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A Curriculum Project on the Utilization of the Rule of Four to Enable Students to Work in Their Zone of Proximal Development

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Abstract

Discussed, developed, and written over the course of roughly two-and-a-half-years, the Common Core State Standards (CCSS) seek to prepare students to be college- and career-ready (K-12 Blueprint, 2014). Released for implementation in classrooms nation-wide in June 2010, the CCSS are claimed to be "a balance of concepts and skills, with content standards that require both conceptual understanding and procedural fluency" (K-12 Blueprint, 2014, p. 3), as well as address the issue of mathematical curricula being described as "a mile wide and an inch deep" (Common Core State Standards Initiative, 2014). In order to abide by the newly adopted curricula, teachers must find ways to help students think more conceptually. This curriculum project on trigonometric functions at the Algebra II level attempts to do just that through the utilization of the Rule of Four in order to support students working within their zone of proximal development.

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Chapter One: Introduction

Overview

In today's turbulent world, it is crucial that those involved with education implement curricula that will best prepare students for their journeys ahead both inside and outside the classroom; the very idea behind the most recently released set of national standards (Common Core State Standards Initiative, 2014). In November of 2007, discussion about developing a new set of common standards began at the Council of Chief State School Officers' (CCSSO) Annual Policy Forum (Common Core State Standards Initiative, 2014), and on June 2nd, 2010, the Common Core State Standards (CCCSS) were released for implementation in classrooms across the country (K-12 Blueprint, 2014). Although the standards that influence the foundation of curriculum have changed, the necessity of meeting students' educational needs has not. Therefore, it is crucial to establish a curriculum that meets such needs of all students while simultaneously working toward the goal of the CCSS; that is, for all students to be college and career ready (Common Core State Standards Initiative, 2014).

Project Description

This curriculum project is designed for Algebra II; specifically the unit of trigonometric functions. It is recommended that the focus is on understanding radian measures, extending the domain of trigonometric functions using the unit circle, and modeling periodic phenomena with trigonometric functions (Common Core State Standards Initiative, 2014). These standards will be met through the utilization of the Rule of Four, which is a strategy teachers can use to teach one concept in multiple formats (Annenberg Learner, 2014).

Rationale

Portraying the same idea with multiple representations is one way in which educators can ensure their students' academic needs are met so learning can occur. Implementing the Rule of Four during instruction should support students problem solving and allow them to choose the solution method that works best. However, there is more to using the Rule of Four than this. Cognitively, students make connections across content topics and representations that allow them to develop a deeper understanding of the material over time (San Francisco Unified School District (SFUSD) Mathematics Department, 2014). What goes on during this cognitive process occurs in the student's zone of proximal development (ZPD). Defined as "… the distance between learners' independent performance and the higher level that can be achieved under the guidance of a more expert partner, such as an adult or more capable peer" (Vygotsky, 1978; Goos et al., 2003, p. 196), ZPD can be the key to unlocking student potential.

Teachers and students involved in Algebra II are not the only groups who would benefit from this curriculum. The ideas behind ZPD and the Rule of Four are beneficial to all mathematics education research and field practitioners because all students have academic needs and need to work in their zone of proximal development. Therefore, the foundation upon which the author develops the curriculum can be taken out of context and applied to any mathematical content.

Chapter Two: Literature Review

In order to understand how to create a curriculum that is aligned with the CCSS, it is helpful to become familiar with the history and development processes of the standards over time, as well as the paradigm shift from the NCTM Standards to the CCSS. Doing so allows for the development of more sound expectations and goals for students to achieve because it provides a better understanding of why the CCSS were developed and written the way they were.

History of the Mathematical Standards

Founded in 1920, the National Council of Teachers of Mathematics (NCTM) was formed partly in response to a movement that would have diminished the role of mathematics in the secondary curriculum (McLeod, 2003).

> Collegiate mathematicians as well as many high school mathematics teachers were determined that mathematics should be a significant part of the school curriculum and that they, rather than educational administrators or other generalists, should be the ones to decide what the mathematics curriculum would include (McLeod, 2003, p. 754).

Decades to follow were filled with the dominant role in mathematics education played by mathematicians; shifting to mathematics educators in the 1970s (McLeod, 2003).

As the 70s progressed, so did the availability of computers. It was soon apparent that the traditional "drill and practice" approach was no longer suitable for adequately preparing students, and that "a much more complex approach would be needed to change school mathematics" (McLeod, 2003, p. 758). With this in mind, a committee was appointed to develop *An Agenda for Action*; a set of guidelines for school mathematics in the 1980s. "The *Agenda* recommended that problem solving be the focus of school mathematics, that basic skills be defined more broadly than in the past, and that calculators and computers be used at all grade levels" (McLeod, 2003, p. 761). Although creating the *Agenda* was a step in the right direction, it became clear that more specificity and structure was needed.

As the 1980s came to an end, various educational professionals gathered to create the *Curriculum and Evaluation Standards for School Mathematics*; finalized in 1989 (National Council of Teachers of Mathematics, 2014). “The document marked a historically important first step by a professional organization to articulate extensive goals for teachers and policymakers in a school discipline” (National Council of Teachers of Mathematics, 2014), and was “intended to be evolutionary in nature, not revolutionary” (McLeod, 2003, p. 773). Additional components to the standards released in 1989 include *Professional Standards for Teaching Mathematics* in 1991 and *Assessment Standards for School Mathematics* in 1995. The last revision to the NCTM Standards was completed in 2000 under the Standards 2000 Project (National Council of Teachers of Mathematics, 2014); however by this point and few years to follow, every state had developed and implemented its own standards for learning (Common Core State Standards Initiative, 2014). This, along with the fact that each state had its own definition for proficiency, forced there to be a lack of standardization across the country; one of the reasons why the states concluded that a new set of national standards needed to be created (Common Core State Standards Initiative, 2014).

In the spring of 2009, roughly two years after discussion of revamping the national standards began, the process of creating the new standards commenced (Common Core State Standards Initiative, 2014). Those who helped develop what became the CCSS consisted of content experts, states, teachers, school administrators, and parents (National Governors Association, 2010). Initially, the standards were divided into the two categories of college- and career-readiness standards and K-12 standards. However, during the creation process the college- and career-readiness standards were developed first, but were then incorporated into the K-12 standards (Common Core State Standards Initiative, 2014).

Released on June 2nd, 2010, the Common Core State Standards are intended to be "a balance of concepts and skills, with content standards that require both conceptual understanding and procedural fluency" (K-12 Blueprint, 2014, p. 3). The claim is that the standards are

> aligned with college and work expectations, are clear, understandable and consistent, include rigorous content and application of knowledge through high-order skills, build upon strengths and lessons of current state standards, informed by other top performing countries so that all students are prepared to succeed in our global economy and society, and are evidence- and research-based (National Governors Association, 2010);

however, the effectiveness of the standards cannot be adequately measured until the first group of students passes through grades K-12. Nevertheless, just four years after being released, "43 states, the Department of Defense Education Activity, Washington, D.C., Guam, the Northern Mariana Islands and the U.S. Virgin Islands have adopted the CCSS" (Common Core State Standards Initiative, 2014). The more states that adopt the CCSS and align curricula to its aims, the easier it will be to implement them nation-wide.

Figure one displays six shifts in mathematics that must occur in order for implementation of the Common Core to be successful. Although educators may have to change their curriculum in order to align their materials with the requirements of the CCSS, there are several strategies and perspectives to consider that will aid this process. One such strategy is being aware of students' zones of proximal development.

Shifts in Mathematics		
Shift 1	Focus	Teachers significantly narrow and deepen the scope of how time and energy is spent in the math classroom. They do so in order to focus deeply on only the concepts that are prioritized in the standards.
Shift 2	Coherence	Principals and teachers carefully connect the learning within and across grades so that students can build new understanding onto foundations built in previous years.
Shift 3	Fluency	Students are expected to have speed and accuracy with simple calculations; teachers structure class time and/or homework time for students to memorize, through repetition, core functions.
Shift 4	Deep Understanding	Students deeply understand and can operate easily within a math concept before moving on. They learn more than the trick to get the answer right. They learn the math.
Shift 5	Application	Students are expected to use math and choose the appropriate concept for application even when they are not prompted to do so.
Shift 6	Dual Intensity	Students are practicing and understanding. There is more than a balance between these two things in the classroom – both are occurring with intensity.

Figure 1 (EngageNY, 2013)

Zone of Proximal Development

Research has shown that as teachers, it is important to get to know students on multiple levels (i.e. ability, social, academic, etc.) because the more teachers know their students, the better they can determine their academic needs and take actions to make sure they are met. One of the many areas in which teachers can get to know their students is on a metacognitive level. Defined as, "… students' awareness of their own cognitive process, and the regulation of these processes in order to achieve a particular goal" (Brown, Bransford, Ferrara and Campione, 1983; Flavell, 1976; Goos, Galbraith, and Renshaw, 2003, p. 193), metacognition is a crucial concept to keep in mind when determining the best practices to meet the needs of all students involved. This gives reason as to why the zone of proximal development (ZPD) is an important tactic to consider using in the classroom.

The question that first arises is, 'how can teachers meet students' metacognitive needs?' Goos et al. (2003) provides an answer and cites,

> In mathematical problem solving, regulation of cognition involves such activities as planning an overall course of action, selecting specific strategies, monitoring progress, assessing results, and revising plans and strategies if necessary (Garofalo and Lester, 1985; p. 193).

Although this can be time consuming, it benefits teachers and students alike; not only will teachers be able to find ways to assess and push student thinking further, but if done correctly, students will find their *own* ways to push their thinking further. This potentially aligns with the ideas behind the CCSS because such thinking may allow students to get closer to being college- and career-ready.

The ZPD Model has four stages (Siyepu, 2013), the first of them dealing with the development of basic understanding by relying on others (i.e. the instructor). It is natural to learn by observing others, however it is important that students are given the chance to complete tasks on their own so a deeper understanding can be attained and the possibility for higher-level thinking can occur. The second stage of ZPD is where the transition to independent practice begins. Here, students connect to their prior knowledge in order to help direct them on what to do next. "The ZPD occurs between the first and second stages. Learners practice alone, which implies that they perform certain activities without assistance. However, they are not at a stage of perfect proficiency and require some assistance sometimes" (Siyepu, 2013, p. 6).

Performance is developed and independence is achieved in stage three, while stage four consists of those who "… are at the de-automatisation of performance that leads to the process of repeating a function, each time applying it to the results of the previous stage through the ZPD" (Siyepu, 2013, p. 6). A key observation to make about ZPD is that when taken out of the context of a classroom, it can be used to learn any task. "Lifelong learning by any individual is made up of the same regulated ZPD sequences, from other-assistance to self-assistance recurring over and over again for the development of new capacities" (Gallimore & Tharp, 1990; Siyepu, 2013, p. 6). This, too, potentially supports students being college- and career-ready.

In a two-year study where classroom practices and student interactions were monitored, high school students were given mathematical problems to solve for which they did not have a predetermined solution method (Goos et al., 2003). The exchanges the students shared while collaborating on the solution processes were recorded and transcribed so they could be analyzed. Goos et al. (2003) determined that, "… the discussion around, and generated by, individual metacognitive acts is crucial to the success of the mathematical enterprise" (p. 213), as well as finding "challenge as a stimulus for mathematical thinking" (p. 218). Collectively, it can be concluded that challenging students helps stimulate their brains so they can begin to not only problem solve, but do so in a way that enables them to monitor and self-correct their own solution processes. As a result, they are working within their zone of proximal development. The study places a heavy emphasis on student-student interactions, however the role of the teacher is equally vital. "Critics of group work often claim that students' thinking can become confused without the teacher's guidance, and this observation raises questions as to whether, when, and how the teacher should intervene to redirect students' efforts to more productive ends" (Goos et al., 2003, p. 220). This skill is something that will come with practice for the teacher. Students cannot always be left on their own to solve a problem, but at the same time cannot be led through each and every question if they are to achieve higher-level thinking. It is in these types of situations for which inquiry based approaches would work best. Instead of suggesting what struggling students should do next, *ask* them what *they* think they should do next. This way, the teacher can assess their thinking and determine the next step without confusing the student or preventing them from discovering something on their own (Goos et al., 2003).

As with most cases in education, learning to facilitate classes in which students (and teachers) are aware of their metacognition will take time and practice to master. It can be viewed as a balancing act of teacher involvement. "The scaffolding process is best understood as involving mutual adjustment and appropriation of ideas rather than a simple transfer of information and skills from teacher to learner" (et al., 1993; Packer, 1993; Wertsch, 1984; Goos et al., 2003, p. 195). It is one thing to instruct students on how to solve a problem, but being aware of their metacognition, as well as their zones of proximal development, helps them determine *why* they are thinking in the ways they are. This only helps push their ideas further and to more expansive levels, which they can collaboratively share with others so students' needs are met and goals can be achieved. One strategy mathematics teachers can use to teach any concept while incorporating these ideas is through the utilization of the Rule of Four.

The Rule of Four

The Rule of Four is a technique teachers can use to teach any mathematical content to various groups of students. It is an extension of the Rule of Three, which was developed by the Calculus Consortium at Harvard University (Gleason & Hughes-Hallett, 1992). Consisting of geometric, numeric, analytic, and verbal representations, the Rule of Four provides multiple and distinct solution methods to any given problem.

Geometric representations consist of diagrams, figures, graphs, models, pictures, coordinate planes, etc. Often referred to as "graphic representations," they help to add clarity to the content (Cal Teach, 2014). Numeric (or number) representations include raw data, lists, tables, etc. and refer to organizing data for display. Although seemingly simple, it can have a huge impact.

> In the real world, data is presented numerically more often than in any other form. That's because data is initially generated as a collection of raw numbers. Only with manipulation does it become a graph, picture, algebraic equations, or paragraph. Students need to understand numbers in their raw form. Otherwise, they will not have the flexibility to present them accurately in other arrangements (Cal Teach, 2014, p. 2).

This is perfectly aligned with the CCSS in the sense that it may help students become college- and career-ready.

Analytic representations refer to algebraic and symbolic representations, which makes it possible for the expression of complex thoughts into words and symbols (Cal Teach, 2014). For example, by the Pythagorean Theorem, the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two legs. Under this rule, however, it can simply be written as $a^2 + b^2 = c^2$. If left in its written form, it would fall under the last representation set; verbal. Including both spoken and written forms, verbal representations include lectures, classroom discussions, textbooks, class notes, handouts, etc. (Cal Teach, 2014). Figure two is a diagram that simplifies the four components of the Rule of Four.

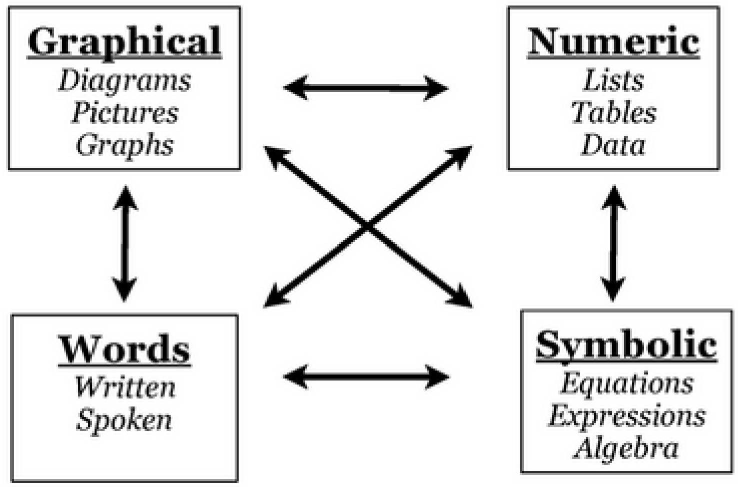


Figure 2 (SFUSD Mathematics Department, 2014)

Although it seems time consuming, implementing the Rule of Four helps educators meet their students' academic needs; thus it should be viewed as an investment.

> Because students have various learning styles, preferences, and dispositions, teachers who use the Rule of Four facilitate learning for all students by providing several presentations of the same concept… and all students, regardless of their dispositions for learning, will benefit from seeing various representations and identifying the interconnections among them (Annenberg Learner, 2014).

From the perspective of the students, they may not be accustomed with representing their work in such a variety of ways. In alignment with stages one and two of the ZPD model,

> Explicit modeling from the teacher and giving students many opportunities to practice representing their work in multiple ways is an effective way to teach students to think about their math work in this fashion. It is equally important that students make

connections among the representations. These connections lead to a deeper understanding over time (SFUSD Mathematics Department, 2014).

Chapter Three: Curriculum Project Design

Overview

The following curriculum project is designed for implementation in an Algebra II mathematics class, specifically on the unit of trigonometric functions. According to the New York State Common Core (2014), by this point students have had some introduction to sine, cosine, and tangent ratios for right triangles in their Geometry course most likely taken the previous year. Furthermore, students were required to derive and use the formula for the area of an arbitrary triangle, as well as prove and apply the Laws of Sine and Cosine. This unit builds upon these conjectures and provides opportunity for deeper understanding of trigonometry, which will be expanded further if the student chooses to advance to Pre-Calculus.

Unit Timeline

Date	Day	Topic
	1	An Introduction to Trigonometry and Radian Measures
	2	Special Triangles and the Unit Circle
	3	Reference Angles
	4	Graphing Exploration
	5	Tangent
	6	Co-functions
	7	Graphing Sine and Cosine
	8	Graphing Sine and Cosine, continued
	9	Modeling with Trig Functions
	10	Review
	11	Unit Test

Common Core State Standards

The following CCSS are addressed in this unit (New York State Common Core, 2014). Note that each lesson plan lists the standard(s) to be covered for that particular lesson, as well as any additional resources used to create the materials for that day.

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.1

  Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2

  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

- ✓ CCSS.MATH.CONTENT.HSF.TF.B.5

  Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline

- ✓ CCSS.MATH.CONTENT.HSS.ID.B.6.A

  Fit a function to the data; use functions fitted to data to solve problems in the context of data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models

- ✓ CCSS.MATH.CONTENT.HSF.IF.C.7.E

  Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude

Day One: An Introduction to Trigonometry and Radian Measures

Objectives

- ✓ Students will understand some origin of trigonometry and how it may have shaped the foundation it has been built upon (geometrically)
- ✓ Students will explore what a radian angle is, as well as what 1 radian is equal to (geometrically, algebraically, numerically)
- ✓ Students will derive the proportion used to convert radians and degrees (algebraically, numerically)

NOTE: The guided notes format supports the written verbal component of the Rule of Four, while class discussions support the spoken verbal component. This is true for all lessons throughout this unit.

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.1
  Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle

Materials

- ✓ Notes packet

Before Phase (5-10 min): Have an open discussion about what students recall about trigonometry from geometry. Write down what students say on the board, and discuss what will be expanded on during this unit. This will help students recall information from last year and make connections with the new material.

During Phase (40 min): Complete pages 1 – 3 in the guided notes packet

- ✓ Page 1: Trigonometry and its origins
- ✓ Page 2: Basics of graphing an angle and coterminal angles
- ✓ Page 3: Radians and deriving the conversion proportion

After Phase: Worksheet 1 can be given as homework, class work, or be picked apart for an exit ticket.

Modifications: If there are students who have an IEP that require calculators, try to supply basic four-function calculators rather than graphing calculators (unless it is noted that a graphing calculator will be required). This is true for all of the lessons in the unit.

References: https://www.engageny.org/resource/algebra-ii-module-2; Ellis, 2012

****Note:** The complete reference is listed in “References” section on page 79.
This is true for all further lesson references.

Name____ Date_____________
Algebra II Trig Functions

Module 2: Trigonometric Functions

Day One: An Introduction to Trigonometry and Radian Measures

- What is **trigonometry**?

Trigonometry is the study of _______________________. It explores how ________________ affects ________________ and/or __________________.

- What are its origins?

Ancient scholars in Babylon and India conjectured that celestial motion was _____________; the sun and the stars orbited the earth in a circular fashion. The earth was presumed the center of the sun's orbit. The quadrant numbering in a coordinate system is consistent with the counterclockwise motion of the sun, which rises from the east and sets in the west.

What would this look like graphically?

The 6^{th} century Indian scholar Aryabhata created the first ______________________, using a measurement he called __________. The purpose of his table was to calculate the position of the sun, the stars, and the planets.

- **The basics of graphing an angle**

Initial Side:____

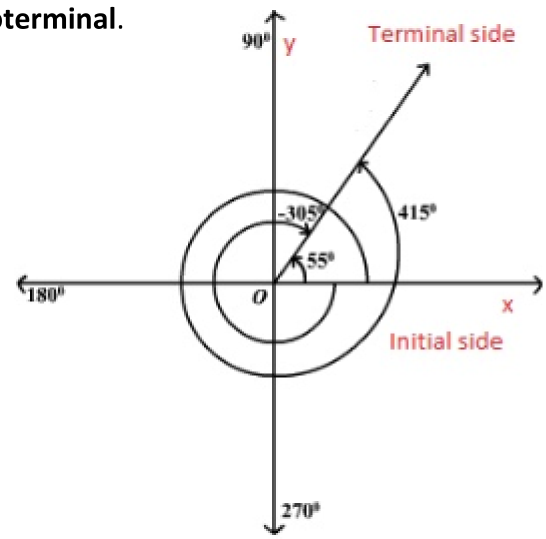
Terminal Side:____

Coterminal angles:____

____

Notice from the diagram, 55°, -305°, and 415° are all **coterminal**.

What does this look like **algebraically?**



Notice how the angles are sketched!

Positive angles travel ____________________________

Negative angles travel ____________________________

Practice: Sketch the angles below. Then determine 2 additional angles that are coterminal.

120°   -120°   -730°   300°

- **Radians**

A **radian angle** is ____

____

____

One radian (I rad) is ____

____

____

One radian is literally "____________________________________" around a circle

Radian measure is determined by____

What angle, in radians, corresponds to 360°? Why?

We can use this result to convert between radians and degrees.
What proportions should we use?

Radians → Degrees: Degrees → Radians:

Practice: Convert the following degree measure to radians, or radian measures to degrees.

135° $\frac{3\pi}{2}$ -330° $\frac{7\pi}{4}$

Name____ Due Date________________

Algebra II Trig Functions

Worksheet 1

1. Convert each angle in degrees to radians. Leave your answer in terms of π.
   a. $18°$ b. $76°$ c. $-40°$ d. $200°$

2. Convert each angle in radians to degrees. Round to the nearest hundredth.
   a. 2 rad b. $\frac{\pi}{13}$ c. $\frac{3\pi}{17}$ d. $-\frac{2\pi}{5}$

3. Sketch each of the angles below and label the quadrant where the terminal side is. Then determine 2 coterminal angles for each and sketch them on the same set of axes.

   a. 46°

   b. -120°

   c. $\frac{3\pi}{5}$

   d. $-\frac{\pi}{3}$

Day Two: Special Triangles and the Unit Circle

Objectives

- ✓ Students will recall the trigonometric ratios they learned in geometry to determine the special triangle ratios (graphically, numerically, algebraically)
- ✓ Students will use the special triangles to construct a unit circle (graphically, numerically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

Materials

- ✓ Notes packet

Before Phase (5 min): Use the "recall" at the top of page 5 to generate a short discussion about the trig ratios and how they will be very useful throughout this unit.

During Phase (45 min): Complete pages 5 – 7 in the guided notes packet

- ✓ Page 5: Special triangles
- ✓ Page 6: The unit circle
- ✓ Page 7: Constructing the unit circle with the special triangles

After Phase: Worksheet 2 can be given as homework, class work, or be picked apart for an exit ticket.

Modifications: If students do not finish the unit circle construction activity in class, it can be assigned as homework. The triangles used to construct the circle are traced from the special triangles on page 5.

References: Ellis, 2012

Day Two: Special Triangles and the Unit Circle

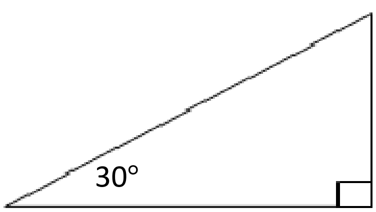
- **Recall:** In geometry, you derived trig ratios based on right triangles. What were they?

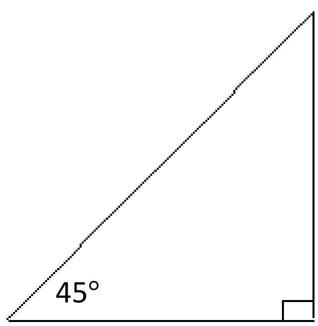
$\sin\theta =$ $\qquad\qquad\qquad$ $\cos\theta =$ $\qquad\qquad\qquad$ $\tan\theta =$

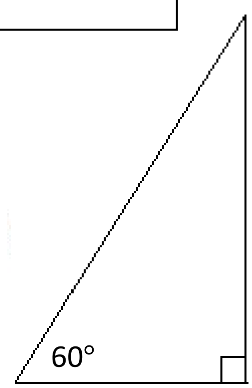
In trigonometry, we have ________________________________ that help us find ____________ ________________________ of trig functions. These triangles contain _____________ _________________, which are _______ (   ), _______ (   ), and _______ (   ). We call these ______________________________ because they yield ____________________ _______________________.

- So what are these special triangles?

Ratio of Sides for 30° - 60° - 90°	Ratio of Sides for 45° - 45° - 90°







sin30 =

cos30 =

tan30 =

sin45 =

cos45 =

tan45=

sin60 =

cos60 =

tan60 =

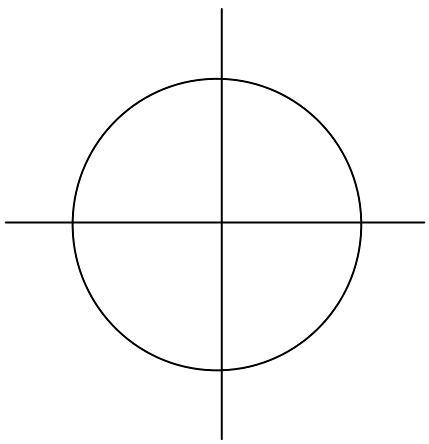
Special triangles are extremely helpful when dealing with the ______________________________.

- ➢ Wait, what is the **unit circle?**

The **unit circle** is a ____

____.

At any point on the unit circle, we can ____



Hypotenuse=

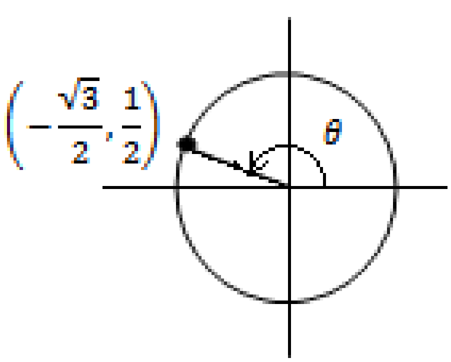
Width =

Height =

Therefore, a point on the unit circle directly gives us

the ________________ and _____________________ of the angle.

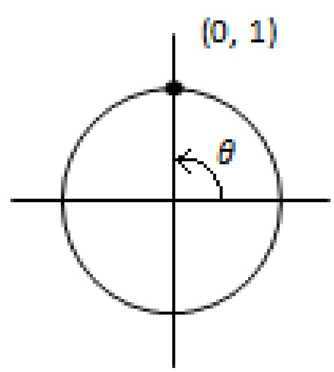
Practice: Determine $\sin\theta$ and $\cos\theta$.



