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Computational Fluency: The Key to Multiplicative Success

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Children across the country are struggling to meet their classrooms’ and states’ required standards of mathematics knowledge. The unfortunate truth, according to many educational professionals, is that, “high-stakes testing has forced schools to push aside subjects like history, science, music, and art in a scramble to avoid the embarrassing consequences of not making ‘adequate yearly progress' in mathematics” (Steen, 2007). Not only are state tests causing problems across subjects, they are making teachers frantic with the pressures of integrating the new Common Core State Standards. The majority of people, when asked how they feel about mathematics, will often say that they do not like mathematics and will admit it was their hardest subject in school. A lot of students struggle their whole lives to get a good grasp on mathematics, while others are able to learn on their own. Why is it so hard for some students to master basic mathematics skills? What do the students who master their basic mathematics skills easily have that others do not? Has instruction lost its focus due to all the high-stakes testing?

One of the most important basic mathematics skills is multiplication. Multiplication can be defined as repeated addition or “a groups of b” for a × b. Students learn multiplication starting typically in third and fourth grade, but the concept is presented informally beforehand. In many different areas of mathematics, students wonder where they can apply what they have learned into their own lives. A few of the major applications of multiplication involve areas, arrangements, combinations, and groupings. Many everyday jobs involve multiplication; a simple rearrangement of a room can involve multiplication or at least a little mathematical thinking. Going to the grocery store can be turned into a multiplication problem simply by having to find the cost when buying multiples of items or when an item is on sale for a percent off. Although multiplication is useful and important, by middle school, even high school for some, many students have not yet mastered their multiplication facts because they lack the strategies to help them figure the facts out. They are missing a significant piece that would give them a solid foundation in multiplication, computational fluency.

Computational fluency is the ability to use strategies to figure out mathematics facts quickly and efficiently. A student who has computational fluency would be able to use strategies together with the facts he or she knows to figure out a more challenging problem. For example, in one classroom, a teacher showed her students had achieved computational fluency by having them multiply extremely high numbers mentally. The teacher started with 5×6 and gradually increased the numbers until she gave the students the problem, 12×251. Amazingly, because the students had computational fluency, they were able to solve the problem in their heads and explain how and why they used the strategies they chose (Wickett, 2003). Students who do not have this skill often have trouble solving more challenging problems because they have to focus on recalling multiplication facts and performing calculations, in addition to using problem solving skills. The students who do not achieve computational fluency in the third grade are at a huge disadvantage when compared to their peers. It is also extremely discouraging for a student to realize that their peers have a skill that they just do not comprehend. There are ways to obtain computational fluency, though. Educators need to ensure that their students gain computational fluency and have a solid understanding of their multiplication facts by using the best teaching methods for building computational fluency.

Although there are many ways to achieve computational fluency in multiplication, there are practices that are more effective than others. According to many educational experts, there are three stages students must go through to master multiplication facts:

- Phase 1: Counting Strategies
- Phase 2: Reasoning Strategies

- Phase 3: Mastery

(Baroody, Bajwa, & Eiland, 2009). Counting strategies involve using objects or manipulatives to solve problems. Reasoning strategies involve using facts that are known to discover products to unknown facts. Mastery refers to producing answers from memory. These stages are also sometimes referred to as the enactive, iconic, and symbolic representation stages (Bruner, 1966). No matter what they are called, experts agree on the content of each stage and that a child needs to go through each to achieve computational fluency. The first stage always involves concrete objects so that the child can physically manipulate the quantities presented to them. The second involves using drawings and reasoning to move away from objects and towards strategies. The third stage is mastery of multiplication and using one's memory to produce the correct fact.

The best practices for computational fluency relate directly to these stages. Manipulatives, pictures, and drawings are recommended to start off with because these methods coincide with the first stage of learning development, the counting phase. Informal and formal experiences with word problems are also necessary in the first stage of learning, especially when shown with objects and pictures. The use of specific multiplication strategies relates directly to the second phase because they encourage students to use known facts to discover new facts. Think alouds that showcase metacognitive strategies, or explanations of how to think when solving a problem, mixed with class discussions of strategy provide a good base during the second stage and in the third stage, where students begin to move towards working on mental mathematics. Surprisingly enough, practice does have its place in the classroom, when used in games, peer tutoring, and taped problems. Practice is stressed only in the third stage of learning multiplication because it takes away from the importance of the other strategies. Practice is best used in moderation, since in traditional education, it has been overused through flash cards and drills of multiplication facts.

Over use of rote memorization could be one of the reasons students fail to develop computational fluency. Rote memorization is the process of memorizing content and reproducing it. Rather than focusing on understanding, it involves just spitting out answers, similar to the use of flash cards. Having no prior knowledge on a topic makes it hard to make connections to because there is nothing the student has learned previously to link it to. The educator's concern in rote memorization is about the product of the calculations, not the process (Brownell, 1944). This is so dangerous for students to get into the habit of because they memorize facts for tests and then the information is gradually forgotten since the facts have nothing meaningful to connect to. In school systems around the world, many problems have been identified; the way education is orientated toward memorization is one of the larger problems: "The 'efficiency' of this kind of teaching is low, since it is based primarily on mechanical memorization of huge amounts of information and algorithms to find solutions to standard problems" (Dubova, 2014). Memorization appears effective when taking a test and will show knowledge gained, but students who merely memorized the multiplication facts really do not have the understanding they need to progress with ease and confidence. Teachers need to aid their students in making meaningful connections that can lead them to computational fluency.

Although teachers have state standards they need their students to meet, it is important that they do not forget the true purpose of teaching, specifically in mathematics. The whole purpose of school is to help students become avid learners and functional citizens (Dubova, 2014). Computational fluency, because it promotes deeper understanding of multiplication, prepares students to be more capable adults as far as mathematics goes. Giving students a solid foundation in multiplication is important to helping them develop computational fluency. Students will learn nothing if teachers just continue to give them things to memorize without helping them understand why these things are important and what they actually mean: “[l]earning depends on the active engagement of the learner. It is what the learner does that is learned, not what the teacher does” (Anderson, 2009). Multiplication gives students a base for the rest of their mathematical education. Furthermore, if they lack computational fluency, they will forever struggle to figure basic mathematics problems out. Teachers need to orient their instruction towards students to help them make connections and have meaningful understandings. Teachers also need to be aware of the best practices that can aid students in gaining computational fluency. The Common Core State Standards encourage and require students to develop deeper understandings, so by teaching computational fluency, teachers will be meeting the Common Core State Standards. The most essential question we need to answer now is: What are the best practices for helping students gain computational fluency in multiplication, and how do we incorporate them into the classroom so that our students gain computational fluency and grasp multiplication facts with ease?

Objects, Pictures, and Beginning Experiences

All children should start off their mathematical journey with some form of concrete objects or a physical representation of the problem (Bruner, 1966). Beginning with learning to count and continuing with each new mathematics skill, using concrete objects is highly important to support the child as they learn. Multiplication, since it involves grouping, is effortlessly represented by concrete objects and easier to understand when a child can physically see the manipulation of the objects. Manipulatives are a form of concrete representation and include any type of object a child can physically manipulate. Pictures and drawings should come after objects, or manipulatives, since children can still manipulate them, but not as much with their hands as with their pencils and minds. Pictures can be manipulated with a pencil and eraser and children can easily scribble out, circle, or add to parts of drawings to help them see groupings better. Pictures are considered a form of semi-concrete representation because they are not quite concrete manipulatives, but they are still able to be manipulated. These are used after concrete materials because they fall into the natural pattern of learning.

Area and array models are two forms of pictorial representations that help children accurately depict multiplication problems. When using manipulatives and drawings, children can discover strategies and patterns that help them more quickly solve multiplication problems. Symbolic representations of problems come after using concrete and semi-concrete materials, so students need a chance to use these materials first, before mentally solving problems. Manipulatives, pictorial representations, and drawings are incredibly important experiences to a student's multiplicative development, especially when used in systematic order. Objects and pictures allow students to form a base of understanding as to what multiplication is which, in turn helps them begin to develop computational fluency.

