (categorically) equivalent to the category of Prime Event Structures. In this section we only prove some technical lemmas for OTS, OTS and PES (the category of Prime Event Structures) are equivalent categories to be used in the proofs of the main results of the next sections.

Definition 3.1. (OTS Category)
Let OTS denote the category consisting of

- *objects*: transition systems $TS = (s, E, T, s^{in})$ satisfying axioms **S1**, **S2**, and **T1** of Definition 1.4 and
  axiom 0 : $\forall e \in E.\ \exists s \in S.\ [\uparrow s \in R_{TS}$ and $°\uparrow s = \{e\}]$, and

- *morphisms:* transition system morphisms as defined in Definition 1.5.

□

Proposition 3.2. OTS is a full subcategory of ETS.

Proof.

Follows immediately from Theorem 1.7, because **axiom 0** trivially implies (by Proposition 1.3) **E1** and **E2**.

One might say that OTS is obtained from ETS by a strengthening of axioms **E1** and **E2**. **E1** and **E2** may be interpreted as "each event is characterized by its nonempty set of pre-regions (or, of course, equivalently its set of post-regions)". **Axiom 0** may be interpreted as "each event is charaterized by one single post-reqion(or equivalently pre-region) of a particularly simple form (equal to $\uparrow s$ for some $s \in S$)" . However, this seemingly innocent strengthening implies some dramatic restrictions on the kind of allowable transition systems.

Lemma 3.3. Let $TS = (S, E, T, s^{in})$ be an OTS object. Then $\xrightarrow{*} \subseteq S \times S$ is a partial order with s^{in} as the least element.

Proof.

Transitivity and reflexivity of $\xrightarrow{*}$ follow from definition. We must only prove antisymmetry. Take any $(s', e, s'') \in T$. From **axiom 0** we have a region $R = \uparrow s$ for some $s \in S$ such that $^\circ R = \{e\}$, i.e., $s' \notin R$, $s'' \in R$. Clearly, this implies $s' \notin \uparrow s''$, from which antisymmetry of $\xrightarrow{*}$ follows. Minimality of s^{in} w.r.t. $\xrightarrow{*}$ follows directly from axiom **S1**. □

Lemma 3.4. Let $TS = (S, E, T, s'^{in})$ be an OTS-object. Assume $\uparrow s$ and $\uparrow s'$ are both regions of TS such that $^\circ \uparrow s = {}^\circ \uparrow s' = \{e\}$. Then $s = s'$.

Proof.

Consider $\uparrow s$.

Since $\uparrow s \in R_{TS}$, **S1** implies that $s^{in} \notin \uparrow s$. But $s \in \uparrow s$ and so, because $\{e\} = {}^\circ \uparrow s$, there exist $\bar{s}, \bar{\bar{s}} \in S$ such that $\bar{s} \notin \uparrow s$, $\bar{\bar{s}} \in \uparrow s$ and $s^{in} \xrightarrow{*} \bar{s} \xrightarrow{e} \bar{\bar{s}} \xrightarrow{*} s$. Since $\{e\} = {}^\circ \uparrow s'$, it must be that $s' \xrightarrow{*} \bar{\bar{s}}$ and consequently $s' \xrightarrow{*} s$.

By symmetric arguments we get $s \xrightarrow{*} s'$.

Hence, by Lemma 3.3, $s = s'$. □

So, from Lemma 3.4, we may talk about *the* state s satisfying the property of **axiom 0** for a given e of an OTS object. We shall use the notation $s_e, e \in E$, for this particular state. Obviously from the definition of $^\circ R$, this association is injective in the sense that $s_e = s_{e'} \Rightarrow e = e'$. So, we may think of s_e as "the state representation of e".

Based on this, one may ask if there is also a natural way to talk about the states of an OTS-object in terms of its events. One obvious idea seems to be to associate with a state s the set of events e for which s belongs to the characteristic region $\uparrow s_e$.

Definition 3.5. Let $TS = (S, E, T, s^{in})$ be an OTS-object. Let past : $S \to 2^E$ be the function defined as $past(s) = \{e \mid s \in \uparrow s_e\}$. □

The use o f the word "*past*" is justified by the following lemma.

Lemma 3.6. Let $TS = (S, E, T, s)$ be an OTS object.

(a) $past(s^{in}) = \emptyset$, and

(b) for every $(s, e, s') \in T$, $past(s) \subset past(s') = past(s) \cup \{e\}$.

For every computation of the form $s^{in} = s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} s_2 \cdots \xrightarrow{e_n} s_n = s$ we have

(c) $1 \le i < j \le n \Rightarrow e_i \ne e_j$ and

(d) $\{e_i \mid 1 \le i \le n\} = past(s)$.

Proof.

Clearly **(c)** and **(d)** follow from **(a)** and **(b)**. Assume $s^{in} \in \uparrow s_e$. From axiom **S1** we get $\uparrow s_e = S$, contradicting $\uparrow s_e$ being a nontrivial region. Hence we conclude **(a)**.

Consider an arbitrary $(s, e, s') \in T$.

Obviously $s' \in \uparrow s$, and so $past(s) \subseteq past(s')$.

Since $\uparrow s_e$ is a region such that $^\circ \uparrow s_e = \{e\}$, $s \notin {}^\circ \uparrow s_e$ and $s' \in {}^\circ \uparrow s_e$. Hence $e \in past(s') \backslash past(s)$.

Now let $e' \in past(s')$ be such that $e \ne e'$. Since $e' \in past(s')$, $s' \in \uparrow s_{e'}$. Since $e \ne e'$ and $^\circ \uparrow s_{e'} = \{e'\}$, it must be that $s \in \uparrow s_{e'}$ which implies that $e' \in past(s)$. Consequently $past(s') \setminus past(s) = \{e\}$, and so **(b)** holds.

Lemma 3.7. Let $TS = (S, E, T, s^{in})$ be an OTS object. The function past from Definition 3.5 is injective.

Proof.

Let $s \in S$, and let R be any region of TS. Then from Lemma 3.6 (**(c)** and **(d)**)) we get

$(*)$ $s \in R$ iff either $(s^{in} \in R$ and$|R^\circ \cap past(s)| = |{}^\circ R \cap past(s)|)$
$\qquad$ or $(s^{in} \notin R$ and$|R^\circ \cap past(s)| + 1 = |{}^\circ R \cap past(s)|)$,

where $|M|$ denotes the cardinality of a set M.

From this we clearly get for two states s and s' that

$$past(s) = past(s') \Rightarrow \forall R \in R_{TS}. \ [s \in R \text{ iff } s' \in R].$$

But then by axiom **S2** (Definition 1.4) we conclude that $s = s'$ $\qquad\qquad$ $\square$.

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4 Equivalence between OTS and PES

In this section we prove that there is a very strong relationship between the two categories OTS and PES; they are basically one and the same thing in the sense that they are categorically equivalent. So, one might conclude that the axioms of OTS-objects identify the transition system version of prime event structures.

It was indicated already in [NPW] that one may view a PES-object as a transition system, where the states correspond to configurations, and transitions to the $\xrightarrow{e}\prec$ relations mentioned previously. We start by proving that the idea may be formalized in the form of a functor $T : PES \to OTS$.

Theorem 4.1. T defined as follows is a functor from PES to OTS:

- *On objects*: $T(ES = (E, \leq, \#)) = (FC(ES), E, \longrightarrow\!\!\!\prec, \emptyset)$.

- *On morphisms*: Let η be a PES-morphism from ES_0 to ES_1. Then $T(\eta) = (f, \eta)$, where $\forall c_0 \in FC(ES_0).\ f(c_0) = \eta(c_0)$.

Proof.

The only non-trivial part is to see that $T(ES)$ as defined satisfies the axioms for OTS objects.