θ = 150°

sin θ=

cos θ=



θ = 90°

sin θ=

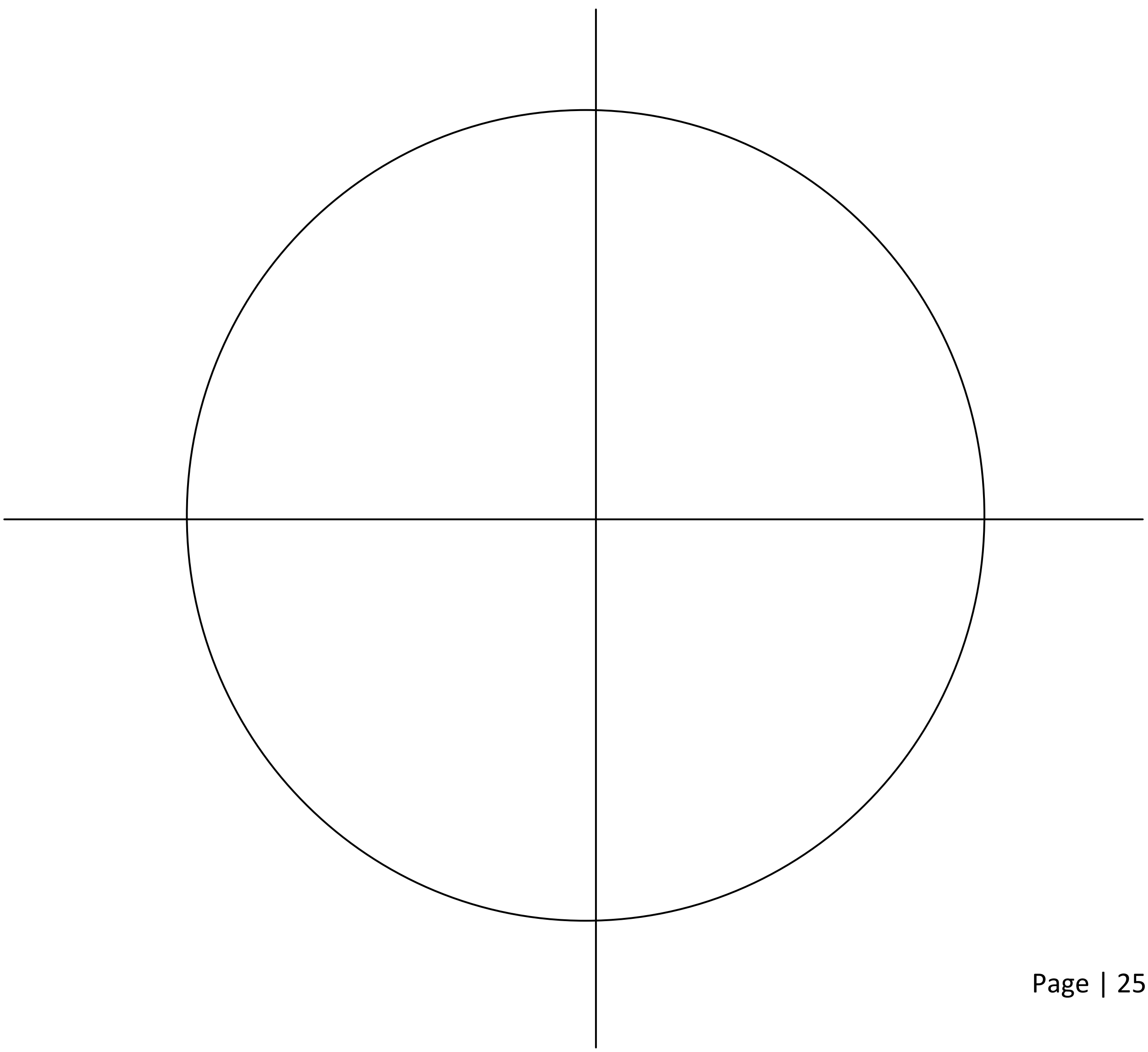
cos θ=

- Constructing the **unit circle** with **special triangles**

Trace the triangles from page 5 to mark…

- ✓ Quadrant ____: _________ (________), _________ (________), _________ (_______)
- ✓ Quadrant ____: _________ (________), _________ (________), _________ (_______)
- ✓ Quadrant ____: _________ (________), _________ (________), _________ (_______)
- ✓ Quadrant ____: _________ (________), _________ (________), _________ (_______)

*****Make sure you are placing the triangles so the right angle lies on the x axis and the hypotenuse extends from the origin (think of how the sun moves)!!*****



Name____ Due Date_______________
Algebra II Trig Functions

Worksheet 2

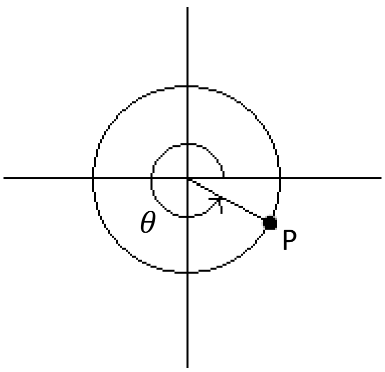
1. Fill out the tables below based on the unit circle you constructed in class.

θ (degrees)	0°	30°	45°	60°	90°	120°	135°	150°
θ (radians)								
$\sin\theta$								
$\cos\theta$								

θ (degrees)	180°	210°	225°	240°	270°	300°	315°	330°	360°
θ (radians)									
$\sin\theta$									
$\cos\theta$									

2. Fill in any missing information below. Each problem should have:
    - ✓ A sketch of the angle
    - ✓ A value for θ
    - ✓ A coordinate for the point on the unit circle (denoted by P)
    - ✓ A value for sin θ (as a fraction)
    - ✓ A value for cos θ (as a fraction)

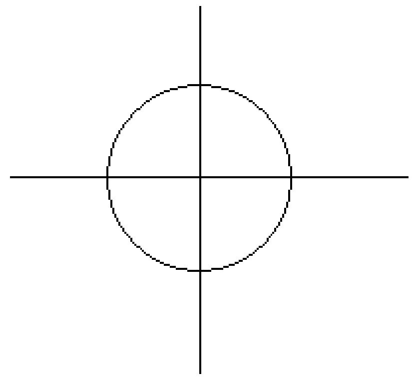
a.



$\theta = \frac{11\pi}{6}$ P =

sin θ = cos θ =

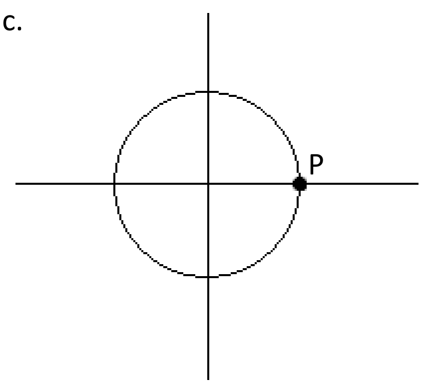
b.



θ = P = $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

sin θ = cos θ =

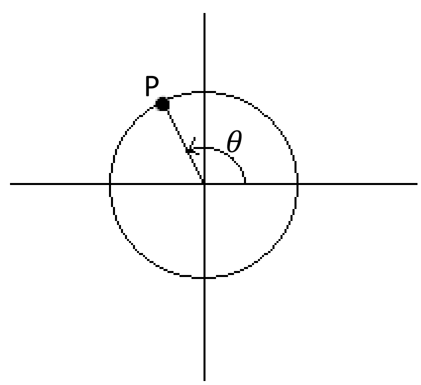
c.



θ = P =

sin θ = cos θ =

d.



θ = P =

sin θ = $\frac{\sqrt{3}}{2}$ cos θ =

Day Three: Reference Angles

Objectives

- ✓ Students will connect how special triangles can be used to find reference angles (graphically, algebraically)
- ✓ Students will use reference angles to determine exact values (graphically, algebraically, numerically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

Materials

- ✓ Notes packet

Before Phase (5 min): Use the “recall” at the top of page 10 to generate a short discussion about the special triangle ratios and how they will be very useful throughout this unit.

During Phase (40 min): Complete pages 10 – 12 in the guided notes packet

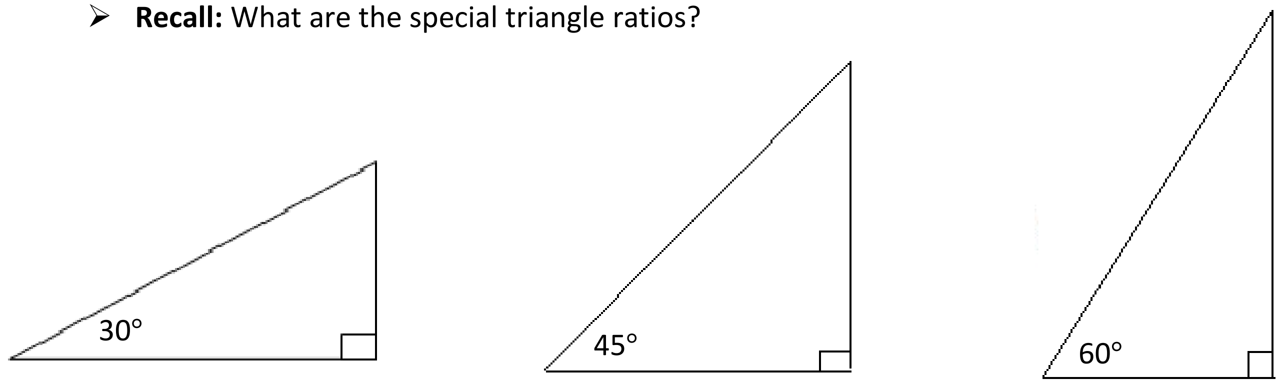
- ✓ Page 10: Reference angles
- ✓ Page 11: Using reference angles and special triangles to find an exact value
- ✓ Page 12: Practice

After Phase: Worksheet 3 can be given as homework, class work, or be picked apart for an exit ticket.

Modifications: For the practice examples on page 12, use your best judgment for either working as a class or letting students work independently/small groups (2 – 3). To help differentiate this portion of the lesson, lead a small group discussion for those who are struggling in one part of the room and allow others who feel confident to work on their own/in a small group.

Day Three: Reference Angles

- **Recall:** What are the special triangle ratios?



- So why are **special triangles** important?

Special triangles can help us determine ____________________________________, which can help us calculate the ____ of an angle.

A **reference angle** is ____

____

Algebraically, this looks like:

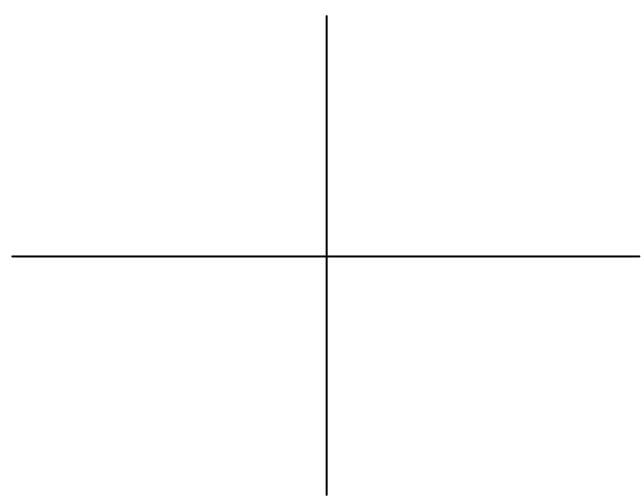
Quadrant I: reference angle =_______________

Quadrant II: reference angle =______________

Quadrant III: reference angle =______________

Quadrant IV: reference angle =______________

Graphically, this looks like:



- So how do we use **reference angles with special triangles**?

1.____

____

2.____

____

3.____

____

(x, y) =

(x, y) =

(x, y) =

(x, y) =

Practice: Find the exact value of each of the following:

1. $\sin 135°$

2. $\cos \frac{7\pi}{6}$.

3. sin(-315).

4. cos 300°.

5. tan$\frac{5\pi}{6}$

Name____ Due Date________________

Algebra II Trig Functions

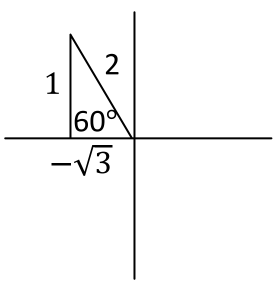
Worksheet 3

1. Determine the reference angle for each of the following. Then find the exact value.

a. 330° b. -330° c. $\frac{2\pi}{3}$ d. $-\frac{3\pi}{4}$

2. Find the one error in each of the following solutions of finding an exact value. Correct it and provide a correct response.

a. cos120°



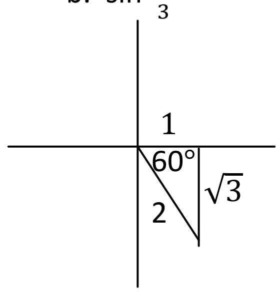
120° → Q 2

ref angle = 180° - 120 = 60°

In Q 2: cos60° = $\frac{A}{H}$ = $\frac{-\sqrt{3}}{2}$

→ cos120° = $\frac{-\sqrt{3}}{2}$

b. sin $\frac{5\pi}{3}$



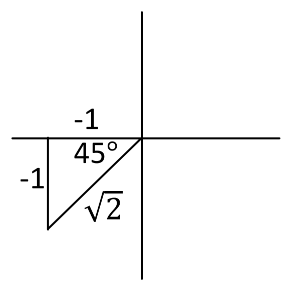
$\frac{5\pi}{3}$ = 300° → Q 4

ref angle = 360° - 300 = 60°

In Q 4: sin60° = $\frac{O}{H}$ = $\frac{\sqrt{3}}{2}$

→ sin300° = $\frac{\sqrt{3}}{2}$

c. tan225°



225° → Q 3

ref angle = 225° - 180 = 45°

In Q 4: tan45° = $\frac{O}{H}$ = $\frac{-1}{\sqrt{2}}$ = $-\frac{\sqrt{2}}{2}$

→ tan225° = $-\frac{\sqrt{2}}{2}$

Day Four: Graphing Exploration

Objectives

- ✓ Students will explore the relationship between special triangles, the unit circle, and the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ (graphically, numerically, verbally)
- ✓ Students will begin to explore periodicity (graphically, algebraically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

Materials

- ✓ Graphing Exploration Packet

Before Phase (5-10 min):

- ✓ Lead a brief discussion about the unit circle and special triangles, and how the values can also be represented graphically.
- ✓ Briefly run through the exploration with the class

During Phase (30 – 40 min):

- ✓ Have students get into groups of 2 – 3 to complete the exploration
  - o Students will fill out a table based off the unit circle they constructed
  - o Students will produce one cycle of $f(x) = \sin x$ and $f(x) = \cos x$ based off this table, which is connected to one rotation around the unit circle
  - o Students graph what they think the graphs will look like after 2 rotations
  - o Students explain why these graphs are infinite
  - o Students produce graphs for 1 backwards rotation around the unit circle to conclude they are infinite in both directions
  - o Students state the domain and range of $f(x) = \sin x$ and $f(x) = \cos x$
  - o Students explore the patterns periodic functions
- ✓ Circulate throughout the room and help when needed

After Phase: Worksheet 4 can be given as homework, class work, or be picked apart for an exit ticket.

Modifications: Worksheet 4 contains error analysis questions involving different solution methods. If students finish the exploration early, run through some of these examples as a class to reinforce various solution methods pertaining to the Rule of Four.

References: obtained graphs from Google image search.

Name____ Date_________________
Algebra II Trig Functions

Graphing Exploration

We have been studying how to find exact values using special triangles and the unit circle, however there are other options! This exploration will help you create the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ so you can see how they relate to the unit circle.

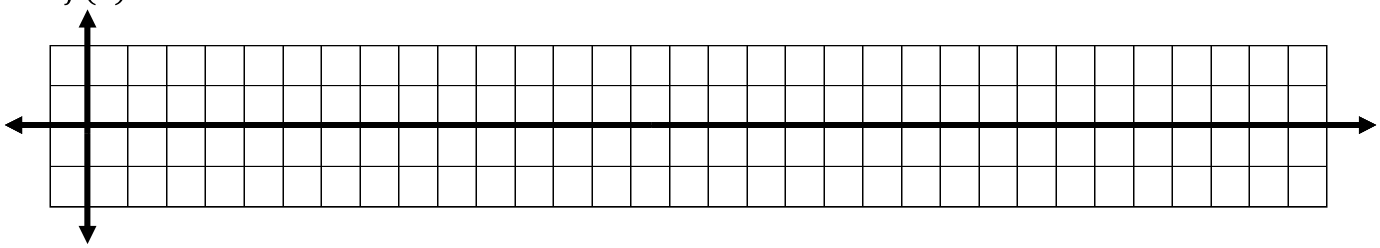
- ✓ Fill out the tables below based off the unit circle you constructed in class.
- ✓ Mark where each of the quadrants are on the table.

θ (degrees)	0°	30°	45°	60°	90°	120°	135°	150°
θ (radians)								
$\sin\theta$								
$\cos\theta$								

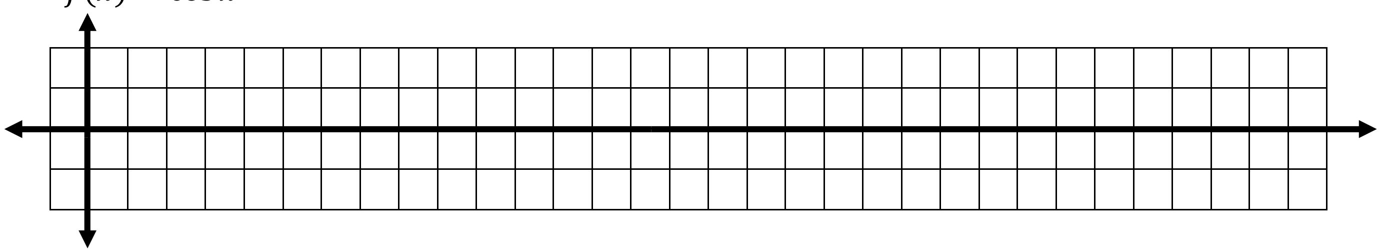
θ (degrees)	180°	210°	225°	240°	270°	300°	315°	330°	360°
θ (radians)									
$\sin\theta$									
$\cos\theta$									

What would these look like **graphically**?

$f(x) = \sin x$



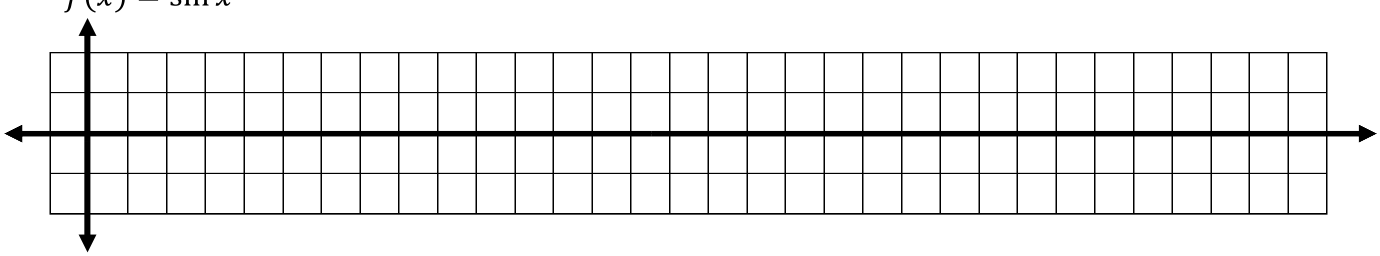
$f(x) = \cos x$



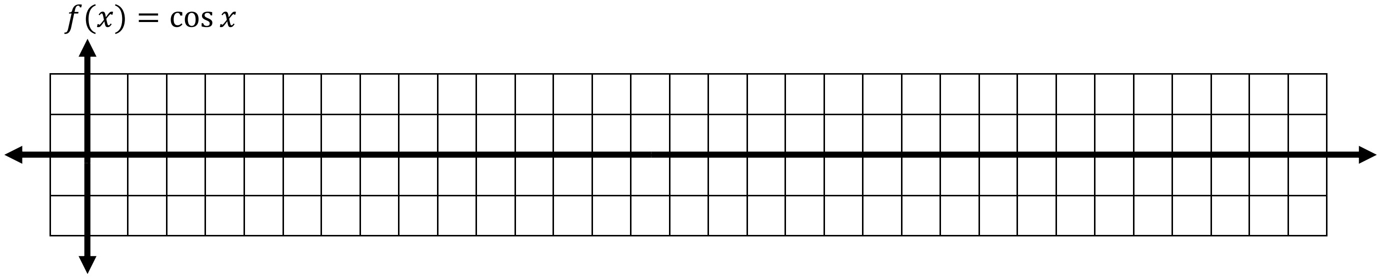
The graphs you just made represent one cycle of each function, which can be represented by one rotation around the unit circle. How many cycles do you think you would see if you went around the unit circle 2 times? Graph them below.

- ✓ Mark where each of the quadrants are on your graphs

$f(x) = \sin x$



$f(x) = \cos x$

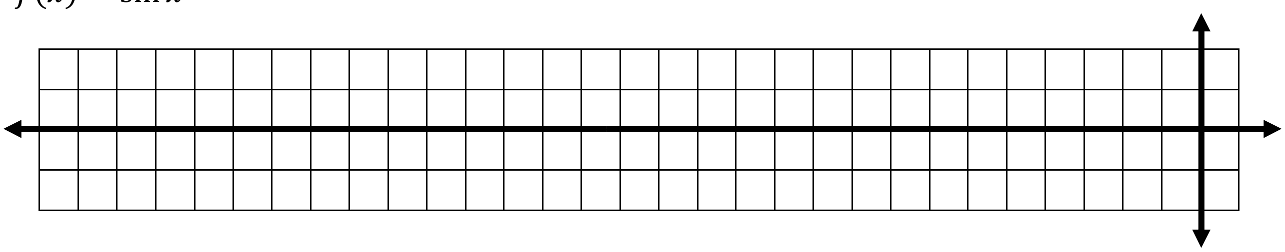


How many cycles do you think you would see if you traveled around the unit circle 3 times? 4 times? Do you think this go on infinitely? Why?

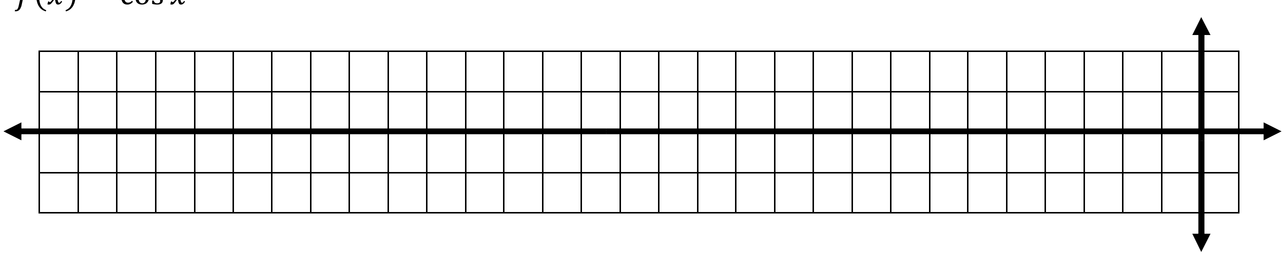
Now graph what $f(x) = \sin x$ and $f(x) = \cos x$ would look like if you traveled BACKWARDS around the unit circle one time.

- ✓ Mark where each of the quadrants are on your graphs

$f(x) = \sin x$



$f(x) = \cos x$



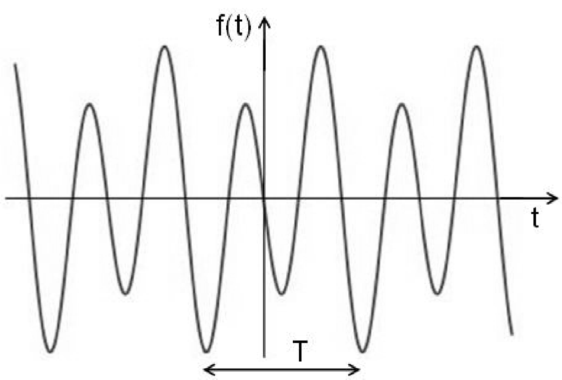
Based on what happened when you graphed multiple cycles of $y = \sin\theta$ and $y = \cos\theta$ in the positive direction, do you think you can graph infinite cycles in the negative direction? Why?

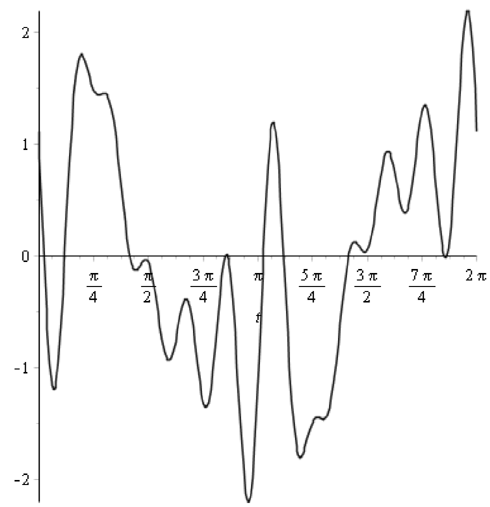
State the domain and range for $f(x) = \sin x$ and $f(x) = \cos x$.

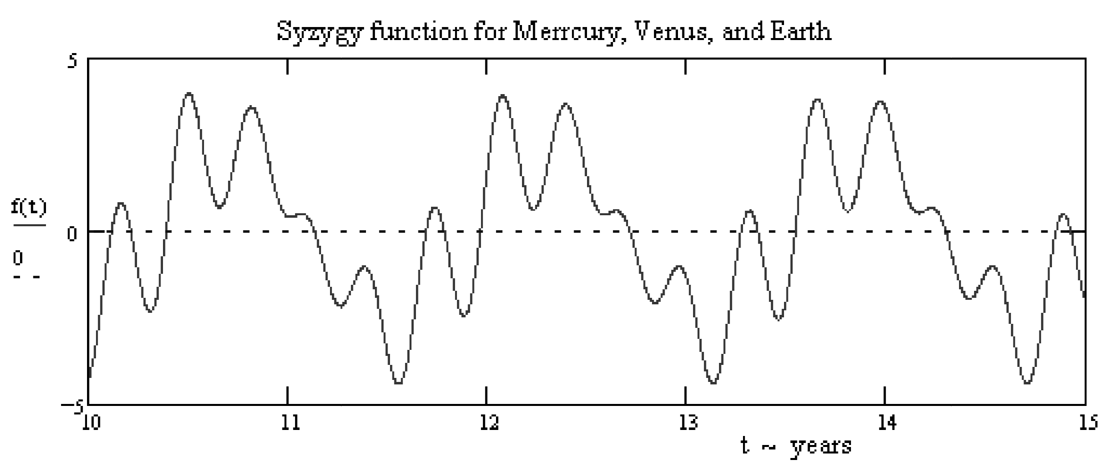
Periodic is a term we use to describe a function that repeats values at fixed intervals. Are sine and cosine periodic?

Algebraically, this means for any period $P > 0$ (where the **period** is the length of one curve), the function f contains $x + P$ whenever it contains x, and $f(x + P) = f(x)$ for all real numbers in its domain.

Determine which of the functions below are **periodic**.







Name____ Due Date________________

Algebra II Trig Functions

Worksheet 4

We have learned multiple ways to determine the sine and cosine of a given value. We can use the unit circle with special triangles, the exact values chart (ref. angles 30°, 45°, and 60°), and the graphs of $y = \sin x$ and $y = \cos x$. Solve each of the following using a solution method of your choice. Show your work or explain how you got your answer.

1. Find the one error in each of the following solutions of finding an exact value. Correct it and provide a correct response.

a. $\sin 0° + \cos 30°$

The graph of $y = \sin x$ starts at 1, so sin0° = 1

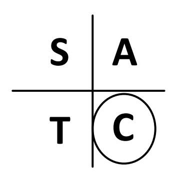
From the exact values chart, cos30° = $\frac{\sqrt{2}}{2}$

Therefore sin0° + cos30° = $1 + \frac{\sqrt{2}}{2} = \frac{2}{2} + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}$

b. $\cos\frac{5\pi}{3} \div \cos\frac{3\pi}{2}$

$\cos\frac{5\pi}{3}$ = cos300° $\cos\frac{3\pi}{2}$ = cos270°

300° → Q 4 → ref angle = 60° → $\cos 60° = \frac{1}{2}$



In Q 4 → cosine is positive → cos300° = $\frac{1}{2}$

270° is on the unit circle at the point (0, -1)

cos270° = y-value = -1

→ $\cos\frac{5\pi}{3} \div \cos\frac{3\pi}{2} = \frac{1}{2} \div (-1) = -\frac{1}{2}$

2. Find the exact value of each of the following.

a. $\sin 90° + \cos 210°$

b. $\frac{\cos 0}{\sin 0}$

c. $\sin\frac{11\pi}{6} - \cos\frac{2\pi}{3}$

d. $\cos\left(\frac{5\pi}{4}\right)\sin\left(\frac{7\pi}{4}\right)$

e. $\sin^2(150°)$

**Hint: $\sin^2(x) = (\sin x)^2$

e. $(\cos 405°)(\cos 225°)$

Day Five: Tangent

Objectives

- ✓ Students derive the ratio for tangent in terms of sine and cosine (graphically, algebraically, numerically)
- ✓ Students derive that tangent can be viewed as the slope of the triangle on a unit circle (graphically, algebraically)
- ✓ Students understand why it is called "tangent" (graphically, algebraically)
- ✓ Students use tangent to determine exact values (numerically, algebraically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

Materials

- ✓ Notes packet

Before Phase (5-10 min): Use the "recall" at the top of page 14 to help set up the triangle used to derive tangent.

During Phase (40 min): Complete pages 14 – 16 in the guided notes packet

- ✓ Page 14: Defining tangent as $\frac{\sin\theta}{\cos\theta}$
- ✓ Page 15: Tangent as the slope of the hypotenuse and the graph of $f(x) = \tan x$
- ✓ Page 16: Why we call it "tangent" and practice

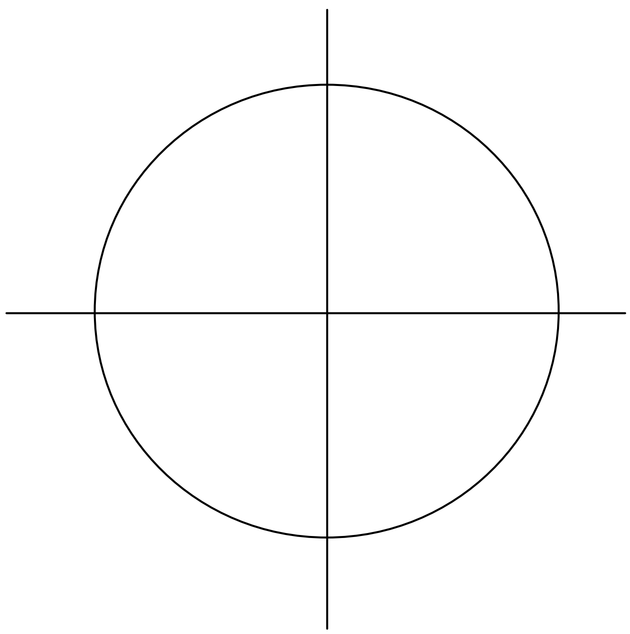
After Phase: Worksheet 5 can be given as homework, class work, or be picked apart for an exit ticket.

Modifications: When writing the domain of $f(x) = \tan x$, review of how to determine complex domains may be necessary. Reinforce checking the domain by plugging in a value for *k* and seeing if it holds true.

References: https://www.engageny.org/resource/algebra-ii-module-2

Day Five: Tangent

- **Recall**
  1. The trig ratio for tangent is _______________
  2. The x-value on the unit circle is the value of _________________
  3. The y-value on the unit circle is the value of _________________



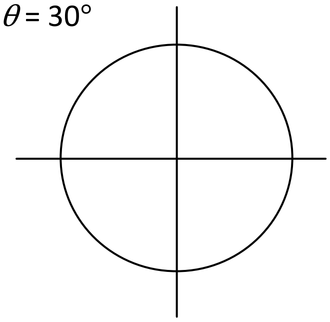
tan θ =___________

- What does this look like **numerically**?

