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A Curriculum Project on Quadratics Aligned to the Common Core State Standards

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by

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Abstract

This project illustrates a process of designing curriculum for the study of quadratics in Algebra 1, in alignment with the Common Core State Standards in mathematics. The process incorporates the body of research on student conceptions of quadratics and available techniques and technologies for the enhancement of student conceptions. The Algebra 1 course trajectory recommended by the Common Core is analyzed in light of the existing research, unpacked into learning objectives, and restructured to address key conceptual roadblocks synthesized from the research. The resulting curriculum capitalizes on technology, rich problem contexts, and students' prior knowledge of linear relationships in order to build student concepts of quadratic relationships. The curriculum is sequenced from graphical to symbolic representations, wherein visual and dynamic models provide meaning to symbolic structures and manipulations. A six week instructional calendar and supporting materials are provided to support implementation of this curriculum project.

Chapter One: Introduction

This curriculum project was designed with practical purposes in mind. The demanding pace of the algebra curriculum makes it difficult to transition from the National Council of Teachers of Mathematics (NCTM) standards to the Common Core State Standards (CCSS) while personalizing plans with high integrity. Thus the purpose in designing this curriculum project is to offer students learning experiences that will allow them to put their own signatures on mathematics.

With the advent of the CCSS in Mathematics, there is more focus on essential understandings. The CCSS Movement is a nationwide initiative to increase performance in United States (US) schools, which requires rigorous learning standards with greater coherence and conceptual depth (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage). Once the new standards are put into place, classroom practice should be aligned more closely to the best research-based methods.

High school teachers' unfamiliarity with the new standards presents a new challenge in designing instruction. High school mathematics teachers may find the algebra Common Core Revision of an Algebra unit exceedingly difficult to work with. Thus this thesis provides as an opportunity to explore the algebra Common Core standards in greater detail, which may help teachers to successfully implement these standards into their classroom instruction.

The topic of study is quadratic equations and functions, referred to more generally as quadratics. A quadratic function is represented graphically as a parabola.

A parabola is a more complicated curve than the straight line graph that students have previously worked with in Algebra. Real world phenomena modeled by quadratic functions are also more complicated. These applications involve rates of change that increase or decrease over certain intervals, as opposed to the constant rates of change that students are familiar with. The associated algebra techniques are rigorous. Computations involve more advanced arithmetic, more symbolic abstraction, and many more steps to reach solutions.

Quadratics is widely considered, by teachers and students alike, to be the most difficult topic encountered in Algebra 1 (Almy, 2011). The study of quadratics in Algebra may provide roadblocks for many students who are interested in science, technology, engineering and mathematics (STEM) careers. Proficiency in algebra is required in most general college entrance exams and in certification exams for many vocational fields. Algebra inefficiency may also be the gate keeper that keeps students out of STEM education and career fields.

Effective curriculum may allow students to further appreciate the beauty of mathematics and to gain a deeper understanding of the world around them. An understanding of quadratic functions may allow students to explore monumental concepts like motion and gravity, instantaneous rates of change, and be better prepared to learn the fundamental ideas of calculus. Quadratic functions are a gateway into understanding increasingly dynamic and applicable mathematical models (Eraslan, 2005).

The goal of this project is to develop effective unit plans for teaching quadratics. The unit design is aligned with the CCSS in Mathematics, the existing research on how students learn quadratics, and considers what techniques and technologies are available to improve student learning. It is imperative that the new standards are viewed in light of student conceptions of quadratics. With the increased conceptual rigor of the Common Core, it is also crucial to research the best methods and resources available to illuminate concepts for students.

In designing this unit plan, the intended audience is for teachers of mathematics. Therefore the instructional implications of the unit plans will be discussed. The plan will match specific problem situations, solution techniques, and student conceptions with target objectives. Implications of the unit design process will be discussed throughout the paper, as they unfold through the various components of the contributing research. This way, the reader can generalize my design process to other topics in mathematics education.