Manipulatives

Manipulatives should be the first mathematical tool children learning multiplication, or any skill, come into contact with. Manipulatives may include blocks, counters, poker chips, coins, fingers, post it notes, or any physical object that a student could use to represent a quantity. Most people have seen children start learning to count by counting physical objects; multiplication education starts in the same exact way. According to Brownell: "Processes are of at least equal importance with products" (Brownell, 1944). The process in this specific case is how one is using manipulatives to solve a multiplication problem. The product is the resulting answer. Since new concepts are hard to internalize right away, the manipulatives give a concrete show of the process students use to reach an answer. Often, teachers will start by modeling how to use manipulatives and then students can work together to figure out how to use them to represent quantities. Even though the children learning multiplication are learning it in third and fourth grade, manipulatives should still be the starting point.

A study was conducted with older students who had learning disabilities and they were given prompts each time they did not respond to a multiplication fact quickly (Williams & Collins, 1994). The prompts included manipulatives such as, poker chips, number lines, and using their fingers. All of these can be physically manipulated and follow the natural pattern of learning. The students preformed better once given the prompts because they could physically see what they were trying to compute. Using manipulatives helped these students make concrete connections to their prior knowledge and improved their understanding of multiplication. Manipulatives need to be used by educators in a way that is succinct so that students can easily follow how to use them and imitate that use in their own problem solving.

When starting to teach multiplication, it should be introduced as repeated addition and as groups of equal size. This is something students can clearly show with manipulatives. For example, if students are trying to solve 2×3 then it should be introduced as $3 + 3$. Using groups naturally leads to repeated addition and it would be best to start with so as not to overwhelm students, but students still need to be shown that multiplication is related to addition. Students can make groups of three using their objects and since the problem is, 2×3 they would show two groups. Now the students could either add $3 + 3$ or count their objects and they would determine that their answer is six. Students learn strategies to solve mathematics problems in three phases. The first phase is the counting phase, in which students solve problems by using manipulatives. The second phase is reasoning strategies, which involve using approaches students have come up with based on known facts to more quickly discover answers. The third phase is mastery of multiplication, in which students are able to quickly solve facts mentally. Being able to physically see what is happening with multiplication is important: "In brief, phases 1 and 2 are essential for laying the conceptual groundwork—the discovery of patterns and relationships—and providing the reasoning strategies that underlie the attainment of computational fluency with the basic combinations in phase 3" (Baroody, 2006). Phase 1, in this case, is referring to counting actual objects. Counting will eventually become skip counting, or counting by adding a number repeatedly, and students will also see relationships in their groups that will become faster strategies to attaining the product. After formally working on problems with manipulatives for a while, students will begin to notice patterns that give them new more efficient strategies for solving multiplication problems, but first, they should try drawings as a means to advance their knowledge.

New pieces of technology can provide opportunities for practice with a different type of manipulative for students. For example, Wall, Beatty, and Rogers (2015) reviewed many different types of iPad applications, also referred to as apps, in search of one that would be beneficial for teaching multiplication. Eventually, they decided the most effective application of all the ones they looked at was a free application called Fruit Plate Math. In this application, pieces of fruit can be arranged in equal groups, which shows the definition of multiplication. The game also has many fun features that display important properties of multiplication: "With Fruit Plate Math, the *Array* view of the *Basic Multiplication* tab allows students to use a slider to change the number of rows and columns in the array. Moving the slider simultaneously modifies the array, enabling an immediate visualization that is impossible with concrete manipulatives"

(Wall, Beatty, & Rogers, 2015). This application allows manipulation of visuals that highlights different ways to define multiplication more clearly. The application also shows skip counting as well. Moreover, feedback is provided as the student interacts with the content, which is also important because it helps students learn to correct their errors as they experiment with multiplication. Although this is only one application, there are many out there that are also free and can be used in a technology friendly classroom. In another study, Zhang (2015) found that multiplication applications, such as the ones available on iPads, can help close the mathematics achievement gap between students. Interactive whiteboards can also provide great feedback for students beginning to experiment with multiplication and its definition.

The development of computational fluency is supported by the use of manipulatives because concrete representations help build basic understanding of the definition of multiplication. Manipulatives allow students to connect to prior knowledge like addition and because they are connecting to prior knowledge, this new information is much more meaningful to students. Thus, working with manipulatives is essential to the development of computational fluency with multiplication.

Pictures and Drawings

Pictures and drawings work well with using manipulatives because the movement from concrete objects to semi-concrete objects is quite natural to students. These two types of representations are linked together by their concreteness, which provides visual aide without going too far into the abstract and forcing students to attempt mental mathematics before they are ready. Although drawings are not as concrete as manipulatives, it is still easier to represent a problem pictorially than to solve it mentally. Pictures and drawings are particularly helpful representations of multiplication because they can help address misconceptions children might have. In a study comparing Japanese mathematics textbooks and American mathematics textbooks, it was discovered that the Japanese textbooks addressed more misconceptions with pictures (Watanabe, 2003). Japanese textbooks put pictures of both examples and non-examples of multiplication, so when students are learning what it means to split objects into groups of equal size, they also experience instances where it cannot be done. This is important because misconceptions can arise from incorrect generalizations based on the limited number of examples that students do see. Thus, pictures of non-examples can prevent incorrect generalizations. For instance, many students misconceive that multiplication always creates a larger product than its factors and this can be addressed with pictures as well. When multiplying by fractions or decimals, multiplication can produce smaller amounts. By seeing examples of drawings of all kinds of multiplication representations, and quantities that cannot be represented by the multiplication of two whole numbers, students' conceptions have more depth. Another effective practice Japan uses with drawings is writing matching number sentences. Since students can see the drawings and use them to explain the problem, it is easier for them to create a number sentence to match their picture (Watanabe, 2003). With practice, drawings will gradually become more organized and concise. Moreover, students will become quicker at multiplying the more they draw. Thus, drawings continue to foster the understanding that manipulatives began and this strengthens the early development of computational fluency even more. Certain types of drawings and pictures are more effective at developing computational fluency in multiplication than others.

Area and Array Models

Within the assortment of drawings that can be used to represent multiplication, there are a couple of methods that are particularly concise and display some of the most important properties of multiplication, thus further developing strong computational fluency skills. These methods are referred to as area and array models. Figure one shows an example of an array model, which is concerned with an arrangement of objects, such as desks, in rows and columns and it is related to multiplication in that the number of rows and columns are multiplied together to get the total number of objects. Figure 1 shows the example $3 \times 3 = 9$. Figure 2 shows an example of an area model of the problem 5×7. An area model is concerned with the actual space inside of a rectangular shaped object, which is found by multiplying the length of the sides times the width of the sides. According to Barmby, Harries, Higgins, and Suggate (2009), array models are very beneficial in helping students show multiplication problems in a more concrete way: "The array representation encourages pupils to develop their thinking about multiplication as a binary operation with rows and columns representing the two inputs. Both the commutative and the distributive properties of multiplication are more evident in this representation." (p. 224). Area and array models are very simple for students to use because they are easy to create and use as a visual aid. Both models can easily provide visual examples of the distributive property of multiplication over addition and the commutative property of multiplication. The distributive property of multiplication over addition says that, one can break up one of the factors in a problem and multiply the other factor by each of the parts and still get the same answer. The distributive property of multiplication over addition would look something like this: 10×5 is the same as $(5\times5) + (5\times 5)$ because 10 can be broken up into $(5+5)$. Figure 3 illustrates the distributive property of multiplication over addition using the problem 2×7. In figure 3, the 7 is broken into two parts: 3 and 4 which together, add up to seven. The commutative property of multiplication means that the two numbers being multiplied together do not have to be in a certain order and can be flipped to get the same product. For example, 5×2 creates the same product as 2×5. Figure 4 shows an example of 5×2 and 2×5 and how by the commutative property, these two facts have the same product. These facts can be shown to students through an area or array problem because they can actually divide up the area or array to show these properties.