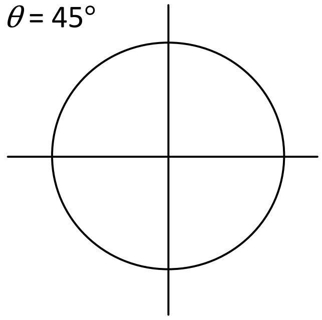
θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	270° or $\frac{3\pi}{2}$	360° or 2π
sin θ							
cos θ							
tan θ							

We just concluded tan θ =_________. What is another term for this ratio?____________________

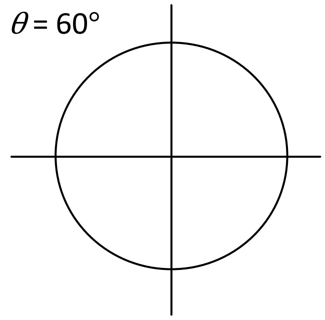
$\theta = 30°$



$\theta = 45°$

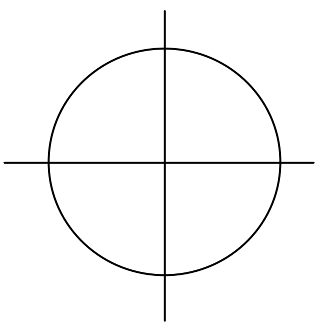


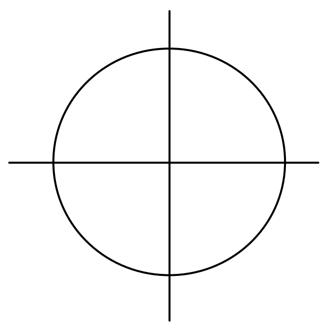
$\theta = 60°$



The slope when $\vartheta = 0°$ is ________________ The slope when $\vartheta = 90°$ is ________________

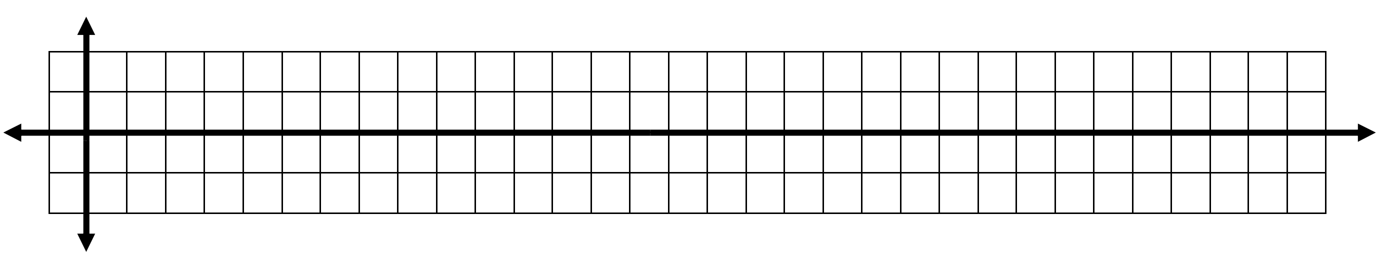
- **Geometrically**, what can you conclude about tan0 and tan90?



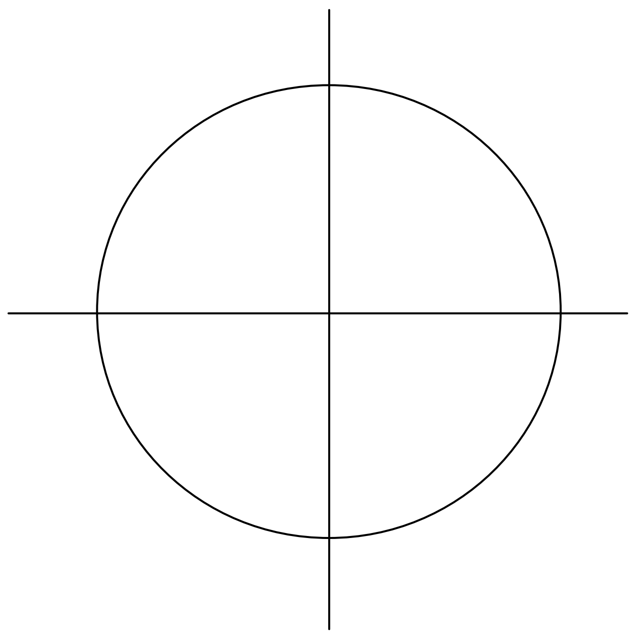


- What does this look like **graphically**?

$f(x) = \tan x$ Domain: Range:



- So why do we call it "tangent"?



The value of tangent also represents the

length of ______________________________

____

____

____

____

Practice: Determine the exact value of each of the following.

1. tan45°

2. tanπ

3. tan330°

4. tan $\frac{4\pi}{3}$

Name____ Due Date________________

Algebra II Trig Functions

Worksheet 5

1. If $\sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$, then $\cos\theta$ = ?

2. If $\sin\theta < 0$ and $\cos\theta < 0$, then θ is in which quadrant?

3. If $\tan\theta = -\frac{\sqrt{3}}{3}$ and $\sin\theta$ is positive, then θ is in which quadrant?

4. Evaluate the following:
   a. sin60° + tan45°
   b. (tan150°)(cos-120°)
   c. $\tan\frac{7\pi}{6} \div \sin^2\left(\frac{4\pi}{3}\right)$
   d. $(\cos\frac{11\pi}{6} + \sin\frac{3\pi}{4})(\tan\pi - \sin\frac{5\pi}{3})$

Day Six: Co-functions

Objectives

- ✓ Students make the connection between triangles of the unit circle and co-functions (graphically, algebraically)
- ✓ Students will observe the reciprocal relationship via their graphs (graphically)
- ✓ Students will use co-functions to determine exact values (algebraically, numerically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

Materials

- ✓ Notes packet

Before Phase (5 min): Generate a brief discussion to determine what students know about co-functions. They have not yet been introduced to them, so the discussion will most likely be very brief.

During Phase (45-50 min): Complete pages 18 – 21 in the guided notes packet

- Page 18: Intro. to the co-functions
- Page 19: Cosecant
- Page 20: Secant
- Page 21: Cotangent

After Phase: Worksheet 6 can be given as homework, class work, or be picked apart for an exit ticket.

Modifications: Since this a longer lesson, certain parts may need to be skipped. Be sure to at least get through page 18, the tables of values, and the examples of determining exact values using co-functions.

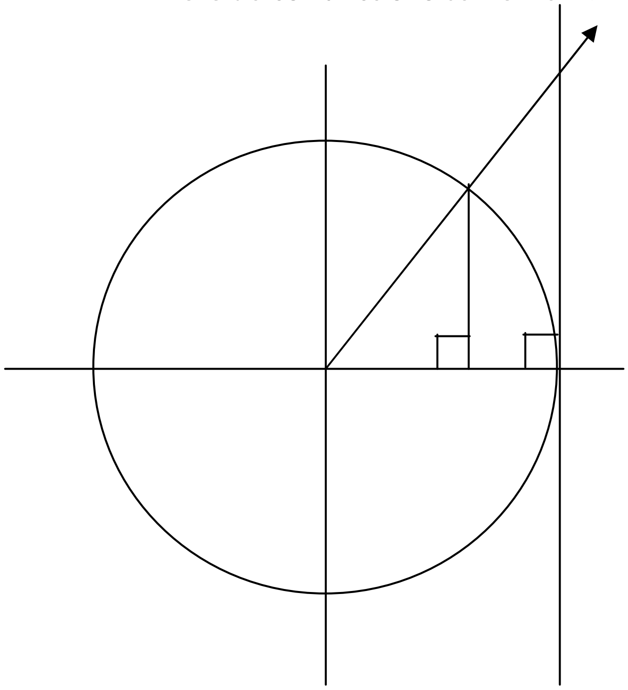
References: https://www.engageny.org/resource/algebra-ii-module-2

Day Six: Co-functions

In addition to sine, cosine, and tangent, there are 3 co-functions: ______________________,

_____________________________, and _____________________________.

- Where do **co-functions** come from?



We call ____.

Similarly, we call _______________________________________, and ____________________

_________________________. The co-functions are also referred to as ___________________

____

Does taking the reciprocal change the sign of a function? ______________

Sine is positive in quadrants _________ → _________ is positive in quadrants______________

Cosine is positive in quadrants _________ → _________ is positive in quadrants____________

Tangent is positive in quadrants _________ → _________ is positive in quadrants___________

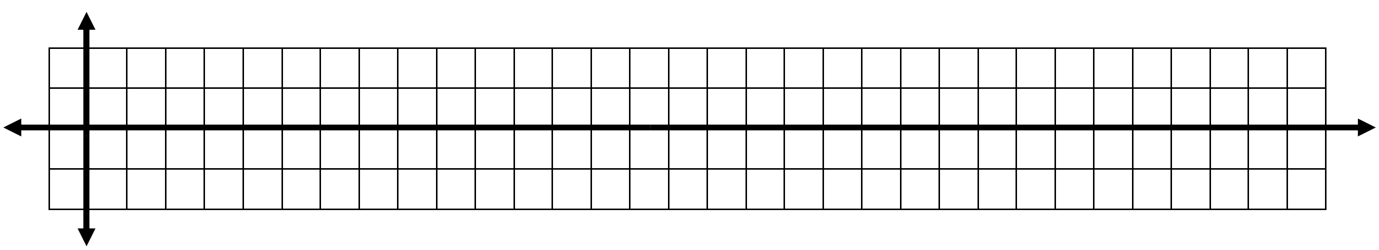
➢ What does cscθ look like **numerically**?

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
sin θ								
csc θ								

Practice: Evaluate the following. Write your answer as a single fraction when necessary.

$\csc 225°$ $\qquad$ $\csc\frac{\pi}{6} + \sin\frac{2\pi}{3}$ $\qquad$ $\csc^2 300°$

➢ What does cscθ look like **graphically?**



	Domain	Range
$f(x) = \sin x$		
$f(x) = \csc x$		

➢ What does $\sec\theta$ look like **numerically**?

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
$\sin\theta$								
$\csc\theta$								

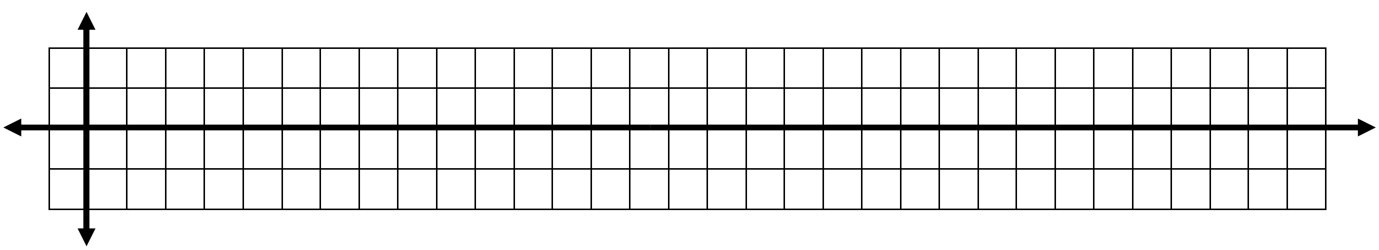
Practice: Evaluate the following. Write your answer as a single fraction when necessary.

$\sec 225°$

$\sec\frac{\pi}{6} + \cos\frac{3\pi}{2}$

$\sec^2 300°$

➢ What does $f(x) = \sec x$ look like **graphically?**



	Domain	Range
$f(x) = \cos x$		
$f(x) = \sec x$		

➢ What does $\cot\theta$ look like **numerically**?

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
sin θ								
csc θ								

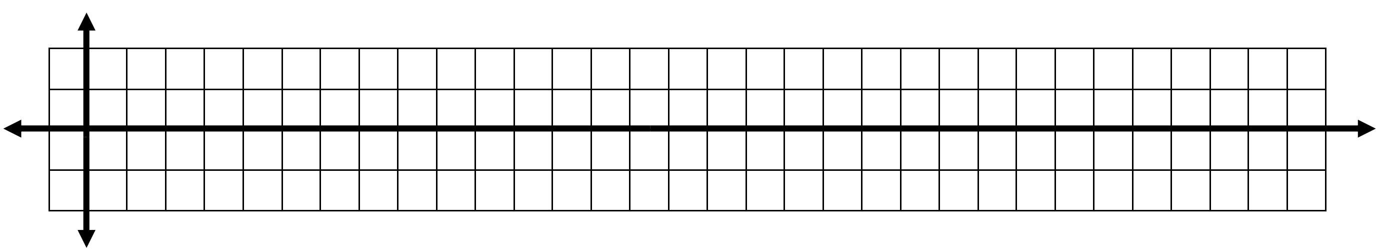
Practice: Evaluate the following. Write your answer as a single fraction when necessary.

$\cot 225°$

$\cot\frac{\pi}{6} + \tan\frac{5\pi}{6}$

$\cot^2 300°$

➢ What does $f(x) = \cot x$ look like **graphically?**



	Domain	Range
$f(x) = \tan x$		
$f(x) = \cot x$		

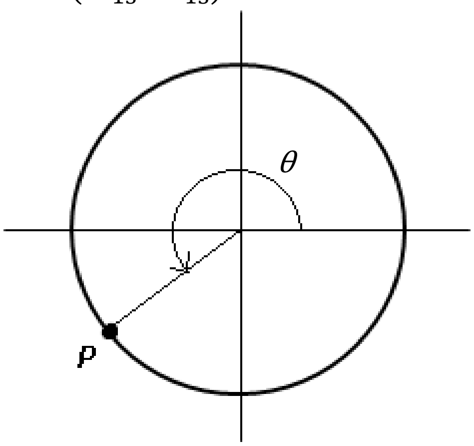
Name____ Due Date_________________

Algebra II Trig Functions

Worksheet 6

1. Given the unit circle with point *P*, state the following:

$P = \left(-\frac{5}{13}, -\frac{12}{13}\right)$

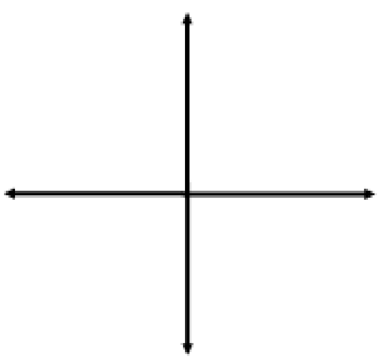


$\sin\theta =$ ________ $\csc\theta =$ _______

$\cos\theta =$ _________ $\sec\theta =$ ________

$\tan\theta =$ ________ $\cot\theta =$ ________

2. Find the exact value of each of the six trig functions given $\sin\theta = \frac{1}{3}$ and $\cos\theta < 0$.



$\sin\theta =$ ________ $\csc\theta =$ _______

$\cos\theta =$ _________ $\sec\theta =$ ________

$\tan\theta =$ ________ $\cot\theta =$ ________

3. Evaluate the following. Evaluate the following. Write your answer as a single fraction when necessary.

a. sin180° - csc45° b. $\cos\frac{\pi}{6} + \cot 2\pi$ c. $\tan^2\left(\frac{\pi}{3}\right) \div \sec\frac{11\pi}{6}$

Days Seven and Eight: Graphing Sine and Cosine

Objectives

- ✓ Students define amplitude, period, frequency, phase shift, and midline (and graphing interval) (graphically, algebraically)
- ✓ Students observe that sine and cosine differ by a phase shift (graphically)
- ✓ Students will explore $f(x) = \sin x$ and $f(x) = \cos x$ after multiple transformations (graphically, algebraically, numerically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.IF.C.7.E
  Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude

Materials

- ✓ Notes packet

Before Phase (5-10): Use the "recall" at the top of page 23 to connect back to the graphing exploration.

During Phase (35-40 min/day): Complete pages 23 – 27 in the guided notes packet

- Page 23: Sine and cosine are a phase shift apart
- Page 24: Form and components of a sinusoidal function and amplitude changes
- Page 25: Changes in frequency/period and the relationship they share
- Page 26: Phase and vertical shifts and practice
- Page 27: Practice

After Phase: Worksheet 7 can be given as homework, class work, or be picked apart for an exit ticket.

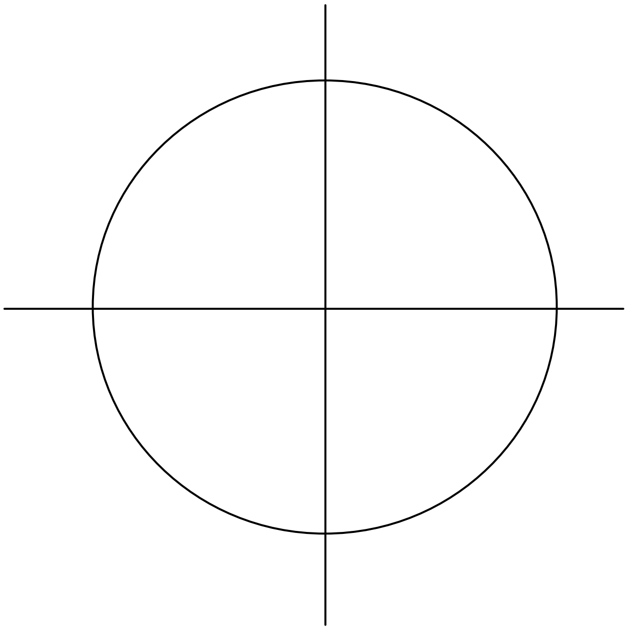
Modifications: If the majority of students are struggling, keep them as a whole-group and work through the examples together. Once they start becoming more confident, have them begin to work on their own or in partners. For those who are still struggling, lead a small group discussion in another part of the room.

References: https://www.engageny.org/resource/algebra-ii-module-2; Ellis, 2012

Days Seven and Eight: Graphing Sine and Cosine

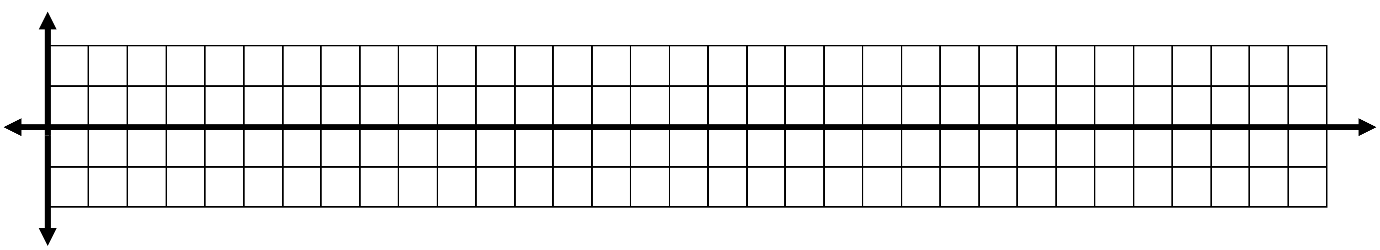
- **Recall:** Remember when we used the unit circle to create the graph $f(x) = \sin x$?

  What do you think are the easiest values to use when graphing sine and cosine?



θ					
$\sin\theta$					

θ					
$\cos\theta$					



- What can we conclude about the relationship between $f(x) = \sin x$ and $f(x) = \cos \theta$?

- Because of this, let's just focus on ________________________________ for now.

➢ **Recall:** A **periodic function** is ____

____

A **sinusoidal function** is ____

____

Where

➔ $|A|$ is called the ______________________ of the function

➔ $\frac{2\pi}{|w|}$ is called the _______________________ of the function

➔ $\frac{|w|}{2\pi}$ is called the _______________________ of the function

➔ h is called the ________________________ of the function

➔ the graph of $y = k$ is called the ____________________ of the function

- k is also called the _________________________________

➔ The **amplitude (A)** is the ____

____

➢ What does **amplitude** look like **numerically**?

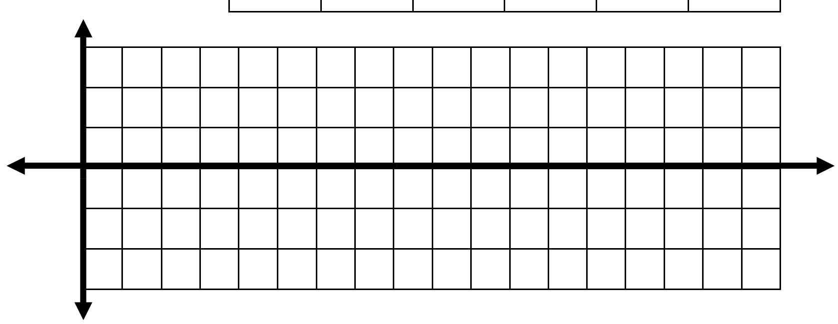
θ					
$\sin\theta$					

θ					
$\sin\theta$					

➢ **Graphically**?

$f(x) = \sin x$ (dotted)

$f(x) = 3\sin x$



➔ The **period (P)** is the ____

____

➔ The **frequency (F)** is the ____

____

➔ Helpful step: **Graphing Interval**

The **graphing interval** helps with ____

____

➢ What do the **period** and **frequency** look like **numerically**?

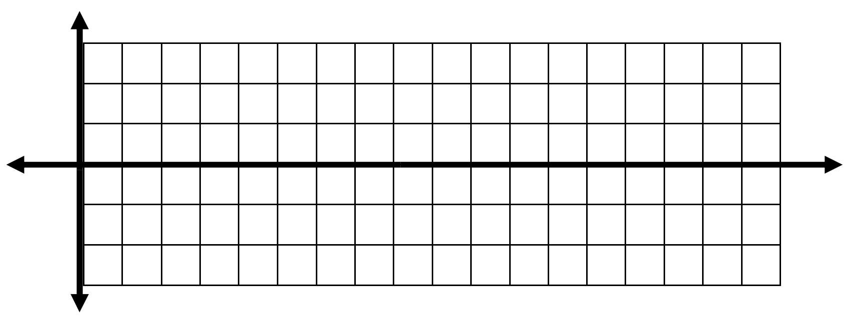
θ					
$\sin\theta$					

θ					
$\sin\theta$					

➢ **Graphically**?

$f(x) = \sin x$ (dotted)

$f(x) = \sin 2x$



What is the relationship between the **period** and **frequency**?

➔ Another term for “phase shift” is ____.
What do we have to be careful of?

➔ While $y = k$ is the ________________, k represents the ________________________________.

When graphing transformations, ____!!!

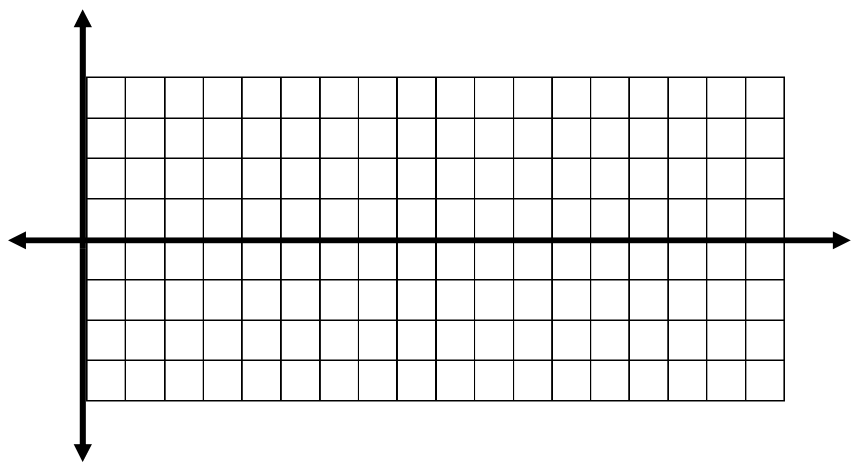
Practice: Graph the following

1. $f(x) = 2\sin x + 1$

A = Midline:

F = *P* =

Order of transformations:

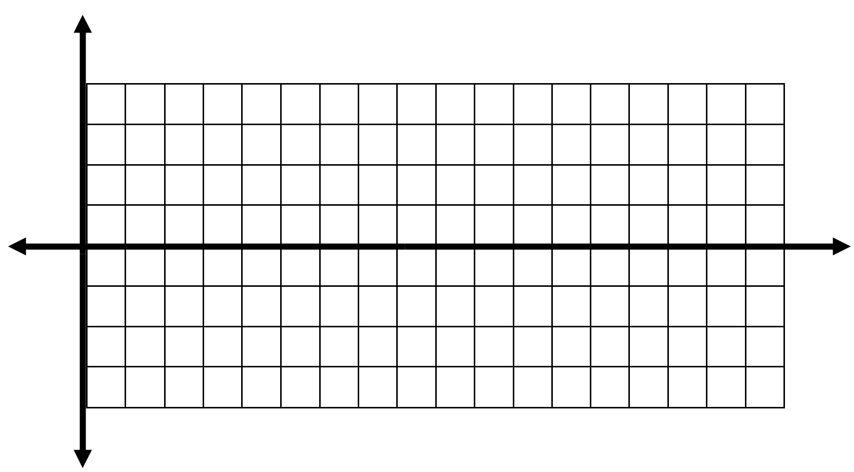


2. $f(x) = \sin\left(\frac{x}{2}\right) - 2$

A = Midline:

F = *P* =

Order of transformations:

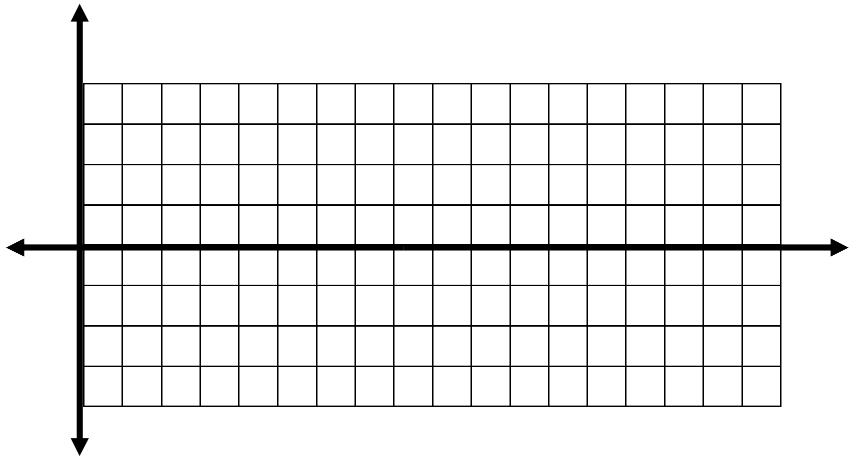


3. $f(x) = 3\sin(2x + \frac{\pi}{2}) - 1$

A = Midline:

F = *P* =

Order of transformations:

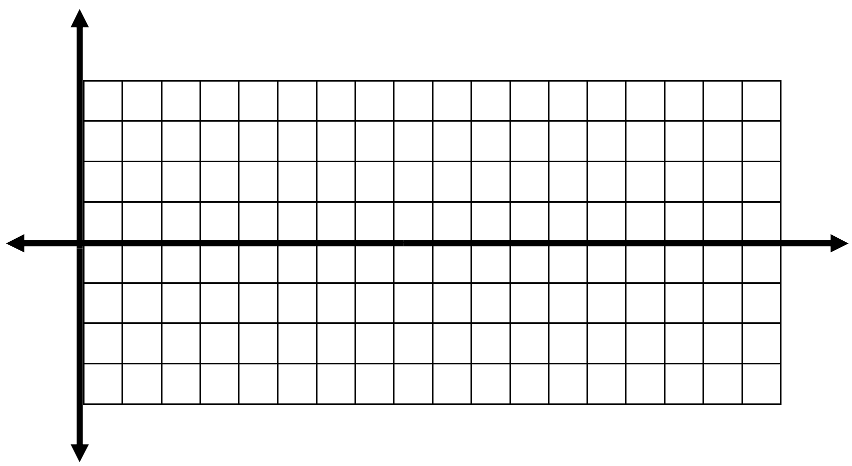


4. $f(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right) + 1$

A = Midline:

F = *P* =

Order of transformations:

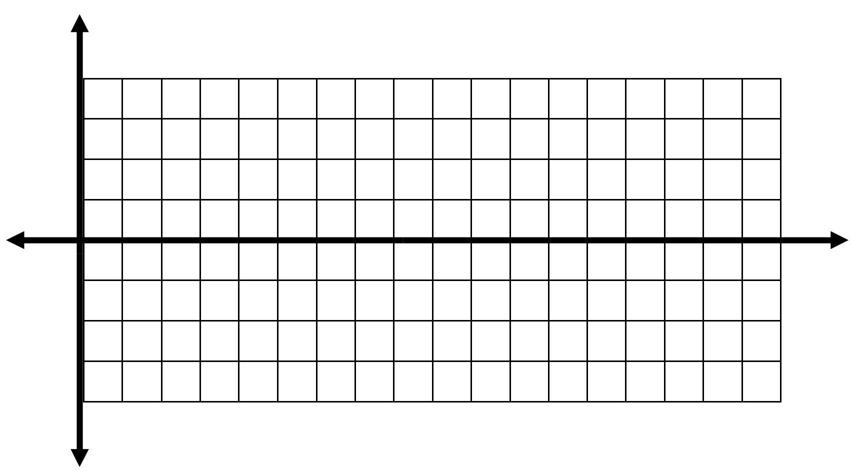


5. $f(x) = \cos(3x + \pi) + 2$

A = Midline:

F = *P* =

Order of transformations:

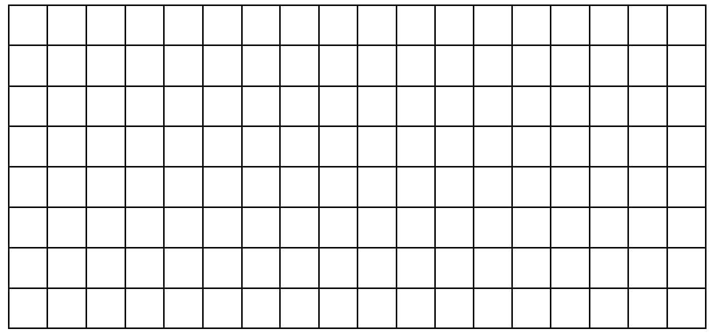


Name____ Due Date________________
Algebra II Trig Functions

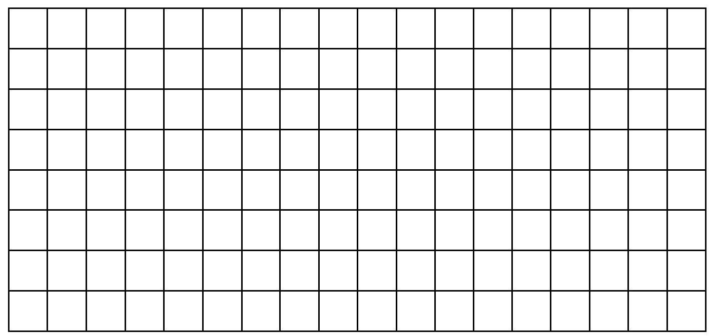
Worksheet 7

1. Graph the following:

   –



2. –



3. Which of the following would match the graph of – – ?

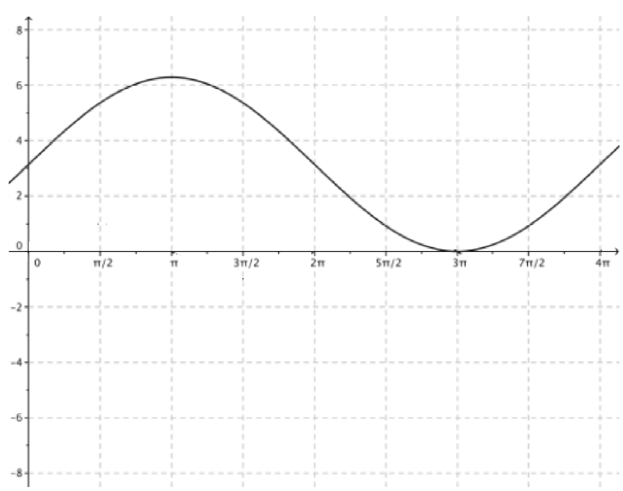
a. – – b. – c. –

4. Fill in the missing information below based off the given graph.
   **Hint: It may help to draw the parent function; i.e.

