# Coordinate system connected with lower extremity and its application to knee impulsive force induced by landing after vertical fall

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**Abstract**

New coordinate system connected with the lower extremity is contrived and the transformation rule to such a coordinate system from any Cartesian coordinate system is derived. By applying those to a knee impulsive force induced when a subject lands after the vertical fall, the adduction direction component of such an impulsive force, which the subject straightforwardly experiences, is analyzed.

**Keywords:** coordinate system, orthogonal transformation, lower extremity, knee impulsive force, landing after vertical fall

## 1. Introduction

In order to analyze the motion of an object whose movement is mechanically and/or spatially constrained, physicists use a special coordinate system [1-5]. For example, the motion of a free pendulum may be well treated by making use of the polar coordinate system [2-3]. In such a coordinate system, the valuables of the position of the object are reduced to *two* angles $\theta$ and $\varphi$, since the motion of the object is constrained on a sphere and the radius is invariant.

In contrast to usual medical analyses with physical ones, it seems that the analyses of human movements in biomechanics are, sometimes, unconditionally accomplished with Cartesian coordinate systems in which the position of an object are represented by ($x$, $y$, $z$)-coordinate [6], in spite of the fact that the movements of the objects are constrained by the gravity, the ground, the joints and others. The purpose of this paper is to propose the most suitable variable for analyses of the lower extremities.

In section 2, we will formally describe the general theory of the orthogonal coordinate systems and the orthogonal transformations in three dimensions. In section 3 and 4, we will contrive the coordinate system connected with the lower extremity and derive the transformation rule to such a coordinate system from any Cartesian coordinate system. Section 5 will be devoted to an application of section 3 and 4 to analyze the knee impulsive force when a subject lands after the vertical fall. In section 6, the discussions and the conclusions in this paper will be described.

## 2. Formal theory of orthogonal transformation

In this section, we use Einstein's summation convention to indices. Namely, a repeated index appeared in a term implies summation over the whole range of the index.

An orthogonal transformation maps an orthogonal coordinate system (O; $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) to another orthogonal coordinate system (O; $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$). The vectors $\mathbf{e}_a$ and $\mathbf{e}'_a$ are orthogonal bases in each coordinate system and satisfy the conditions

$$\mathbf{e}_a \cdot \mathbf{e}_b = \mathbf{e}'_a \cdot \mathbf{e}'_b = \delta_{ab}, \qquad (1)$$

where $\delta_{ab}$ is Klonecker's delta and the indices $a$, $b$, ... take the values 1, 2 and 3. On such coordinates, any vector $\mathbf{v}$ in space is represented as $v_a \mathbf{e}_a$ and $v'_a \mathbf{e}'_a$, respectively, that is,

$$\mathbf{v} = v_a \mathbf{e}_a = v'_a \mathbf{e}'_a. \qquad (2)$$

The real numbers $v_a$ and $v'_a$ are the components of $\mathbf{v}$ in the coordinate system (O; $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) and (O; $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$),

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respectively.

PROPOSITION 1. *Let us define* $t_{ab}$ *as*

$$t_{ab} = \mathbf{e}'_a \cdot \mathbf{e}_b . \quad (3)$$

*Then the matrix given by* $(t_{ab})$ *is an orthogonal matrix, that is,*

$$t_{ac} t_{bc} = t_{ca} t_{cb} = \delta_{ab} \quad (4)$$

*or*

$$t_{ba} = t^{-1}{}_{ab} . \quad (5)$$

*Proof.* Note that, from (2), $\mathbf{e}'_a$ can be represented as

$$\mathbf{e}'_a = \alpha_{ab} \mathbf{e}_b .$$

From this expression and (1), we obtain

$$\alpha_{ab} = \mathbf{e}'_a \cdot \mathbf{e}_b$$

and

$$\alpha_{ac}\alpha_{bc} = \alpha_{ac}(\mathbf{e}'_b \cdot \mathbf{e}_c) = (\alpha_{ac}\mathbf{e}_c)\cdot \mathbf{e}'_b = \mathbf{e}'_a \cdot \mathbf{e}'_b = \delta_{ab} .$$

On the other hand, $\mathbf{e}_a$ can be written as

$$\mathbf{e}_a = \mathbf{e}'_b \, \beta_{ba} ,$$

from (2), where

$$\beta_{ab} = \mathbf{e}'_a \cdot \mathbf{e}_b .$$

From this expression and (1), we obtain also

$$\beta_{ca}\beta_{ca} = \beta_{ca}(\mathbf{e}'_c \cdot \mathbf{e}_b) = (\mathbf{e}'_c \, \beta_{ca})\cdot \mathbf{e}_b = \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab} .$$

Noting that $t_{ab} = \alpha_{ab} = \beta_{ab}$, we have

$$t_{ac} t_{bc} = t_{ca} t_{cb} = \delta_{ab} .$$

Thus this proposition is true. ■

From the proof of PROPOSITION 1, we have

$$\mathbf{e}'_a = t_{ab} \mathbf{e}_b , \quad (6a)$$

or

$$\mathbf{e}_a = \mathbf{e}'_b \, t_{ba} . \quad (6b)$$

From the equations (6), we can obtain the following true proposition.