Chapter Two: Literature Review

The Common Core

The mission statement of the CCSS initiative commits to a clear plan for students' college and career readiness at a globally competitive level, and reads:

> The Common Core State Standards provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them. The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage)

The CCSS initiative is a combined effort of politicians, corporate leaders, research groups, educators, and academic organizations across the nation. The Common Core follows the principles of the standards-based movement launched by the National Council of Teachers of Mathematics (NCTM) back in 1989 (revised in 2000). This new leap in the evolution of education standards is, as expressed, in response to the demands of a global economy. Over the last decade, students in the United States have performed below average on international achievement tests in mathematics, as measured by mean scores (National Center for Education Statistics, Webpage). The percentage of U.S. students scoring at advanced levels also falls below average ranking in international comparisons (Hanushek, Peterson, Woessmann, 2010). In both measures, U.S. students rank about average on comparative science tests. These statistics raise concern about how well our students will be prepared for a STEM major or to perform in the STEM fields after graduation; and consequently, how our nation will be prepared to compete in a global market.

Currently, forty eight states, including New York, have adopted the CCSS. States must follow the common set of learning standards for grades K-12 in English Language Arts and Mathematics, but may add up to 15% more standards to each subject. Participating states are currently phasing into the new standards and full implementation is expected by 2015 (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage).

The Common Core Standards in Mathematics are committed to promote both procedural and conceptual learning, as established by the preceding NCTM standards. In addition, the Common Core aims for "greater focus and coherence" (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, p. 3) of standards around key mathematical content and practices. This later goal targets deeper understanding and real world application of the key topics, rather than cursory study of "mile wide, inch deep" curriculum. The aim towards greater focus and coherence is supported by research of mathematics programs in higher performing countries (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The Common Core specifies eight general mathematical practices that students are expected to develop across the curriculum. These are based on the existing NCTM process standards, plus the mathematical proficiency strands developed by the National Research Council. When the word "understand" appears in the content standards, it is a cue to incorporate one or more of these mathematical practices.

There are eight standards for mathematical practice, and they are: (1) For students to make sense of problems and to persevere in solving them; (2) for students to reason abstractly and quantitatively; (3) for students to construct viable arguments and critique the reasoning of others; (4) for students to model with mathematics; (5) for students to use appropriate tools strategically; (6) for students to attend to precision; (7) for students to look for and make use of structure; and (8) for students to look for and express regularity in repeated reasoning. (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, p. 5) Also, the CCSS outline Key Points in the mathematics standards which stress both procedural and conceptual learning within meaningful contexts. Thus students should gain proficiency in applying techniques and may retain what they have learned (Calais, 2006). The CCSS requires students learn to apply mathematics and to model real world problems and use the results to inform decisions. Similarly, students are challenged to apply mathematics to novel situations, as is expected in math related careers (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage).

Rather than being organized into course trajectories, the NCTM high school mathematics standards are organized into six conceptual domains: number and quantity, algebra, functions, modeling, geometry, and statistics and probability (National Council of Teachers of Mathematics, 2000). The primary domains that pertain to the unit of study in this paper are the algebra standards and the functions standards. Some standards in the number and quantity domain also apply. See Table 2 on p.57 for these standards.

The intention of organizing the standards by concept, rather than by course, is to promote coherence. These concepts are developed across all grades and courses. Specific standards may even be practiced in more than one course. For example, the quadratic formula might be used in both Algebra 1 and Algebra 2. Here is an example standard that would span both courses:

> A.REI.4 Solve quadratic equations in one variable:
>
> a. Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
>
> b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers *a* and *b* (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, p.56).

The layout-by-concept may be very difficult to work with in designing a quadratics unit. This type of instruction design requires the mapping of standards from three different areas, without knowing which of the standards are intended for Algebra 1 and which are intended for Algebra 2, while also trying to incorporate the standards for mathematical practice. Since all states are using a common set of standards, traditional trajectories that once varied by state cannot be addressed specifically by topic. So it can be expected that some topics will be added and some will disappear, but without much certainty of what topics those might be.

Be aware that there is a separate document that can help. Common Core State Standards for Mathematics Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010) outlines the recommended course trajectories for implementing the Common Core Standards. For example, the recommended pathway for Algebra 1 is mapped out into five units with quadratics appearing in the fourth and fifth units. A summary of the pathway also indicates that the quadratic formula and completing the square to solve quadratic equations, both traditionally taught in Algebra 2, are now recommended for inclusion in Algebra 1 (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The quadratics units of the recommended pathway may be considered as rigorous (Zaslavsky, 1997; Bossé and Nandakumar, 2005; Eraslan, 2005; Hutchings and McCuaig, 2008; Zakaria, 2010). The standards effectively incorporate modeling and comparisons of functions to build the concepts of quadratic relationships (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010). However, students might have difficulty learning some of the skills that are embedded in these standards, without those skills being practiced in isolation. Factoring, which takes considerable practice to master, is combined with the techniques of completing the square and the quadratic formula to solve equations. Also, abstract techniques are presented before supporting graphical models are used in the recommended pathway.