(S1) $\uparrow\emptyset = FC(ES)$ in $T(ES)$.
Follows from Proposition 2.3.

(S2) $R_c = R_{c'} \Rightarrow c = c'$ in $T(ES)$, where $c, c' \in FC(ES)$.
Assume $c \neq c'$, e.g., there exists $e \in c$, $e \notin c'$. It i s easy to see that $R_e \stackrel{\text{def}}{=} \{x \in FC(ES) \mid e \in x\}$ is a region of $T(ES)$ (such that $^\circ R_e = \{e\}$ and $R_e^\circ = \emptyset$). Clearly $c \in R$, $c' \notin R_e$.

(T1) $^\circ e \subseteq R_c \Rightarrow \exists c'.\ [c \xrightarrow{e}\prec c'$ in $T(ES)$, $e \in E$, $c \in FC(ES)]$.

Obviously all one must prove is that from the assumption $c \in FC(ES)$ and $^\circ e \subseteq R$ in $T(ES)$ we get $c \cup \{e\} \in FC(ES)$. (From $^\circ e \subseteq R$ and the fact that $FC(ES) \backslash R_e$ is a region we at once get $e \notin c$ (R_e is the region constructed above)).

$c \cup \{e\}$ can fail to be a configuration for two reasons.

Case 1. $c \cup \{e\}$ is not downwards closed, i.e., there exists $e' < e$ such that $e' \notin c \cup \{e\}$, i.e., $e' \notin c$. From $e' < e$ it is easy to see that $R = \{x \in FC(ES) \mid e' \in x, e \notin x\}$ is a region of $T(ES)$ such that $R \in^\circ e$. But we have also $R \notin R_c$ (since $e' \notin c$). Thus we get contradiction to our assumption $^\circ e \subseteq R_c$.

Case 2. $c \cup \{e\}$ is not conflict free, i. e., there exists $e' \in c$ such that $e \# e'$ (remember c is configuration). From $e \# e'$, it is again easy to see that $R = \{x \in FC(ES) \mid e \notin x \text{ and } e' \notin x\}$ is a region of $T(ES)$ such that $R \in^\circ e$ (and $R \in^\circ e'$). But since $e' \in c$ we also have $R \notin R_c$, and hence again a contradiction to our assumption $^\circ e \subseteq R_c$.

(0) $\forall e \in E.\ \exists c \in FC(ES).\ [(\uparrow c \in R_{T(ES)} \text{ and } ^\circ\uparrow c = \{e\}) \text{ in } T(ES)]$.

Given $e \in E$, define $c_e \overset{\text{def}}{=} [e]$. It follows immediately that $[e] \in FC(ES)$. It follows from proposition 2.3 that $\uparrow[e] = \{x \in FC(ES) \mid e \in x\} = R_e$ as in the proof of **S2** above. Now we have already seen that $R_e \in R_{T(ES)}$ and $^\circ R_e = \{e\}$. $\qquad\square$

Theorem 4.2. The functor T determines an equivalence of categories between PES and OTS.

Proof.

It follows from [Mac], theorem 4.4.1 that it is sufficient to prove that T is full and faithful, and that for every OTS object TS there exists a PES-object ES such that TS is isomorphic to $T(ES)$. These three facts are proved in three separate lemmas in the following. $\qquad\square$

Lemma 4.3. T is ful.

Proof.

Given two prime event structures $ES_i = (E_i, \leq_i, \#_i)$, $i = 0, 1$ and an OTS-morphism (f, η) from $T(ES_0)$ to $T(ES_1)$, we must prove that there exists a PES-morphism $\hat{\eta}$ from ES_0 to ES_1 such that $T(\hat{\eta}) = (f, \eta)$. Since (f, η) is an OTS-morphism, we have from the definition of T that η is a partial function from E_0 to E_1. Suppose η is itself an PES-morphism from ES_0 to ES_1. Then, once again by the definition of T, $T(\eta) = (g, \eta)$ is an OTS-morphism from $T(ES_0)$ to $T(ES_1)$ where $g : FC(ES_0)$ to $FC(ES_1)$ is given by $g(c) = \eta(c)$ for every $c \in FC(ES_0)$. But from [NRT2] it follows that if (f_1, η_1) and (f_2, η_2) are a pair of OTS-morphisms from TS to TS'

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then $\eta_1 = \eta_2$ implies $f_1 = f_2$. Now (f, η) and (g, η) are a pair of morphisms from $T(ES_0)$ to $T(ES_1)$. Hence we can conclude that $f = g$ and this would establish the fullness of T.

Thus it suffices to prove that η is a PES-morphism from ES_0 to ES_1. So, to prove that η must be a PES-morphism from ES_0 to ES_1, we make use of Lemma 2.4, i.e., we show that property $(*)$ of Definition 2.2 is satisfied for every $c \in FC(ES_0)$. By simple induction on the size of c we can show that $f(c) = \eta(c)$ and since $f : FC(ES_0) \to FC(ES_1)$ we have that $\eta(c) \in C(ES_1)$. Secondly, assume $e, e' \in c$, $e \neq e'$ and that $\eta(e)$ and $\eta(e')$ are both defined. From Proposition 2.3 we may assume configurations $c' \xrightarrow{\ e\ }\!\!\prec c''$ such that $e' \in c'$. From the arguments above we have $f(c') = \eta(c')$, i.e., $\eta(e') \in f(c')$. But since (f, η) is a morphism and $\eta(e)$ defined we have $f(c') \xrightarrow{\ \eta(e)\ }\!\!\prec f(c'')$ in ES_1 but this implies $\eta(e) \notin f(c')$, i.e., $\eta(e) \neq \eta(e')$ as required.

$\square$

Lemma 4.4. T is faithful.

Proof.

Let η, η' be two PES morphisms from ES_0 to ES_1. We must prove that $\eta \neq \eta'$ implies that $T(\eta) \neq T(\eta')$. But this follows from the definition of T. $\square$

Lemma 4.5. For every OTS-object TS there exists an PES-object ES such that TS and $T(ES)$ are isomorphic.

Proof.

Given an OTS-object $TS = (S, E, T, s^{in})$ we define $\zeta(TS) = (E, \leq, \#)$ where $\forall e, e' \in E$. $[e \leq e'$ iff $s_e \xrightarrow{*} s_{e'}$ in TS and $e \# e'$ iff $(\uparrow s_e \cap \uparrow s_{e'}) = \emptyset$ in $TS]$ where s_e and $s_{e'}$ are the unique states associated with e and e' respectively according to Lemma 3.1. First, we must prove that $\zeta(TS)$ is a prime event structure. Lemma 3.1 tells us that $\leq$ is a partial order and from definition we get that $\#$ is a symmetric relation such that $\leq \cap \# = \emptyset$. $\#$ is also clearly inherited by $\leq$ in the sense of **A1** of Definition 2.1, and finally **A2** of Definition 2.1 follows from Lemma 3.6. So, $\eta(TS)$ is a prime event structure.

Next we prove that $(past, id_E)$ is the required isomorphism between TS and $T(\eta(TS))$, where $past : S \to 2^E$ is defined, in Definition 3.5, and $id_E : E \to E$ is the identity function. Clearly, $past$ as defined is a function from S to $FC(\eta(TS))$ (left for the reader to see) and it follows from Lemma 3.6 that $past$ is a TS-morphism. From Lemma 3.7 it follows that $past$ is injective, and hence has a partial inverse $past^{-1}$. From Lemma 4.6 (to follow) we conclude that $past^{-1}$ is a total function on $FC(\eta(TS))$ and that $(past^{-1}, id)$ is the categorical inverse of $(past, id)$. This concludes the proof of Lemma 4.5. □

Lemma 4.6. Let $TS = (S, E, T, s^{in}) \in OTS$. Then for every finite configuration $c \in FC(\zeta(TS))$

(a) $\exists s_c \in S.\ past(s_c) = c$,

(b) $\forall c^1 \xrightarrow{\ e\ }\!\!\!< c$ in $\zeta(TS).\ s_{c^1} \xrightarrow{e} s_c$ in TS.