PROPOSITION 2. *The equation*

$$v'_a = t_{ab} v_b \quad (7)$$

*holds for any vector* **v** *of which the components are* $v_a$ *and* $v'_a$ *in* (O; $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) *and* (O; $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$), *respectively.*

*Proof.* From (2) and (6), we have

$$v'_a \, \mathbf{e}'_a = v_a \mathbf{e}_a = v_a \mathbf{e}'_b \, t_{ba} = t_{ab} v_b \mathbf{e}'_a .$$

From the left and right hand sides of this expression, we obtain

$$(v'_a - t_{ab} v_b)\mathbf{e}'_a = \mathbf{0} .$$

In order that this equation holds for any $\mathbf{e}'_a$, those coefficients must vanish. Therefore we obtain

$$v'_a = t_{ab} v_b .$$

Thus this proposition is true. ■

The equation (7) provides us the orthogonal transformation rule for components $v_a$ of any vector **v** from the coordinate system (O; $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) to (O; $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$). The transformation rule (7) can be easily extended to any tensor. Since the tensor space is represented by the direct product $\mathbf{V} \otimes \mathbf{V} \otimes \cdots \otimes \mathbf{V}$ of the vector space $\mathbf{V}$, the components $w_{ab\cdots c}$ of any tensor $\mathbf{w}$* is transformed by the rule

$$w'_{ab\cdots c} = t_{ad} t_{be} \cdots t_{cf} w_{de\cdots f} . \quad (8)$$

As an example of above the transformation, let us consider the transformation from a Cartesian coordinate system (O; **i**, **j**, **k**) to the polar coordinate system $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$. As is well known, any Cartesian coordinate $(x,\ y,\ z)$ corresponds to the polar coordinate $(r, \theta, \varphi)$ as follows;

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} . \quad (9)$$

Taking the total differentiations of those, we obtain

$$\begin{cases} \mathrm{d}x = \sin\theta\cos\varphi\ \mathrm{d}r + r\cos\theta\cos\varphi\ \mathrm{d}\theta - r\sin\theta\sin\varphi\ \mathrm{d}\varphi \\ \mathrm{d}y = \sin\theta\sin\varphi\ \mathrm{d}r + r\cos\theta\sin\varphi\ \mathrm{d}\theta + r\sin\theta\cos\varphi\ \mathrm{d}\varphi \\ \mathrm{d}z = \cos\theta\ \mathrm{d}r - r\sin\theta\ \mathrm{d}\theta . \end{cases} \quad (10)$$

Since the bases correspond to the line element in the coordinate system, we have

$$\mathbf{i} \leftrightarrow \mathrm{d}x, \quad \mathbf{j} \leftrightarrow \mathrm{d}y, \quad \mathbf{k} \leftrightarrow \mathrm{d}z , \quad (11)$$

and

* Namely, the tensor **w** is represented by
$\mathbf{w} = w_{ab\cdots c}\mathbf{e}_a \otimes \mathbf{e}_b \otimes \cdots \otimes \mathbf{e}_c = w'_{ab\cdots c}\,\mathbf{e}'_a \otimes \mathbf{e}'_b \otimes \cdots \otimes \mathbf{e}'_c .$

$$\mathbf{e}_r \leftrightarrow \mathrm{d}r, \quad \mathbf{e}_\theta \leftrightarrow r\mathrm{d}\theta, \quad \mathbf{e}_\varphi \leftrightarrow r\sin\theta\,\mathrm{d}\varphi \,. \tag{12}$$

From (10), (11) and (12), we obtain

$$\begin{cases} \mathbf{i} = \sin\theta\cos\varphi\ \mathbf{e}_r + \cos\theta\cos\varphi\ \mathbf{e}_\theta - \sin\varphi\ \mathbf{e}_\varphi \\ \mathbf{j} = \sin\theta\sin\varphi\ \mathbf{e}_r + \cos\theta\sin\varphi\ \mathbf{e}_\theta + \cos\varphi\ \mathbf{e}_\varphi \\ \mathbf{k} = \cos\theta\ \mathbf{e}_r - \sin\theta\ \mathbf{e}_\theta \end{cases} . \tag{13}$$

Comparing (13) with (6b), we find the following transformation matrix $(t_{ab})$ which maps the Cartesian coordinate system (O; **i**, **j**, **k**) to the polar coordinate system (O; $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$);

$$(t_{ab}) = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}. \tag{14}$$

Furthermore, from (7), acting the matrix (14) to components $(v_x, v_y, v_z)$ of any vector **v** in (O; **i**, **j**, **k**), we can obtain its components $(v_r, v_\theta, v_\varphi)$ in (O; $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$) as follows;

$$\begin{cases} v_r = \sin\theta\cos\varphi\ v_x + \sin\theta\sin\varphi\ v_y + \cos\theta\ v_z \\ v_\theta = \cos\theta\cos\varphi\ v_x + \cos\theta\sin\varphi\ v_y - \sin\theta\ v_z \\ v_\varphi = -\sin\varphi\ v_x + \cos\varphi\ v_y \end{cases} . \tag{15}$$