A = Midline:

F = *P* =

Equation:



Day Nine: Modeling with Trig Functions

Objectives

- ✓ Students will explore how to model real-life scenarios using sinusoidal functions (graphically, numerically, algebraically)

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.B.5
  Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline
- ✓ CCSS.MATH.CONTENT.HSS.ID.B.6.A
  Fit a function to the data; use functions fitted to data to solve problems in the context of data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models

Materials

- ✓ Modeling with Trig Functions packet

Before Phase (5 min): Ask the class if they can think of any real-life situations that can be modeled using sine or cosine functions.

During Phase (45 min)

- ✓ There are 3 modeling problems to be completed. Do the first as a class, and let students complete the last 2 in groups of 2 – 3
  - Problem 1 is about modeling the height of a paddle on a paddle wheel for the S.S. Beaver. The questions will guide the class on how to determine each component of a sinusoidal function in order to create the model for this scenario.
  - Problem 2 is similar to problem 1 but is about the height of a Ferris wheel car. Have students try this one on their own using the questions from problem 1 to guide their solution process.
  - Problem 3 is about temperature changes inside and outside an igloo. The questions will guide students through the solution process.
- ✓ Circulate throughout the room and help when needed

After Phase: If there is time left, generate a whole-class discussion about how to generalize the solution process for modeling with trig functions

Modifications: If the majority of students are struggling, keep them as a whole-group and work through the examples together. Once they start becoming more confident, have them begin to work on their own or in partners. For those who are still struggling, lead a small group discussion in another part of the room.

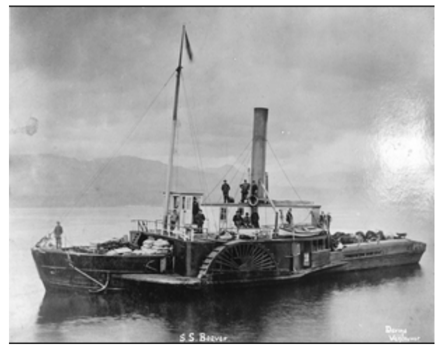
References: http://www.classzone.com/eservices/home/pdf/student/LA214EAD.pd

Name____ Date________________
Algebra II Trig Functions

Modeling with Trig Functions

We have been discussing the characteristics of sinusoidal functions. Now it is time to see how they can be put to use in modeling situations!

1. The paddle wheel of the S.S. Beaver was 13 feet in diameter and revolved 30 times per minute when moving at top speed. Using this speed and starting from a point at the very top of the wheel, write a model for the height *h* (in feet) of the end of a paddle relative to the water's surface as a function of the time *t* (in seconds). Note that the paddle is 2 feet below the water's surface at its lowest point.



   a. What is the range of the paddle wheel's height?

   How can you use this to determine the midline and amplitude of the model? What shift does the midline help you determine?

   b. The model has to be in units of feet per second. How long does it take for one cycle of the wheel? What does this help you find?

   c. How can you determine whether to use sine or cosine?
   Hint: think about the line behavior after the starting point

   d. What is the final sinusoidal model for this scenario?

2. A Ferris wheel with a radius of 25 feet is rotating at a rate of 3 revolutions per minute. When $t = 0$, a chair starts at the lowest point on the wheel, which is 5 feet above the ground. Write a model for the height h (in feet) of the chair as a function of the time t (in seconds).
   **Hint: If you are stuck, use the questions from question (1) to help guide you through the solution process.

3. Eskimos use igloos as temporary shelter from harsh winter weather. Below is a table displaying the temperature both inside and outside the igloo over a typical day.

Time	**8AM**	**10AM**	**12PM**	**2PM**	**4PM**	**6PM**
Outside Temp. (°F)	-20	-15	-12	-10	-13	-16
Inside Temp. (°F)	22	23	25	26	25.5	24

Time	**8PM**	**10PM**	**12AM**	**2AM**	**4AM**	**6AM**	**8AM**
Outside Temp. (°F)	-19	-26	-28	-30	-29	-24	-21
Inside Temp. (°F)	23	22	20.5	20	21	21.5	23

a. At what time does the lowest temperature occur? The highest? What do these represent?

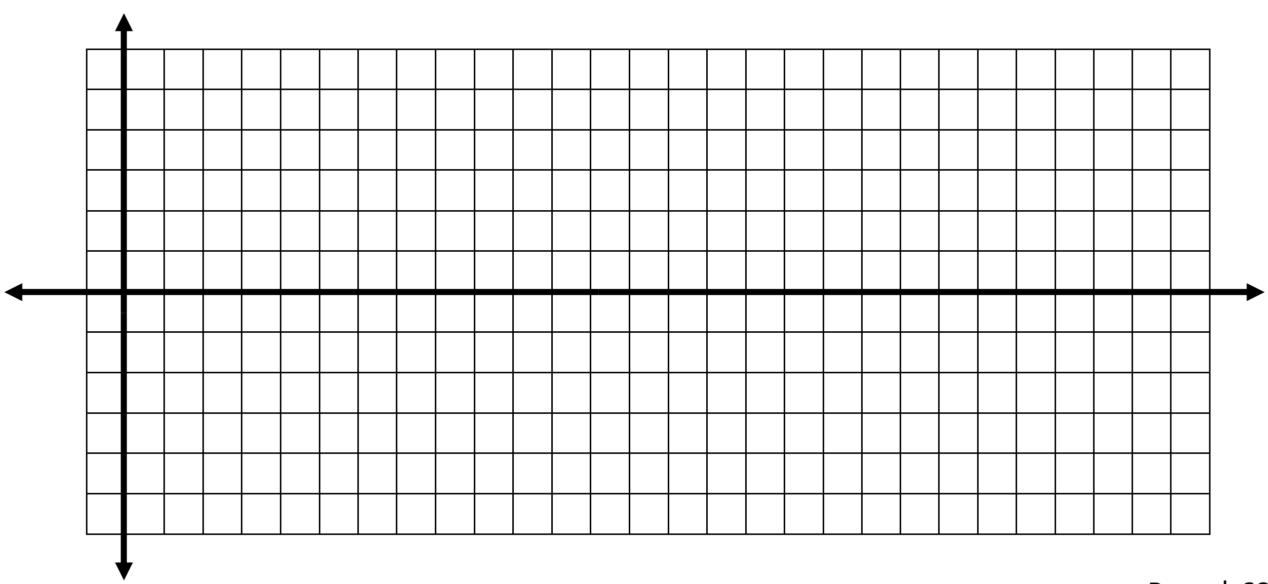
b. For a sinusoidal function of the form $f(x) = A\sin\big(w(x-h)\big) + k$, for which variables can you determine the values for from (a) for each function? State these values.

c. How can you determine the frequency and period (units!!)? Is this value the same for both functions? Why/why not?

d. Determine if you will need to use sine or cosine. Explain your reasoning.

e. Write a sinusoidal model for both the outside temperature T_1 and inside temperature T_2 (in degrees Fahrenheit) as a function of the time of day t (in hours since midnight).

f. Plot the data on the axes below. Does the model you created make sense for this graph?



Day Ten: Review

Objectives

- ✓ Students practice the concepts learned throughout the unit before being tested on them

CCS Standard(s)

- ✓ CCSS.MATH.CONTENT.HSF.TF.A.1
  Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle
- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle
- ✓ CCSS.MATH.CONTENT.HSF.TF.B.5
  Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline
- ✓ CCSS.MATH.CONTENT.HSS.ID.B.6.A
  Fit a function to the data; use functions fitted to data to solve problems in the context of data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models
- ✓ CCSS.MATH.CONTENT.HSF.IF.C.7.E
  Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude

Materials

- ✓ Review Packet

Before Phase (5 min): Ask the class if they have questions on anything from the unit that need to be addressed before completing the review packet.

During Phase (45 min): Allow students to work in partners to complete the test review packet

After Phase: If students do not finish the review, assign it for homework.

Modifications: Have the answer key available for those who like to check their answers. This way you will have more time to circulate the room and spend more time with the students. This class can also be used as a catch-up day.

References: http://math.usask.ca/emr/examples/mwtf_eg1.html; Ellis, 2012

Name____ Date________________
Algebra II Trig Functions

Trig Functions Test Review

4. Convert each angle in degrees to radians. Leave your answer in terms of π. Make sure your answer is in the interval $0 \leq \theta < 2\pi$.

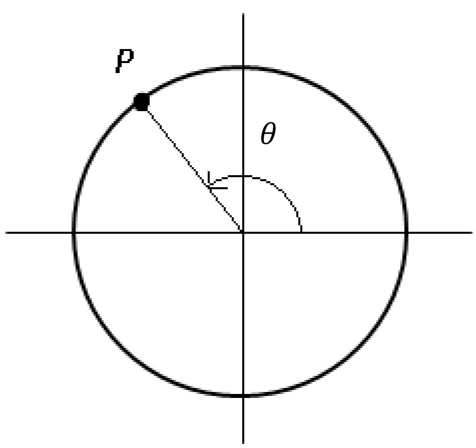
   b. $15°$ b. $120°$ c. $-40°$ d. $495°$

5. Convert each angle in radians to degrees. Round to the nearest hundredth when necessary. Make sure your answer is in the interval $0 \leq \theta \leq 360°$.

   b. $\frac{10\pi}{3}$ b. $\frac{\pi}{13}$ c. $\frac{3\pi}{4}$ d. $-\frac{2\pi}{5}$

6. Given the unit circle with point *P*, state the following. Rationalize all denominators and simplify ratios completely.

$P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

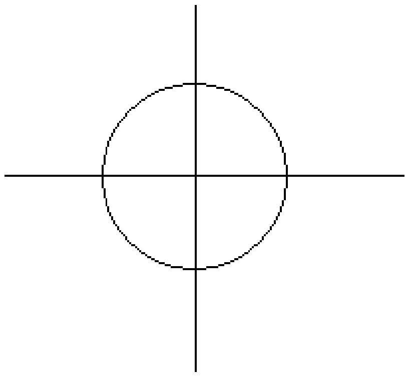


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

7. Provide the missing information below, determine the point *P* on the unit circle, and sketch θ.



$\sin\theta =$

$\csc\theta =$

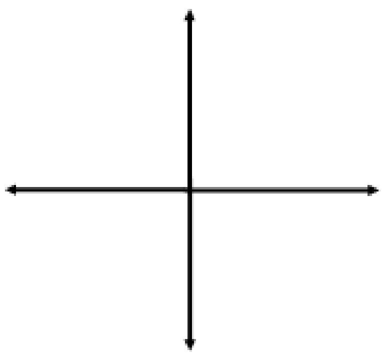
$\cos\theta =$

$\sec\theta = -2$

$\tan\theta = -\sqrt{3}$

$\cot\theta =$

8. Determine the exact value of each trig ratio if (6, -8) is on the terminal side of θ.



$\sin\theta =$

$\csc\theta =$

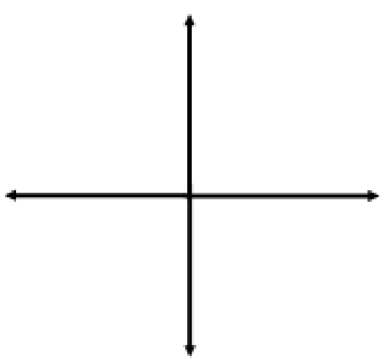
$\cos\theta =$

$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

9. Find the exact value of each of the six trig functions given $\cos\theta = -\frac{5}{13}$ and $\csc\theta < 0$. Rationalize all denominators and simplify ratios completely.



$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

$\sec\theta =$

$\tan\theta =$

$\cot\theta =$

10. Determine the exact value of each of the following.

a. $\sin 120°$

b. $\sec(-315\,°)$

c. $\csc\frac{\pi}{6}$

d. $\cos\frac{5\pi}{4}$

e. $-\sin 270°$

f. $\cot 90°$

11. Evaluate each of the following.

a. $\sin 180° - \csc 45°$

b. $\tan^2\frac{\pi}{6} - \cot\frac{\pi}{4}$

c. $\cos(-120°) + \csc 150°$

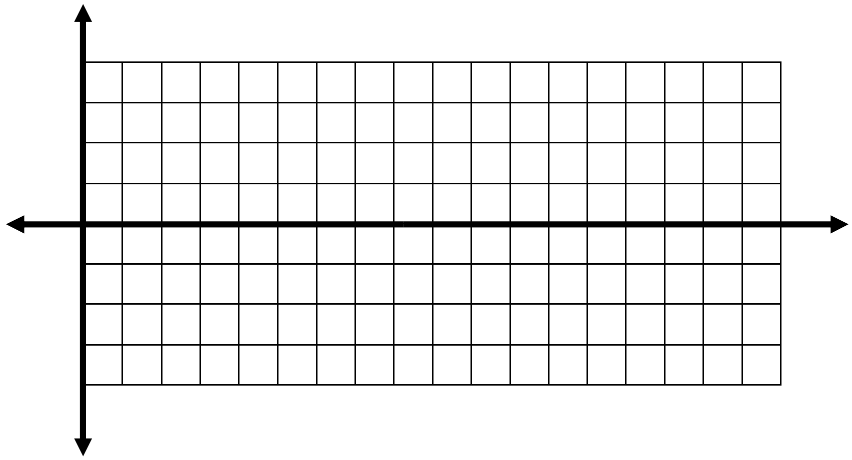
d. $(\sec\frac{5\pi}{4})\left(\sin - \left(\frac{\pi}{4}\right)\right)$

e. $(\csc^2 300°)\,(\tan 330°)$

f. $\tan\frac{2\pi}{3} \div \cot\frac{5\pi}{6}$

12. Determine the missing information below, and then graph each function.

a. $f(x) = \frac{1}{2}\sin 4x - 1$

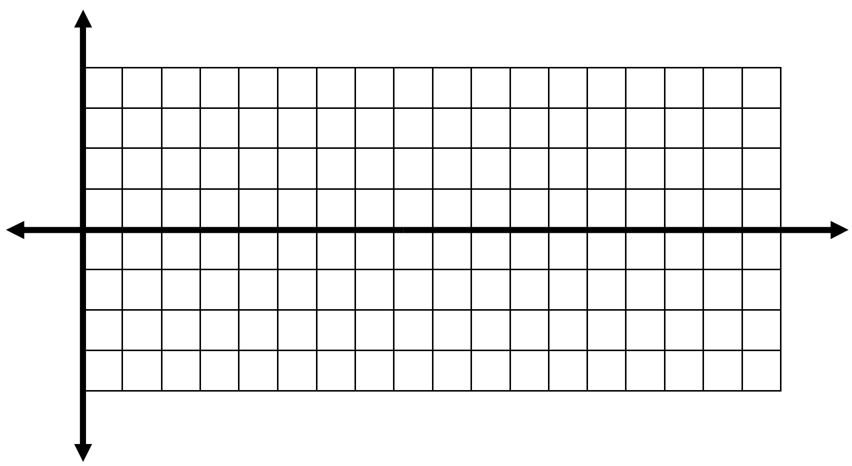


A =   Midline:

F =   *P* =

Order of transformations:

b. $f(x) = 3\cos\frac{1}{2}(x - \pi)$

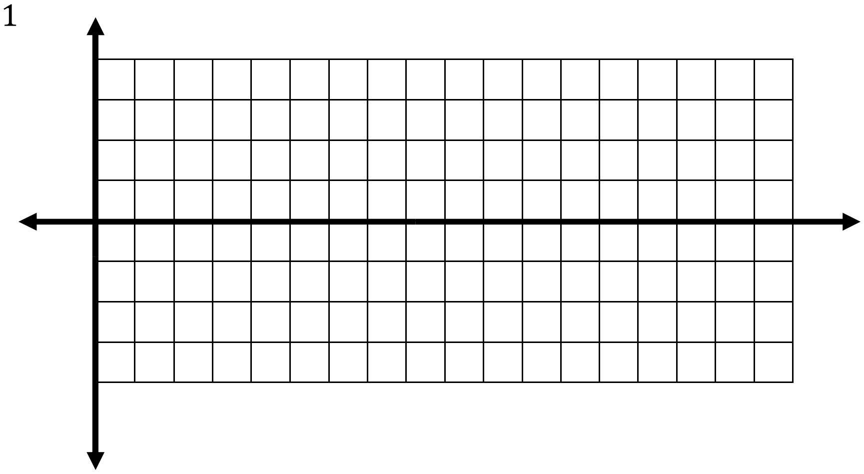


A =   Midline:

F =   *P* =

Order of transformations:

c. $f(x) = 2\sin(2x + \pi) + 1$



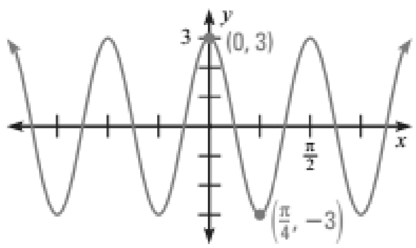
A =   Midline:

F =   *P* =

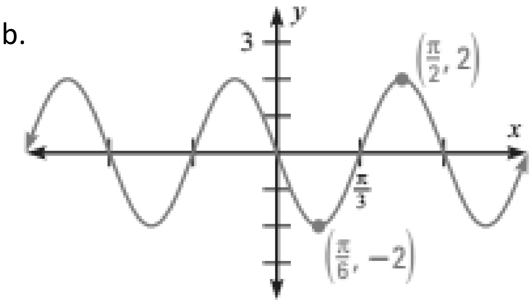
Order of transformations:

13. Write 2 functions for each of the following (one sine and one cosine).

a.



b.



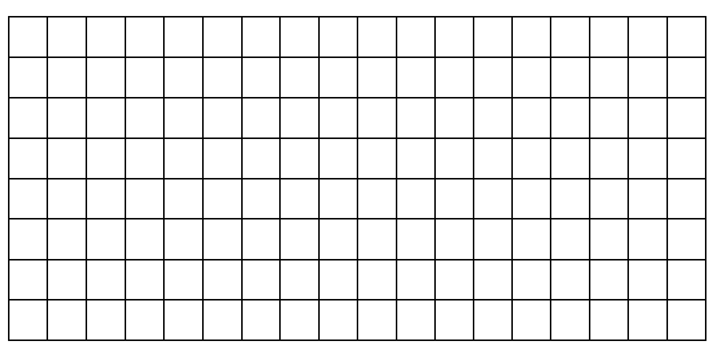
14. Below is a table of temperatures recorded weekly over the course of a year.

a. Form a sinusoidal function, *T*, that best models the changes in temperature, *t*.

Week	**0**	**4**	**8**	**12**	**16**	**20**	**24**
Temp (°F)	43	54	62	71	68	61	51

Week	**28**	**32**	**36**	**40**	**44**	**48**	**52**
Temp (°F)	36	19	13	11	14	30	41

a. Plot the data, and determine a function that fits the data graphically. Does this match your answer from (a)?



Day Eleven: Test

Objectives
- ✓ Students will demonstrate their knowledge of the unit

CCS Standard(s)
- ✓ CCSS.MATH.CONTENT.HSF.TF.A.1
  Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle
- ✓ CCSS.MATH.CONTENT.HSF.TF.A.2
  Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle
- ✓ CCSS.MATH.CONTENT.HSF.TF.B.5
  Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline
- ✓ CCSS.MATH.CONTENT.HSS.ID.B.6.A
  Fit a function to the data; use functions fitted to data to solve problems in the context of data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models
- ✓ CCSS.MATH.CONTENT.HSF.IF.C.7.E
  Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude

Materials
- ✓ Test

Before Phase (3-5 min): Ask if there are any last minute questions before handing out the test.

During Phase (45-50 min): Have students complete the test

After Phase: As students finish, make sure they stay quiet for those who are still taking the test.

Modifications: Be aware of students with IEPs and their needs according to the document.

References: Ellis, 2012

Name____ Date________________

Algebra II Trig Functions

Trig Functions Test

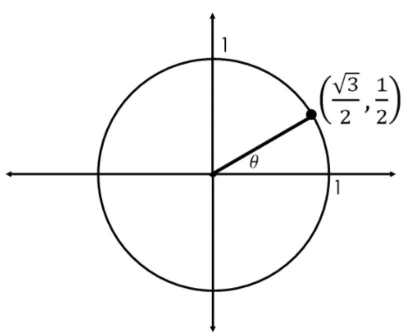
Directions: Answer the following questions the best you can. Part one is a multiple choice section of 10 questions (worth 2 points each), and part two has 3 extended response questions. For part 2, you MUST show your work to receive full credit. Good luck!!!

Part One: Multiple Choice

1. What is 200° in radians?

   a. — b. — c. — d. —

2. Consider the diagram below. What is the value of



   a. b. —

   c. – d. —

3. If and , which quadrant does the terminal side of θ fall?

   a. 1 b. 2 c. 3 d. 4

4. What is the reference angle for — ?

   a. – b. – c. – d. –

5. What is the exact value of — ?

a. — b. — c. d.

6. What is the phase shift of the function – – ?

a. Right – b. Left – c. Right π d. Left π

7. What is the amplitude of the function ?

a. b. 3 c. – d. 2

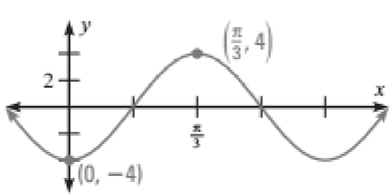
8. What is the frequency of the function – – ?

a. 6π b. — c. d. —

9. What is the period of the function – – ?

a. 6π b. — c. d. —

10. Which could be the sinusoidal function for the graph?

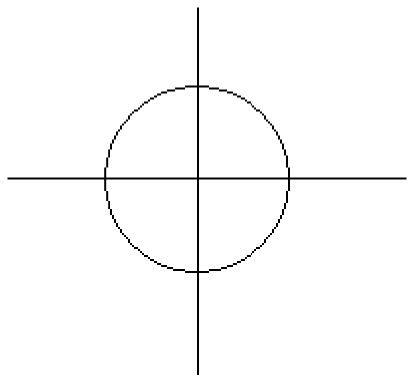


a. – b. –

c. – d. –

Part Two: Extended Response

15. Provide the missing information below, determine the point *P* on the unit circle, and sketch θ. Rationalize all denominators and simplify ratios completely. (6 points)



$\sin\theta =$ $\qquad$ $\csc\theta = -2$

$\cos\theta =$ $\qquad$ $\sec\theta =$

$\tan\theta =$ $\qquad$ $\cot\theta = -\sqrt{3}$

16. Determine the exact value for each of the following. Express your answer as a single fraction (when applicable) and rationalize all denominators. (2 points each)

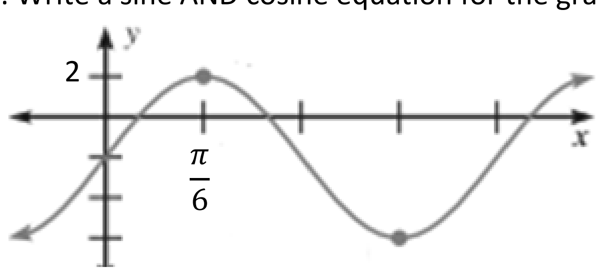
a. $\sin 90° - \cot 180°$

b. $\sec^2\left(\frac{\pi}{3}\right) + \tan\left(\frac{5\pi}{6}\right)$

c. $(\csc\frac{2\pi}{3})(\sec\frac{3\pi}{2})$

d. $\tan\frac{3\pi}{4} \div \cos\frac{5\pi}{4}$

17. Write a sine AND cosine equation for the graph below. (4 points)



18. Graph the function $f(x) = 3\sin\left(\frac{1}{2}(x - \pi)\right) + 1$ over the interval $[0, 2\pi]$. State the amplitude, period, and frequency, as well as the phase shift and vertical shift. (8 points)

Amplitude:_________

Period:_____________

Frequency:_________

Phase Shift:_________

Vertical Shit:________

19. A Ferris wheel with a diameter of 40 feet is rotating at a rate of 4 revolutions per minute. When *t* = 0, a chair starts at the lowest point on the wheel, which is 5 feet above the ground. Write a model for the height *h* (in feet) of the chair as a function of the time *t* (in seconds). (10 points)

Chapter Four: Validity of Curriculum Project

This curriculum project was submitted for feedback to a veteran teacher at a suburban school district located in upstate New York. The author knows this teacher from one of her student teaching placements, and is confident in the teacher's abilities; specifically because he (the teacher) initially taught Algebra II/Trigonometry (now called Algebra II) at the start of his teaching career, and has since moved up to teaching Pre-Calculus and double-accelerated mathematics. The knowledge and experience of this teacher is what validates the author's curriculum as his feedback speaks to its content validity (Clause, 2003).

The teacher provided two portions of feedback. The first is a general statement about the overall curriculum design, while the second is more specific and speaks to particular portions of the project. His first, general statement follows:

> *As I was the entire experience that we worked together, I am very impressed!! This is a very well thought out, thorough, and strongly designed curriculum design. I made some comments, suggestions, and minor changes to your curriculum design. I attached the document. All my suggestions are in red. Most likely, you could not make those changes and still have a very strong design. The lessons themselves (including the worksheets, closed notes, and lesson keys) are exemplary. They look as if a very experienced teacher had put these together. I would be very interested to see how this would play out with the kids. I like the structure, design, and pacing of the unit.*
>
> *Unfortunately, I did not make many changes for you. And there was not much room to make suggestions. It was too good and much too solid to provide as much feedback as I think that you were hoping to get. I spent a few days looking over everything, pacing through the lessons, and considering student time tracking. VERY NICE JOB!! From my standpoint, this would be a very successful design implementing student success.*
>
> *I really enjoyed the graphing exploration. What a very cool design!*
>
> *My only criticism is the unit exam. I think it is very good, however, what is the time frame that you are planning on using to give it? Also, you have a strong multiple choice section and your free response is okay. I attached an assessment that we use for A2/Trig regents here. It is a slightly different format, it has some different questioning techniques, and this is an appropriate 40 minute test. I am not saying that yours is bad - not by any*

means. I just wanted to give you some more options to see how a test could be derived with similar topics.

Again, overall - VERY IMPRESSED! Nice job!!

His second portion of feedback was noted on the lesson plan, when applicable (what he stated was in red, here is italicized). That is,

Day One

- ✓ He suggested adding an objective: *Students will be introduced to the origins of trigonometry and understand how it may have shaped the foundation it has been built upon (geometrically)*
- ✓ **Be careful using words like "Understand some of…" – The objective is to get the kids to build and understand as much as possible. I agree, most kids will only understand* <u>*SOME*</u>*, but never admit it!*

Day Two

- ✓ He suggested to reconfigure one objective into two objectives
  - *Students will recall, strengthen, and work with the trigonometric ratios they learned in geometry*
  - *Students will use the trigonometric ratios to determine the special triangle ratios (graphically, numerically, algebraically)*
- ✓ **Breaking the objectives up gives a little more emphasis of the importance of working with previously learned concepts and how it will influence where you are headed.*

Day Six

- ✓ *This is a great day – For future work, from my experiences: co-functions blend very nicely with the basic trig ratios (Day 1) as well as the special rights and Unit circle. This could* *have the potential that it feels like you are stepping back, then moving forward again. I do like the design of the lesson, I feel that it flows well with what you have designed throughout the unit. However, co-functions sometimes flow very nicely with trig ratios and their direct relation is sometimes more easily understood.*

Days Seven and Eight

- ✓ *You are planning on demonstrating that Sine and Cosine come from the unit circle. You demonstrate the creation of both curves from the unit circle very well in your lesson. This is a very important connection that needs to be made for the students. It ties all the special right, ratio, and co-function work that you have previously done with where you are going throughout the rest of the unit. I see the phase shifts (which is awesome), but make sure you emphasize the connection of the graphs with the unit circle with your objectives.*

Day Nine

- ✓ *This is an awesome lesson!! You can really use technology to hit this home with the students. Use this opportunity to include videos and captivate the students through more non-traditional forms of education. This would be a great lesson to experiment, explore, and make meaningful connections!*

Chapter Five: Discussion, Summary, and Reflection

The overall aim of this curriculum project was to develop a unit that is not only aligned with the CCSS, but also incorporates the Rule of Four in order to ensure students are working within their zone of proximal development. "The idea behind the Rule of Four is that students learn in different ways… all students should learn various modes of representation, but each students typically has an innate strength in one of these four areas" (Cal Teach, 2014, p.2). The inclusion of the Rule of Four separates the author's work from existing resources because it takes into consideration that there are various solution processes to any mathematical problem, which in turn allows students the opportunity to choose which solution method(s) works best for them. Doing so enables students to work in their ZPD.