Unfortunately because many teachers focus on memorization rather than understanding, students do not tend to explore the properties of multiplication fully: “Only a minority of upper primary children could use commutative properties, and ‘few’ 8 to 9 year olds could draw on distributive properties, in order to solve problems involving multiplicative situations” (Barmby, Harries, Higgins, & Suggate, 2009). The commutative property of multiplication although it seems obvious, is not actually obvious to a child: “For example, when the numbers are swapped in the diagrams, the representations will look quite different. It is not immediately obvious why the commutative law should apply” (Barmby, Harries, Higgins, & Suggate, 2009). The more practice seeing array and area models, the more students will start to see why these properties make sense. For example if starting with the problem 3×7, we can show it as two arrangements, three rows and seven columns or seven rows and three columns. Both represent the same product, but are visually different and provide a different look at the problem (See Figure 4). Barmby, Harries, Higgins, and Suggate (2009) stated that: “We observed that the array provided a representation from which children could simply ‘count’ the result of the calculation, as well as a range of other calculation strategies” (p. 235). Further benefits of the array representation are that mental math is not required when using it and it provides a good base for students to see multiplication as repeated groups. These models are useful for many students at varying levels, not just the advanced or low, when solving multiplication problems. According to researchers, the best way to practice unknown facts is to use arrays for visual emphasis: “As the teacher points to different arrays, the student can be asked to find the product. For selected pairs of numbers, ask a student to explain his or her thinking. Once a fact becomes solidly understood, the array may be removed from the set" (Flowers & Rubenstein, 2010). Additionally, area and array models provide students with a solid method for making sense of the properties of multiplication.

Word Problems, the Informal and Formal Hero

Word problems are dreaded by people of all ages, but are also highly effective as a teaching tool. Word problems are feared because most people learn to memorize multiplication facts, so applying these facts to a word problem is complicated because they do not have a true understanding of multiplication and how it works in a real situation. Word problems are best paired with manipulatives, drawings, and pictures. With word problems, students can apply their method of choice for representing the problem when they go to solve it. Word problems can give a context to multiplication and help clarify the concept of multiplication. According to research: "Children who start with problem situations, directly model situations to these problems…[T]heir development of computational fluency and their acquisition of problem-solving skills are intertwined as both develop with understanding" (Fuson, 2003). As students use more advanced strategies to solve word problems, their understanding of multiplication grows. The major reason people tend to struggle with word problems has to do with their level of understanding of the definition of multiplication. Someone with true computational fluency in multiplication can evaluate a word problem and pick a matching multiplication fact by using reasoning. Computational fluency can be strengthened through the use of word problems because as students practice with them their understandings deepen.

Word problems can be used in many ways in the classroom. In one study, children were read a book about pigs wearing scarves and hats. These children had not been introduced to multiplication previously and were only in second grade. This is considered an informal multiplication experience because these children had no formal experience with multiplication and were not informed that they were using it. The children completed a task without ever knowing they were multiplying. They were given paper pigs, hats, and scarves of three different colors pink, blue, and yellow and were told to use the objects to show how many outfits Jillian (the girl from the book) could make for her pigs (Betts & Crampton, 2011). This is actually a simple multiplication problem: A student with multiplication experience would realize that this problem is simply "3 ×3" because there are three colors for the hats and three for the scarves. However, for the students in this study this was a more challenging task. Some students began organizing their pigs into pigs with yellow hats, pink hats, and blue hats so that their pictures were systematic and the answer easy to find. Students who did this easily realized if they missed a combination because of their organization. What really helped these students was being able to construct their own understanding of multiplication based on this informal experience. In this case, multiplication would be referred to as combinations of hats and scarves since the students do not yet know the word multiplication. Giving a context to multiplication is necessary for understanding, even if students do not yet know the word multiplication. Understanding of the actual operation helps students work fluently with it: "Understanding is also crucial. We know that the greater the degree of understanding the less practice that is required to obtain fluency and to sustain the change in strategy use" (Bobis, 2007).

Sharing is often involved in multiplication word problems, especially in equal group problems. In these types of problems, students are trying to make equal sized groups from a quantity and often this involves sharing amongst numbers of people. According to one study, children as young as four could understand multiplication word problems without knowing multiplication and without being given any sort of cues (De Brauwer & Fias, 2009). These children had an understanding of sharing and how to distribute objects to groups equally, but did not know the word multiplication. A word problem provides an example that even the very young can understand, and that is why it is important to introduce word problems involving multiplication in an informal way before introducing multiplication formally as an operation at all. Some research suggests that students can work on discovering their own ideas about multiplication through word problems (Cifarelli & Wheatley, 1979). Word problems provide an easy gateway into multiplication from addition because they do not have to be introduced as strictly multiplication facts to be memorized. Connecting to addition makes the information more meaningful and easier to comprehend, which makes computational fluency much more possible for students.

Word problems can also be used in a more formal multiplication setting. As students progress from using concrete objects to pictures to abstract representations of numbers, word problems can continuously be used. Teachers may need to help students progress through those stages: “The challenge for teachers is to encourage the development of, and consistent use of, more efficient and appropriate strategies for solving mathematical problems without it being ‘too hard’ for children” (Bobis, 2007). A student can begin solving word problems using manipulatives and become comfortable using the manipulatives in these situations. Gradually, they should be encouraged to move to drawings and eventually more abstract methods. The word problem given does not have to change, though; what does change is the student’s pattern of thinking and the way they understand the multiplication. Teachers need to model new strategies for students to try with word problems to help them progress through the stages of representation.

One of the best ways to make word problems accessible to all students is to teach mnemonics for problem solving steps that can be transferred to multiple scenarios. Word problems in a more formal multiplication situation can be tricky so strategies need to be clear to students so they can use them efficiently: “Each strategy step [mnemonic step] should prompt the student to perform an overt action, such as write the answer in the answer space; use a cognitive or metacognitive technique, such as paraphrase the problem question; or apply a rule, such as use the rounding rule” (Miller & Stringfellow, 2011). If the mnemonic is clear in what it requires, then students better understand how to apply it to the problem to get the correct product. The mnemonic should make the problem goal clear and easy to understand. According to Miller and Stringfellow (2011), there is a popular mnemonic that works well with all operations, including multiplication, known as the DRAW strategy. DRAW stands for Discover the sign, Read the problem, Answer/draw or check, and Write the answer. In the case of multiplication, students would start by discovering the sign (multiplication), then they would read the problem and choose an appropriate strategy for solving it. At this point, students could use concrete or semi concrete materials before answering the problem and writing the answer. If a teacher wants students to use a mnemonic, then they need to explicitly teach it to students. The strategy needs to be modeled and experimented with so that students can understand why it works and learn how to use it correctly. Research stresses the importance of being able to use a strategy well: “Students must learn the memory tool with automaticity in order to apply it successfully to the mathematics problems they are trying to solve” (Miller & Stringfellow, 2011).

Another trick students can use with word problems to help them be more effective is to look for key words within the problem. As students read multiplication word problems they will notice patterns. Some of the word problems will be looking for the total number of objects, others the number of groups, and others the number of objects in each group. The key is to look at the wording of the problem. If students become accustomed to wording then they can more easily understand what a word problem is looking for and then solve it in a way that makes sense to them. Table 1 provides examples of the four types of multiplication word problems and key words to look for in each.

Word problems are important to a student's multiplicative education and more importantly, to their development of computational fluency. When used from the start, they are actually the opposite of confusing. If word problems are not used in the beginning of learning multiplication, they become an add on lesson at the end of a unit and do not help further understandings, but cause confusion in the aftermath of learning multiplication. Word problems can be solved with different strategies, so they never become too easy or too hard to solve. Computational fluency is supported strongly through this because students pick more efficient strategies to solve word problems as their understanding grows, which is exactly what computational fluency involves. Word problems have a bad reputation solely because most people focus more on memorization of facts and do not understand how multiplication could be transferred to real life. Instead, word problems give context to a concept that can seem very abstract and for that reason word problems are the unsung hero of multiplication.