In general, the determinant of the matrix $(t_{ab})$ defined by (3) takes the value +1 or -1, that is,

$$\det(t_{ab}) = \varepsilon_{abc} t_{a1} t_{b2} t_{c3} = \pm 1 \,, \tag{16}$$

where $\varepsilon_{abc}$ is Levi-Civita's symbol;

$$\varepsilon_{abc} = -\varepsilon_{bac} = -\varepsilon_{acb}\,, \quad \varepsilon_{123} = 1. \tag{17}$$

This implies that the set of transformations defined by (7) includes the reflections of coordinate systems. Namely, an orthogonal transformation of which matrix is denoted by $(t_{ab})$ includes the reflection if $\det(t_{ab}) = -1$. If we do not wish to treat any reflections, we must carefully take our coordinate systems. When the set of orthogonal transformations does not include any reflections of coordinate systems*, such transformations can be represented by the rotations of coordinate systems.

## 3. Coordinate system connected with a lower extremity

It seems that, in order to represent the configuration of a (right or left) lower extremity simplified by a rod, we can use the polar coordinate system described in section 2. However, since the lower extremity is connected with the upper extremity and the foot with the knee joint and the ankle joint, respectively, the movement of the rod by which the lower extremity is simplified is constrained by the rods corresponding to the upper extremity and the foot, respectively. Thus it is not appropriate to represent the configuration of the lower extremity by the polar coordinate system.

In order to well represent the constrained movement of the lower extremity, we define a new orthogonal coordinate system in this section. Let us consider an orthogonal coordinate system (O; **l**, **m**, **n**), where **l**, **m** and **n** are the unit vectors which are normal to the frontal plane, the sagittal plane and the transverse plane, respectively, and are frontward, the leftward and upward, respectively. Moreover, let $\xi$, $\eta$ and $\zeta$ be the axes parallel to **l**, **m** and **n**, respectively, and $\Pi$ be $(\eta, \zeta)$-plane (or the frontal plane).

We label the two joints of the lower extremity with K and A, where K is the knee joint and A is the ankle, and write the vector $\overrightarrow{\mathrm{AK}}$ as follows;

$$\overrightarrow{\mathrm{AK}} = \mathbf{r} = \xi\,\mathbf{l} + \eta\,\mathbf{m} + \zeta\,\mathbf{n}. \tag{18}$$

Then for the projective vector **r*** of **r** to $\Pi$, we have

$$\mathbf{r}^* = \eta\,\mathbf{m} + \zeta\,\mathbf{n}. \tag{19}$$

Let $\psi$ be the angle between the fixed vector **n** and the radius vector **r***. The angle $\psi$ takes the positive value if it is the counterclockwise angle around the rotation axis **l** and the negative value if it is the clockwise angle around the same rotation axis. Then we have the following true proposition.

PROPOSITION 3. *The angle* $\psi$ *is given by*

$$\psi = \sin^{-1}\left( -\frac{\eta}{\sqrt{\eta^2 + \zeta^2}} \right), \tag{20}$$

*if it takes the range*

$$-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}. \tag{21}$$

* The quantities provided by the vector products are not invariant to the reflection. For example, the angular velocity, the angular momentum, the moment of force and others are such quantities.

*Proof.* From (19), we have

$$\mathbf{n}\times\mathbf{r}^* = \eta\mathbf{n}\times\mathbf{m}+\zeta\mathbf{n}\times\mathbf{n} = -\eta\mathbf{l}.$$

On the other hand, from the definition of the angle $\psi$, we have

$$\mathbf{r}^* = |\mathbf{r}^*|\cos\psi\,\mathbf{n} - |\mathbf{r}^*|\sin\psi\,\mathbf{m},$$

and

$$\begin{aligned}\mathbf{n}\times\mathbf{r}^* &= |\mathbf{r}^*|\cos\psi\,(\mathbf{n}\times\mathbf{n}) - |\mathbf{r}^*|\sin\psi\,(\mathbf{n}\times\mathbf{m})\\ &= |\mathbf{r}^*|\sin\psi\,\mathbf{l}.\end{aligned}$$

Therefore, we obtain

$$\sin\psi = \frac{-\eta}{|\mathbf{r}^*|} = -\frac{\eta}{\sqrt{\eta^2+\zeta^2}}.$$

Since the principal value of arcsine takes the range (21), the solution of this equation is given by (20). Thus this proposition is true. ■

We define a normal unit vector $\mathbf{e}_\psi$ to $\mathbf{l}$ on $\Pi$ as follows;

$$\mathbf{e}_\psi = \frac{\mathbf{l}\times\mathbf{r}^*}{|\mathbf{l}\times\mathbf{r}^*|} = \frac{-\zeta\mathbf{m}+\eta\mathbf{n}}{\sqrt{\eta^2+\zeta^2}}. \quad (22)$$

Note that $\mathbf{e}_\psi$ is pointed to the positive direction of $\psi$.