Proof.

We prove the lemma by induction on the size of configuration c.

$c = \emptyset$. Clearly $past(s^{in}) = \emptyset$, and **(b)** is trivially satisfied.

$c \neq \emptyset$. Let c^1 and e be such that $c^1 \xrightarrow{\ e\ }\!\!\!< c$ in $\zeta(TS)$. Then clearly e is maximal w.r.t. in c. We consider two subcases:

Case 1. $\forall e' \in c^1.\ e' \leq e$.

In this case we have $c = \{e' \in E \mid e' \leq e\}$ (remember c is a configuration). Hence $past(s) = c$ where s_e is the unique state associated with e from Lemma 3.4. From axiom **S1**, we must have a state s' and event e' such that $s' \xrightarrow{e} s_e$ in TS. From **axiom 0** we get from the assumption $e' \neq e$ that $s' \in \uparrow s_e$ — contradicting Lemma 3.3. So $e = e'$, and from Lemma 3.6 we get $past(s') = past(s_e) \backslash \{e\} = c^1$, and hence from injectivity of past, $s' = s_{c^1}$.

Case 2. $\exists e' \in c^1.\ e' \not\leq e$. We can assume without loss of generality that e' is a maximal element (under $\leq$) of c^1. It follows from Proposition 2.3 that in $\eta(TS)$ we then have that $c \backslash \{e, e'\}$, $c \backslash \{e'\}$, and $c \backslash \{e\} = c'$ all belong to $FC(\eta(TS))$.
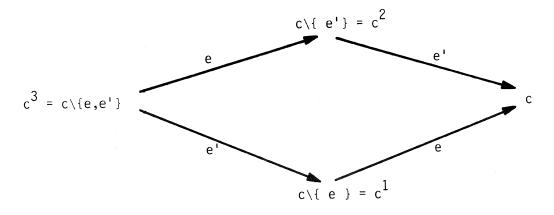
Figure 1:

So, from induction hypothesis we must have states s^1, s^2 and s^3 such that $past(s^i) = c^i$, $i = 1, 2, 3$, and $c^1 = c \setminus \{e\}$, $c^2 = c \setminus \{e'\}$, and $c^3 = c \setminus \{e, e'\}$.
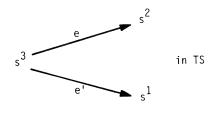

Figure 2:

Now, from our assumptions we have that e and e' are neither related by $\leq$ or $\#$, and hence from Lemma 4.7 (to follow) we get $(^{\circ}e \cup e^{\circ}) \cap (^{\circ}e' \cup e'^{\circ}) = \emptyset$ and hence from axiom **T1** we have that there must exist $s \in S$ such that $s^1 \xrightarrow{e} s$. But now clearly from Lemma 3.6 we get $past(s) = \{e\} \cup past(s^1) = c^1 \cup \{e\} = c$, and this concludes our proof.

$\square$

Lemma 4.7. Let $TS = (S, E, T, s^{in}) \in OTS$, and let $e_0, e_1 \in E$ be two

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events not related by either $\leq$ or $\#$ as defined in the proof of Lemma 4.5. Then $({}^\circ e_0 \cup e_0^\circ) \cap ({}^\circ e_1 \cup e_1^\circ) = \emptyset$ in TS.

Proof.

From the assumptions of the lemma it follows that we have a state s such that $s_{e_0} \xrightarrow{*} s$ and $s_{e_1} \xrightarrow{*} s$ in TS, where s_{e_0}, s_{e_1} are the unique states associated with e_0, e_1, from Lemma 3.4. Choose s to be a minimal (w.r.t. *past*) state satisfying this property. We want to argue that we must have states s_0 and s_1 such that the situation shown in Figure 3 obtains.

Assume that no such s_0 exists. Since $s_{e_0} \xrightarrow{*} s$, we must have from Lemma 3.6 that any computation in TS to s^{in} must contain exactly one e_0-occurrence. Now, consider any computation of the form $s^{in} \xrightarrow{*} s_{e_1} \xrightarrow{*} s$. Such a computation cannot have an e_0-occurrence before s_{e_1} since this would imply $s_{e_0} \xrightarrow{*} s_{e_1}$ contradicting our assumption that e_0 and e_1 are not $\leq$-related. So, we must have states s' and s'' such that $s^{in} \xrightarrow{*} s_{e_1} \xrightarrow{*} s' \xrightarrow{e_0} s'' \xrightarrow{*} s$. But now $s_{e_1} \xrightarrow{*} s''$ and also from **axiom 0**, $s_{e_0} \xrightarrow{*} s''$ and hence from the minimality of s, we get $s = s''$.
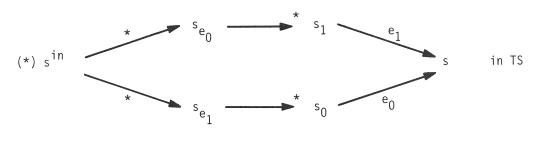

Figure 3:

Now, based on Figure 3, we want to argue for the conclusion of the lemma. Assume $R \in ({}^\circ e_0 \cup e_0^\circ) \cap ({}^\circ e_1 \cup e_1^\circ) \neq \emptyset$.
Case 1. $R \in ({}^\circ e_0 \cap e_0^\circ) \cup ({}^\circ e_1 \cap e_1^\circ)$.
This assumption leads to the immediate contradiction $s \in R \Leftrightarrow s \notin R$.
Case 2. $R \in {}^\circ e_0 \cap {}^\circ e_1$.
From the arguments of the proof of Lemma 4.6, Case 1 we must have $s_{e_0} \notin R$ (from assumption $R \in {}^\circ e_0$), and hence we must have (from the assumption $R \in {}^\circ e_1$) that $s_1 \in R$ and hence the existence of some $s', s'' \in S$ and e_2 such

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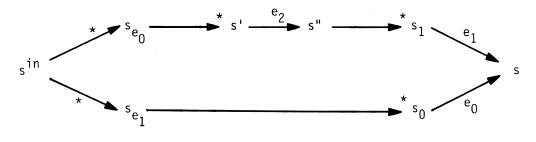that $R \in e_2^\circ$ and the situation shown in Figure 4 obtains.

Figure 4:

Now, from **axiom 0** we know that $s'' \in \uparrow s_{e_2}$ and hence $s \in (\uparrow s_{e_0} \cap \uparrow s_{e_1} \cap \uparrow s_{e_2})$, so we cannot have neither $e_2 \# e_0$ nor $e_2 \# e_1$. But from the existence of $R \in e_2^\circ \cap (^\circ e_0 \cap e_1^\circ)$ we must have from Case 1 of this proof that e_2 must be $\leq$-related to both e_0 and e_1.

Assume $e_2 < e_0$. This implies from definition, $s_{e_0} \in \uparrow s_{e_2}$ and hence $s' \in \uparrow s_{e_2}$ contradicting the fact that $\uparrow s_{e_2}$ is a post-region of e_2. So, we must have $e_0 < e_2$.

Assume $e_1 < e_2$. This implies from definition, $s_{e_2} \in \uparrow s_{e_1}$ and hence $s'' \in \uparrow s_{e_2}$ (besause $\uparrow s_{e_2}$ is post-region of e_2) and also $s_1 \uparrow s''$ (see Figur 4.4), we get $s_1 \uparrow s_{e_1}$, contradicting the fact that $\uparrow s_{e_1}$ is a post-region of e_1. So, we must have $e_2 < e_1$.