Multiplication Strategies

Strategies are methods students use to solve multiplication problems. Generally these strategies are found by the students themselves and shared through discussion. Strategies include skip counting, multiplying by ten, doubling, near tens, and near doubles. For example, when multiplying by two, students may notice that you are doubling what you have. These strategies correspond with the second phase of learning, reasoning strategies. Strategies can be figured out with concrete objects and transferred to more abstract representations. Success using strategies, such as these, motivates students to continue to practice and eventually master multiplication facts.

Skip counting is one of many strategic ways of representing multiplication. Since multiplication can be defined as repeated addition, skip counting makes sense developmentally because it is essentially a process of adding the same number repeatedly. Skip counting can occur with any set of numbers. Children generally learn that it is faster to count by 2s or 3s rather than 1s and it becomes a quick short cut to finding products because skip counting is a way to express the results of repeated addition. With skip counting, fingers can be used so that students are not deviating too far from concrete representation to more abstract representations.

Doubles are a very important strategy for students because it allows them to solve more facts efficiently once they understand them: “Using several even factors, such as 2s, 4s, and 8s, can be natural starting points for doubles. Once the 3s are known, the 6s and the 12s can be placed in the mix.” (Flowers & Rubenstein, 2010). For example, if I had the problem 6×7 I could use the fact 3×7 and say well if the product of that is 21, and 6 is two times 3, then 6×7 must be 42 because two times 21 is 42. Doubling and using strategies involving doubles, helps students better understand multiplication and compute problems at a faster rate, which is the essence of computational fluency.

Doubles strategies are a gateway to other methods of solving problems: “Knowing that a number in the counting sequences is 1 more than the previous number (e.g., ‘‘four’’ is one more than ‘‘three’’) can enable a child to use a known ‘‘doubles’’ combination to logically deduce the sum of a ‘‘near double’’’ (Baroody, 2009). If a student is able to double multiplication problems (as discussed above), then they will be able to use these doubles as reference points of solving other facts. For example, if a student knows that the product of 2×3 is 6, then by the doubles strategy they know that the product of 4×3 is 12 because 4 is double 2 (to figure out the product a student would realize 12 is double 6). This then becomes a new fact that a student can easily reference mentally. Next, they can say if I know 4×3 is 12, then 5×3 is 15 because 5 is one more set of three than four sets of three. According to Baroody (1985): "As with any worthwhile knowledge, meaningful memorization of basic combinations can reduce the amount of time and practice needed to achieve mastery, maintain efficiency and facilitate application of extant knowledge to unknown or unpracticed combinations" (as cited in Baroody, 2009). Knowing what are called derived facts, or facts that a student can figure out using a strategy, helps students become fluent in multiplication. The more they use derived facts, the more familiar the strategies become and eventually the answers to the multiplication facts become second nature. Students achieve mastery by using such strategies frequently.

Multiplying by ten is also an important strategy. Students often notice when multiplying by ten that you can simply add a zero to the end of the other factor to get your product. Some teachers refer to this as adding magic zeros (Landry, 2015). Using the distributive property of multiplication over addition, students can break factors up that contain a ten (like for example, 13 contains a 10) and multiply by that first and add a magic zero to that part of the product. This provides an easy starting point for many other multi-digit strategies. These are only some of the strategies for multiplying, but discussion provides a way to highlight different ways of thinking about multiplication in class.

Bringing it Together with Discussion

Similar to word problems, discussion is a neglected part of most students' mathematics education. Discussion is the process of talking about something, typically in order to reach a decision or to exchange ideas. Many teachers do not realize the benefits of talking to students about how to solve problems. Discussion can be paired with any of the stages of representation. No matter where a child is in their understanding of multiplication, they can benefit from discussion. By strengthening their understanding and strategy use, discussion also strengthens computational fluency. According to Brownell (1944), there is always more than one way to find an answer to a mathematical problem. Discussion provides an outlet for the different ways of thinking to be heard, and moreover, seeing different ways of approaching problems can be helpful to students who are struggling. In a study of mathematics textbooks from the United States and Japan, it was found that the United States, as well as Japan, encourages the use of a lot of strategies for solving multiplication rather than focusing in on one method (Watanabe, 2003). Both countries want students to explore multiplication and different ways to solve problems in depth. Discussion can bring these strategies into the open for a lot of students. Discussion can also occur in many different settings: “Dialogues are more engaging and enable students to develop understanding, as well as to check their understanding. These conversations may occur between teachers and students, individually or collectively, or among students themselves” (Anderson, 2009). Discussion is especially helpful for English Language Learners (ELLs), who might not have a good handle on English yet and struggle to express their thoughts in words (Bresser, 2003). Discussion of any type can be important to learning and broadening strategies.

Often teachers have students engage in number talks, a form of discussion, in which students use the relationships between numbers to solve different mathematical problems. Number talks can be particularly useful with multiplication because it brings to light a lot of the properties and patterns, which will be elaborated on in the sections to follow. Discussion is an important tool for helping all students develop computational fluency because it provides deeper understandings. Discussion is helpful for a number of reasons, such as, helping ELLs develop language skills, seeing a variety of strategies, and recognizing patterns and properties of operations.

Number Talks: A Method for Discussing Strategies

A number talk is when students discuss with their peers and the teacher the relationships between numbers and operations to solve mathematical problems. Number talks can be extremely beneficial and do not have to be incredibly formal. Wickett (2003), a third and fourth grade teacher, did an experiment using an informal number talk and multiplication in her classroom. She had the students start by multiplying five by six and gradually moved up to 251 times 12. The students were able to do the calculations because she planned the steps leading up to it, so that each problem built on the last and helped the students develop deeper understandings. At the end, she had students explain what they did to get their answers to provide examples for those that struggled. She also showcased students who used different strategies to illustrate that when students understand the process; there are many paths they can take to get to the product. These students had what is called, number sense and computational fluency. Number sense refers to a student’s ability to work fluidly and correctly with numbers. A child who has strong number sense can perform mental math without a problem. For example, a student with number sense knows that the number five can be split up into two and three or four and one depending on how they intend to use it. Students in their class used a range of strategies as they attempted to multiply 251 times 12 in their heads. Multiplying by ten and adding a zero was a popular strategy along with doubling answers from previous problems. These students understood a lot of the basic concepts of multiplication and number order, so they were able to correctly do the mental computations. The teacher, in this case also planned which problems to use so that students would gravitate towards more efficient strategies.

Number Talks and ELLs

Number talks can really benefit ELLs. The opposite might be thought to be true because ELLs do not have the English skills of their peers, but in reality hearing other students express their ideas helps ELLs. It is also helpful because ELLs have a chance to try to express their ideas and their peers and the teacher can offer support when necessary. Number talks are actually beneficial to them in that they lead to practice in two of the main areas of English: speaking and listening. According to Bresser (2003), communication is extremely important for these students, especially with words that have multiple meanings. The teacher's job is to facilitate the discussion so that confusing words like left, column, row, are clear in the context they are being used. For instance, left could mean direction so the teacher has to take special care when students are explaining their strategies orally to ensure everyone understands what is being said.

For ELLs, discussion is extremely helpful as long as it includes retellings of strategies in different ways so that ELLs can hear another way of explaining the strategy. Vocabulary is always helpful for ELLs so if discussion is being combined with a word problem, a vocabulary sheet for some students might be advantageous. One strategy within discussion that assists ELLs greatly is the think-pair-share strategy (Bresser, 2003). This means you think about the problem individually and then pair up and discuss it with a partner first. After discussing it with a partner and clarifying ideas, then it is shared with the rest of the class. This gives ELLs a chance to clarify problems and meanings before they have to go in front of the entire classroom. Discussion benefits ELLs, especially when developing computational fluency, because it provides practice with communicating ideas.