PROPOSITION 4. *The unit vector* $\mathbf{e}_\psi$ *defined by* (22) *is orthogonal to the vector* $\mathbf{r}$ *given by* (18)*, or*

$$\mathbf{e}_\psi\cdot\mathbf{r} = 0 \quad (23)$$

*Proof.* From (18) and (19), we have

$$\mathbf{e}_\psi\cdot\mathbf{r} = \mathbf{e}_\psi\cdot(\xi\,\mathbf{l}+\mathbf{r}^*) = \xi\,\mathbf{e}_\psi\cdot\mathbf{l}+\mathbf{e}_\psi\cdot\mathbf{r}^*.$$

From the definition of $\mathbf{e}_\psi$, we have clearly $\mathbf{e}_\psi\cdot\mathbf{r}^* = 0$. Moreover, from (22) and

$$(\mathbf{l}\times\mathbf{r}^*)\cdot\mathbf{l} = (\mathbf{l}\times\mathbf{l})\cdot\mathbf{r}^* = 0,$$

we have $\mathbf{e}_\psi\cdot\mathbf{l} = 0$. Thus the equation (23) holds. Therefore this proposition is true. ■

Let $\rho$ be the angle between the fixed vector $\mathbf{r}^*$ and the radius vectors $\mathbf{r}$. From PROPOSITION 4, it is shown that the angle $\rho$ takes the positive value if it is the counterclockwise angle around the rotation axis $-\mathbf{e}_\psi$ and the negative value if it is the clockwise angle around the same rotation axis. Then we have the following true proposition.

PROPOSITION 5. *The angle* $\rho$ *is given by*

$$\rho = \sin^{-1}\left(\frac{\xi}{\sqrt{\xi^2+\eta^2+\zeta^2}}\right), \quad (24)$$

*if it takes the range*

$$-\frac{\pi}{2} \le \rho \le \frac{\pi}{2}. \quad (25)$$

*Proof.* From (18), (19) and (22), we have

$$\begin{aligned}\mathbf{r}^*\times\mathbf{r} &= (\eta\,\mathbf{m}+\zeta\,\mathbf{n})\times(\xi\,\mathbf{l}+\eta\,\mathbf{m}+\zeta\,\mathbf{n})\\ &= \xi\zeta\,\mathbf{m}-\eta\xi\,\mathbf{n} = \xi(\zeta\,\mathbf{m}-\eta\,\mathbf{n}) = \xi|\mathbf{r}^*|(-\mathbf{e}_\psi).\end{aligned}$$

On the other hand, from the definition of the vector product, we have

$$\mathbf{r}^*\times\mathbf{r} = |\mathbf{r}^*||\mathbf{r}|\sin\rho\,(-\mathbf{e}_\psi).$$

From these two expressions, we obtain

$$\sin\rho = \frac{\xi}{|\mathbf{r}|} = \frac{\xi}{\sqrt{\xi^2+\eta^2+\zeta^2}}.$$

Since the principal value of arcsine takes the range (25), the solution of this equation is given by (24). Thus this proposition is true. ■

We define a unit vector $\mathbf{e}_r$ by

$$\mathbf{e}_r = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\xi\mathbf{l}+\eta\mathbf{m}+\zeta\mathbf{n}}{\sqrt{\xi^2+\eta^2+\zeta^2}}, \quad (26)$$

From (23), it is clearly orthogonal to $\mathbf{e}_\psi$, i.e.

$$\mathbf{e}_r\cdot\mathbf{e}_\psi = 0. \quad (27)$$

Furthermore, we define another unit vector $\mathbf{e}_\rho$ by

$$\mathbf{e}_\rho = \mathbf{e}_\psi\times\mathbf{e}_r. \quad (28)$$

From (27) and (28), we see that the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\psi$ and $\mathbf{e}_\rho$ are mutually orthogonal, that is,

$$\begin{cases}|\mathbf{e}_r| = |\mathbf{e}_\rho| = |\mathbf{e}_\psi| = 1,\\ \mathbf{e}_r\cdot\mathbf{e}_\rho = \mathbf{e}_\rho\cdot\mathbf{e}_\psi = \mathbf{e}_\psi\cdot\mathbf{e}_r = 0.\end{cases} \quad (29)$$

Thus it is clear that the following true proposition holds.

PROPOSITION 6. *The set* $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$ *constructs an orthogonal coordinate system.*

Note that the set of the origin and the bases in PROPOSITION 6 is not $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_\rho)$ but $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$. It is the reason that, as shown in the next section, the transformation to $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_\rho)$ from (O; **l**, **m**, **n**) does not include any reflection.

The restrictions (25) and (21) to the angles $\rho$ and $\psi$, respectively, do not induce any problems if the foot connected with it contacts with the ground, since the lower extremity never be parallel to the ground. Therefore, it is appropriate that we suppose the coordinate system $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$ defined in PROPOSITION 6.