But now obviously $e_0 < e_2$ and $e_2 < e_1$ imply $e_0 < e_1$ contradicting our assumption that e_0 and e_1 are not $\leq$-related. All in all, we have contradicted the assumption of Case 2.

Case 3. $R \in e_0^\circ \cap e_1^\circ$.
In this case we would have $\bar{R}$ (the complement of R) belonging to $^\circ e_0 \cap {}^\circ e_1$ - thus this case is reduced to Case 2 .

Since these three cases exhaust the assumption $R \in (^\circ e_0 \cup e_0^\circ) \cap (^\circ e_1 \cup e_1^\circ)$ we have proved Lemma 4.7 and hence our main Theorem 4.2. □

5 Unfoldings of Elementrary Transition Systems

One of the nice aspects of net theory is that it provides a uniform formalism in which both distributed *systems* and their *behaviours* can be defined. For instance, one may define the behaviour of an elementary net system in terms of its unfolding. The unfolding is simply an elementary net system called an occurrence net. Hence occurrence nets can be defined as a subcategory of the category of elementary net systems. Furthermore, the operation of unfolding of an elementary net system (extended in a natural way to a functor) was shown by Winskel to be not an arbitrary functor (from the category of elementary net systems to the subcategory of occurrence nets) but infact, the right adjoint to the inclusion functor from occurrence nets to elementary net systems. Unfortunately the category in which this result was proved has a notion of a net morphism which differs from (and which is in some sense is weaker than) the net morphisms we have used to establish a co-reflection between the category of elementary transitions systems (considered here) and a category of elementary net systems [NRT1]. However, well-understood (co-reflective) relationship to the category of occurence nets considered by Winskel [W] . Moreover, by the strong result of the previous section this category of prime event structures is the "same" as the subcategory of occurrence transition systems. Hence one could hope that the inclusion functor from OTS to ETS would have a right adjoint resembling the unfoldings of elementary net systems.

This hope is based on the fact that prime event structures are more abstract than occurrence nets [W] and hence by the result of the previous section, occurrence transition systems are more abstract than occurrence nets. On the other hand, elementary transition systems are more abstract than elementary net systems [NRT1]. Thus, at this more abstract level one might be able to avoid the technical difficulties that arise when we try to relate occurrence nets to elementary net systems in the presence of the strong net morphims that we insist on. The aim of this section is to show that this hope is entirely justified.

We shall use the theory of trace languages — originating from the work of Mazurkiewicz [Maz] — to define unfoldings of elementary transition systems.

We will show that this "unfold" map produces occurrence transition systems and it can be smoothly extended to become a functor from ETS to OTS. More importantly we will prove that this functor is the right adjoint of the inclusion functor from OTS to ETS.

In the literature, a number of authors have indecently shown that a strong relationship exists between trace languages and prime event structures [RT, Sh, B]. In what follows we will appeal to a number of technical results that arise in the process of establishing that trace languages yield prime event structures. We will not give detailed proofs of these results since they can be found in or can be easily extracted from [RT]. For background material on trace languages the reader is referred to [AR, Maz].

Until further notice, fix an elementary transition system $TS = (S, E, T, s_{in})$. Then FS_{TS} the set of *firing sequences* of TS and the relation $[>_{TS} \subseteq \{s_{in}\} \times FS_{TS} \times S$ are given inductively by:

- $\Lambda \in FS_{TS}$ and $s_{in}[\Lambda >_{TS} s_{in}$.

- If $\rho \in FS_{TS}$, $s_{in}[\rho >_{TS} s$ and $(s, e, s') \in T$, then $\rho e \in FS_{TS}$ and $s_{in}[\rho e >_{TS} s'$.

Where TS is clear from the context we will write FS instead of FS_{TS} and $[>$ instead of $[>_{TS}$.

In fact, we will follow this convention for a number of relations that we will soon define relative to TS. The independence relation $I_{TS} \subseteq E \times E$ associated with TS is given by:

$$I_{TS} = \{(e_1, e_2) \mid ({}^{\circ}e_1 \cup e_1^{\circ}) \cap ({}^{\circ}e_2 \cup e_2^{\circ}) = \emptyset\}$$

Clearly I_{TS} is irreflexive and symmetric and hence induces an equivalence relation (see [Maz]) over E^*. This equivalence relation will in fact be a congruence w.r.t. the operation of concatenation over E^*. To be specific $\doteq_{I_{TS}}$ (written for convenience as $\doteq_{TS}$) is the subset of $E^* \times E^*$ given by:

$\sigma \doteq_{TS} \sigma'$ iff $\exists \sigma_1, \sigma2 \in E^*$. $\exists (e, e') \in I_{TS}$. $[\sigma = \sigma_1 \, e \, e' \, \sigma_2$ and $\sigma' = \sigma_1 \, e' \, e \, \sigma_2]$.

The equivalence relation we want is denoted as $=_{TS}$ and it is the reflexive transitive closure of $\doteq_{TS}$. In other words, $=_{TS} = (\doteq_{TS})^*$. For $\sigma \in E^*$

we let $[\sigma]_{TS}$ denote the equivalence class containing σ and call it a *trace*. Formally, $[\sigma]_{TS} = \{\sigma' \mid \sigma =_{TS} \sigma'\}$. As remarked earlier, we will often write $[\sigma]$ instead of $[\sigma]_{TS}$. Unless otherwise stated, in what follows we let ρ, ρ', ρ'' with or without subscripts to range over FS; we let $\sigma, \sigma', \sigma''$ with or without subscripts to range over E^*; we let e, e', e'', e_1, e_2 to range over E. The result we mention next is a well-known and very useful characterization of the relation $=_{TS}$ (see, for instance, [AR] for a proof).

In stating the result we will use the following notations. For $e \in E$, $\#_e(\sigma)$ is the number of times the symbol e appears in σ. For $X \subseteq E$, $Proj_X(\sigma)$ is the sequence obtained by erasing from σ all appearances of non-members of X.

In other words,

- $Proj_X(\Lambda) = \Lambda$.

- $Proj_X(\sigma e) = \begin{cases} Proj_X(\sigma)e, & \text{if } e \in X, \\ Proj_X(\sigma), & \text{otherwise.} \end{cases}$

Proposition 5.1. $\sigma_1 =_{TS} \sigma_2$ iff the following two conditions are satisfied.

(i) $\forall e \in E.\ \#_e(\sigma_1) = \#_e(\sigma_2)$.

(ii) $\forall (e, e') \in (E \times E) - I_{TS}.\ Proj_{\{e,e'\}}(\sigma_1) = Proj_{\{e,e'\}}(\sigma_2)$. $\qquad\square$

Next we recall the standard ordering over the traces generated by $=_{TS}$.

$[\sigma] \leq_{TS} [\sigma']$ iff $\exists \sigma''.\ \sigma\sigma'' =_{TS} \sigma'$. It is easy to check that $\leq$ is a partial ordering relation with $[\Lambda] = \{\Lambda\}$ as the least element. $[\sigma] \sqcup [\sigma']$ will denote the least upper bound of $[\sigma]$ and $[\sigma']$ under $\leq$, if it exists.

Given our purposes, a relation closely related to $\leq$ and denoted as $\longrightarrow_{TS}$ will turn out to very useful to have around. $\longrightarrow_{TS} \subseteq E^* \times E^*$ is given by:

$$\sigma \longrightarrow_{TS} \sigma' \text{ iff } \exists \sigma''.\ \sigma\sigma'' =_{TS} \sigma'.$$

The next set of observations are easy to verify.

Proposition 5.2.