Overall, discussion is important because it makes students active in their own learning and gives the opportunity for sharing of important strategies. Ripp (2014) stated that: “Throughout the first week of school, I encourage students to speak up and add their ideas to the class discussion. By promoting this early, students get used to being part of the discussion…”. By encouraging discussion from early on, it becomes second nature and thinking deeply about multiplication, or any topic, becomes easier for students. Discussion brings a lot of strategies to light and gives opportunities for participation to all types of learners, including students with learning disabilities. For these reasons, discussion makes computational fluency much more possible for a wide range of students.

Think Alouds

Discussions and word problems are often successful at helping students develop computational fluency when paired with another learning tool, think alouds. A think aloud can be either when a teacher shows students how to approach and solve a problem or when students demonstrate to other students how to solve a problem. Think alouds are not step by step explanations of how to solve a problem, but a verbalization of the conceptual processes and thought processes that one should go through when approaching a problem. When doing a think aloud, a lot more explanation of why steps are performed is involved then just instruction to proceed through them. It is usually used in conjunction with modeling, but true think alouds involve talking out the thinking processes one should go through as they work through the problem, not just pure modeling of the steps needed to solve a problem. According to Vernille (2002): “Students learn mathematics best when there is more time for lengthy verbal explanations of solution strategies for problems (as cited by Anderson, 2009). In other words, when teachers take the time to explain multiplication strategies to students, students develop deeper understandings. These strategies can be generated by the teacher or by the students themselves, but what is important is that the metacognitive steps are explained. Metacognition is the process by which one examines their own thought process. When using think alouds, the person thinking aloud needs to explain how they thought through the problem as they attempted to solve it.

When starting off new types of problems, for example word problems as suggested by Fuson (2003), teachers can start by modeling the process and explaining the steps as they go through a think aloud. Students can generate strategies as well, as long as they share the mental steps, in order for others to understand how they reached their product. Think alouds provide an introduction to basic strategies that can lead to other more advanced strategies. Students can use the examples given during think alouds to strengthen their own understanding of multiplication and begin to use more advanced strategies to find products (Bobis, 2007).

Think alouds can also be used in many different multiplication situations beginning even with informal experiences. In the study conducted by Betts and Crampton (2011), they exposed children to word problems involving multiplication without exposing them to the actual term multiplication. Although the instructors in this situation did not directly model and tell the students how to solve the word problem, there are hints of a think aloud through the sharing of student strategies. Asking questions can help students organize their thinking better. Teachers need to plan the problems and questions they ask about them so that students will model strategies on their own. Questioning can be used with more formal teacher think alouds, too. It involves the students more in the process of solving. Think alouds should be done quite frequently when introducing new strategies to students.

Think alouds led by the teacher should be incorporated in beginning multiplication instruction. Step by step instruction that includes metacognitive steps needs to occur often. According to Cook and Dossey (1982), about twenty minutes a day was found to be effective in their study. Students retained more information with more frequent elaboration of thinking processes. Thornton (1978) directly connects think alouds to the word encouraging. By this, she means that teaching metacognitive steps encourages students to try new strategies. The more modeling of how to think while solving multiplication problems that occurs, the more motivated students become to try new these strategies since they now have more background information and insight on how to think.

Think alouds do not come from textbooks, but from the teacher or another student, who can provide verbal approximations of how one thinks when evaluating a multiplication problem. In a study comparing the textbooks of Japan and the United States, it was found that the texts in the United States contain fewer lessons on multiplication (Watanabe, 2003). In the United States, due to the shortage of lessons within their textbooks, students often face connection issues from second to third grade. In order to make this transition smoother, so that an understanding of multiplication properly develops, think alouds need to be done. It is the teacher's responsibility to provide instruction that connects across grade levels and explicitly shows students these links. Think alouds help students to mimic the thinking process of others, which leads to the development of their own ways of mathematical thinking.

Think alouds match up perfectly with the natural progression of learning children should experience. They can be used at any stage of representation, whether students are using manipulatives or working completely in the abstract. A teacher can just as easily show how to use manipualtives as they can verbally explain the thinking process behind solving a problem. A teacher explaining their thinking provides an example of how to later participate in a discussion about strategies. Students might not be sure how to explain their thinking about multiplication, but after seeing a teacher model how to explain, students should be able to, with practice, come up with a coherent explanation. Stated by Bay-Williams and Kling (2014): “The key is to help students see the possibilities and then let them *choose*…”(p. 240). Due to their usefulness across the stages, think alouds help students move forward in their thinking about multiplication.

Overall, think alouds are very useful for teaching multiplication. They should be used when teaching multiplication because they promote all the stages students go through and help students develop their own thinking processes. Think alouds by the teacher need to occur more frequently at the beginning of instruction, even daily, and then gradually can be used only when necessary. Think alouds by the students should occur more frequently as times goes on because students should become more capable of vocalizing the process by which they solve multiplication problems. They are constantly a tool that is successful in developing deeper understanding of multiplication and therefore should be used whenever necessary. It is up to teachers to help students fill the points of confusion in their mathematics education and think alouds bridge a lot of gaps by showing step by step how to approach different multiplication problems. By bridging gaps in learning and thinking, teachers build up students’ computational fluency. Think alouds are a necessary method for helping students develop true understanding of multiplication because they help fill in gaps and direct students towards using strategies correctly through showing how a person would think through a problem.

Where Practice Fits In

Practice can be extremely useful when it is used at the right time and as a supportive technique to other methods. According to many experts, practice has a time and place: “Overall though, the empirical evidence does not clearly support the proposition that massive practice is the key to combination mastery" (Baroody, Bajwa, & Eiland, 2009). Computational fluency does not require a focus on practice in the sense of memorizing flashcards because learning a small number of computation strategies takes the place of memorizing a large number of multiplication facts. Practice can help with familiarizing students with strategies and increasing the speed (fluency) at which they use them. It can also be useful after students have gained some computational fluency because after students have learned strategies, practicing of those strategies can help lead to quicker recollection of multiplication facts.

Practicing multiplication facts does not have to involve using flash cards and repeating facts over and over. This type of practice is not beneficial for improving students' understanding of multiplication. The best ways to practice multiplication facts include games, peer tutoring, and taped problems. Practice, or as it is sometimes called repetition, has a time and place in multiplication education, and it is important to include after students have a solid understanding of the operation but need more experience with their facts.

Games

Games allow for practice in a manner that does not seem tedious. Students take the games as a fun challenge and enjoy trying to get as many multiplication facts right as possible. Games provide a way for students to work on the aspect of speed within their computational fluency while having fun as well. One example of such a game is called Rio, designed by Kamii and Anderson (2003). In Rio, three students play together with ten cardboard tiles, fifteen multicolored chips, and a ten sided die. The cardboard tiles vary depending on the multiplication facts being used: "For the table of 4s, for example, we wrote the ten products (4, 8, 12, 16, 20, 24, 28, 32, 36, and 40) on the tiles. These tiles are scattered in the middle of the table, and each player takes five chips of the same color" (Kamii & Anderson, 2003). A student rolls the die and if they get a 7 then in the case of using products of 4 they would put their chip on the 28. If students get repeats of products and chips are already on that tile, they have to take the chip. The goal of the game is to get rid of all of your chips. This game is good for small groups and can be exciting for students as they try to identify multiplication products and get rid of their chips (Kamii & Anderson, 2003).