From the proof of PROPOSITION 3, we have

$$\mathbf{r}^* = |\mathbf{r}^*|\cos\psi\,\mathbf{n} - |\mathbf{r}^*|\sin\psi\,\mathbf{m}. \tag{30}$$

Note that the vector **n** and **r*** lie on the plane $\Pi$ (or the frontal plane), **r*** is the projective vector of **r** to $\Pi$ and $\mathbf{r} = \overrightarrow{\mathrm{AK}}$ is the vector along a lower extremity with the ankle A and the knee joint K. Therefore, from (30), we understand that the angle $\psi$ represents the transverse inclination of the lower extremity with the fixed point A. Namely, if $\psi$ takes the positive value, the left (or the right) lower extremity inclines to the inside (or the outside). Conversely, if $\psi$ takes the negative value, the left (or the right) lower extremity inclines to the outside (or the inside).

On the other hand, from the proof of PROPOSITION 5, we have

$$\frac{\mathbf{r}^*}{|\mathbf{r}^*|} \times \frac{\mathbf{r}}{|\mathbf{r}|} = \sin\rho\,(-\mathbf{e}_\psi). \tag{31}$$

From (31), we understand that $\rho$ represents inclination of the vector **r** along the lower extremity to the front and the rear from **r*** with the fixed point A. Namely, if $\rho$ takes the positive value, the lower extremity inclines to the front from **r*** Conversely, if $\rho$ takes the negative value, the lower extremity inclines to the rear from **r***.

The above discussions are suggested that, by making use of the orthogonal coordinate system $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$, we can appropriately represent each component of any vector with respect to the lower extremity. For example, let the components of some force **f** acting on a knee joint from with respect to such a coordinate system be $f_r$, $f_\rho$ and $f_\psi$, respectively. Then we can regard $f_r$ as the element of **f** along the lower extremity, $f_\rho$ as the element of **f** making the lower extremity incline to the front or the rear and $f_\psi$ as the element of **f** making the lower extremity expand or close.

## 4. Transformation rule to the coordinate system connected a lower extremity

Let us consider the transformation rule to map (O; **l**, **m**, **n**) to $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$. The transformation matrix $(t_{ab})$ must be an orthogonal matrix. From PRPPOSITION 1, such a matrix can be formally represented by

$$(t_{ab}) = \begin{pmatrix} \mathbf{e}_r\cdot\mathbf{l} & \mathbf{e}_r\cdot\mathbf{m} & \mathbf{e}_r\cdot\mathbf{n} \\ \mathbf{e}_\rho\cdot\mathbf{l} & \mathbf{e}_\rho\cdot\mathbf{m} & \mathbf{e}_\rho\cdot\mathbf{n} \\ \mathbf{e}_\psi\cdot\mathbf{l} & \mathbf{e}_\psi\cdot\mathbf{m} & \mathbf{e}_\psi\cdot\mathbf{n} \end{pmatrix}. \tag{32a}$$

Therefore, from PROPOSITION 2, the components of any vector

$$\mathbf{v} = v_1\mathbf{l} + v_2\mathbf{m} + v_3\mathbf{n} = v_r\mathbf{e}_r + v_\rho\mathbf{e}_\rho + v_\psi\mathbf{e}_\psi \tag{33}$$

is mapped from (O; **l**, **m**, **n**) to $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$ according to the transformation rule

$$\begin{cases} v_r = (\mathbf{e}_r\cdot\mathbf{l})v_1 + (\mathbf{e}_r\cdot\mathbf{m})v_2 + (\mathbf{e}_r\cdot\mathbf{n})v_3 \\ v_\rho = (\mathbf{e}_\rho\cdot\mathbf{l})v_1 + (\mathbf{e}_\rho\cdot\mathbf{m})v_2 + (\mathbf{e}_\rho\cdot\mathbf{n})v_3 \\ v_\psi = (\mathbf{e}_\psi\cdot\mathbf{l})v_1 + (\mathbf{e}_\psi\cdot\mathbf{m})v_2 + (\mathbf{e}_\psi\cdot\mathbf{n})v_3 \end{cases}. \tag{34a}$$

Since $\mathbf{e}_r$, $\mathbf{e}_\rho$, $\mathbf{e}_\psi$, $\rho$ and $\psi$ are completely determined by (26), (28), (22), (24) and (20), respectively, we can obtain $v_r$, $v_\rho$ and $v_\psi$ by applying to the transformation (34a) to the given $v_1$, $v_2$ and $v_3$.

From the discussions of POROPOSITION 5 and 6 and the expressions (22), (26), (28) and (29), we can also rewrite the components of the matrix $(t_{ab})$ by the expressions of only $\rho$ and $\psi$. For that purpose, we must rewrite the bases $\mathbf{e}_r$, $\mathbf{e}_\psi$ and $\mathbf{e}_\rho$ by the expressions of **l**, **m**, **n**, $\rho$ and $\psi$.