(i) $[\sigma] \le [\sigma']$ iff $\sigma \longrightarrow \sigma'$. Thus is a pre-order the equivalence relation induced by which is exactly $=_{TS}$.

(ii) $\forall \rho \in FS.\ [\rho] \subseteq FS$.

(iii) Suppose $\rho e,\ \rho e' \in FS$ with $(e, e') \in I_{TS}$. Then $\rho e e',\ \rho e' e \in FS$. $\qquad\square$

Part **(iii)** of this result leans on the fact that TS, being elementary, satisfies the axiom **T1**.

The set $\{[\rho] \mid \rho \in FS\}$ will serve as the set of states of $Uf(TS)$, the unfolding of TS, that we wish to construct. To identify the events of $Uf(TS)$ we must work with the *prime intervals* generated by TS denoted as PI_{TS}. It is the subset of $\Sigma^* \times \Sigma^*$ given by $PI_{TS} \overset{\text{def}}{=} \{(\sigma, \sigma') \mid \exists e \in E.\ \sigma e =_{TS} \sigma'\}$. Next we define the map $\varphi_{TS} : PI \to E$ as follows:

$$\forall (\sigma, \sigma') \in PI.\ \varphi(\sigma, \sigma') = e \text{ provided } \sigma e =_{TS} \sigma'.$$

Now suppose that $\sigma e =_{TS} \sigma e'$. Then according to Proposition 5.1, $e = e'$. Hence φ is well-defined. This map — or more precisely, our extension of this map to certain equivalence classes of prime intervals — will turn out to be crucial for linking up the behvaiour of $Uf(TS)$ to that of TS; but we still need to identify the events of $Uf(TS)$.

To this end, define the relation $\alpha_{TS} \subseteq PI \times PI$ by:

$$(\sigma_1, \sigma_1')\ \alpha_{TS}\ (\sigma_2, \sigma_2') \text{ iff } \exists \sigma.\ [\sigma_1 \sigma =_{TS} \sigma_2 \text{ and } \sigma_1' \sigma =_{TS} \sigma_2'].$$

$Set \approx_{TS} = (\alpha_{TS} U(\alpha_{TS})^{-1})^*$. Clearly $\approx_{TS}$ is an equivalence relation over PI. In what follows, we denote by $< \sigma, \sigma' >_{TS}$ the equivalence class of prime intervals containing the prime interval (σ, σ'). Again using Proposition 5.1 and the definitions, the next set of observations are easy to verify.

Proposition 5.3.

(i) α_{TS} is a pre-order.

(ii) Suppose $(\sigma_1, \sigma_1'),\ (\sigma_2, \sigma_2') \in PI$. Then $(\sigma_1, \sigma_1')\ \alpha_{TS}\ (\sigma_2, \sigma_2')$, and (σ_2, σ_2') $\alpha_{TS}\ (\sigma_1, \sigma_1')$ iff $\varphi(\sigma_1, \sigma_1') = \varphi(\sigma_2, \sigma_2')$ and $\sigma_1 =_{TS} \sigma_2$.

(iii) Suppose $(\sigma_1, \sigma_1')\ \alpha_{TS}\ (\sigma_2, \sigma_2')$. Then $\varphi(\sigma_1, \sigma_1') = \varphi(\sigma_2, \sigma_2')$. $\qquad\square$

Extend φ to $\approx_{TS}$-equivalence classes of prime intervals as follows (by abuse of notation, this extension will also be denoted as φ) :

$$\forall (\sigma_1, \sigma_1') \in PI.\ \varphi(<\sigma_1, \sigma_1'>) = \varphi(\sigma_1, \sigma_1').$$

According to Proposition 5.3, this extension of φ is also well-defined. Some of the equivalence classes of prime intervals will serve as the events of $Uf(TS)$.

Definition 5.4. $Uf(TS)$, the *unfolding* of TS, is the transition system

$$
\begin{aligned}
Uf(TS) &= (\hat{S}, \hat{E}, \hat{T}, \hat{s}_{in})\ \text{where}\\
\hat{S} &= \{[\rho] \mid \rho \in FS\},\\
\hat{E} &= \{<\rho, \rho'>\mid \rho, \rho' \in FS \text{ and } (\rho, \rho') \in PI\},\\
\hat{T} &= \{([\rho], <\rho, \rho'>, [\rho']) \mid <\rho, \rho'> \in \hat{E}\},\ \text{and}\\
\hat{s}_{in} &= \{\Lambda\}.
\end{aligned}
$$

$\qquad\square$

Our first task will be to prove that $Uf(TS)$ is an occurrence transition system. As mentioned earlier, in doing so, we will appeal to a number of technical results without giving proofs. These proofs can be found in or can be easily extracted from [RT]. However we will provide sufficient information so that an enterprising reader can work out the details for herself/himself.

Lemma 5.5.

(i) Suppose $\sigma e_1 \sigma_1 =_{TS} \sigma e_2 \sigma_2$, with $e_1 \neq e_2$. Then $(e_1, e_2) \in I_{TS}$. Moreover there exists σ' such that $\sigma e_1 \sigma_1 =_{TS} \sigma e_1 e_2 \sigma_1 =_{TS} \sigma e_2 e_1 \sigma' =_{TS} \sigma e_2 \sigma_2$. Consequently, $[\sigma e_1] \sqcup [\sigma e_2] = [\sigma e_1 e_2]$.

(ii) Suppose $\sigma_1 \longrightarrow \sigma$ and $\sigma_2 \longrightarrow \sigma$. Then $[\sigma_1] \sqcup [\sigma_2]$ exists.

(iii) Suppose $\rho \longrightarrow \sigma$ and $\rho' \longrightarrow \sigma$ (with $\rho, \rho' \in FS$, $\sigma \in E^*$). Then $[\rho] \sqcup [\rho'] \in \hat{S}$. $\qquad\square$

The property captured in part **(i)** of this result is the so-called forward diamond property. The relevant situation is shown in Figure 5. The proof follows easily by repeated applications of Proposition 5.1. Part **(ii)** of the result foll ows by repeated appl ications of part (i) of the result. Part **(iii)**
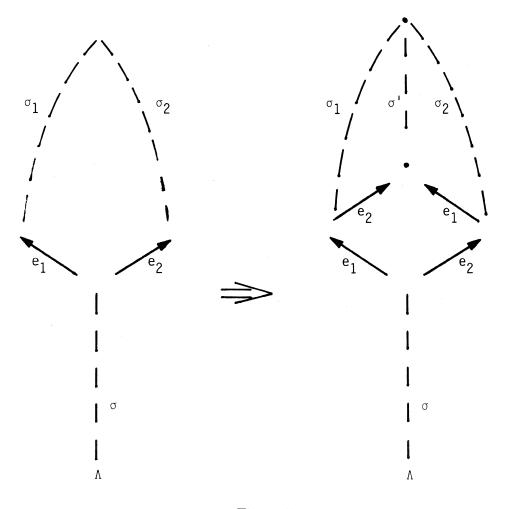
Figure 5:

of the result follows from part **(ii)** and repeated applications of Proposition 5.2.

Lemma 5.6. Suppose $\sigma_1 e_1 =_{TS} \sigma_2 e_2$ with $e_1 \neq e_2$. Then $(e_1, e_2) \in I_{TS}$. Moreover there exists σ such that $\sigma e_2 =_{TS} \sigma_1$ and $\sigma e_1 =_{TS} \sigma_2$. $\qquad\square$

This is the so-called backward diamond property. This result also follows easily through repeate applications of Proposition 5.1. The relevant situation is shown in Figure 6.