Salute, another game, is similar to the popular game "headbands", but with numbers. This game uses a deck of cards, which can be split up so students are focusing on fewer facts at a time. Three students play this game, the dealer and the two players. The dealer gives cards to the players who then say "salute" and put the card on their head so that the other player can see it, but they cannot. The dealer tells both players the products of their two numbers and the two players have to compete to name the factor of the product that is on their head. Whoever says the correct factor first, gets the two cards. The winner is determined by how many cards the student has at the end (Kamii &Anderson, 2003). This game is especially beneficial because the amount of facts can be adjusted by adding cards or taking cards away from the deck. Being able to control the facts students practice is helpful with games because teachers can pinpoint which facts students need the most practice with and make sure those are the facts students are practicing during these games.

Four in-a-row is another multiplication game that provides ample practice and entertainment. It involves a table/board similar to a bingo card with numbers on the bottom outside the square, as shown in figure five. Paper clips are placed on the numbers outside the table by one student and the other student whose turn it is finds the product of those numbers and places a chip. The students switch roles now and the paper clips are moved. The goal is to get a horizontal or vertical row of chips like in bingo. This game, like Salute, can be modified to include different sets of facts (Kamii & Anderson, 2003). Winning touch is another game that students can play with tables (see figure 6 for tables/boards). This one involves receiving tiles, which are the products, and putting the tiles so that the numbers at the top of the row and column are the correct factors that make that product. All of the games mentioned above were very effective in enhancing understandings and it was also found that teachers joining in could further motivate the students: "When the teacher played every day with small groups of children, they received a stronger message: that games are important enough for the teacher to play" (Kamii & Anderson, 2003). Without the teacher showing interest in the games, students will not see the games as beneficial.

Most of the above games do not require a lot of speed, but other games depend on speed to be won. Speed games can be very helpful or very dangerous when used in the classroom. Speed games can create the same environment that a timed test would create, which often results in anxiety and fear towards multiplication. A lot of research has been done on the psychological effects of timed tests: "Evidence strongly suggests that timed tests cause the early onset of math anxiety for students across the achievement range" (Boaler, 2014). To avoid this anxiety and creating a classroom full of fear, teachers need to be careful where and when they use speed games. If the class is strong with their multiplication facts, it can be fun to incorporate them occasionally as a treat or fun friendly competition. If the class is mixed abilities, as most classrooms generally are, it is important to keep in mind the feelings and anxieties that might arise if students are put in these situations. Being sure to explain to students that it is all right to be wrong and that this game is simply for practice is important: "In too many math classrooms, students believe that their role is to perform—to show they know math and can answer questions correctly—rather than to learn" (Boaler, 2014). Speed games, if used, need to be used in a way that promotes practice not anxiety or negative competition.

"Around the world" is an example of one of the games that require speed. This game is generally played as a whole class. Students compete with one another to say the product of a given fact first. The winner moves on to compete with the next student. Students can be lined up for this or in desks, as long as the way students move around the room is stated by the teacher before starting the game. The winner needs to move around to the end of the line or around the room completely getting facts right. "Around the world" is essentially a one on one challenge, where the winner moves on to the next round and goes against each other student. When playing this game students need to be reminded that it is merely practice with facts. If a classroom is not strong in multiplication facts and has a lot of nervous learners, this game is best to be avoided.

Another speed game is multiplication war, which involves a deck of cards. The students split the deck and turn over their top cards and try to say the product first. Whoever says it first wins the cards. The goal is to gain as many cards as possible. Unlike around the world, multiplication war only requires two players. This game is better than Around the World because students can be paired based on abilities so that they are equally matched as they play. This way one student does not constantly win all the time and students feel comfortable. The last speed game is Arithmetiles, which is similar to checkers. Players move chips around the board and attempt to collect chips for correctly stating products (Kamii & Anderson, 2003). Again, this game is best when used with ability pairings. As long as these games are used in a positive manner to demonstrate practice, they will be helpful to students. Teachers need to evaluate their classrooms before using speed games though.

All of these games can be beneficial when used in at least the middle stage (reasoning strategies stage) of multiplication, where students have background knowledge of some simple facts and are working on strategies. Speed games work best later on in learning, when students are trying to master facts, because students have the understanding needed to solve the problems and are trying to increase their speed at recalling quick strategies (Miller & Stringfellow, 2011). The other games promote practice of strategies, but do not necessarily demand speed to win. Overall, games provide an entertaining way of practicing multiplication. It gives a new setting to a familiar operation and allows students to converse with others as they try to gain a better understanding of multiplication. Only a few games are listed above, but other games can be just as beneficial. The games above can also be modified for differing levels, which makes them appropriate for any classroom with diverse learners. The possibilities for multiplication games are endless. Games, although they require recall of facts, are the perfect way to incorporate practice without impairing students' deeper understandings.

Peer Tutoring

Another form of practice that can be beneficial is peer tutoring. Peer tutoring involves students working together to solve problems. If a student is quizzing another student on a problem and the student being quizzed gets stuck, the other student can provide feedback to help the student remember the fact for next time. The students that are serving as tutors are trained by the teacher so that they can be more effective tutors. Peer tutoring can provide instruction to a wide range of students because students that have lower multiplication abilities can receive tutoring and high ability students can tutor so that they get a more solid base in their our understanding (Maheady & Gard, 2010). Peer tutoring can be used as much or as little as a teacher decides necessary.

Student tutors during training are told to give positive feedback to their tutees. This helps the tutees, who might lack confidence in their abilities to multiply quickly, become more confident. It also creates a good relationship between the two students so that they feel comfortable discussing strategies with one another. Flashcards can be used effectively when paired with peer tutoring. In a summary of studies, many of the students being tutored or doing the tutoring felt excited about the peer tutoring process and referred to it as a “tutoring game” (Maheady & Gard, 2010). Peer tutoring develops positive relationships between students, which provides an environment for comfortable communication about mathematics.

Communication of strategies is important for students to broaden their understanding. According to Bruner (1960): “[i]n seeking to transmit our understanding of such structure to another person, there is the problem of finding the language and ideas that the other person would be able to use…” (p. 333). In other words, how we explain something to someone else matters. If the explanation given does not make sense to the student, in this case the tutee, then it will not help their understanding of multiplication. Students tend to share similar language to one another, though, because they have similar thinking patterns. Teachers tend to use more complex language, so peer tutoring is beneficial in that the language used is more student friendly because it is being used by other students. The structure of the peer tutoring can be altered to suit different classrooms in order to promote effective communication.

Peer tutoring does not have to be class wide. According to a summary of sources, there can be variations of peer tutoring including: class wide, tutoring teams, and peer assisted learning strategies (Hawkins, Musti-Rao, Hughes, Berry, & Mcguire, 2009). Essentially, the rules are consistent across the forms. Students must give positive reinforcement and require training to tutor. Tutoring teams involve tag teams of tutors who work together to tutor. The other two forms of tutoring styles include, basic class wide tutoring with rotations among tutors and tutees. The main deciding factor of how to use peer tutoring in the classroom is the needs of the students which, are observed by the teacher, who ultimately decides how to best meet the needs in the classroom. The teacher decides on how long pairs or groups work together and how to decide who works with who. Teachers need to spend time on these decisions in order to give students the best support. If students do not work well with a peer they are less likely to gain understanding of multiplication from the session.

According to Cubukcu (2012), peer tutoring is important for students to strengthen their understanding of multiplication. Student-student interaction is extremely beneficial for deeper understandings. Computational fluency in multiplication requires students to be able to quickly and efficiently evaluate and solve multiplication problems and this can be worked on through peer tutoring.

Taped Problems

Another way to use practice as an effective tool is through taped problems. Taped problems involve trying to beat a tape or audio recording of facts. This is like a game for a lot of students because they see it as a challenge. Taped problems are a type of time delayed procedure in which an audio recording says facts and then gives a certain amount of time before giving the answer. It discourages finger counting and less effective strategies because the tape that is being played does not give students enough time to engage in longer strategies. By not giving students ample time to use less effective strategies, taped problems encourage quicker strategies. In a study comparing taped problems to using flash cards and covering the cards and uncovering them, taped problems was proven to be more effective at helping students develop further understandings (Poncy, Skinner, & Mccallum, 2012). Although both methods were coupled with positive reinforcement when students got correct answers, taped problems were still more successful. Taped problems are useful for many reasons, including the immediate feedback.