PROPOSITION 7. *The unit vectors* $\mathbf{e}_r$, $\mathbf{e}_\rho$ *and* $\mathbf{e}_\psi$ *provided by* (26), (28) *and* (22)*, respectively, can be rewritten by*

$$\begin{cases} \mathbf{e}_r = \sin\rho\,\mathbf{l} - \cos\rho\sin\psi\,\mathbf{m} + \cos\rho\cos\psi\,\mathbf{n} \\ \mathbf{e}_\rho = -\cos\rho\,\mathbf{l} - \sin\rho\sin\psi\,\mathbf{m} + \sin\rho\cos\psi\,\mathbf{n} \\ \mathbf{e}_\psi = -\cos\psi\,\mathbf{m} - \sin\psi\,\mathbf{n} \end{cases}. \tag{34}$$

*Proof.* Firstly, note that $\psi$ is the angle between **r*** and **n** and the expression (19) holds. Thus, from

$$\mathbf{n}\cdot\mathbf{r}^* = |\mathbf{n}||\mathbf{r}^*|\cos\psi = |\mathbf{r}^*|\cos\psi$$
$$= \eta(\mathbf{n}\cdot\mathbf{m}) + \zeta(\mathbf{n}\cdot\mathbf{n}) = \zeta,$$

we have

$$\frac{\zeta}{|\mathbf{r}^*|} = \cos\psi.$$

From this expression, (30) and (22), we obtain the third expression in (34), that is,

$$\mathbf{e}_\psi = -\cos\psi\,\mathbf{m} - \sin\psi\,\mathbf{n}.$$

Secondly, from (26), the proof of PROPOSITION 3 and the first expression in this proof, we have

$$\mathbf{e}_r = \frac{1}{|\mathbf{r}|}(\xi\mathbf{l} + \eta\mathbf{m} + \zeta\mathbf{n}) = \sin\rho\,\mathbf{l} + \frac{|\mathbf{r}^*|}{|\mathbf{r}|}\left(\frac{\eta}{|\mathbf{r}^*|}\mathbf{m} + \frac{\zeta}{|\mathbf{r}^*|}\mathbf{n}\right)$$
$$= \sin\rho\,\mathbf{l} + \frac{|\mathbf{r}^*|}{|\mathbf{r}|}(-\sin\psi\,\mathbf{m} + \cos\psi\,\mathbf{n}).$$

Note that, since $\rho$ is the angle between $\mathbf{r}^*$ and $\mathbf{r}$ and $\mathbf{r}^*\cdot\mathbf{r}$ can be expressed by

$$\mathbf{r}^*\cdot\mathbf{r} = (\eta\mathbf{m} + \zeta\mathbf{n})\cdot(\xi\mathbf{l} + \eta\mathbf{m} + \zeta\mathbf{n}) = \eta^2 + \zeta^2 = |\mathbf{r}^*|^2,$$

we have

$$\cos\rho = \frac{\mathbf{r}^*\cdot\mathbf{r}}{|\mathbf{r}^*||\mathbf{r}|} = \frac{|\mathbf{r}^*|}{|\mathbf{r}|}.$$

Therefore, we obtain the first expression in (34), that is,

$$\mathbf{e}_r = \sin\rho\,\mathbf{l} + \cos\rho\sin\psi\,\mathbf{m} + \cos\rho\cos\psi\,\mathbf{n}.$$

Lastly, from (28), the first and third expressions of (33), we obtain the second expression in (34), that is,

$$\mathbf{e}_\rho = (-\cos\psi\,\mathbf{m} + \sin\psi\,\mathbf{n})$$
$$\times(\sin\rho\,\mathbf{l} + \cos\rho\sin\psi\,\mathbf{m} + \cos\rho\cos\psi\,\mathbf{n})$$
$$= -\cos\rho\,\mathbf{l} - \sin\rho\sin\psi\,\mathbf{m} + \sin\rho\cos\psi\,\mathbf{n}.$$

Thus this proposition is true. ■

Substituting (34) into each component of (32a), we obtain

$$(t_{ab}) = \begin{pmatrix} \sin\rho & -\cos\rho\sin\psi & \cos\rho\cos\psi \\ -\cos\rho & -\sin\rho\sin\psi & \sin\rho\cos\psi \\ 0 & -\cos\psi & -\sin\psi \end{pmatrix}. \quad (32b)$$

Moreover, we can represent the transformation rule (34a) as follows;

$$\begin{cases} v_r = \sin\rho\, v_1 - \cos\rho\sin\psi\, v_2 + \cos\rho\cos\psi\, v_3 \\ v_\rho = -\cos\rho\, v_1 - \sin\rho\sin\psi\, v_2 + \sin\rho\cos\psi\, v_3 \\ v_\psi = -\cos\psi\, v_2 - \sin\psi\, v_3 \end{cases}. \quad (34b)$$

PROPOSITION 8. *The orthogonal transformation* (34a) or (34b) *from* (O; **l**, **m**, **n**) *to* (O; $\mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi$) *does not include any reflection.*