Considering students' ZPD is crucial because "teaching and learning in the ZPD entails moving students past their present capabilities towards new forms of reasoning and action" (Goos et al., 2003, p. 218). Although the ZPD model can be applied to any learning situation, it is especially important in mathematics because of the various learning styles and paces of students. Furthermore, in addition to the six shifts previously discussed in chapter two that New York State recognizes, the CCSS identifies three key shifts that must occur for successful implementation. These three shifts are (1) greater focus on fewer topics, (2) coherence via linking topics and thinking across grades, and (3) rigor by pursuing conceptual understanding, procedural skills and fluency, and application with equal intensity (Common Core State Standards Initiative, 2014). Recognizing these shifts and incorporating them in the classroom may be difficult at first, however using the Rule of Four to pertain to students' ZPD may ease the process.

For future work, the area of this curriculum project that would need the most revision before implementation in a classroom is the unit test. After taking the feedback into consideration and analyzing the test that the teacher who reviewed the curriculum provided, the author determined that students may have trouble finishing the test in the allotted time if that time is less than one hour. The test may be more appropriate for class periods exceeding this time frame. Furthermore, some of the extended response questions could be reworded. An example of this would be writing “If $f(\theta) = \sec^2\theta + \tan\theta$, find the exact value of $f\left(\frac{\pi}{3}\right)$” rather than simply stating in the directions to find the exact value of an expression.

Additionally, it should be noted that there is a bit more material to be covered in the trigonometric functions module of the Common Core, however it can be viewed as a separate unit. This material deals with the concept of proving and applying trigonometric identities, which has a greater emphasis on algebraic proofs and computations rather than focusing on radian measures, the unit circle, modeling periodic phenomena, and graphing transformations (New York State Common Core, 2014). Keeping this as a separate unit is beneficial because it helps to chunk the concepts that are more closely related; making it easier for teachers and students alike.

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Appendix

The answer keys for the class notes packet, worksheets, graphing exploration, modeling with trig functions, test review, and test can be found on the following pages. Note that the keys for the worksheets are included with its corresponding materials for the day. For example, day four is the graphing exploration, which is separate from the class notes packet. Therefore the key for worksheet four directly follows the key for the graphing exploration (i.e. they are grouped by days, not by assignment type).

Name____KEY______________________________________ Date______________

Algebra II Trig Functions

Module 2: Trigonometric Functions

Day One: An Introduction to Trigonometry and Radian Measures

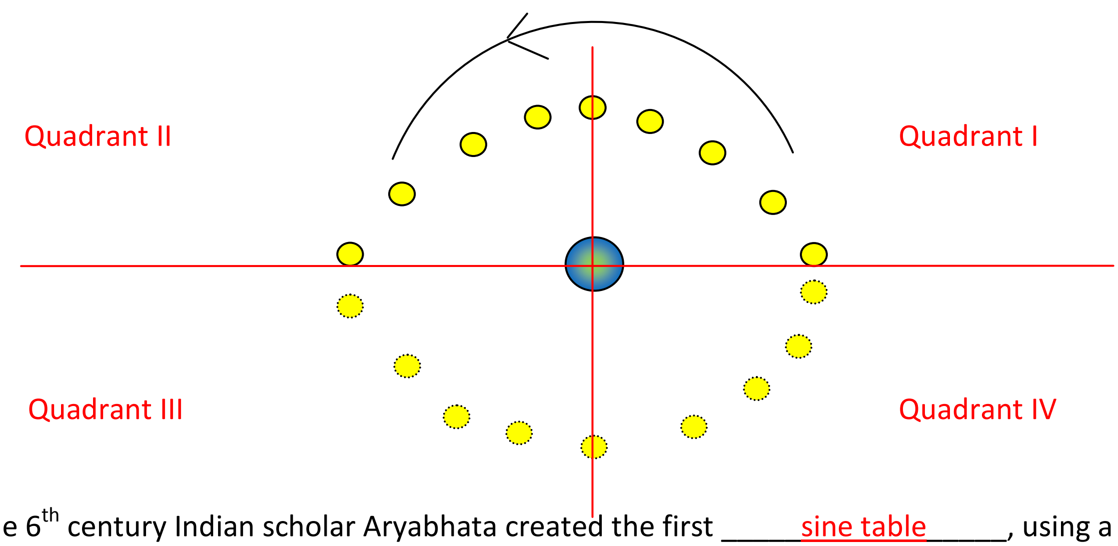
- What is **trigonometry**?

Trigonometry is the study of ______triangles_________. It explores how ___angles______ affects ___height______ and/or _____width_______.

- What are its origins?

Ancient scholars in Babylon and India conjectured that celestial motion was ___circular___; the sun and the stars orbited the earth in a circular fashion. The earth was presumed the center of the sun's orbit. The quadrant numbering in a coordinate system is consistent with the counterclockwise motion of the sun, which rises from the east and sets in the west.

What would this look like graphically?



The 6th century Indian scholar Aryabhata created the first _____sine table_____, using a measurement he called ____jya______. The purpose of his table was to calculate the position of the sun, the stars, and the planets.

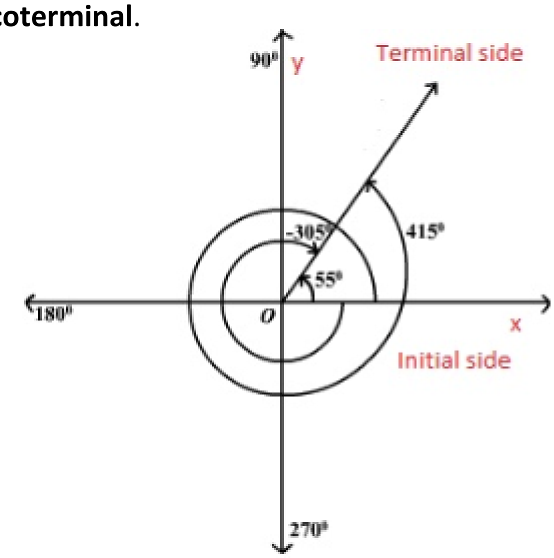
- **The basics of graphing an angle**

Initial Side: __the ray where measurement of an angle starts__________________________

Terminal Side: ___the ray where measurement of an angle stops ____________________

Coterminal angles: _angles with the same initial and terminal sides, but possibly different rotations____

Notice from the diagram, 55°, -305°, and 415° are all **coterminal**.



What does this look like **algebraically?**

360° in one full rotation

Therefore degrees must differ by multiples of 360°

415° - 360° = 55°

-305° + 360° = 55°

Thus, they are coterminal

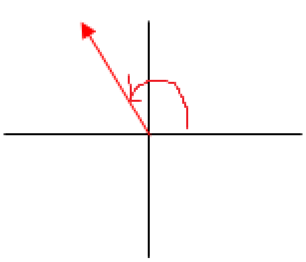
Notice how the angles are sketched!

Positive angles travel __counterclockwise______________

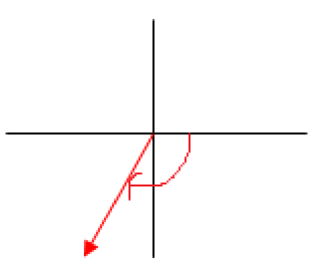
Negative angles travel ___clockwise____________________

Practice: Sketch the angles below. Then determine 2 additional angles that are coterminal.
Note: coterminal angle results will vary.

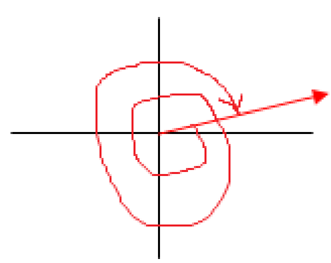
120°



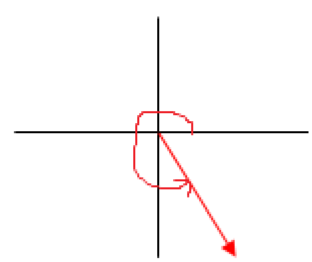
-120°



-730°



300°



- **Radians**

A **radian angle** is __the angle subtended by an arc of a circle that is equal in length to the radius of the circle. The angle is typically in terms of π__

One radian (I rad) is __a unit of rotational measure given by a rotation by a radian angle__

One radian is literally "__one radius__" around a circle

Radian measure is determined by $\frac{\text{length of intercepted arc}}{\text{radius}}$

What angle, in radians, corresponds to 360°? Why? 2π

360° = full circle

Travel around the entire circle → circumference = 2πr

By definition of radians → $\frac{\text{length of intercepted arc}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi$ Therefore, 360° = 2π → 180° = π

We can use this result to convert between radians and degrees.
What proportions should we use?

Radians → Degrees: $\frac{180°}{\pi}$ Degrees → Radians: $\frac{\pi}{180°}$

Practice: Convert the following degree measure to radians, or radian measures to degrees.

135°	$\frac{6\pi}{11}$	-330°	$\frac{7\pi}{4}$
$\frac{3\pi}{4}$	154.29°	$-\frac{11\pi}{6}$	315°

Name KEY Due Date________________

Algebra II Trig Functions

Worksheet 1

20. Convert each angle in degrees to radians. Leave your answer in terms of π.

c. $18°$	b. $76°$	c. $-40°$	d. $200°$
$\frac{\pi}{10}$	$\frac{19\pi}{45}$	$-\frac{2\pi}{9}$	$\frac{10\pi}{9}$

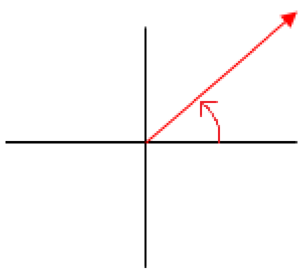
21. Convert each angle in radians to degrees. Round to the nearest hundredth.

c. 2 rad	b. $\frac{\pi}{13}$	c. $\frac{3\pi}{17}$	d. $-\frac{2\pi}{5}$
114.59°	13.85°	31.76°	-72°

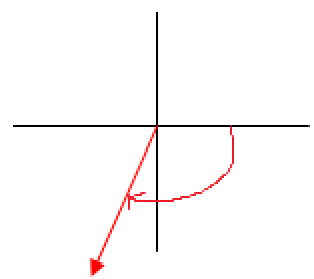
22. Sketch each of the angles below and label the quadrant where the terminal side is. Then determine 2 coterminal angles for each and sketch them on the same set of axes.

Note: Coterminal results will vary

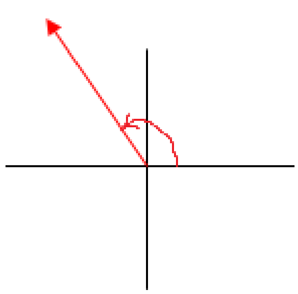
b. 46°



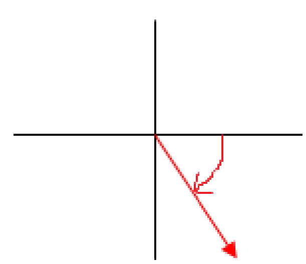
b. -120°



c. $\frac{3\pi}{5}$ 108°



d. $-\frac{\pi}{3}$ -60°



Day Two: Special Triangles and the Unit Circle

- **Recall:** In geometry, you derived trig ratios based on right triangles. What were they?

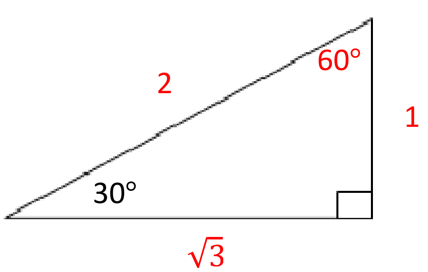
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\quad\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

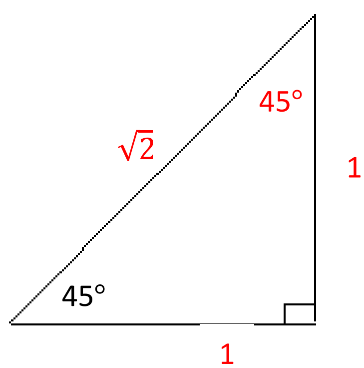
In trigonometry, we have ______special triangles__________ that help us find ___exact___ ________values____________ of trig functions. These triangles contain ___special___ ___angles_________, which are __30°__ ($\frac{\pi}{6}$), __45°__ ($\frac{\pi}{4}$), and __60°__ ($\frac{\pi}{6}$).

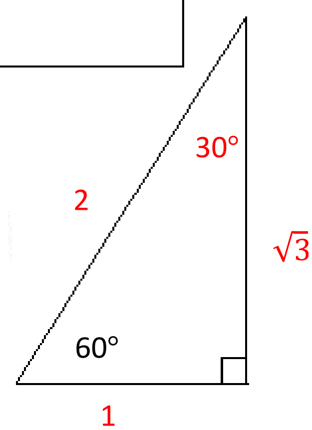
We call these _________reference angles_____ because they yield _____exact________ ________values________.

- So what are these special triangles?

Ratio of Sides for 30° - 60° - 90°	Ratio of Sides for 45° - 45° - 90°
$1 : \sqrt{3} : 2$	$1 : 1 : \sqrt{2}$







sin30 = $\frac{1}{2}$

cos30 = $\frac{\sqrt{3}}{2}$

tan30 = $\frac{\sqrt{3}}{3}$

sin45 = $\frac{\sqrt{2}}{2}$

cos45 = $\frac{\sqrt{2}}{2}$

tan45 = 1

sin60 = $\frac{\sqrt{3}}{2}$

cos60 = $\frac{1}{2}$

tan60 = $\sqrt{3}$

- So how do we use **special triangles**?

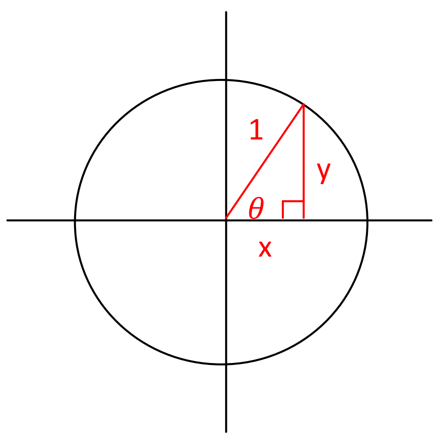
Special triangles are very extremely helpful when dealing with the ______unit circle_________.

- Wait, what is the **unit circle?**

The **unit circle** is a ___circle of radius 1 that is centered at the origin__________________________.

____.

At any point on the unit circle, we can ___draw a triangle___________________



Hypotenuse= 1

Width = cos θ= x

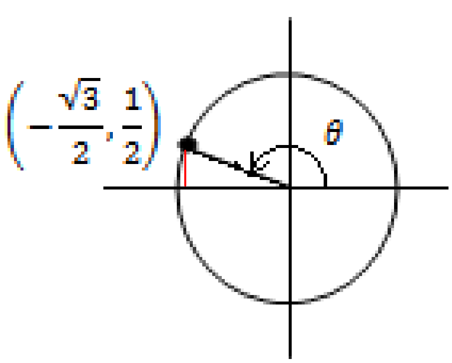
Height = sin θ = y

Note: You can draw more than one triangle so students see the pattern and generalize with x and y

Therefore, a point on the unit circle directly gives us

the ___sine (y)________ and _____cosine (x)_________ of the angle.

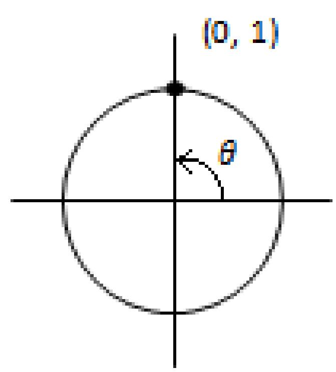
Practice: Determine sin θ and cos θ.



θ = 150°

sin θ= $\frac{1}{2}$

cos θ= $-\frac{\sqrt{3}}{2}$



θ = 90°

sin θ= 1

cos θ= 0

- Constructing the **unit circle** with **special triangles**

Trace the triangles from page 5 to mark…

> Ask the class what angles they think will be marked in the other quadrants after filling out the row for Quadrant 1. They will be doing reference angles without even knowing it; leading into Day 3.

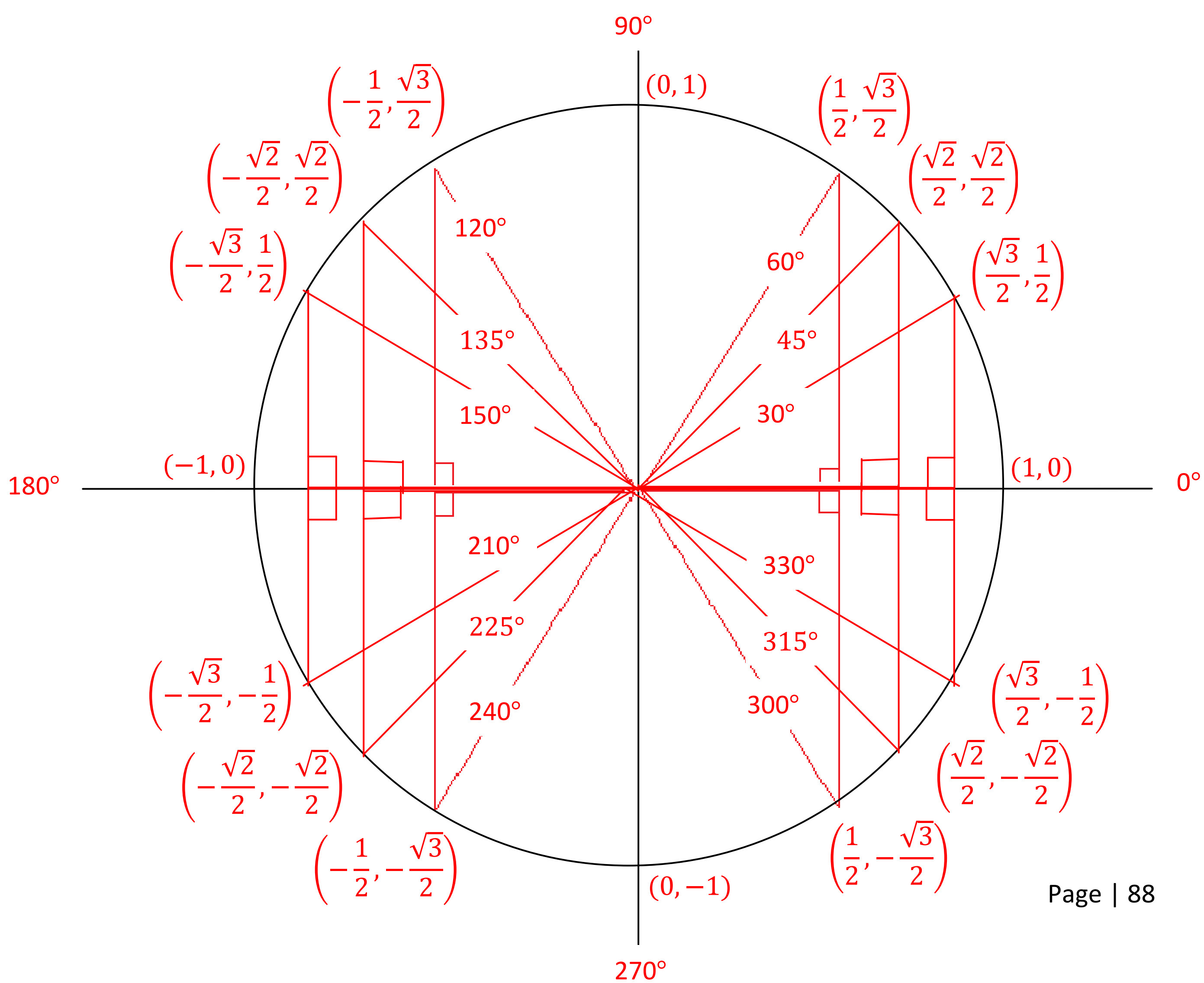
✓ Quadrant I: 30° ($\frac{\pi}{6}$), 45° ($\frac{\pi}{4}$), 60° ($\frac{\pi}{3}$)

✓ Quadrant II: 120° ($\frac{2\pi}{3}$), 135° ($\frac{3\pi}{4}$), 150° ($\frac{5\pi}{6}$)

✓ Quadrant III: 210° ($\frac{7\pi}{6}$), 225° ($\frac{5\pi}{4}$), 240° ($\frac{4\pi}{3}$)

✓ Quadrant IV: 300° ($\frac{5\pi}{3}$), 315° ($\frac{7\pi}{4}$), 330° ($\frac{11\pi}{6}$)

*****Make sure you are placing the triangles so the right angle lies on the x axis and the hypotenuse extends from the origin (think of how the sun moves)!!*****



Name **KEY** ________________________________ Due Date________________
Algebra II Trig Functions

Worksheet 2

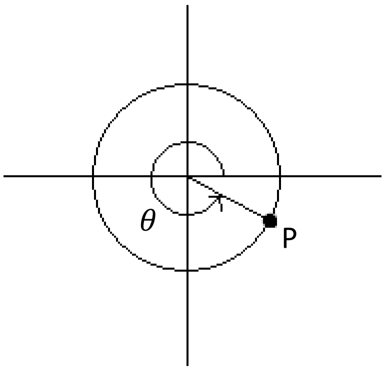
3. Fill out the tables below based on the unit circle you constructed in class.

θ (degrees)	0°	30°	45°	60°	90°	120°	135°	150°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$

θ (degrees)	180°	210°	225°	240°	270°	300°	315°	330°	360°
θ (radians)	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin θ	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos θ	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

4. Fill in any missing information below. Each problem should have:
    a. A sketch of the angle
    b. A value for θ
    c. A coordinate for the point on the unit circle (denoted by P)
    d. A value for sin θ (as a fraction)
    e. A value for cos θ (as a fraction)

a.



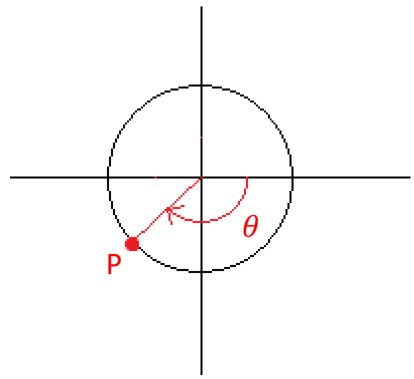
$\theta = \frac{11\pi}{6}$

P = $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

sin θ = $-\frac{1}{2}$

cos θ = $\frac{\sqrt{3}}{2}$

b.



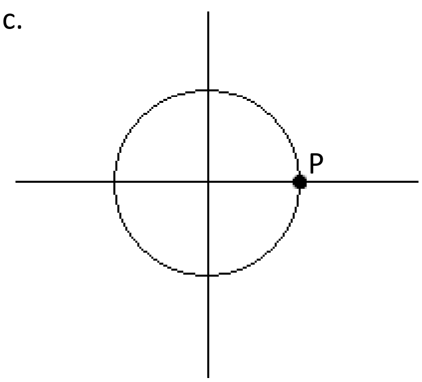
θ = -135° or $-\frac{3\pi}{4}$

P = $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

sin θ = $-\frac{\sqrt{2}}{2}$

cos θ = $-\frac{\sqrt{2}}{2}$

c.



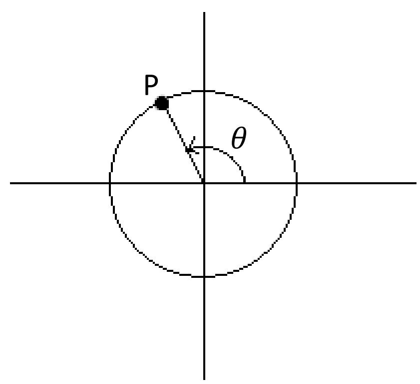
θ = 0° or 0

P = (0, 1)

sin θ = 0

cos θ = 1

d.



θ = 120° or $\frac{2\pi}{3}$

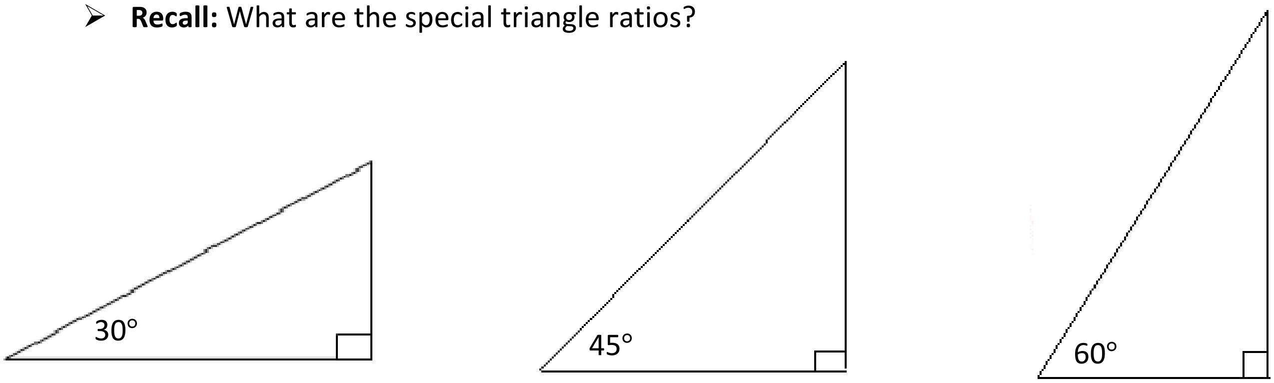
P = $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

sin θ = $\frac{\sqrt{3}}{2}$

cos θ = $-\frac{1}{2}$

Day Three: Reference Angles

➢ **Recall:** What are the special triangle ratios?



➢ So how do we use these **special triangles**?

Special triangles can help us determine ______reference angles______, which can help us calculate the ______exact value______ of an angle.

A **reference angle** is An angle, θ, that the terminal side of a given angle makes with the x-axis.

$0 < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$

Algebraically, this looks like:

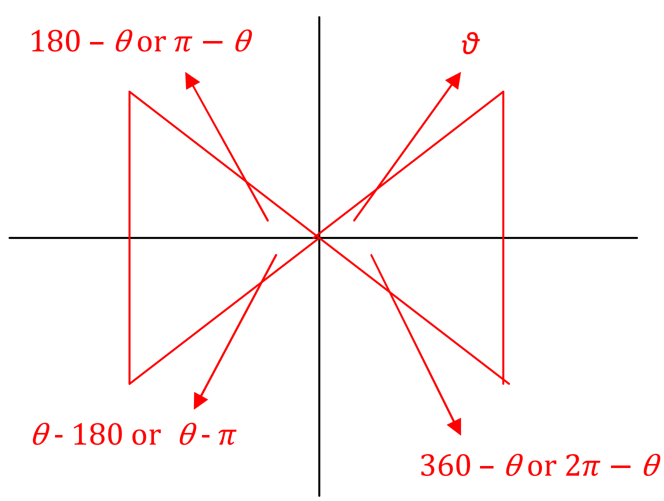
Quadrant I: reference angle = θ

Quadrant II: reference angle = $180 - \theta$ or $\pi - \theta$

Quadrant III: reference angle = $\theta - 180$ or $\theta - \pi$

Quadrant IV: reference angle = $360 - \theta$ or $2\pi - \theta$

Graphically, this looks like:



➢ So how do we use **reference angles with special triangles**?

1. Sketch the angle and make a triangle with the terminal side and x-axis

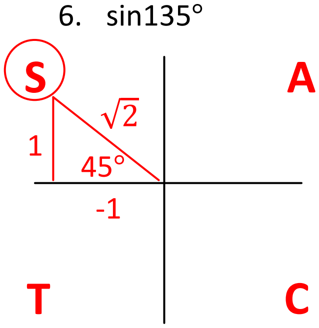
2. Determine reference (ref) angle and label sides of triangle. Negatives?!

3. Use trig ratios to determine value of ref angle. Double check sign with chart

S	**A**
(x, y) = (- , +)	(x, y) = (+, +)
(x, y) = (- , -)	(x, y) = (+, -)
T	**C**

Practice: Find the exact value of each of the following:

6. sin135°

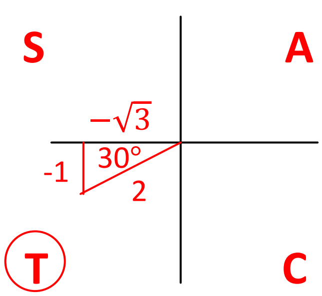


135° → Q 2

ref angle = 180° - 135 = 45°

In Q 2: $\sin 45° = \frac{O}{H} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ → $\sin 135° = \frac{\sqrt{2}}{2}$

7. $\cos \frac{7\pi}{6}$.