For introducing the next result we need a notation. This notation will

Figure 6:

be used extensively in the sequel. Let $(\sigma, \sigma') \in PI$. Then Base $(< \sigma, \sigma' >) \subseteq < \sigma, \sigma' >$ is the set:

$$\{(\sigma_0, \sigma_0') \mid (\sigma_0, \sigma_0') \in < \sigma, \sigma' > \text{ and } \forall (\sigma_1, \sigma_1') \in < \sigma, \sigma' > \cdot (\sigma_0, \sigma_0') \, \alpha_{TS} \, (\sigma_1, \sigma_1')\}.$$

Recall that according to Proposition 5.2, if $(\sigma_1, \sigma_1'), (\sigma_2, \sigma_2') \in Base(< \sigma, \sigma' >)$, then $\sigma_1 =_{TS} \sigma_2$ and $\sigma_1' =_{TS} \sigma_2'$. Hence $Base(< \sigma, \sigma' >)$ identifies in some sense the "least" elements of $< \sigma, \sigma' >$ under α_{TS} modulo the equivalence relation $=_{TS}$.

Lemma 5.7.

(i) $\forall (\sigma, \sigma') \in PI. \ Base(< \sigma, \sigma' >) \neq \emptyset.$

(ii) $\forall \hat{e} \in \hat{E}. \ Base(\hat{e}) \subseteq FS \times FS.$ $\qquad\qquad\qquad\square$

The first part of the result follows fairly easily from Lemma 5.6. The main observation exploiting Lemma 5.6 (and the definition of $\approx_{TS}$) can be depicted graphically as shown in Figure 7.

The second part of the result follows from the first part and the observation that FS is prefix-closed. Thanks to Lemma 5.7 we can *injectively*
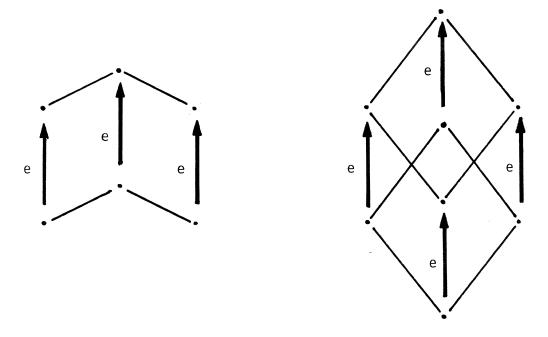
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Figure 7:

associate with each element of $\hat{S}$ (in $Uf(TS)$) a set of events in E; the events that have 'occurred so far". To see this, define $Ev : FS \to P(E)$ (to be soon extended to $\hat{S}$!) as: $\forall \rho \in FS.\ Ev(\rho) = \{\hat{e} \mid \exists(\rho_1, \rho_1') \in \hat{e}.\ \rho_1' \to \rho\}$. To be precise, we must define $Ev(\rho)$ as $\{\hat{e} \mid \exists(\sigma_1, \sigma_1') \in \hat{e}.\ \sigma_1' \to \rho\}$. But, once again, the fact that FS is prefix-closed guarantees that our definition captures the intended meaning. Ev is extended to a map — also denoted as Ev by abuse of notation — from $\hat{S}$ to $P(\hat{E})$ via:

$$\forall \rho \in FS.\ Ev([\rho]) = Ev(\rho).$$

It is easy to verify that this extension is well-defined.

Lemma 5.8.

(i) Suppose $\rho e \in FS$. Then $< \rho, \rho e > \notin Ev(\rho)$. Moreover $Ev(\rho e) = Ev(\rho) \cup \{< \rho, \rho e >\}$.

(ii) $\forall \rho, \rho' \in FS.\ \rho \to \rho'$ iff $Ev(\rho) \subseteq Ev(\rho')$. Hence $\rho =_{TS} \rho'$ iff $Ev(\rho) = Ev(\rho')$. Thus $Ev : \hat{S} \to P(\hat{E})$ is injective.

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(iii) Suppose $([\rho], \hat{e}, [\rho']) \in \hat{T}$ (in $Uf(S)$.) Then $\hat{e} \notin Ev([\rho])$. Moreover, $Ev([\rho']) = Ev([\rho]) \cup \{\hat{e}\}$.

(iv) Suppose $[\sigma] \sqcup [\sigma']$ exists. Then $Ev([\sigma] \sqcup [\sigma'] = Ev([\sigma]) \cup Ev([\sigma'])$. $\qquad\square$

This result follows from Lemma 5.5 and Lemma 5.7. The details are a bit tedious but straight forward. This completes the chain of technical results we shall borrow from the literature. We now turn to the task of proving that $Uf(TS)$ is an occurrence transition system.

Recall the functor T going from PES to OTS. We will show that there exists a prime event structure ES such that $T(ES)$ and $Uf(TS)$ are isomorphic transition systems (relative to the notion of morphisms specified in Definition 1.5).

Since $T(ES)$ is an OTS-object we would have then established that $Uf(TS)$ is also an OTS-object. Define $ES = (\hat{E}, \leq, \#)$ where $\leq, \# \subseteq \hat{E} \times \hat{E}$ are defined as follows:

(i) $\hat{e}_1 \leq \hat{e}_2$ iff $\forall(\rho, \rho') \in Base(\hat{e}_2). \hat{e}_1 \in Ev(\rho')$,

(ii) $\hat{e}_1 \# \hat{e}_2$, iff there does *not* exits $\rho \in FS$ such that $\hat{e}_1 \in Ev(\rho)$ and $\hat{e}_2 \in Ev(\rho)$.

It is easy to verify that ES is indeed a prime event structure in the sense of Definition 2.1. Recall that $T(ES) = (FC(ES), \hat{E}, \mathbin{\longrightarrow\!\!\!\!\!<}, \emptyset)$.

The proof of the fact that $T(ES)$ and $Uf(TS)$ are isomorphic can be split into two steps.

Lemma 5.9. Let $\hat{e}_1, \hat{e}_2 \in \hat{E}$ be such that not ($\hat{e}_1 \leq \hat{e}_2$ or $\hat{e}_2 \leq \hat{e}_1$ or $\hat{e}_1 \# \hat{e}_2$. Then $(\varphi(\hat{e}_1), \varphi(\hat{e}_2)) \in I_{TS}$.

Proof.

Let $(\rho_i, \rho_i') \in Base(\hat{e}_i)$ and $\varphi(\hat{e}_i) = e_i$ for $i = 1, 2$. Since neither $\hat{e}_1 \leq \hat{e}_2$ nor $\hat{e}_2 \leq \hat{e}_1$ it must be the case that $[\rho_1']$ and $[\rho_2']$ are incomparable. Since it is not the case that $\hat{e}_1 \# \hat{e}_2$, there exists $\rho \in FS$ such that $\hat{e}_1, \hat{e}_2 \in Ev(\rho)$. Consequently $\rho_1' \to \rho$ and $\rho_2' \to \rho$. Hence $[\rho_1] \sqcup [\rho_2], [\rho_1] \sqcup [\rho_2'], [\rho_1'] \sqcup [\rho_2]$ and $[\rho_1'] \sqcup [\rho_2']$ all exist. Let $\rho_{11} \in [\rho_1] \sqcup [\rho_2], \rho_{12}' \in [\rho_1'] \sqcup [\rho_2], \rho_{21}' \in [\rho_1] \sqcup [\rho_2']$ and $\rho_{22} \in [\rho_1'] \sqcup [\rho_2']$. From previous results, it is easy to verify the following:

(i) $Ev(\rho'_{12}) = Ev(\rho_{11}) \cup \{\hat{e}_1\}$ and $Ev(\rho'_{21}) = Ev(\rho_{11}) \cup \{\hat{e}_2\}$,

(ii) $Ev(\rho'_{12}) \cup \{\hat{e}_2\} = Ev(\rho_{11}) \cup \{\hat{e}_1, \hat{e}_2\} = Ev(\rho'_{12}) \cup \{\hat{e}_1\}$,

(iii) $\rho_{11}e_1 =_{TS} \rho'_{12}$ and $\rho_{11}e_2 =_{TS} \rho'_{21}$ and $\rho'_{12}e_2 =_{TS} \rho_{22} =_{TS} \rho'_{21}e_1$.