Receiving immediate feedback means that the computer or tape gives the answer within a certain amount of time, allowing students to self correct if necessary. This is especially helpful with students who have disabilities because it frees up mental resources and allows them to focus more on the process they are using to come to an answer rather than the product (Gersten & Chard, 1999). Due to the feedback from the technology, the students can make corrections and not worry about what others might think of them if they make a mistake. Students can be taught to work the tape recorder or computer on their own and so this can be practiced during free time in the classroom as well. Knowledge of facts is required for these types of problems, thus it is best for practicing fact recall after the use of strategies become familiar. Once students have gained a handle on a few different strategies, taped problems make sense to begin practicing with because it strengthens and increases the pace with which students use strategies. Taped problems provide a judgment free way of practicing new strategies.

Taped problems provide a way for students to practice their strategies for solving multiplication facts and increase their speed. Practice is not something teachers should rely on as their single means of teaching, but it can help students increase their understanding and fluency. Used when students already have a basic understanding of multiplication, games, peer tutoring, taped problems, and technology can help students truly master multiplication facts.

Conclusion

Computational fluency in multiplication should be the goal of every third and fourth grade mathematics educator. It is hard though when teachers feel constricted by state testing as stated by Jones, Hoffman, Assaf, and Paris (1999): “In terms of what they teach, teachers reported that accountability has led them to emphasize specific information that will be tested and to neglect material involving higher-order thinking and problem solving (As referenced by Anderson, 2009). By emphasizing test information and not problem solving or higher-order thinking, teachers are preventing their students from having a deeper understanding of the material. Blaming testing for not teaching deeper understanding of multiplication to students is wrong though: “Accountability has become a scapegoat that allows teachers to continue to teach as they always have, rather than to teach in ways that elementary students need to be taught if they’re to learn well and be academically successful in the long term” (Anderson, 2009). Testing is no reason to focus on memorization. If teachers aim at computational fluency, students will perform even better on tests because they will not only know the multiplication facts, but also be able to apply in new situations.

When trying to establish an environment that promotes computational fluency, teachers are fortunate to now have the Common Core State Standards in place. If educators know the standards and plan instruction so that it meets those standards, they should be successful: “We have to know our end goals, or standards, so that we can think backwards and visualize the most exciting path to getting there” (Ripp, 2011). Knowing the audience (the students) and the end goal, teachers need to plan instruction so that the students are engaged and the standards are being meet. Obviously, not every single lesson can be equally exciting, games are often more exciting than a simple demonstration, but it is acceptable to have less exciting lessons mixed in (Ripp, 2011). The Common Core State Standards not only provide a guideline to teaching multiplication, but promote the deep understanding that computational fluency implies. The adoption of standards has the public in outrage, but the designers of Common Core are trying to point out an important point by promoting deeper understanding: “Virtually every subject taught in school is amenable to some use of quantitative or logical arguments that tie evidence to conclusions" (Steen, 2007). If students have deeper understanding and we teach so that connection to the real world is visible, they will value their own mathematics education much more.

The best teaching practices that promote computational fluency also follow the phases in which students learn mathematical skills. Phase one includes using counting strategies combined with concrete objects. Objects, pictures, and word problems fit into this phase and help students move towards computational fluency. Phase two involves learning reasoning strategies and moving away from the use of concrete objects. Strategies strengthen computational fluency, where discussion and think alouds prepare students for the next phase by using reasoning strategies to develop fluency. The third phase, mastery, requires a lot of practice because by the time students complete this phase they have mastered their multiplication facts and can recall them quickly (Baroody, 2006). Ideally in the mastery phase, students also obtain computational fluency because the goal is to get them to have a deeper understanding of multiplication that aids them in recalling the facts. When these practices are put in this order, they line up perfectly with the phases students go through. If used out of order, these practices will not be nearly as effective and students will have a low chance of obtaining computational fluency. When using the phases as a guide for implementing instructional practices, teachers give students a better chance of obtaining computational fluency and reaching their full mathematical potential.

All signs seem to point to computational fluency as the focus of multiplicative education, but how to get there can seem time consuming and tough. As a result of this literature review, the way to get there is clear. As demonstrated in the previous sections; teachers should use: manipulatives, drawings (including area and array models), word problems, discussions, think alouds, games, peer tutoring, taped problems, and even technology. All these things will positively affect students and help them achieve computational fluency. However, teachers are still worried about how to fit this all in. As previously stated a lot of these instructional practices benefit ELLs (Bresser, 2003) and students with disabilities and/or varying levels of ability (Kamii & Anderson, 2003). By using all these practices combined, the diverse needs of learners are being meet. Several of these methods can be altered, if necessary, to make them more beneficial, such as the practice games, which can be used with different sized groups. By using manipulatives, pictures, drawings, word problems, discussions, think alouds, games, peer tutoring, and taped problems, students will be able to learn more easily. An understanding of multiplication starts to develop from students' prior knowledge and using concrete representations and then gradually moves to higher-levels of representation.

Overall, using all of the instructional practices mentioned throughout this text to help students obtain computational fluency can seem tedious, but they work well with many of the other educational factors. Computational fluency is so important for confidence building and for helping students reach their full mathematical potential. If teachers design their multiplication lessons using these manipulatives, pictures, drawings, word problems, discussions, think alouds, games, peer tutoring, and taped problems techniques, in the order they are introduced, their students will be much more successful. Students with computational fluency in multiplication are more successful in future mathematics courses because of this skill. Computational fluency also helps students in the real world as consumers. Computational fluency sets the stage for a student's future and by helping students to obtain fluency, teachers will be giving their students an education that will set students up as more capable mathematicians and informed citizens.

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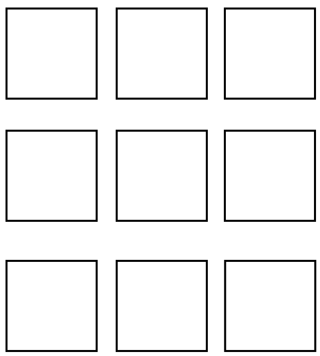
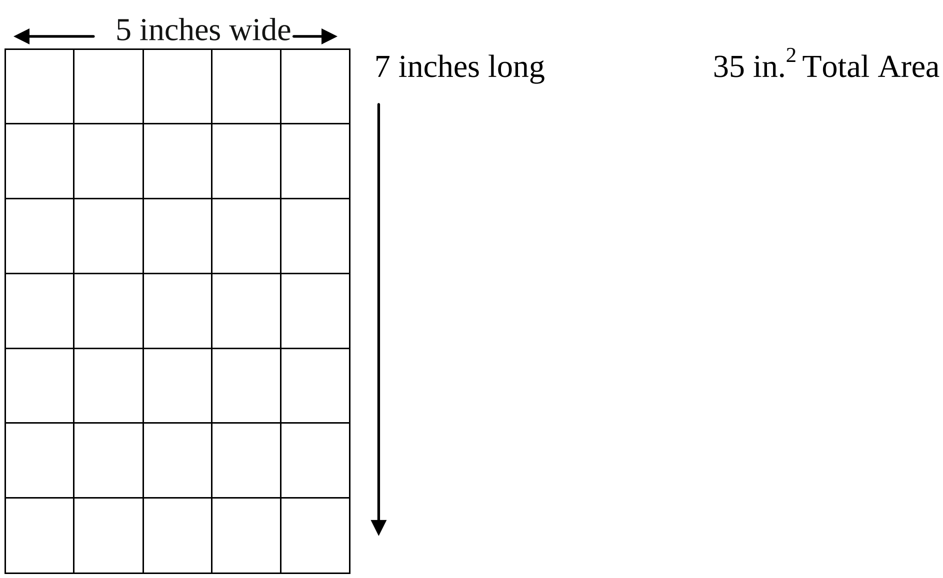
Appendices

<u>Figure 1:</u> This is an example of an array model. This shows an arrangement of desks in a classroom. The total number of desks could be found by multiplying the number of rows by the numbers of columns: $3 \times 3 = 9$



<u>Figure 2:</u> This is an example of an area model for the problem 5×7. The width of the rectangle is 5 inches and the length is seven inches. To find the total area the length and width are multiplied to get 35 inches squared.