*Proof.* Taking the determinant of the transformation matrix (32b), we have

$$\det(t_{ab}) = \begin{vmatrix} \sin\rho & -\cos\rho\sin\psi & \cos\rho\cos\psi \\ -\cos\rho & -\sin\rho\sin\psi & \sin\rho\cos\psi \\ 0 & -\cos\psi & -\sin\psi \end{vmatrix}$$
$$= \sin^2\rho\sin^2\psi + \cos^2\rho\cos^2\psi$$
$$+ \sin^2\rho\cos^2\psi + \cos^2\rho\sin^2\psi$$
$$= \sin^2\rho\,(\sin^2\psi + \cos^2\psi)$$
$$+ \cos^2\rho\,(\cos^2\psi + \sin^2\psi) = 1.$$

Thus this proposition is true. ■

## 5. Application to knee impulsive force induced by landing after vertical fall

When a subject lands on his feet after the vertical fall, the impulsive force acts on every joint of his body. We are particularly interested in the impulsive force acting on the knee joint. Such an impulsive force (vector) **R** which occurs at the time $t_1$ and vanishes at the time $t_2$ is given by the following equation;

$$\mathbf{R} = \mathbf{N} - m_1\left(\frac{\mathbf{v}_1(t_2) - \mathbf{v}_1(t_1)}{t_2 - t_1} + g\mathbf{k}\right)$$
$$- m_2\left(\frac{\mathbf{v}_2(t_2) - \mathbf{v}_2(t_1)}{t_2 - t_1} + g\mathbf{k}\right), \quad (35)$$

where $m_1$, $m_2$, $\mathbf{v}_1$, $\mathbf{v}_2$, **N**, $g$ and **k** are the mass of the foot, the mass of the lower extremity, the velocity of the center of mass of the foot, the velocity of the center of mass of the lower extremity, the grand reaction, the constant of gravitational acceleration and the upward unit vector, respectively. From an experiment, we can immediately obtain the data of the coordinate value of the every joint and the components of the grand reaction **N**. Furthermore, using such data and the appropriate constants [7], we can compute the components of the impulsive force **R** acting to the knee joint. In order to analyze the variable data obtained by the experiment, we data obtained by the experiment, we use *Mathematica** on the computer[8-10].

By making use of the data accumulated by repeating such an experiment, we can compute the sets of the components of **R**. They are the components with respect to a coordinate system (O; **i**, **j**, **k**) determined by the device for the experiment. Therefore, we must firstly transform such components to ones with respect to the coordinate system (O; **l**, **m**, **n**).

The coordinate system (O; **l**, **m**, **n**) is defined as follows. When a subject lands on his feet, we take the unit vector **m** of (O; **l**, **m**, **n**) as

$$\mathbf{m} = \frac{h_x \mathbf{i} + h_y \mathbf{j}}{\sqrt{{h_x}^2 + {h_y}^2}} \ , \tag{36a}$$

where

$$\mathbf{h} = h_x \mathbf{i} + h_y \mathbf{j} + h_z \mathbf{k} = \frac{\overrightarrow{\mathrm{A_R A_L}}}{\left|\overrightarrow{\mathrm{A_R A_L}}\right|} , \tag{37}$$

and $\mathrm{A_R}$ and $\mathrm{A_L}$ are the right and left ankles. Now, we take

$$\mathbf{n} = \mathbf{k}, \quad \mathbf{l} = \mathbf{m} \times \mathbf{k} . \tag{36b}$$

Then, from (3), the transformation matrix $(t_{1ab})$ from (O; **i**, **j**, **k**) to (O; **l**, **m**, **n**) is provided by

$$(t_{1ab}) = \frac{1}{\sqrt{{h_x}^2 + {h_y}^2}} \begin{pmatrix} h_x & -h_y & 0 \\ h_y & h_x & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{37}$$

Furthermore, from (32b), the transformation matrix $(t_{2ab})$ from (O; **l**, **m**, **n**) to the coordinate system $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$ connected with the lower extremity is provided by

$$(t_{2ab}) = \begin{pmatrix} \sin\rho & -\cos\rho \sin\psi & \cos\rho \cos\psi \\ -\cos\rho & -\sin\rho \sin\psi & \sin\rho \cos\psi \\ 0 & -\cos\psi & -\sin\psi \end{pmatrix} . \tag{38}$$

By composing these transformations, we can obtain the components $R_r$, $R_\rho$ and $R_\psi$ of **R** in $(\mathrm{O}; \mathbf{e}_r, \mathbf{e}_\rho, \mathbf{e}_\psi)$ from $R_x$, $R_y$ and $R_z$ of **R** in (O; **i**, **j**, **k**), i.e.

$$R'_a = t_{2ab} t_{1bc} R_c , \tag{39}$$

where $R'_1 = R_r$, $R'_2 = R_\rho$ and $R'_3 = R_\psi$.