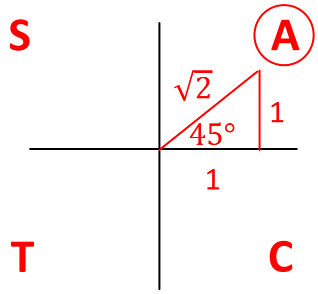


$\frac{7\pi}{6} = 210°$ → Q 3

ref angle = 210° - 180 = 30°

In Q 3: $\cos 30° = \frac{A}{H} = \frac{-\sqrt{3}}{2}$ → $\cos 210° = \cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

8. sin(-315°).

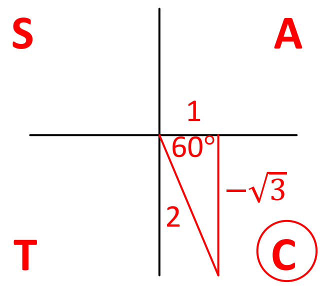


-315° = 45° → Q 1

ref angle = 45°

$\sin 45° = \frac{O}{H} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ → $\sin(-315°) = \frac{\sqrt{2}}{2}$

9. cos 300°.

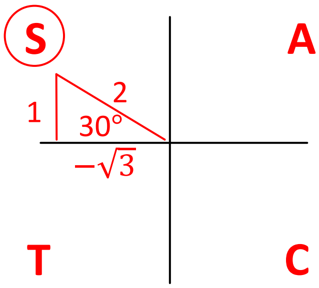


300° → Q 4

ref angle = 360° - 300 = 60°

In Q 4: $\cos 60° = \frac{A}{H} = \frac{1}{2}$ → $\cos 300° = \frac{1}{2}$

10. $\tan\frac{5\pi}{6}$



$\frac{5\pi}{6}$ = 150° → Q 2

ref angle = 180° - 150° = 30°

In Q 2: $\tan 30° = \frac{O}{A} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$ → $\tan 150° = -\frac{\sqrt{3}}{3}$

Name________KEY____ Due Date________________

Algebra II Trig Functions

Worksheet 3

3. Determine the reference angle for each of the following. Then find the exact value.

a. tan330°

ref angle = 30°

exact value = $-\frac{\sqrt{3}}{3}$

b. sin(-330°)

ref angle = 30°

exact value = $\frac{1}{2}$

c. cos $\frac{2\pi}{3}$

ref angle = $\frac{\pi}{3}$

exact value = $-\frac{1}{2}$

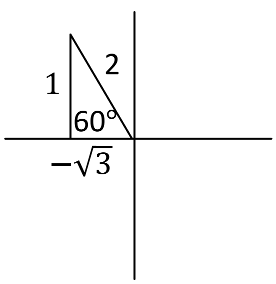
d. $-\frac{3\pi}{4}$

ref angle = $\frac{\pi}{4}$

exact value = $-\frac{\sqrt{2}}{2}$

4. Find the one error in each of the following solutions of finding an exact value. Correct it and provide a correct response.

a. cos120°



120° → Q 2

ref angle = 180° - 120 = 60°

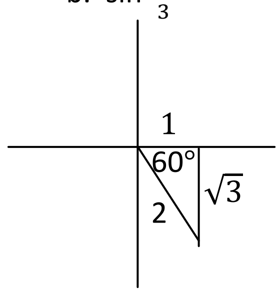
In Q 2: cos60° = $\frac{A}{H}$ = $\frac{-\sqrt{3}}{2}$

→ cos120° = $\frac{-\sqrt{3}}{2}$

Triangle is labeled wrong (switch values of the legs).

cos120° = $-\frac{1}{2}$

b. sin $\frac{5\pi}{3}$



$\frac{5\pi}{3}$ = 300° → Q 4

ref angle = 360° - 300 = 60°

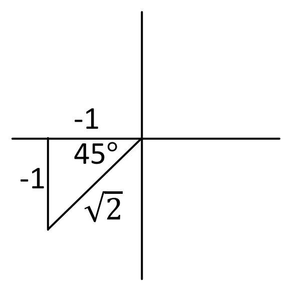
In Q 4: sin60° = $\frac{O}{H}$ = $\frac{\sqrt{3}}{2}$

→ sin300° = $\frac{\sqrt{3}}{2}$

Negatives were forgotten.

sin$\frac{5\pi}{3}$ = $-\frac{\sqrt{3}}{2}$

c. tan225°



225° → Q 3

ref angle = 225° - 180 = 45°

In Q 4: tan45° = $\frac{O}{H}$ = $\frac{-1}{\sqrt{2}}$ = $-\frac{\sqrt{2}}{2}$

→ tan225° = $-\frac{\sqrt{2}}{2}$

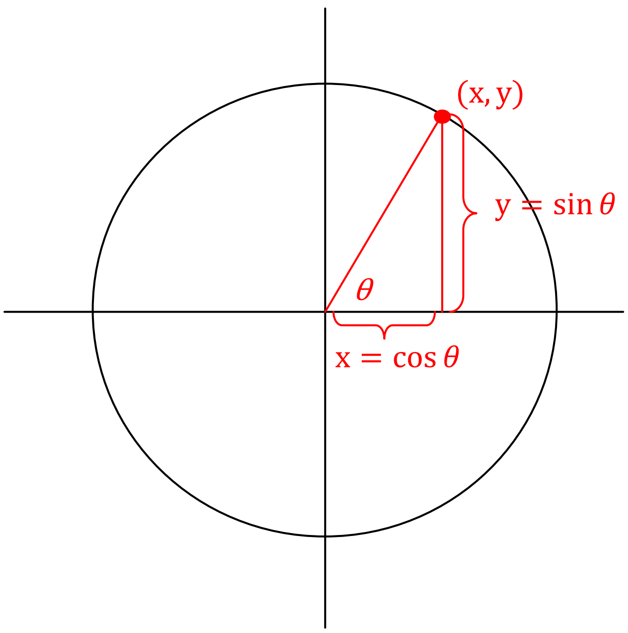
Tangent = $\frac{O}{A}$ not $\frac{O}{H}$.

tan225° = 1

Day Five: Tangent

- **Recall**

4. The trig ratio for tangent is $\frac{\text{opposite}}{\text{adjacent}}$
5. The x-value on the unit circle is the value of cosine
6. The y-value on the unit circle is the value of sine



$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

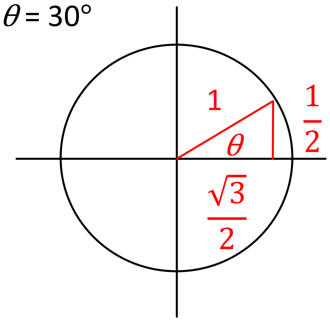
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

- What does this look like **numerically**?

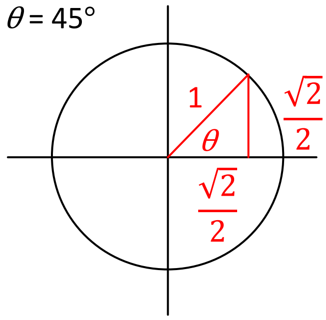
θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	270° or $\frac{3\pi}{2}$	360° or 2π
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	0	1	0	-1
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	1	0	-1	0
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	0	UND.	0	UND.

We just concluded $\tan\theta = \frac{y}{x}$. What is another term for this ratio? $\frac{\text{rise}}{\text{run}} = \text{slope}$

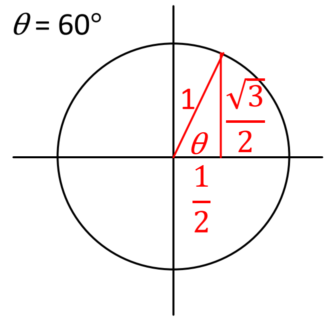
$\theta = 30°$



$\theta = 45°$



$\theta = 60°$



Ratio = $1 : \sqrt{3} : 2$
Unit circle has radius 1
→ Need to rescale
$\div 2 \rightarrow \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

Slope = $\frac{1}{2} \div \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Ratio = $1 : 1 : \sqrt{2}$
Unit circle has radius 1
→ Need to rescale
$\div \sqrt{2} \rightarrow \frac{\sqrt{2}}{2} : \frac{\sqrt{2}}{2} : 1$

Slope = $\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = 1$

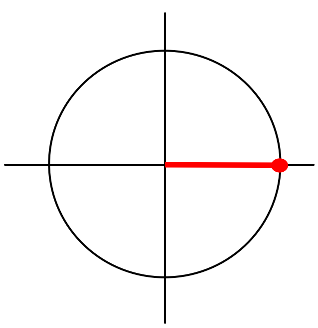
Ratio = $\sqrt{3} : 1 : 2$
Unit circle has radius 1
→ Need to rescale
$\div 2 \rightarrow \frac{\sqrt{3}}{2} : \frac{1}{2} : 1$

Slope = $\frac{\sqrt{3}}{3} \div \frac{1}{2} = \sqrt{3}$

The slope when $\vartheta = 0°$ is horizontal

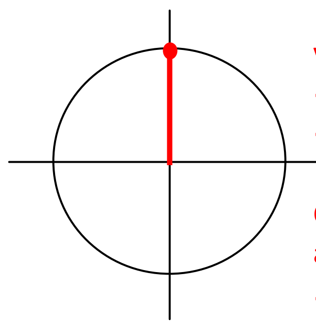
The slope when $\vartheta = 90°$ is vertical

- **Geometrically**, what can you conclude about tan0 and tan90?



Horizontal slope
→ slope = 0
→ tan0 = 0

Check:
at point (1, 0)
→ tan0 = $\frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

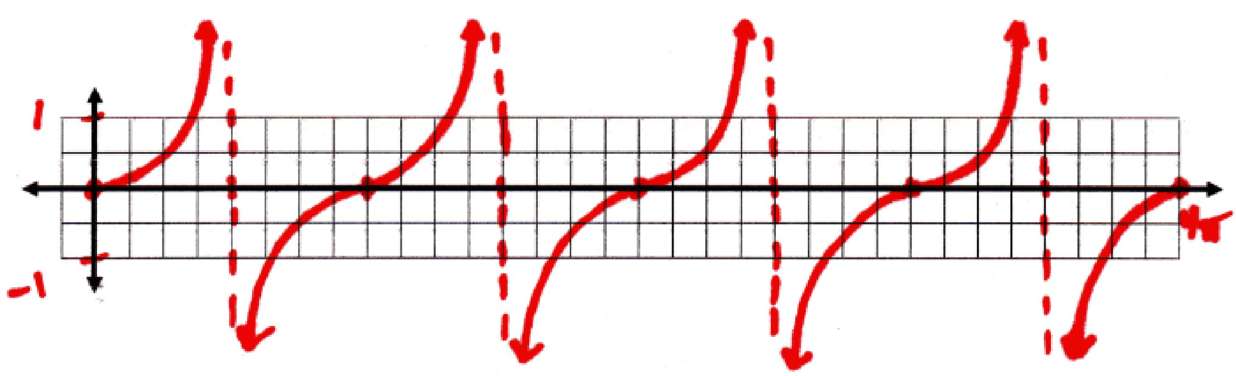


Vertical slope
→ slope = Undefined
→ tan90 = Undefined

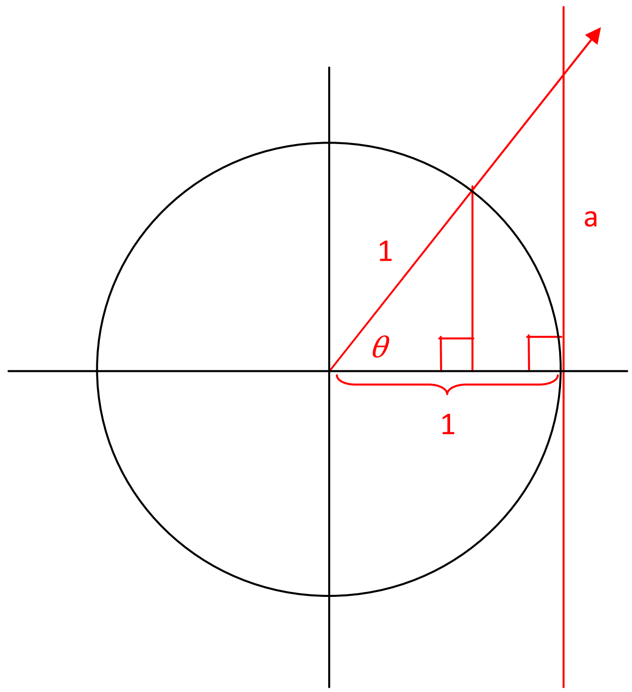
Check:
at point (0, 1)
→ tan90 = $\frac{\sin 90}{\cos 90} = \frac{1}{0}$ = Undefined

- What does this look like **graphically**?

$y = \tan\theta$ Domain: $\{\theta \in \mathbb{R} \mid \theta \neq 90 + 180k, k \in \mathbb{Z}\}$ Range: $\{\mathbb{R}\}$



- So why do we call it "tangent"?



The value of tangent also represents the length of the segment that is tangent to the circle

$$\tan\theta = \frac{O}{A} = \frac{a}{1} = a$$

Practice: Determine the exact value of each of the following. Solution methods may vary.

2. $\tan 45° = 1$

2. $\tan\pi = 0$

3. $\tan 330° = -\frac{\sqrt{3}}{3}$

4. $\tan\frac{4\pi}{3} = \sqrt{3}$

Name______**KEY**______________________________ Due Date________________

Algebra II Trig Functions

Worksheet 5

5. If $\sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$, then $\cos\theta = ?$

$$\frac{1}{2}$$

6. If $\sin\theta = -\frac{\sqrt{2}}{2}$ and $\cos\theta$ is negative, then θ is in which quadrant?

Quadrant III

7. If $\tan\theta = -\frac{\sqrt{3}}{3}$ and $\sin\theta$ is positive, then θ is in which quadrant?

Quadrant II

8. Evaluate the following:

b. $\sin 60° + \tan 45°$

$$\frac{\sqrt{3}}{2} + 1 = \frac{\sqrt{3}+2}{2}$$

b. $(\tan 150°)(\cos -120°)$

$$\left(-\frac{\sqrt{3}}{3}\right)\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{6}$$

c. $\tan\frac{7\pi}{6} \div \sin^2\left(\frac{4\pi}{3}\right)$

$$\frac{\sqrt{3}}{3} \div \frac{3}{4} = \frac{4\sqrt{3}}{9}$$

d. $(\cos 0 - \sin\frac{5\pi}{6})(\tan\pi + \sin\frac{5\pi}{3})$

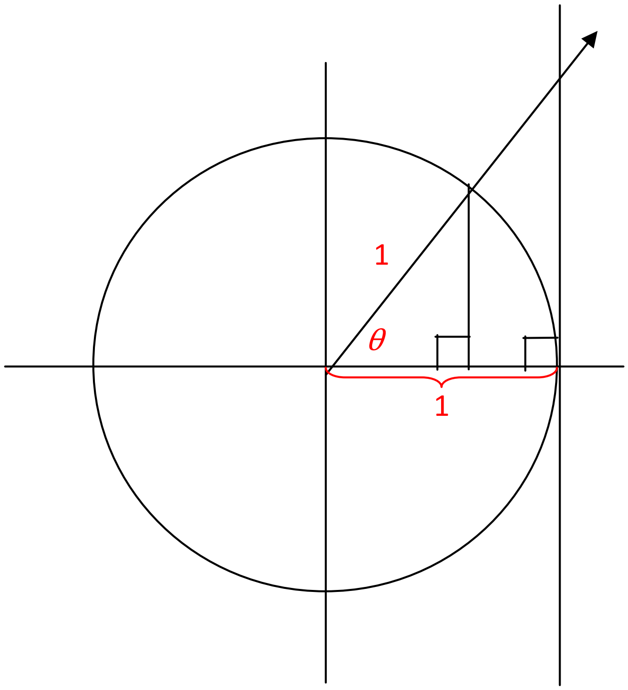
$$\left(1 - \frac{1}{2}\right)\left(0 + \left(-\frac{\sqrt{3}}{2}\right)\right) = \frac{\sqrt{3}}{4}$$

Day Six: Co-functions

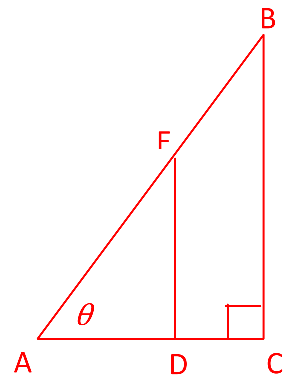
In addition to sine, cosine, and tangent, there are 3 co-functions: ____secant____,

____cosecant____, and ____cotangent____.

- Where do **co-functions** come from?



→



By similar triangles,

$$\frac{\text{AB}}{\text{AC}} = \frac{\text{BC}}{\text{ED}} \rightarrow \frac{\text{AB}}{1} = \frac{\tan\theta}{\sin\theta} = \frac{\sin\theta/\cos\theta}{\sin\theta} = \frac{1}{\cos\theta}$$

$$\rightarrow \text{AB} = \frac{1}{\cos\theta}$$

We call ____$\frac{1}{\cos\theta}$ secant (sec)____.

Similarly, we call ____$\frac{1}{\sin\theta}$ cosecant (csc)____, and ____$\frac{1}{\tan\theta}$ cotangent (cot)____

________. The co-functions are also referred to as ____reciprocal trig functions____

____

Does taking the reciprocal change the sign of a function? ____No____

Sine is positive in quadrants ____1 & 2____ → ____csc____ is positive in quadrants____1 & 2____

Cosine is positive in quadrants ____1 & 4____ → ____sec____ is positive in quadrants____1 & 4____

Tangent is positive in quadrants ____1 & 3____ → ____cot____ is positive in quadrants____1 & 3____

- What does cscθ look like **numerically**?

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
sin θ	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	0	1	0	-1	0
csc θ	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	UND.	0	UND.	-1	UND.

Practice: Evaluate the following. Write your answer as a single fraction when necessary.

$\csc 225°$

$$= \frac{1}{\sin 225°} = -\frac{1}{\frac{1}{\sqrt{2}}}$$

$\rightarrow \csc 225° = -\sqrt{2}$

$\csc\frac{\pi}{6} + \sin\frac{2\pi}{3}$

$$= \frac{1}{\sin\frac{\pi}{6}} + \frac{\sqrt{3}}{2} = \frac{1}{\frac{1}{2}} + \frac{\sqrt{3}}{2}$$

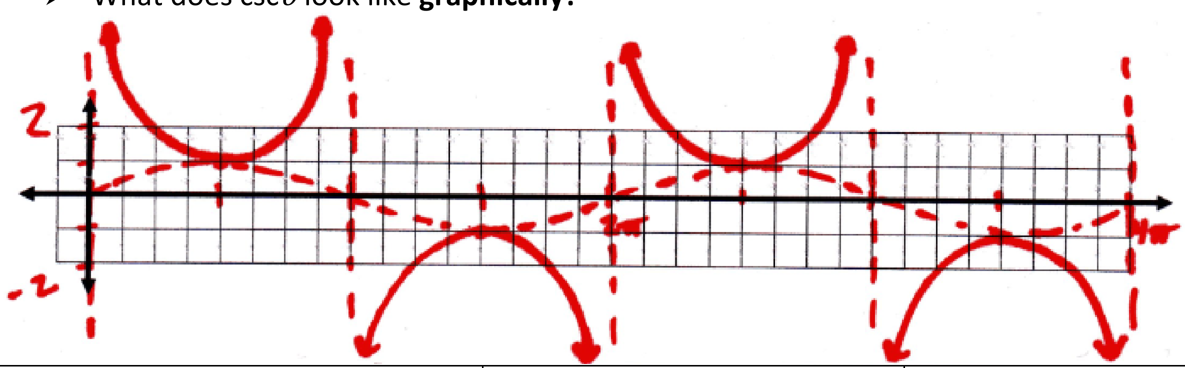
$$= \frac{4+\sqrt{3}}{2}$$

$\csc^2 300°$

$$= \left(\frac{1}{\sin 300°}\right)^2 = \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{4}{3}$$

- What does cscθ look like **graphically?**



	Domain	Range
$f(x) = \sin x$	$\{\mathbb{R}\}$	[-1, 1]
$f(x) = \csc x$	$\{x \epsilon \mathbb{R} \mid x \neq 90 + 180k; k \epsilon \mathbb{Z}\}$	$(-\infty, -1] \text{U} [1, \infty)$

➢ What does secθ look like **numerically**?

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
cosθ	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	1	0	-1	0	1
secθ	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	1	UND.	-1	UND.	1

Practice: Evaluate the following:

$\sec 225°$

$$= \frac{1}{\cos 225°} = -\frac{1}{\frac{1}{\sqrt{2}}}$$

→ $\sec 225° = -\sqrt{2}$

$\sec\frac{\pi}{6} + \cos\frac{3\pi}{2}$

$$= \frac{1}{\cos\frac{\pi}{6}} + \left(-\frac{1}{2}\right) = \frac{1}{\frac{\sqrt{3}}{2}} - 0$$

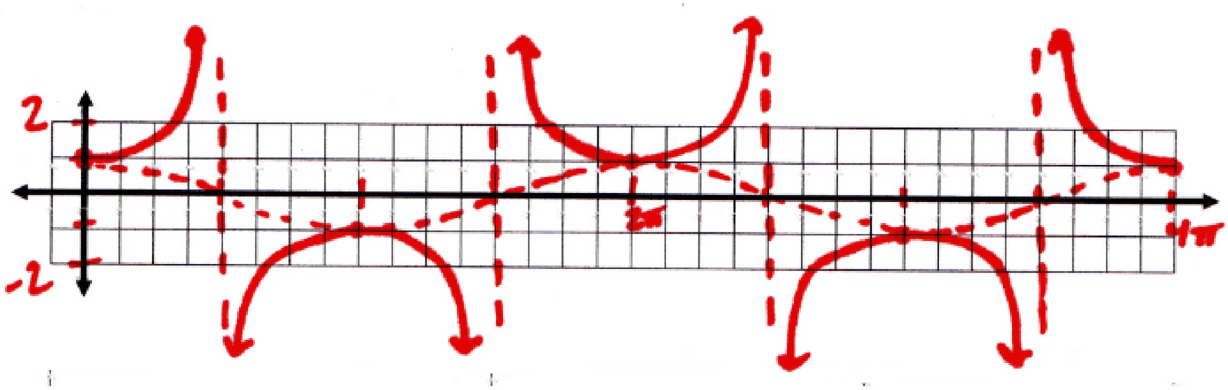
$$= \frac{2\sqrt{3}}{3}$$

$\sec^2 300°$

$$= \frac{1}{(\cos 300°)^2} = \frac{1}{\left(\frac{1}{2}\right)^2}$$

$$= 4$$

➢ What does $f(x) = \sec x$ look like **graphically?**



	Domain	Range
$f(x) = \cos x$	$\{\mathbb{R}\}$	[-1, 1]
$f(x) = \sec x$	$\{x\epsilon\mathbb{R} \mid x \neq 180k; k\epsilon\mathbb{Z}\}$	$(-\infty, -1] \cup [1, \infty)$

- What does $\cot\theta$ look like **numerically**?

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	0°	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	0	UND.	0	UND.	0
$\cot\theta$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	UND.	0	UND.	0	UND.

Practice: Evaluate the following:

$\cot 225°$

$$= \frac{1}{\cos 225°} = -\frac{1}{\frac{1}{\sqrt{2}}}$$

$\rightarrow \sec 225° = -\sqrt{2}$

$\cot\frac{\pi}{6} + \tan\frac{5\pi}{6}$

$$= \frac{1}{\tan\frac{\pi}{6}} + (-\sqrt{3}) = \frac{1}{\frac{\sqrt{3}}{3}} - \frac{1}{\sqrt{3}}$$

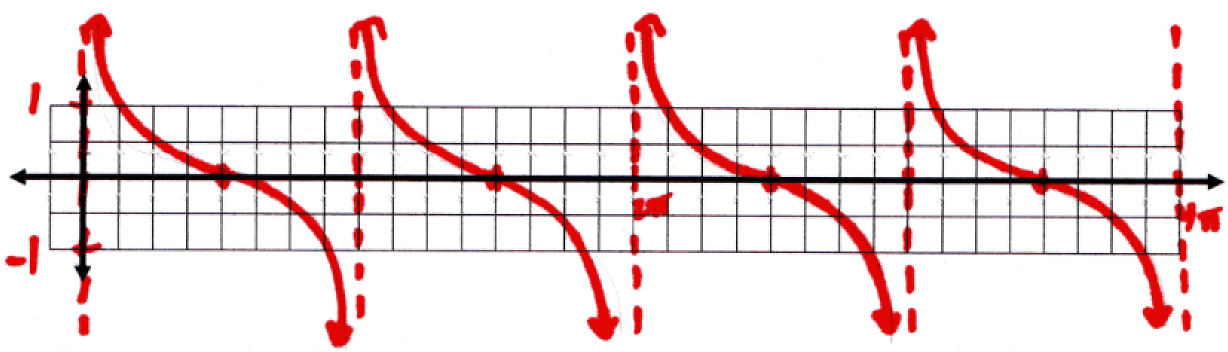
$$= \frac{2\sqrt{3}}{3}$$

$\cot^2 300°$

$$= \frac{1}{(\tan 300°)^2} = \frac{1}{(-\sqrt{3})^2}$$

$$= \frac{1}{3}$$

- What does $f(x) = \cot x$ look like **graphically?**



	Domain	Range
$f(x) = \tan x$	$\{x\epsilon\mathbb{R} \mid x \neq 90 + 180k; k\epsilon\mathbb{Z}\}$	$\{\mathbb{R}\}$
$f(x) = \cot x$	$\{x\epsilon\mathbb{R} \mid x \neq 180k; k\epsilon\mathbb{Z}\}$	$\{\mathbb{R}\}$

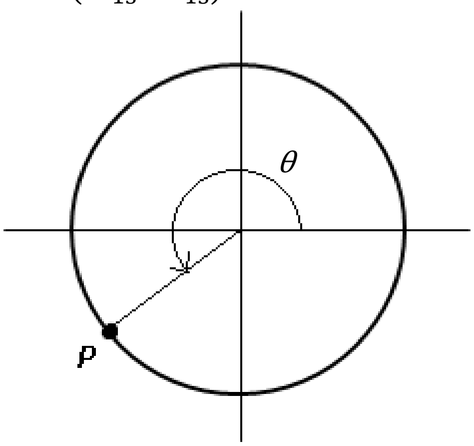
Name KEY
Algebra II

Due Date
Trig Functions

Worksheet 6

4. Given the unit circle with point P, state the following:

$P = \left(-\frac{5}{13}, -\frac{12}{13}\right)$



$\sin\theta =$ $-\frac{12}{13}$

$\csc\theta =$ $-\frac{13}{12}$

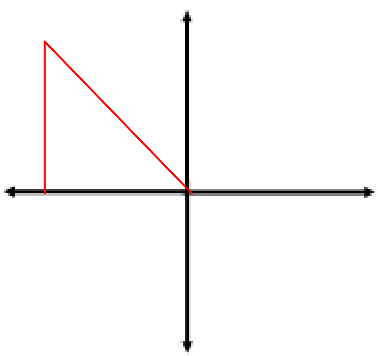
$\cos\theta =$ $-\frac{5}{13}$

$\sec\theta =$ $-\frac{13}{5}$

$\tan\theta =$ $\frac{12}{5}$

$\cot\theta =$ $\frac{5}{12}$

5. Find the exact value of each of the six trig functions given $\sin\theta = \frac{3}{5}$ and $\cos\theta < 0$.



$\sin\theta =$ $\frac{3}{5}$

$\csc\theta =$ $\frac{5}{3}$

$\cos\theta =$ $-\frac{4}{5}$

$\sec\theta =$ $-\frac{5}{4}$

$\tan\theta =$ $-\frac{3}{4}$

$\cot\theta =$ $-\frac{4}{3}$

6. Evaluate the following:

b. sin180° - csc45°

$-\sqrt{2}$

b. $\cos\frac{\pi}{6} + \cot 2\pi$

UND.

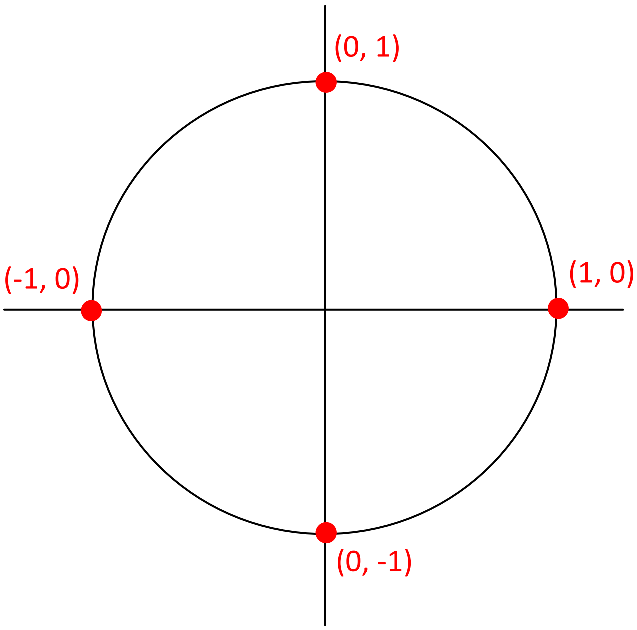
c. $\tan^2\left(\frac{\pi}{3}\right) \div \sec\frac{11\pi}{6}$

$\frac{\sqrt{3}}{6}$

Days Seven and Eight: Graphing Sine and Cosine

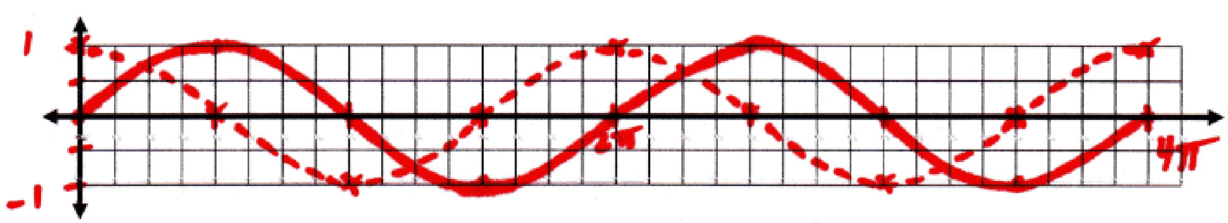
- **Recall:** Remember when we used the unit circle to create the graph $f(x) = \sin x$?