From **(iii)**, it follows at once that $\rho_{11}e_1e_2 =_I \rho_{11}e_2e_1$ which leads to $(e_1, e_2) \in I_{TS}$. $\qquad\square$

Lemma 5.10. The map $Ev : \hat{S} \to P(\hat{E})$ is in fact a bijection from S to $FC(ES)$.

Proof.

From the definition of ES it follows easily that $Ev([\rho]) \in FC(ES)$ for every $\rho \in FS$. This map is injective according to Lemma 5.9. Let $c \in FC(ES)$. We must show that there exits $\rho \in FS$ such that $Ev(\rho) = c$. We proceed by induction on $k = |c|$.

k = 0. Then $c = \emptyset$ and we can set $\rho = \Lambda$.

k > 1. Suppose there exists $\hat{e} \in c$ such that $\hat{e}_1 \leq \hat{e}$ for every $\hat{e}_1 \in c$. (In other words, c has a unique maximal element). Let $(\rho', \rho) \in Base(\hat{e})$. Then it is easy to check, using the definition of ES, that $Ev(\rho) = c$.

So assume that c contains (at least) two distinct maximal elements $\hat{e}_1$, and $\hat{e}_2$. Let $c_0 = c \setminus \{\hat{e}_1, \hat{e}_2\}$, $c_1 = c \setminus \{\hat{e}_2\}$, and $c_2 = c \setminus \{\hat{e}_1\}$. Then by the induction hypothesis there exist $\rho_i \in FS$ such that $Ev(\rho_i) = c_i$ for $i = 0, 1, 2$. It is also clear from Lemma 5.8 that $\rho_0e_1 =_{TS} \rho_1$ and $\rho_0e_2 =_{TS} \rho_2$ where $\varphi(\hat{e}_1) = e_1$ and $\varphi(\hat{e}_2) = e_2$. Clearly not $(\hat{e}_1 \leq \hat{e}_2$ or $\hat{e}_2 \leq \hat{e}_1$ or $e_1 \# e_2)$ holds. Hence by the previous lemma $(e_1, e_2) \in I_{TS}$. According to Proposition 5.2, $\rho_0e_1e_2$, $\rho_0e_2e_1 \in FS$. It is now straight forward to verify that $Ev(\rho_0e_1e_2) = c$.

$\qquad\square$

Theorem 5.11. $Uf(TS)$ is an occurrence transition system.

Proof.

We know that $T(ES) = (FC(ES), \hat{E}, \longrightarrow\!\!\!<, \emptyset)$ is an occurrence transition system where ES is as constructed above. Consider the pair of maps (Ev, id)

where id is the identity map over $\hat{E}$. By the previous lemma Ev is a bijection. It is now easy to verify that $([\rho], \hat{e}, [\rho']) \in \hat{T}$ iff $Ev(\rho) \xrightarrow{\ \hat{e}\ } < Ev(\rho')$. From this it follows that (Ev, id) is a transition system morphism, and hence is in fact an isomorphism. From this it follows that $Uf(TS)$ is also an occurrence transition system. $\square$

To proceed towards the main result we next define the notion of folding as a morphism from $Uf(TS)$ to TS. This map will turn out to be the co-unit of the co-reflection between OTS and ETS that we are trying to establish.

Let TS and $Uf(TS)$ be as defined previously. Let $fold_{TS} = (f, \eta)$ be given by:

(i) $f : \hat{S} \to S$ is such that $\forall \rho \in FS.\ f([\rho]) = s$, where $s_{in}[\rho > s$ in TS.

(ii) $\eta : \hat{E} \to E$ is such that $\forall <\rho, \rho'> \in \hat{E}.\ \eta(<\rho, \rho'>) = \varphi(<\rho, \rho'>)$.

Proposition 5.12. $fold_{TS}$ is a transition system morphism from $Uf(TS)$ to TS.

Proof.

It follows easily from the fact that TS is an ETS-object that f and η are well-defined total functions. It i s then routine to verify that $fold_{TS}$ is indeed a morphism. $\square$

The following lemma will turn out to be useful for proving the main theorem of this section.

Lemma 5.13. Let $TS_0 = (S_0, E_0, T_0, s_0)$ be an occurrence transition system and $TS = (S, E, T, s_{in})$ be an elementary transition system. Let (g, μ) be a morphism from TS_0 to TS. Suppose $s_0[\rho > s$ and $s_0[\rho' > s$ in TS_0 (i.e., ρ and ρ' are two computations – firing sequences – leading to a common states). Then $\mu(\rho) =_{TS} \mu(\rho')$.

Proof. By Lemma 3.6, we know that $|\rho| = |\rho'|$. We now proceed by induction on $k = |\rho|$.

k = 0. Then clearly $\mu(\rho) = \Lambda = \mu(\rho')$.

k > 0. Let $\rho = \rho_1 e$ and $\rho' = \rho_1' e'$. Assume $s_0[\rho_1 > s_1$ and $s_0[\rho_1' > s_1'$. Suppose $e = e'$. Then once again from Lemma 3.6 it follows that

$past(s_1) = past(s_1')$ and hence by Lemma 3.7, it must be the case that $s_1 = s_1'$. Now by the induction hypothesis, $\mu(\rho_1) =_{TS} \mu(\rho_1')$. Clearly, it now follows that $\mu(\rho_1 e) =_{TS} \mu(\rho_1' e')$, since we have assumed $e = e'$.

So suppose that $e \neq e'$. Let $s_0[\rho_1 > s_1$ and $s_0[\rho_1' > s_1'$ as before. Consider the PES-object $\zeta(TS)$ defined in the proof of Lemma 4.5. It now follows directly from the properties of the function $past$ that there must exist a state s' in TS_0 such that the following situation shown in Figure 8 obtains.

Figure 8:

Let $s_0[\rho'' > s'$ as indicated in the figure. Suppose $\mu(e)$ is undefined. Then $\mu(\rho_1 e) = \mu(\rho_1)$. By the induction hypothesis, $\mu(\rho_1) =_{TS} \mu(\rho'' e')$. But then $\mu(\rho'' e) =_{TS} \mu(\rho_1')$ by the induction hypothesis. Since $\mu(e)$ is undefined we get $\mu(\rho'') = \mu(\rho'' e) =_{TS} \mu(\rho_1')$. Consequently $\mu(\rho'' e') =_{TS} \mu(\rho_1', e')$ and we have now $\mu(\rho_1 e) =_{TS} \mu(\rho_1' e')$. By a symmetric argument the result follows if $\mu(e')$ is undefined.

So suppose that both $\mu(e)$ and $\mu(e')$ are defined. First suppose that $\mu(e) = \mu(e')$. Then in TS, we would get $g(s') \xrightarrow{\mu(e)} g(s_1') \xrightarrow{\mu(e)} g(s)$. This is impossible since TS is elementary (see [NRT2]). Thus $\mu(e) \neq \mu(e')$. But then this at once would imply, once again by the fact that TS is elementary that $(e, e') \in I_{TS}$. Hence $\mu(\rho'' ee') = \mu(\rho'' e'e)$. From the induction hypothesis, we get, $\mu(\rho_1) =_{TS} \mu(\rho'' e')$ so that $\mu(\rho_1 e) =_{TS} \mu(\rho'' e'e)$. Similarly $\mu(\rho_1' e') =_{TS} \mu(\rho'' ee')$. The desired conclusion is now immediate. $\square$

We are now prepared to prove the main result. According to [Mac], Theorem 4.1.2, proving that unfold is the right adjoint to the inclusion functor

from OTS to ETS boils down to establishing the following result.