Since any impulsive force rapidly varies, the numerical values indicating its components computed from the data obtained by the experiment will be uneven. In order to solve this problem, we normalize the components of every impulsive force obtained by the experiment and the computation as follows;

$$\overline{R}_r = \frac{R_r}{|\mathbf{R}|}, \quad \overline{R}_\rho = \frac{R_\rho}{|\mathbf{R}|}, \quad \overline{R}_\psi = \frac{R_\psi}{|\mathbf{R}|} . \tag{40}$$

We will estimate, in this paper, the relation between the angle $\psi$ and the normalized component $\overline{R}_\psi$ of the impulsive force.

The height and weight of the subject are 1.7m and 70kg.w, respectively. He lands on his feet after the vertical fall from the stands of which heights are 0.24m and 0.44m. In order to take the adduction directions the positive ones for the both lower extremities, we put $\overline{R}_2 \to -\overline{R}_2$ for the left lower extremity and $\psi \to -\psi$ for the right lower extremity, where $\overline{R}_1$, $\overline{R}_2$ and $\overline{R}_3$ are the components of $\mathbf{R}/|\mathbf{R}|$ with respect to **l**, **m** and **n** in the coordinate system (O; **l**, **m**, **n**), respectively. Therefore, we obtain from (34b)

$$\begin{aligned} \overline{R}_\psi &\to -\cos\psi\,(-\overline{R}_2) - \sin\psi\,\overline{R}_3 \\ &= \cos\psi\,\overline{R}_2 - \sin\psi\,\overline{R}_3 \end{aligned} \tag{41a}$$

for the left extremity under the replacement $\overline{R}_2 \to -\overline{R}_2$ and

$$\begin{aligned} \overline{R}_\psi &\to -\cos(-\psi)\,\overline{R}_2 - \sin(-\psi)\,\overline{R}_3 \\ &= -\cos\psi\,\overline{R}_2 + \sin\psi\,\overline{R}_3 \end{aligned} \tag{41b}$$

for the right extremity under the replacement $\psi \to -\psi$. By the above replacements, $\psi$ and $\overline{R}_\psi$ can be regarded with the adduction angles of the left and right lower extremities and the components of the normalized impulsive forces acting on the left and right knees with respect to the adduction directions, respectively. In Figure 1, we show the relations between $\psi$ and $\overline{R}_\psi$ redefined above. Moreover, in the same Figure, the relation between $\psi$ and $\overline{R}_2$ is also shown, where $\overline{R}_2$ is redefined by applying the above replacement.

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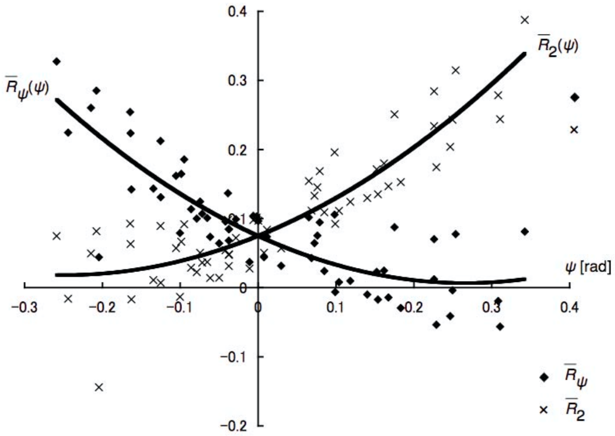


Figure 1. $\overline{R}_2$ and $R_\psi$ with respect to the adduction angle $\psi$.

## 6. Discussions and conclusions

In order to remove the troublesome of the term, we call any normalized impulsive force merely impulsive force, in this section.

Figure 1 does not show the general tendency of the impulsive force that acts on the knee when landing after the vertical fall, since it is not the statistical result obtained by many subjects. Our result shows that the coordinate system and the coordinate transformation play a vital role to research the mechanical effect induced by a human movement, as is below described.

Firstly, let us see the points of $\overline{R}_2$ or the approximate curve $\overline{R}_2(\psi)$ (making use of the method of least squares) in Figure 1. These values are based on the data that are measured by the exterior observer in Cartesian coordinate defined by the device. If one straightforwardly regard $\overline{R}_2$ as the adduction direction component of the knee impulsive force, one may interpret that it positively increases with the positive increase of the adduction angle $\psi$. However, one should note that, since the coordinate system of the observer is exterior to one of the subject, $\overline{R}_2$ is not the adduction impulsive force component which the subject indeed experiences.

Secondly, let us see the points of $\overline{R}_\psi$ or the approximate curve $\overline{R}_\psi(\psi)$ in Figure 1. $\overline{R}_\psi$ is a component of the knee impulsive force in the coordinate system connected with the lower extremity. Namely, the adduction direction component of the knee impulsive force which the subject indeed experiences is $\overline{R}_\psi$. In contrast to $\overline{R}_2$, $\overline{R}_\psi$, which takes the comparatively large value at the negative $\psi$, decreases with the positive increase of the adduction angle $\psi$ and almost vanishes at $\psi \approx 0.25$[rad]. It is difficult to immediately grasp such an effect from the only analyses of experimental data without setting up the appropriate coordinate system connected with the lower extremity and transforming to such a coordinate system from the exterior coordinate system defined by the device.

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