  What do you think are the easiest values to use when graphing sine and cosine?



θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1	0	-1	0

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\theta$	1	0	-1	0	1



- What can we conclude about the relationship between $f(x) = \sin x$ and $f(x) = \cos\theta$?

  They differ by a horizontal shift → $\sin x = \cos\left(x - \frac{\pi}{2}\right)$

- Because of this, let's just focus on $f(x) = \sin x$ for now.

- **Recall:** A **periodic function** is a function that repeats values at fixed intervals.

A **sinusoidal function** is a function that can be written in the form $f(x) = A\sin\big(w(x-h)\big) + k$ for real numbers *A*, *w*, *h*, and *k*.

Where

→ $|A|$ is called the amplitude of the function

→ $\frac{2\pi}{|w|}$ is called the period of the function

→ $\frac{|w|}{2\pi}$ is called the frequency of the function

→ h is called the phase shift of the function

→ the graph of $y = k$ is called the midline of the function

- k is also called the vertical shift

→ The **amplitude (A)** is the Distance between a max point of the graph and the midline

- What does **amplitude** look like **numerically**?

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1	0	-1	0

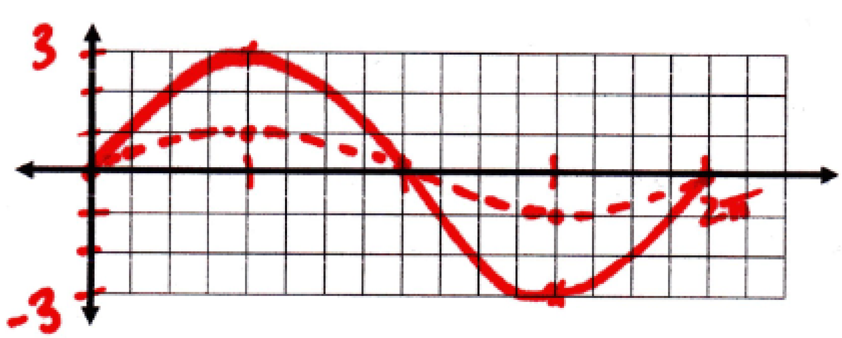
→

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1*A	0	-1*A	0

- **Graphically**?

$f(x) = \sin x$ (dotted)

$f(x) = 3\sin x$



➔ The **period (P)** is the distance between 2 consecutive max (or min) points; length of one wave

➔ The **frequency (F)** is the rate defined by number of cycles per unit of length

➔ Helpful step: **Graphing Interval** $= \frac{90}{F}$ or $\frac{\pi}{2F}$

The **graphing interval** helps with re-scaling the x-axis when there is a change in frequency

➢ What do the **period** and **frequency** look like **numerically**?

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1	0	-1	0

➔

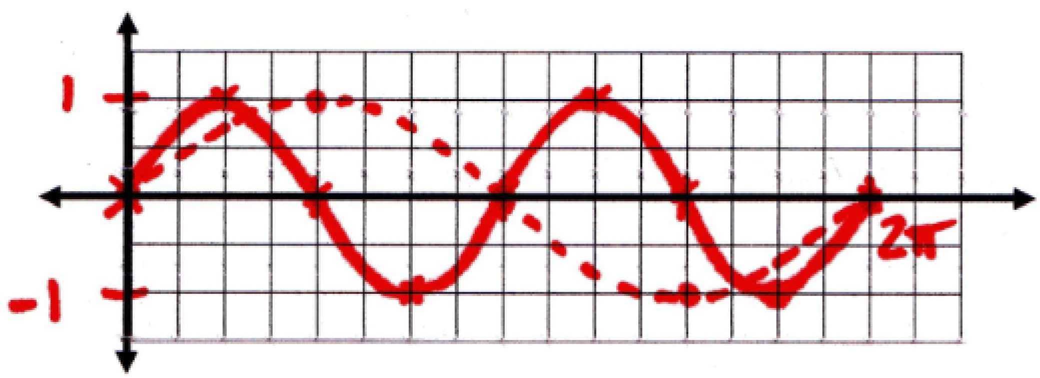
Period

θ	$\frac{0}{F}$	$\frac{\frac{\pi}{2}}{F}$	$\frac{\pi}{F}$	$\frac{\frac{3\pi}{2}}{F}$	$\frac{2\pi}{F}$
$\sin\theta$	0	1	0	-1	0

➢ **Graphically**?

$f(x) = \sin x$ (dotted)

$f(x) = \sin 2x$



What is the relationship between the **period** and **frequency**?

They are inversely related

Another term for “phase shift” is ____horizontal shift____.

What do we have to be careful of?

The sign! Shift in opposite direction!

➔ While $y = k$ is the ____midline____, k represents the ____vertical shift____.

When graphing transformations, ____order matters____!!!

Amplitude (Stretch/Shrink) → Frequency (compress/expand) → Phase & Vertical Shifts
Scale Vertically | Scale Horizontally | Shift

Practice: Graph the following

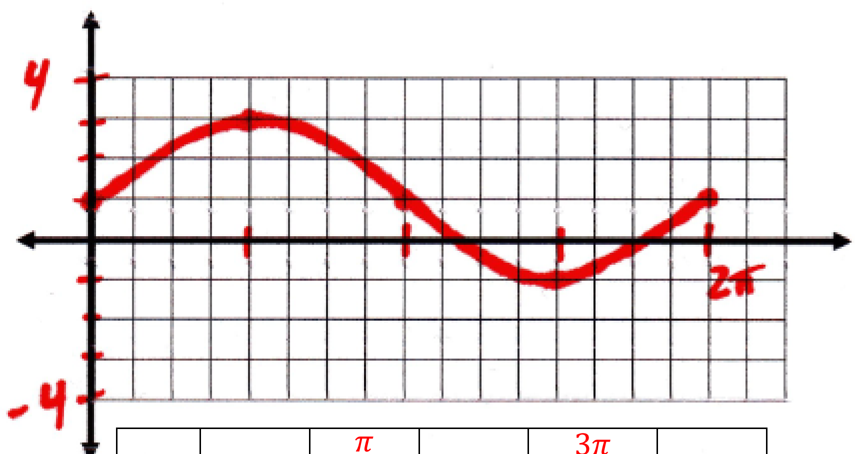
6. $f(x) = 2\sin x + 1$

A = 2 Midline: y = 1

F = $\frac{1}{2\pi}$ *P* = 2π

Order of transformations:

Stretch 2, Uo 1



θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sinθ	0*2=0	1*2=2	0*2=0	-1*2=-2	0*2=0

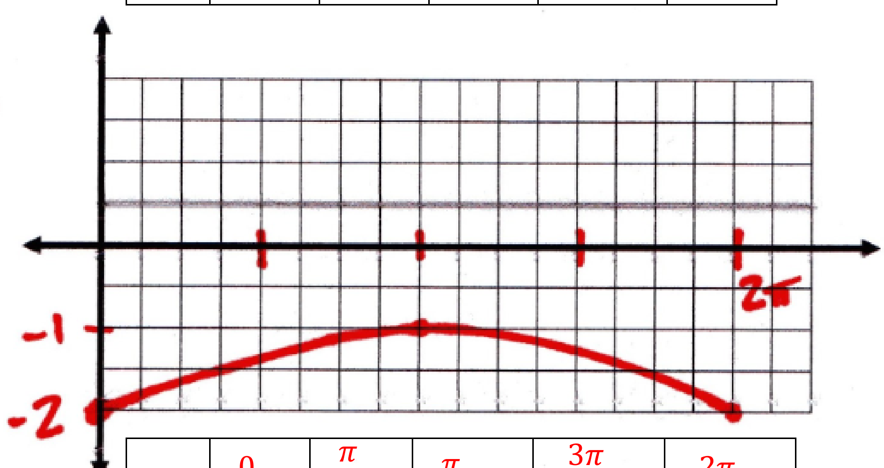
7. $f(x) = \sin\left(\frac{x}{2}\right) - 2$

A = 1 Midline: y = -2

F = $\frac{\frac{1}{2}}{2\pi} = \frac{1}{4\pi}$ *P* = 4π

Order of transformations:

Length doubles (expands), Down 2



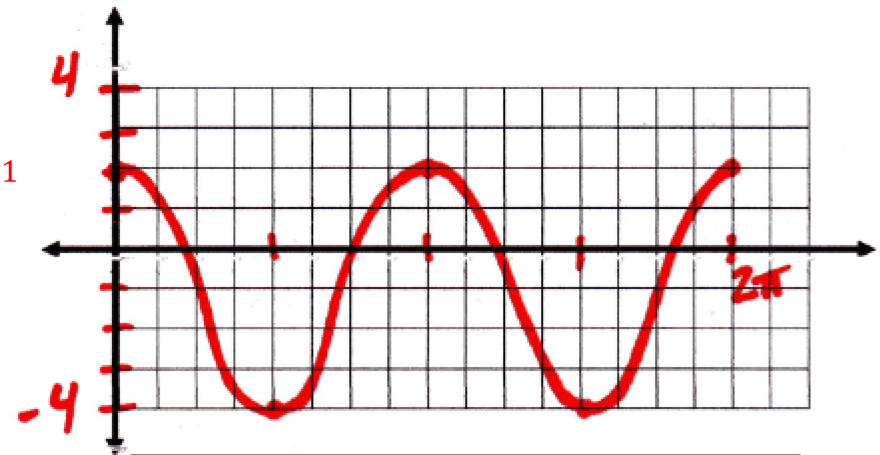
θ	$\frac{0}{\frac{1}{2}} = 0$	$\frac{\frac{\pi}{2}}{\frac{1}{2}} = \pi$	$\frac{\pi}{\frac{1}{2}} = 2\pi$	$\frac{\frac{3\pi}{2}}{\frac{1}{2}} = 3\pi$	$\frac{2\pi}{\frac{1}{2}} = 4\pi$
sinθ	0	1	0	-1	0

8. $f(x) = 3\sin(2x + \frac{\pi}{2}) - 1$

REWRITE!! $f(x) = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$

A = 3 Midline: y = 0

F = $\frac{2}{2\pi} = \frac{1}{\pi}$ *P* = π



Order of transformations:

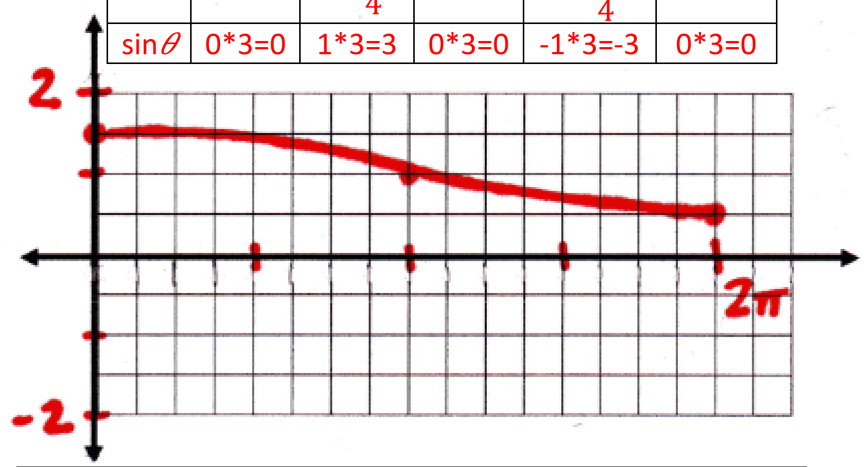
Stretch 3, Length is halved (compresses), LEFT $\frac{\pi}{4}$, down 1

θ	$\frac{0}{2} = 0$	$\frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$	$\frac{2\pi}{2} = \pi$
sinθ	0*3=0	1*3=3	0*3=0	-1*3=-3	0*3=0

9. $f(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right) + 1$

A = $\frac{1}{2}$ Midline: y = 1

F = $\frac{\frac{1}{2}}{2\pi} = \frac{1}{4\pi}$ *P* = 4π



Order of transformations:

Shrink $\frac{1}{2}$, Length is doubled (expands), up 1

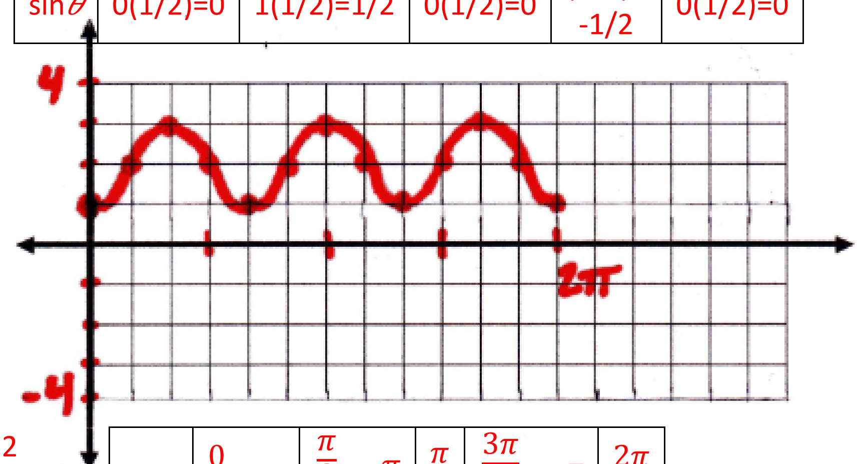
θ	$\frac{0}{\frac{1}{2}} = 0$	$\frac{\frac{\pi}{2}}{\frac{1}{2}} = \pi$	$\frac{\pi}{\frac{1}{2}} = 2\pi$	$\frac{\frac{3\pi}{2}}{\frac{1}{2}} = 3\pi$	$\frac{2\pi}{\frac{1}{2}} = 4\pi$
sinθ	0(1/2)=0	1(1/2)=1/2	0(1/2)=0	(1/2)= -1/2	0(1/2)=0

10. $f(x) = \cos(3x + \pi) + 2$

REWRITE!! $f(x) = \cos\left(3\left(x + \frac{\pi}{3}\right)\right) + 2$

A = 1 Midline: y = 2

F = $\frac{3}{2\pi}$ *P* = $\frac{2\pi}{3}$



Order of transformations:

Length is compressed by $\frac{1}{3}$, LEFT $\frac{\pi}{3}$, up 2

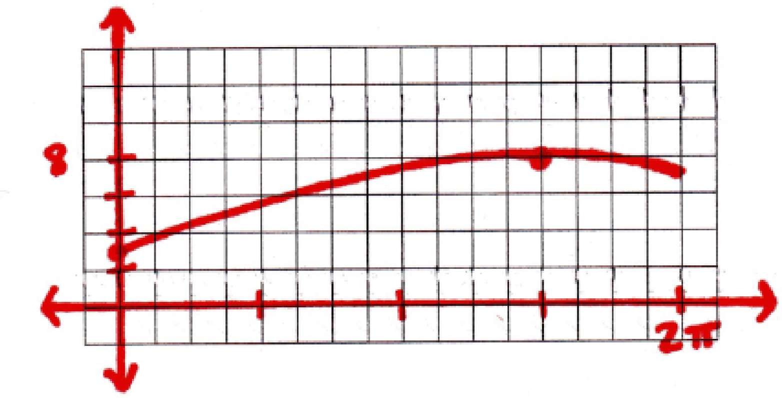
θ	$\frac{0}{3} = 0$	$\frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\frac{3\pi}{2}}{3} = \frac{\pi}{2}$	$\frac{2\pi}{3}$
sinθ	0	1	0	-1	0

Name______KEY_______________________________________ Due Date________________
Algebra II Trig Functions

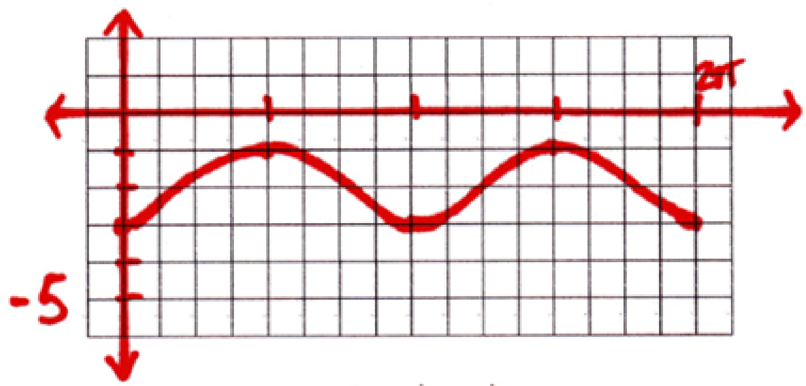
Worksheet 7

5. Graph the following:

–



6. –



7. Which of the following would match the graph of – – ?

a. – – b. – c. –

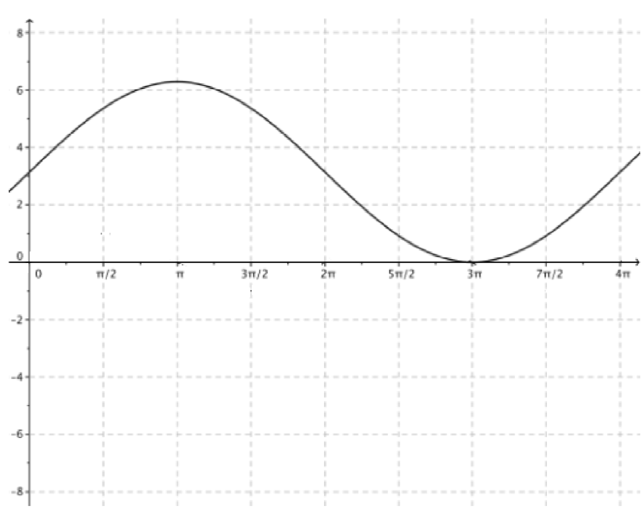
8. Fill in the missing information below based off the given graph.
**Hint: It may help to draw the parent function; i.e.

A = 3 Midline: y = 3

F = — *P* = 4π

Equation:

–



Name KEY Date

Algebra II Trig Functions

Graphing Exploration

We have been studying how to find exact values using special triangles and the unit circle, however there are other options! This exploration will help you create the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ so you can see how they relate to the unit circle.

- ✓ Fill out the tables below based off the unit circle you constructed in class.
- ✓ Mark where each of the quadrants are on the table.

Q 1 Q 2

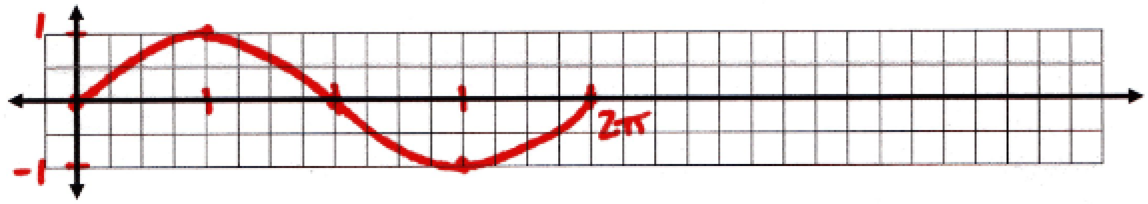
θ (degrees)	0°	30°	45°	60°	90°	120°	135°	150°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$

θ (degrees)	180°	210°	225°	240°	270°	300°	315°	330°	360°
θ (radians)	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

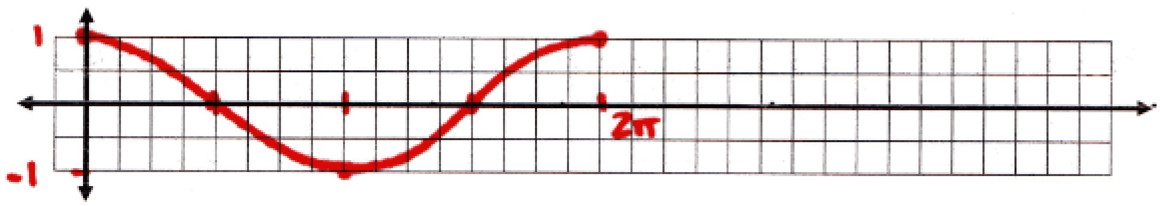
Q 3 Q 4

What would these look like **graphically**?

$f(x) = \sin x$



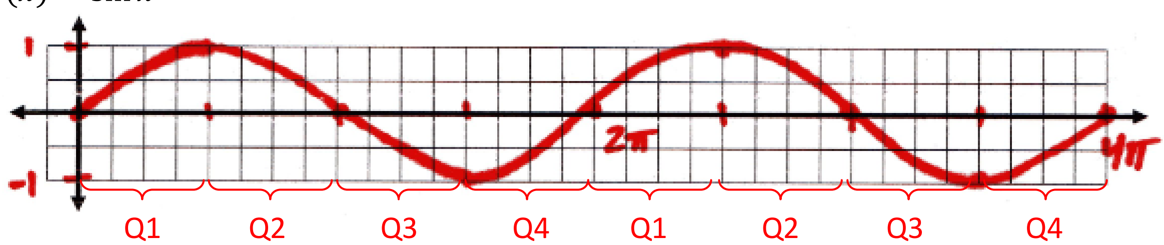
$f(x) = \cos x$



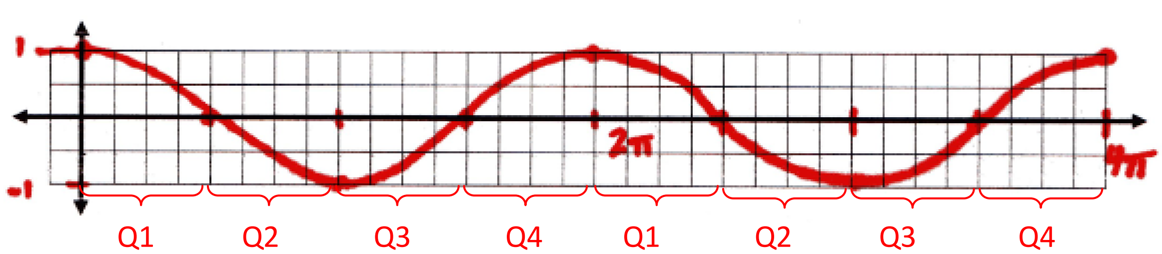
The graphs you just made represent one cycle of each function, which can be represented by one rotation around the unit circle. How many cycles do you think you would see if you went around the unit circle 2 times? Graph them below.

- ✓ Mark where each of the quadrants are on your graphs

$f(x) = \sin x$



$f(x) = \cos x$



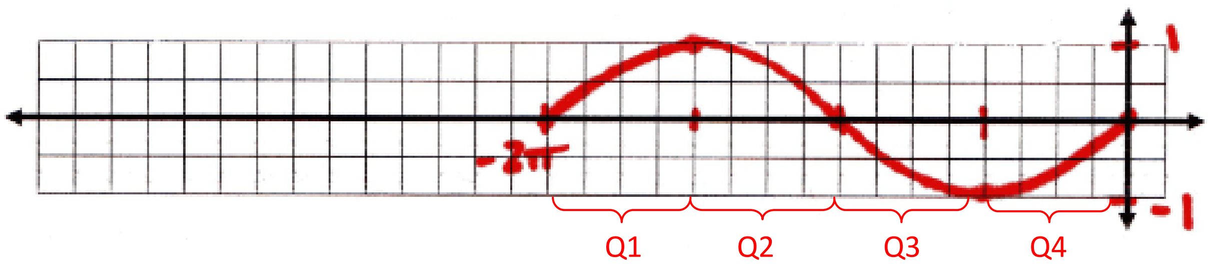
How many cycles do you think you would see if you traveled around the unit circle 3 times? 4 times? Do you think this go on infinitely? Why?

3 cycles; 4 cycles; Yes infinitely; Answers will vary, but may include how you can travel around the unit circle infinitely so the graph is also infinite, the graph is undisturbed over multiple cycles so it can keep going, etc.

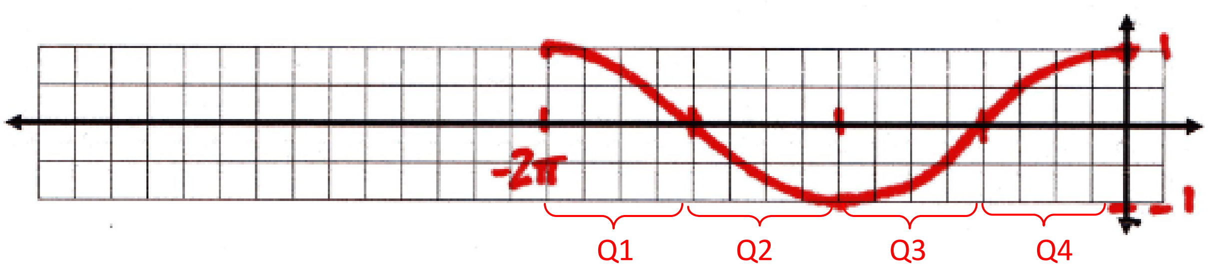
Now graph what $f(x) = \sin x$ and $f(x) = \cos x$ would look like if you traveled BACKWARDS around the unit circle one time.

- ✓ Mark where each of the quadrants are on your graphs

$f(x) = \sin x$



$f(x) = \cos x$



Based on what happened when you graphed multiple cycles of $y = \sin\theta$ and $y = \cos\theta$ in the positive direction, do you think you can graph infinite cycles in the negative direction? Why?

Answers will vary, but may include how you can travel around the unit circle infinitely in either direction so the graph is also infinite; the graph is undisturbed over multiple cycles so it can keep going, etc.

State the domain and range for $f(x) = \sin x$ and $f(x) = \cos x$.

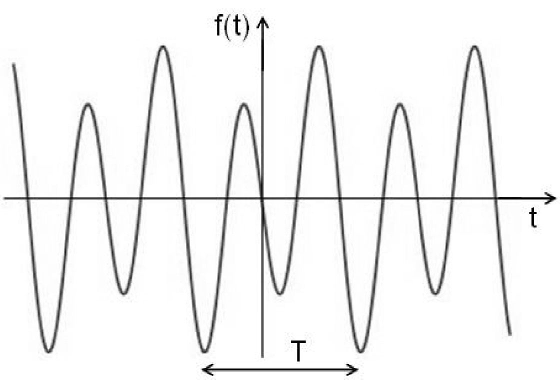
Domain: $\{\mathbb{R}\}$ Range: [-1, 1]

Periodic is a term we use to describe a function that repeats values at fixed intervals. Are sine and cosine periodic?

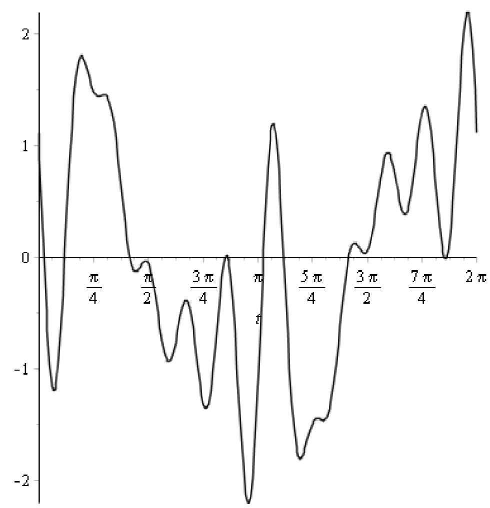
Yes

Algebraically, this means for any period $P > 0$ (where the **period** is the length of one curve), the function f contains $x + P$ whenever it contains x, and $f(x + P) = f(x)$ for all real numbers in its domain.

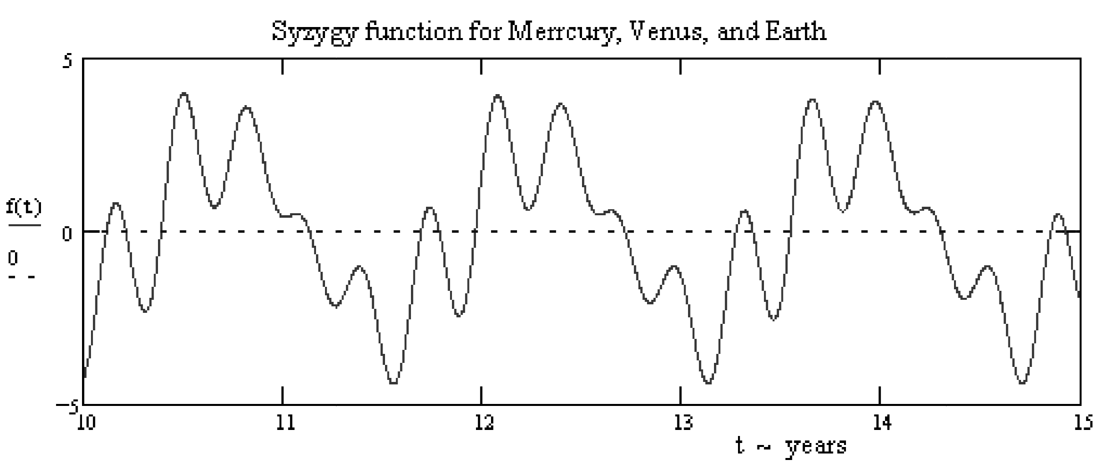
Determine which of the functions below are **periodic**.



Periodic



Not periodic



Periodic

Name KEY Due Date

Algebra II Trig Functions

Worksheet 4

We have learned multiple ways to determine the sine and cosine of a given value. We can use the unit circle with special triangles, the exact values chart (ref. angles 30°, 45°, and 60°), and the graphs of $y = \sin x$ and $y = \cos x$. Solve each of the following using a solution method of your choice. Show your work or explain how you got your answer.