Theorem 5.14. Let TS be an elementary transition system and $Uf(TS)$ and $fold_{TS} = (f, \eta)$ be as defined previously. Suppose $TS_0 = (S_0, E_0, T_0, s_0)$ is an occurrence transition system and (g, μ) is a morphims from TS_0 to TS. Then there exists a unique morphism (h, θ) from TS_0 to $Uf(TS)$ so that the following diagram commutes.
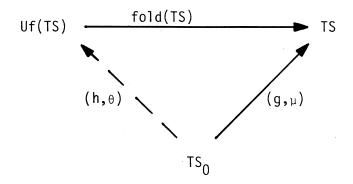

Figure 9:

Proof.

We propose the following definition (h, θ).

$h : S_0 \to \hat{S}$ is given by : $\quad \forall s \in S_0.\ h_s = [\mu(\rho)]_{TS}$ where $s_0[\rho > s$ (in TS_0).
$\theta : E_0 \to \hat{E}$ is given by :

$\forall e \in E_0.\ \theta(e) = \begin{cases} undefined, & \text{if } \mu(e) \text{ is undefined,} \\ < \mu(\rho), \mu(\rho e) >, & \text{otherwise where } s_0[\rho e > s_e \\ & \text{in } TS_0. \end{cases}$

Recall that s_e is the unique state in TS_0 with the property that $\uparrow s_e$ is a non-trivial region with $^\circ(\uparrow s_e) = \{e\}$. By the previous lemma, h and θ are well-defined total and partial functions respectively. We need to prove:

(i) (h, θ) is a morphism from TS_0 to $Uf(TS)$,

(ii) $(f, \eta) \circ (h, \theta) = (g, \mu)$, and

(iii) (h, θ) is unique w.r.t. the properties **(i)** and **(ii)**.

Proof of (i).

Let $e \in E_0$ and $(s', e, s) \in T_0$. Suppose $\theta(e)$ is undefined. We must show that $h(s) = h(s')$. Assume that $s_0[\rho' > s'$ in TS_0. Then $\mu(\rho') =_{TS} \mu(\rho' e)$. Hence $(h(s') = [\mu(\rho')] = [\mu(\rho' e)] = h(s)$ as required.

So suppose that $\theta(e)$ is defined. We then prove that $(h(s'), \theta(e), h(s)) \in \hat{T}$. Let $s_0[\rho_0\ e > s_e$ in TS_0. Then $\theta(e) =< \mu(\rho_0), \mu(\rho_0\ e) >$. Since $e \in Past(s)$, we must have $\rho'' \in E_0^*$ such that $s_e[\rho'' > s$ in TS_0.
We now proceed by induction on $k = |\rho''|$.

k = 0. Then $s = s_e$ and therefore $s_0[\rho_0\ e > s'$ in TS. Consequently $h(s') = [\mu(\rho_0)]$, $h(s) = [\mu(\rho_0\ e)]$, and $\theta(e) =< \mu(\rho_0), \mu(\rho_0\ e) >$. Clearly $(h(s'), \theta(e), h(s)) \in \hat{T}$.

k > 0. Let $\rho'' = \rho_1 e_1$. Then from the results of Section 4 it follows that $(^\circ e \cup e^\circ) \cap (^\circ e_1 \cup e_1^\circ) = \emptyset$ in TS_0 and there exits states s_1 and s_1' such that the situation shown in Figure 10 obtains.
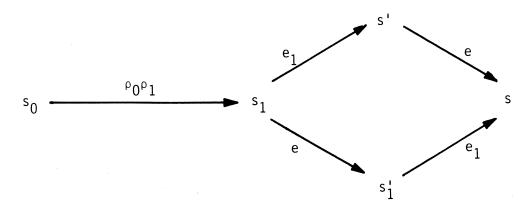

Figure 10:

By the induction hypothesis, $(h(s_1), \theta(e), h(s_1')) \in \hat{T}$. If $\theta(e_1)$ is undefined, then by the previous argument dealing with the case $\theta(e)$ undefined, we must have $h(s_1) = h(s')$ and $h(s_1') = h(s)$. Thus $(h(s'), \theta(e), h(s)) \in \hat{T}$ as required.

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So suppose that $\theta(e_1)$ is defined. Then from the fact that (g, μ) is a morphism from TS_0 into the *elementary* transition system TS, we at once get $(\mu(e), \mu(e')) \in I_{TS}$. Therefore by the definition of the equivalence relation on prime intervals, we get $< \mu(\rho_0\rho_1 e_1), \mu(\rho_0\rho_1 e_1 e) > = < \mu(\rho_0\rho_1), \mu(\rho_0\rho_1 e) > = \theta(e)$ (induction hypothesis). This leads, by the definition of (h, θ) to the desired conclusion that $(h(s'), \theta(e), h(s)) \in \hat{T}$.

Proof of (ii). As observed in [NRT2], to prove that $(f, \eta) \circ (h, \theta) = (g, \mu)$, it suffices to prove that $f \circ h = g$. But this identity follows immediately from the definitions of f and h.

Proof of (iii) Let (h', θ') be some morphism from TS_0 to $Uf(TS)$ which also satisfies the properties **(i)** and **(ii)**.
As observed above, it suffices to show that $h = h'$, because by [NRT2] this would imply $\theta = \theta'$. Let $s \in S_0$ and $s_0[\rho > s$ in TS_0 . We proceed by induction on $k = |\rho|$.

k = 0. Then $s = s_0$ and the two morphisms (h, θ) and (h', θ') must satisfy
$$h(s_0) = [\Lambda] = h'(s_0).$$

k > 0. Let $\rho = \rho_1 e_1$ and $s_0[\rho_1 > s_1$ in TS_0. Then $(s_1, e_1, s) \in T_0$. By the induction hypothesis $h(s_1) = h'(s_1)$.

Suppose $\mu(e_1)$ is undefined. Since $\eta \circ \theta = \mu$ and η is *total*, it must be the case that $\theta(e_1)$ is undefined. Similarly from $\eta \circ \theta' = \mu$ and the totality of η we can conclude that $\theta'(e_1)$ is also undefined. Now (h, θ) being a morphism, we must have $h(s_1) = h(s)$ and similarly $h'(s_1) = h'(s)$. Thus $h(s) = h'(s)$.

So suppose that $\mu(e_1)$ is defined. Then once again from $\eta \circ \theta = \mu = \eta \circ \theta'$ we conclude that both $\theta(e_1)$ and $\theta'(e_1)$ are defined. Since (h, θ) and (h', θ') are morphisms we get $(h(s_1), \theta(e_1), h(s)) \in \hat{T}$ and $(h'(s_1), \theta'(e_1), h'(s)) \in \hat{T}$. Now $h(s_1) = [\mu(\rho_1)]$ by the definition of h and $h(s) = [\mu(\rho_1)\mu(e_1)]$. From property **(ii)** and the definition of $Uf(TS)$, it now follows that $h(s) = [\mu(\rho_1)\eta(\theta(e_1))]$ and $h'(s) = [\mu(\rho_1)\eta(\theta'(e_1))]$. But $\eta(\theta(e_1)) = \eta(\theta'(e_1))$ at once implies that $h(s) = h'(s)$ as required.

Theorem 5.15. The map unfold uniquely extends to a functor which is the right adjoint of the inclusion functor from OTS to ETS, i.e., OTS is a co-reflective full subcategory of ETS.

Proof. Follows easily from the previous theorem according to [Mac]. □

Acknowledgements

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