<u>Figure 3:</u> This figure demonstrates the distributive property of multiplication over addition through an area model. The multiplication fact is 2×7 and to make it simpler the 7 is broken up into 3+4. Next the 2 would be multiplied by both the 3 and the 4 since they make up the 7.

$2 \times 7 = 2 \times (3+4)$ can be written as: $(2\times3) + (2\times4) = 6 + 8 = 14$

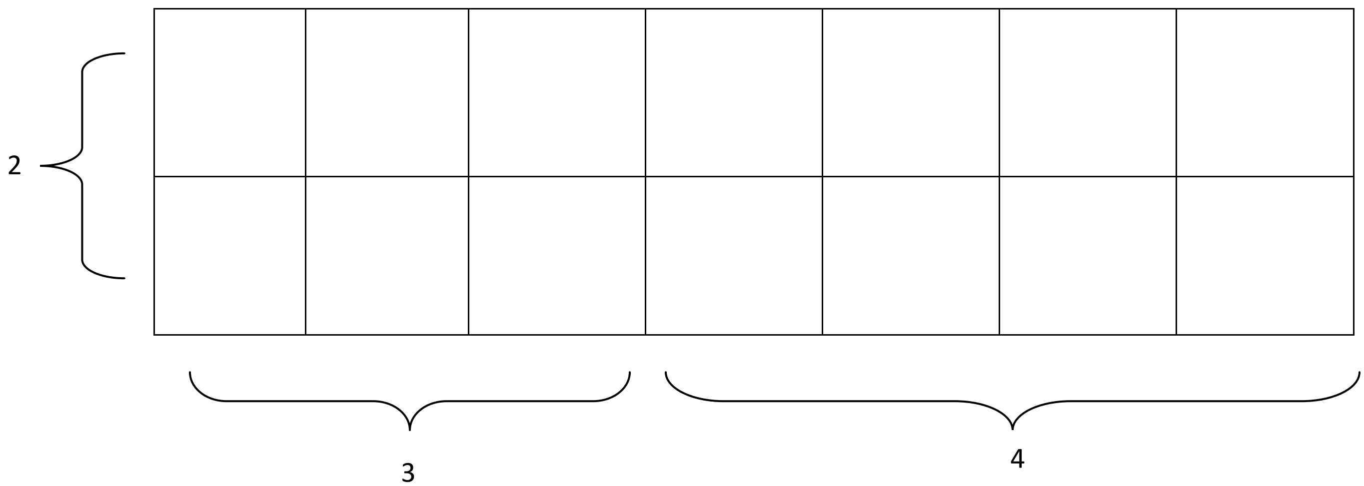


Figure 4: This is the commutative property of multiplication shown through an array model. The problem is 2 × 5 and if you look both arrangements look equal just flipped a different way.

2×5

2 rows × 5 columns = 10 bears

5 rows × 2 columns = 10 bears

<u>Figure 5:</u> Examples of Four-in-a- row game boards taken from: Kamii, C., & Anderson, K. (2003). Multiplication Games: How we made and used them. *Teaching Children Mathematics*, *10*(3), 135-141.

Figure 2

Four-in-a-Row boards

24	9	20	15	30	18
12	30	25	36	24	16
36	15	9	18	20	36
16	36	30	25	12	30
12	20	25	15	24	36
24	16	30	9	25	18

3 4 5 6

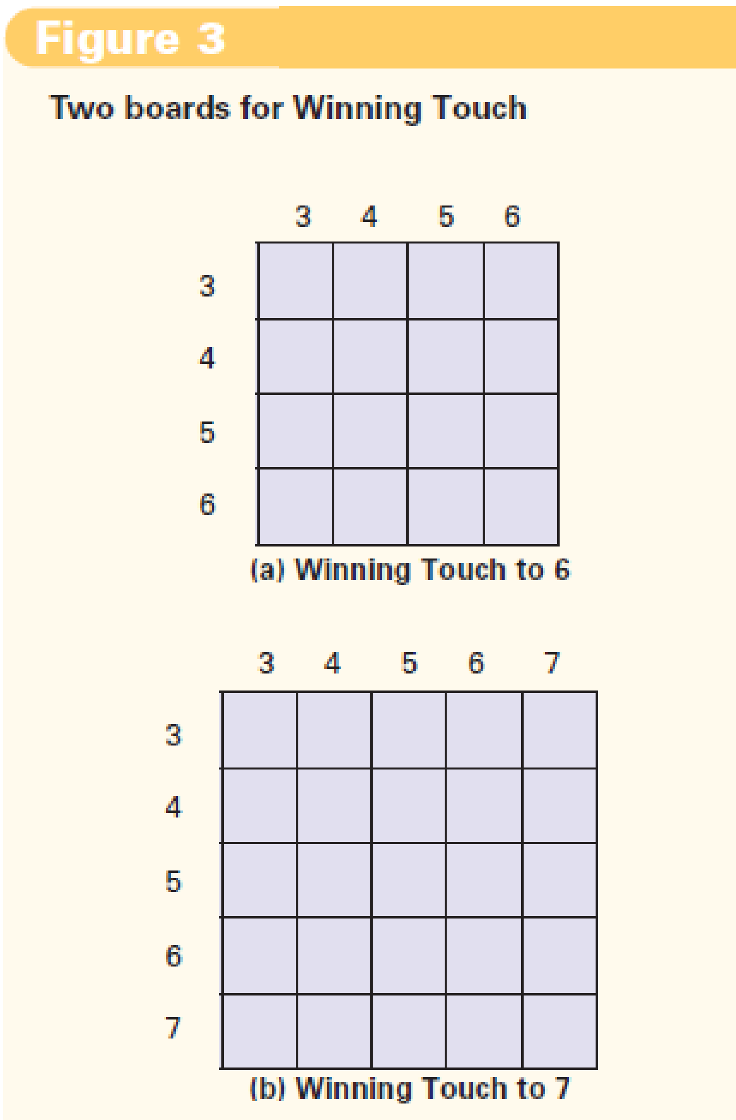
(a) A Four-in-a-Row board with factors 3–6

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

1 2 3 4 5 6 7 8 9

(b) A common Four-in-a-Row board

<u>Figure 6:</u> Examples of game boards for Winning Touch taken from: Kamii, C., & Anderson, K. (2003). Multiplication Games: How we made and used them. *Teaching Children Mathematics, 10*(3), 135-141.



(a) Winning Touch to 6

(b) Winning Touch to 7

Table 1: Examples of Word Problems

Types of Multiplication Problems	Example	Key Words
Comparison Problems	Jennifer can solve three multiplication problems in one minute. Peter can solve multiplication problems five times as fast as Jennifer. How many multiplication problems can Peter solve in one minute?	Five times as fast – means we are multiplying by five.
Equal Groups Problems	Bob has three boxes to put cookies in. He wants to put seven cookies in each box. How many total cookies are there?	Total (this is the product) and seven cookies go in each box (indicates equal groups)
Array Problems	Ms. Lloyd is arranging desks in her classroom. There are six desks in each row and two in each column. How many total desks are there?	Each row and each column (indicate array model will help solve problem) and how many total (tells us to find the product).
Area Problems	George wants to build a rectangular pasture for his	Rectangular (the area of a rectangle is found by

	sheep. He wants the pasture to be 8 feet long and 10 feet wide. What is the total amount of space within his new pasture?	multiplication), 8 ft. long and 10 ft. wide (dimensions of pasture indicate area), Total amount of space (we want the area).