3. Find the one error in each of the following solutions of finding an exact value. Correct it and provide a correct response.

c. $\sin 0° + \cos 30°$

The graph of $y = \sin x$ starts at y = 1, so sin0° = 1

From the exact values chart, cos30° = $\frac{\sqrt{2}}{2}$

Therefore sin0° + cos30° = 1 + $\frac{\sqrt{2}}{2} = \frac{2}{2} + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}$

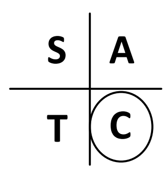
The graph of $y = \sin x$ starts at y = 0, not 1

$\sin 0° + \cos 30° = 0 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

d. $\cos\frac{5\pi}{3} \div \cos\frac{3\pi}{2}$

$\cos\frac{5\pi}{3}$ = cos300° $\cos\frac{3\pi}{2}$ = cos270°

300° → Q 4 → ref angle = 60° → $\cos 60° = \frac{1}{2}$



In Q 4 → cosine is positive → cos300° = $\frac{1}{2}$

270° is on the unit circle at the point (0, -1)

cos270° = y-value = -1

➔ $\cos\frac{5\pi}{3} \div \cos\frac{3\pi}{2} = \frac{1}{2} \div (-1) = -\frac{1}{2}$

cos270° = x-value, not y-value

$\cos\frac{5\pi}{3} \div \cos\frac{3\pi}{2} = \frac{1}{2} \div 0 =$ undefined

4. Find the exact value of each of the following.

b. sin90° + cos210°

$$1+\left(-\frac{\sqrt{3}}{2}\right)=\frac{2-\sqrt{3}}{2}$$

b. $\frac{\cos 0}{\sin 0}$

$\frac{1}{0}$ = undefined

c. $\sin\frac{11\pi}{6}-\cos\frac{2\pi}{3}$

$$-\frac{1}{2}-\left(-\frac{1}{2}\right)=0$$

d. $\cos\left(\frac{5\pi}{4}\right)\sin\left(\frac{7\pi}{4}\right)$

$$\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$$

e. $\sin^2(120°)$
**Hint: $\sin^2(x)=(\sin x)^2$

$$\left(\frac{\sqrt{3}}{2}\right)^2=\frac{3}{4}$$

e. (cos510°)(cos225°)

$$\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{6}}{4}$$

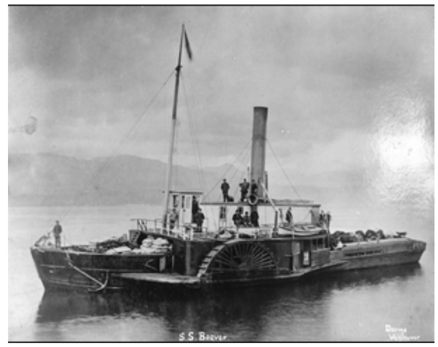
Name________KEY______________________________________ Date________________

Algebra II Trig Functions

Modeling with Trig Functions

We have been discussing the characteristics of sinusoidal functions. Now it is time to see how they can be put to use in modeling situations!

4. The paddle wheel of the S.S. Beaver was 13 feet in diameter and revolved 30 times per minute when moving at top speed. Using this speed and starting from a point at the very top of the wheel, write a model for the height *h* (in feet) of the end of a paddle relative to the water's surface as a function of the time *t* (in seconds). Note that the paddle is 2 feet below the water's surface at its lowest point.



e. What is the range of the paddle wheel's height?

Min height = m = -2

Max height = M = 13 – 2 = 11

Range = [-2, 11]

How can you use this to determine the midline and amplitude of the model? What shift does the midline help you determine?

$\text{midline} = k = \frac{M+m}{2} = \frac{11+(-2)}{2} = \frac{9}{2}$ $\text{ampliude} = A = \frac{M-m}{2} = \frac{11-(-2)}{2} = \frac{13}{2}$

Midline determines vertical shift

f. The model has to be in units of feet per second. How long does it take for one cycle of the wheel? What does this help you find?

Wheel turns at a rate of 30 times per minute → 1 turn per 2 seconds → period = 2sec

$2 = P = \frac{2\pi}{w} \rightarrow w = \pi$

g. How can you determine whether to use sine or cosine?
Hint: think about the line behavior after the starting point

Wheel starts at its highest point → use cosine. Since it starts at the top and decreases in height after *t* = 0, it does not need to be flipped i.e. no negative sign.

h. What is the final sinusoidal model for this scenario?

$f(x) = A\sin\big(w(x-h)\big) + k \;\rightarrow\; f(t) = \frac{13}{2}\cos(\pi t) + \frac{9}{2}$

Note: although the form uses sine, it hold for cosine, too.

5. A Ferris wheel with a radius of 25 feet is rotating at a rate of 3 revolutions per minute. When t = 0, a chair starts at the lowest point on the wheel, which is 5 feet above the ground. Write a model for the height h (in feet) of the chair as a function of the time t (in seconds).
**Hint: If you are stuck, use the questions from question (1) to help guide you through the solution process.

Max = M = 25(2) + 5 = 55 min = m = 5

$$k = \frac{M+m}{2} = \frac{55+5}{2} = 30$$

$$A = \frac{M-m}{2} = \frac{55-5}{2} = 25$$

3 rev/min = 1 rev/20 sec → P = 20 → $20 = \frac{2\pi}{w}$ → $w = \frac{\pi}{10}$

Start at lowest point → cosine, but negative coefficient (start at bottom and go up, opposite of cosine behavior after t = 0)

$$f(t) = -25\cos\left(\frac{\pi}{10}t\right) + 30$$

6. Eskimos use igloos as temporary shelter from harsh winter weather. Below is a table displaying the temperature both inside and outside the igloo over a typical day.

Time	8AM	10AM	12PM	2PM	4PM	6PM
Outside Temp. (°F)	-20	-15	-12	-10	-13	-16
Inside Temp. (°F)	22	23	25	26	25.5	24

Time	8PM	10PM	12AM	2AM	4AM	6AM	8AM
Outside Temp. (°F)	-19	-26	-28	-30	-29	-24	-21
Inside Temp. (°F)	23	22	20.5	20	21	21.5	23

g. At what time does the lowest temperature occur? The highest? What do these represent?

Inside igloo:

Highest = Max = 26

Lowest = min = 20

Outside igloo:

Highest = Max = -10

Lowest = min = -30

h. For a sinusoidal function of the form $f(x) = A\sin\big(w(x-h)\big) + k$, for which variables can you determine the values for from (a) for each function? State these values.

Inside:

$$k = \frac{M+m}{2} = \frac{26+20}{2} = 23$$

$$A = \frac{M-m}{2} = \frac{26-20}{2} = 3$$

Outside:

$$k = \frac{M+m}{2} = \frac{-10+30}{2} = -20$$

$$A = \frac{M-m}{2} = \frac{-10-(-30)}{2} = 10$$

i. How can you determine the frequency and period (units!!)? Is this value the same for both functions? Why/why not?

1 cycle = 1 day = 24 hours → $P = 24 = \frac{2\pi}{w}$ → $w = \frac{\pi}{12}$

Same for both because each model temperature changes over 1 day, meaning they will have the same period and therefore same frequency.

j. Determine if you will need to use sine or cosine. Explain your reasoning.

Use sine. Data starts in the middle and increases after $t = 0$, therefore positive coreflicient.

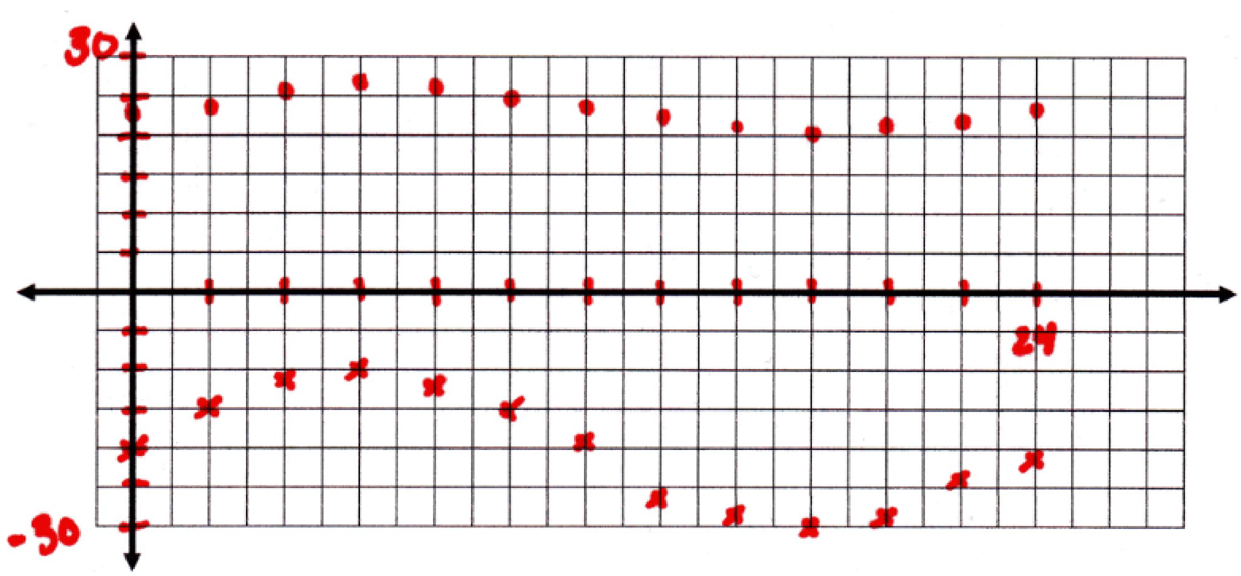
k. Write a sinusoidal model for both the outside temperature T_1 and inside temperature T_2 (in degrees Fahrenheit) as a function of the time of day t (in hours since midnight).

$$T_1(t) = 3\sin\left(\frac{\pi}{12}t\right) + 23$$

$$T_2(t) = 10\sin\left(\frac{\pi}{12}t\right) - 20$$

l. Plot the data on the axes below. Does the model you created make sense for this graph?

● $T_1(t)$ X $T_2(t)$



Name________**KEY**____ Date_________________

Algebra II Trig Functions

Trig Functions Test Review

23. Convert each angle in degrees to radians. Leave your answer in terms of π. Make sure your answer is in the interval $0 \leq \theta < 2\pi$.

d. $15°$ b. $120°$ c. $-40°$ d. $495°$

$\frac{\pi}{12}$ $\frac{2\pi}{3}$ $\frac{16\pi}{9}$ $\frac{3\pi}{4}$

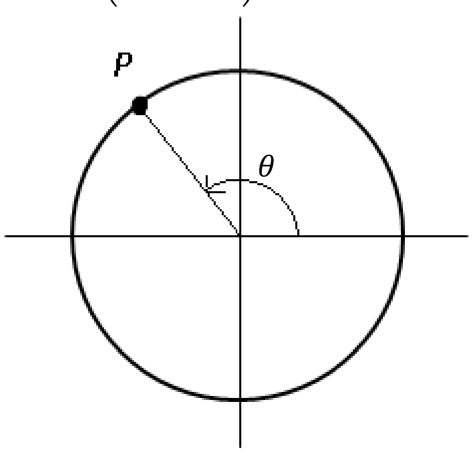
24. Convert each angle in radians to degrees. Round to the nearest hundredth when necessary. Make sure your answer is in the interval $0 \leq \theta \leq 360°$.

d. $\frac{10\pi}{3}$ b. $\frac{\pi}{13}$ c. $\frac{3\pi}{4}$ d. $-\frac{2\pi}{5}$

240° 240° 135° 288°

25. Given the unit circle with point *P*, state the following. Rationalize all denominators and simplify ratios completely.

$P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

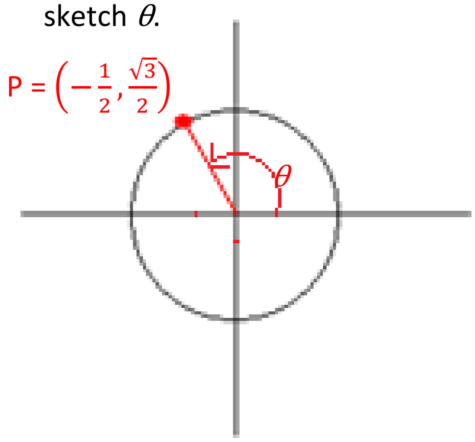


$\sin\theta = \frac{\sqrt{2}}{2}$ $\csc\theta = \sqrt{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$ $\sec\theta = -\sqrt{2}$

$\tan\theta =$ -1 $\cot\theta =$ -1

26. Provide the missing information below, determine the point *P* on the unit circle, and sketch θ.

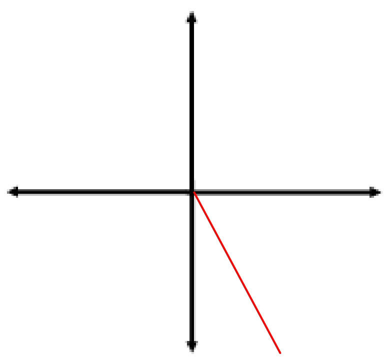


$\sin\theta = \frac{\sqrt{3}}{2}$ $\csc\theta = \frac{2\sqrt{3}}{3}$

$\cos\theta = -\frac{1}{2}$ $\sec\theta = -2$

$\tan\theta = -\sqrt{3}$ $\cot\theta = -\frac{\sqrt{3}}{3}$

27. Determine the exact value of each trig ratio if (6, -8) is on the terminal side of θ.

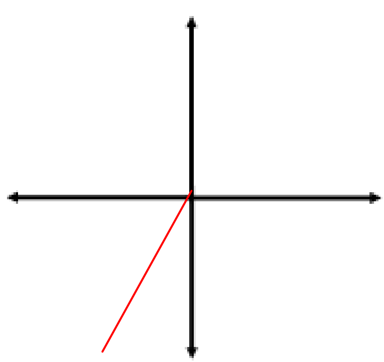


$\sin\theta = -\frac{8}{10} = -\frac{4}{5}$ $\csc\theta = -\frac{10}{8} = -\frac{5}{4}$

$\cos\theta = \frac{6}{10} = \frac{3}{5}$ $\sec\theta = \frac{10}{6} = \frac{5}{3}$

$\tan\theta = -\frac{4}{3}$ $\cot\theta = -\frac{3}{4}$

28. Find the exact value of each of the six trig functions given $\cos\theta = -\frac{5}{13}$ and $\csc\theta < 0$. Rationalize all denominators and simplify ratios completely.



$\sin\theta = -\frac{12}{13}$ $\csc\theta = -\frac{13}{12}$

$\cos\theta = -\frac{5}{13}$ $\sec\theta = -\frac{13}{5}$

$\tan\theta = \frac{12}{5}$ $\cot\theta = \frac{5}{12}$

29. Determine the exact value of each of the following.

b. $\sin 120°$

$\frac{\sqrt{3}}{2}$

b. $\sec(-315\,°)$

$\sqrt{2}$

c. $\csc\frac{\pi}{6}$

2

d. $\cos\frac{5\pi}{4}$

$-\frac{\sqrt{2}}{2}$

e. $-\sin 270°$

1

f. $\cot 90°$

UND.

30. Evaluate each of the following.

a. $\sin 180° - \csc 45°$

$0-\sqrt{2}=-\sqrt{2}$

b. $\tan^2\frac{\pi}{6} - \cot\frac{\pi}{4}$

$\frac{1}{3}-1=-\frac{2}{3}$

c. $\cos(-120°) + \csc 150°$

$-\frac{1}{2}+2=\frac{3}{2}$

d. $(\sec\frac{5\pi}{4})\left(\sin-\left(\frac{\pi}{4}\right)\right)$

$(-\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right)=1$

e. $(\csc^2 300°)\,(\tan 330°)$

$\left(\frac{4}{3}\right)\left(-\frac{\sqrt{3}}{3}\right)=-\frac{4\sqrt{3}}{9}$

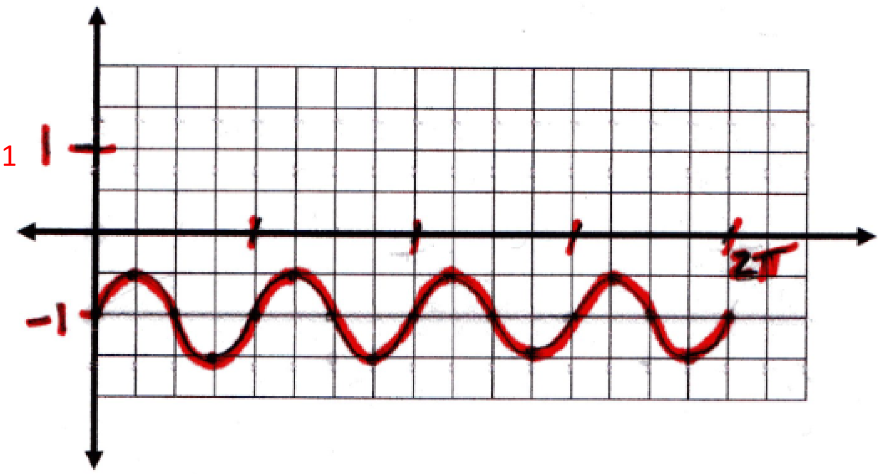
f. $\tan\frac{2\pi}{3} \div \cot\frac{5\pi}{6}$

$\sqrt{3}\div(-\sqrt{3})=-1$

31. Determine the missing information below, and then graph each function.

d. $f(x) = \frac{1}{2}\sin 4x - 1$

A = $\frac{1}{2}$ Midline: y = -1



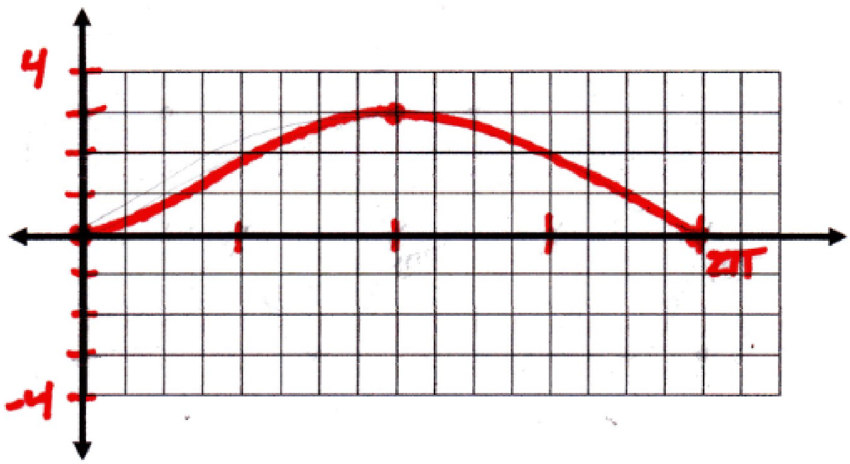
F = $\frac{4}{2\pi} = \frac{2}{\pi}$ *P* = $\frac{\pi}{2}$

Order of transformations:

Shrink $\frac{1}{2}$, length is halved (compressed), down 1

e. $f(x) = 3\cos\frac{1}{2}(x - \pi)$

A = 3 Midline: y = 0



F = $\frac{\frac{1}{2}}{2\pi} = \frac{1}{4\pi}$ *P* = 4π

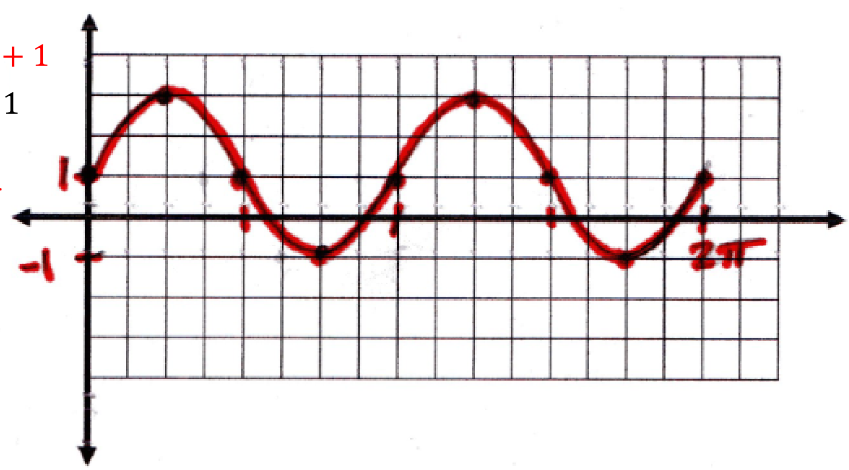
Order of transformations:

Stretch 3, length is doubled (expanded), RIGHT $\frac{\pi}{4}$

REWRITE!! $f(x) = 2\sin(2\left(x + \frac{\pi}{2}\right) + 1$

f. $f(x) = 2\sin(2x + \pi) + 1$

A = 2 Midline: y = 1



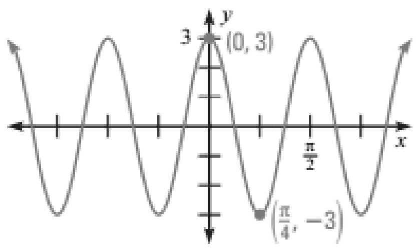
F = $\frac{2}{2\pi} = \frac{1}{\pi}$ *P* = π

Order of transformations:

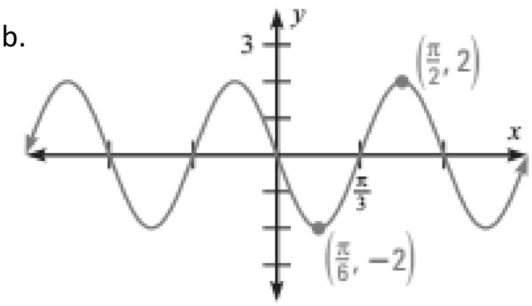
Stretch 2, length is halved (compressed), LEFT $\frac{\pi}{2}$, up 1

Write 2 functions for each of the following (one sine and one cosine).

b.



b.



Note: Answers may vary

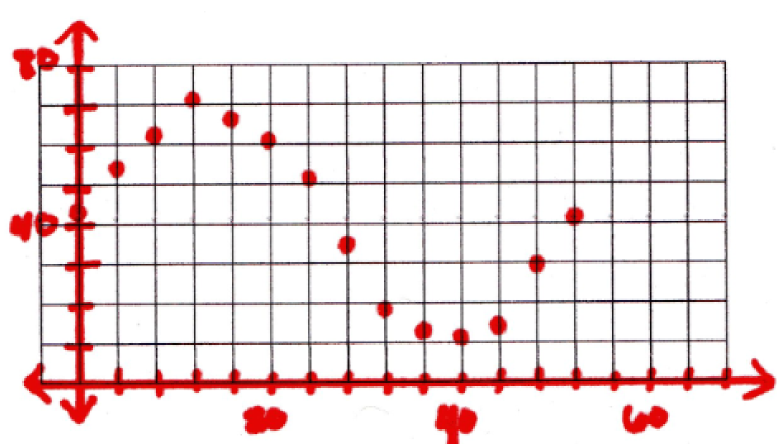
32. Below is a table of temperatures recorded weekly over the course of a year.

b. Form a sinusoidal function, *T*, that best models the changes in temperature, *t*.

Week	**0**	**4**	**8**	**12**	**16**	**20**	**24**
Temp (°F)	43	54	62	71	68	61	51

Week	**28**	**32**	**36**	**40**	**44**	**48**	**52**
Temp (°F)	36	19	13	11	14	30	41

c. Plot the data, and determine a function that fits the data graphically. Does this match your answer from (a)?



Name____KEY____ Date________________

Algebra II Trig Functions

Trig Functions Test

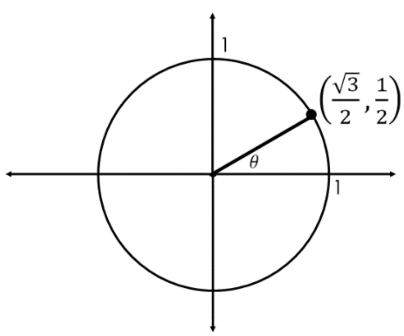
Directions: Answer the following questions the best you can. Part one is a multiple choice section of 10 questions (worth 2 points each), and part two has 3 extended response questions. For part 2, you MUST show your work to receive full credit. Good luck!!!

Part One: Multiple Choice

11. What is 200° in radians?

b. — b. — c. — d. —

12. Consider the diagram below. What is the value of



b. b. —

c. – d. —

13. If and , which quadrant does the terminal side of θ fall?

b. 1 b. 2 c. 3 d. 4

14. What is the reference angle for — ?

a. – b. – c. – d. –

15. What is the exact value of — ?

b. ——   b. ——   c.   d.

16. What is the phase shift of the function – – ?

b. Right –   b. Left –   c. Right π   d. Left π

17. What is the amplitude of the function ?

b.   b. 3   c. –   d. 2

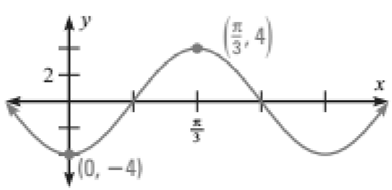
18. What is the frequency of the function – – ?

b. 6π   b. —   c.   d. —

19. What is the period of the function – – ?

a. 6π   b. —   c.   d. —

20. Which could be the sinusoidal function for the graph?

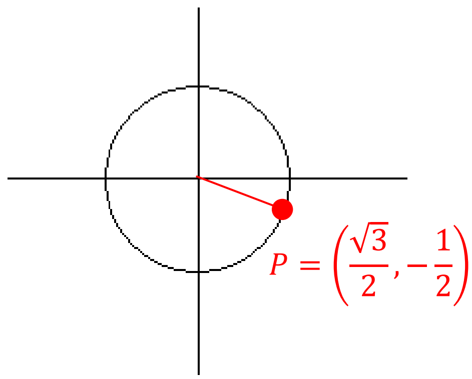


a. –   b. –

c. –   d. –

Part Two: Extended Response

33. Provide the missing information below, determine the point *P* on the unit circle, and sketch θ. Rationalize all denominators and simplify ratios completely. (6 points)



$\sin\theta = -\frac{1}{2}$ $\csc\theta = -2$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\sec\theta = \frac{2\sqrt{3}}{3}$

$\tan\theta = -\frac{\sqrt{3}}{3}$ $\cot\theta = -\sqrt{3}$

34. Determine the exact value for each of the following. Express your answer as a single fraction (when applicable) and rationalize all denominators. (2 points each)

b. $\sin 90° - \cot 180°$

$1 - 0 = 1$

b. $\sec^2\left(\frac{\pi}{3}\right) + \tan\left(\frac{5\pi}{6}\right)$

$4 + \left(-\frac{\sqrt{3}}{3}\right) = \frac{12-\sqrt{3}}{3}$

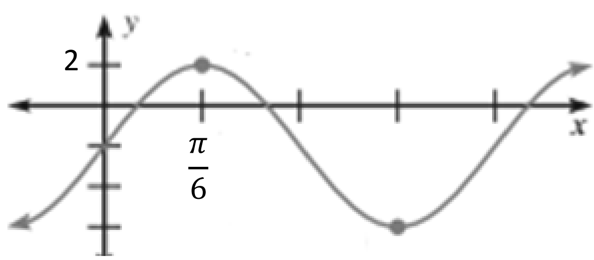
c. $(\csc\frac{2\pi}{3})(\sec\frac{3\pi}{2})$

$\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{1}{0}\right) = \text{UND.}$

d. $\tan\frac{3\pi}{4} \div \cos\frac{5\pi}{4}$

$(-1) \div \left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$

35. Write a sine AND cosine equation for the graph below. (4 points)

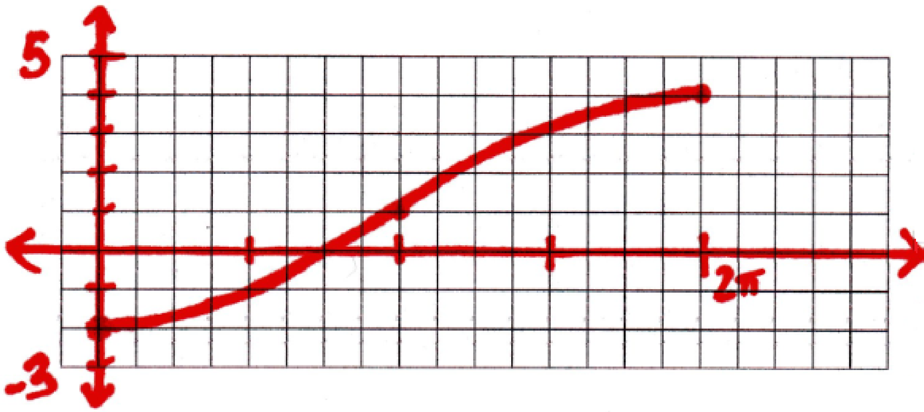


$f(x) = 2\sin(3x) - 1$

$2\cos\left(3\left(x - \frac{\pi}{6}\right)\right) - 1$

Note that answers may vary

36. Graph the function $f(x) = 3\sin\left(\frac{1}{2}(x-\pi)\right) + 1$ over the interval $[0, 2\pi]$. State the amplitude, period, and frequency, as well as the phase shift and vertical shift. (8 points)



Amplitude: 3

Period: 4π

Frequency: $\frac{1}{4\pi}$

Phase Shift: Right π

Vertical Shit: Up 1

37. A Ferris wheel with a diameter of 40 feet is rotating at a rate of 4 revolutions per minute. When *t* = 0, a chair starts at the lowest point on the wheel, which is 5 feet above the ground. Write a model for the height *h* (in feet) of the chair as a function of the time *t* (in seconds). (10 points)

$$f(x) = -20\sin\left(\frac{2\pi}{15}t\right) + 25$$