NCTM (2010) did release a public statement supporting the implementation of the Common Core Standards. The statement was also endorsed by the National Council of Supervisors of Mathematics (NCSM), the Association of State Supervisors of Mathematics (ASSM), and the Association of Mathematics Teacher Educators (AMTE). These organizations embraced the vision of a more focused and coherent set of standards, the blend of procedural and conceptual learning, and the development of standards at the national level. They did make a note, however, encouraging ongoing "research on specific learning trajectories and grade placement of specific content and their implementation, as well as periodic review and revision based on such research." (NCTM, 2010, Webpage) This implies that NCTM does see some rough spots in the pathways.

NCTM (2006) produced a guide, *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence,* which identifies the most important topics and concepts per grade level from the 2000 NCTM standards. The guide also illustrates how the concepts should be vertically developed across grade levels. This publication influenced and supported the development of the Common Core Standards for grades K-8 (Achieve, Inc; 2010). Forthcoming supports from NCTM, for the Common Core high school standards should be a great help.

Hekimoglu and Sloan (2005) point out that the main criticisms of the NCTM standards have been for their departure from traditional instruction to the inclusion of more concept, inquiry, and technology-based learning. As can be seen throughout the Common Core standards, this is the movement that mathematics education must take in order to suit the demands of a more technology-based world. Another important point regards the criticism of the breadth of NCTM standards. The principles and standards were intended to be guidelines, not mandates. It was not an imperative that all the standards must be covered.

Many perceived problems with the old standards do not have much to do with the standards themselves, but with how they are taught and assessed. In both ways, the process strands of the NCTM standards have been largely ignored and the content strands alone have become the basis of practice (Hekimoglu and Sloan, 2005). Successful implementation of the Common Core standards will also depend on how they are assessed. That is what drives instruction. As long as traditional assessments are the primary measure of learning, traditional instruction will likely pervade mathematics education (Hekimoglu and Sloan, 2005). Process goals are intrinsic in the language of many of the Common Core standards. This should provide a safeguard. It is questionable though, at this point, whether the new standards are indeed more focused. The standards will need to be unpacked to levels of similar practicality to the NCTM standards in order to find out.

Both the CCSS document and NCTM *Standards* mention that ongoing research is necessary to effectively implement and shape the standards. In particular, research needs to explore the development of student understanding as it relates to the sequencing of learning experiences and the challenges inherent in the quadratics standards. Since the Common Core aligns the states to the same standards, there may be greater opportunities for this type of research. The following sections of this paper will be looking at the existing research in these areas.

Student Misconceptions

Students face many challenges in learning to interpret quadratic functions. A key element of designing effective curriculum is to look at student thinking and anticipate difficulties that need to be addressed. This section will outline student reactions to specific content, procedural difficulties, conceptual obstacles, and possible causes of student misconceptions.

Zaslavsky (1997) conducted the most comprehensive study of student misconceptions of quadratics, with a sample size of 800 students! Zaslavsky (1997) categorized common misconceptions into seven key areas. The first of these areas to present a cognitive obstacle to students is the graphical representation of a quadratic function.

Students often fail, upon visual inspection, to realize the domain of a vertically oriented parabola is infinite; and that each point on a parabola is determined by a specific input and output from the expression that defines the function. Students also assume equivalence between quadratic equations and quadratic functions that "look" the same. For example, $x^2 + 2x - 8 = 0$ is equivalent to $2x^2 + 4 - 16 = 0$, but the function $y = x^2 + 2x - 8$ is not equivalent to $y = 2x^2 + 4x - 16$. Students don't make this distinction. These misconceptions should raise major concern, since they involve the defining aspects of quadratic functions.

Students often try to generalize procedures associated with linear functions to quadratics. For example, students will calculate slope to find the leading coefficient of a quadratic. This is no surprise, since both families of functions share the coefficient symbols, *a* and *b*. Students often assume they mean the same thing, regardless of the situation. This misconception becomes even more pervasive when any of the coefficients is equal to zero. When a term is "missing" from the standard form, students often fail to recognize a function as being quadratic. Students also tend to misapply the visible coefficients in finding the vertex, the y-coordinate, and the axis of symmetry.

Another common error is over-emphasis of the x-coordinate. Students often fail to provide the y-coordinates of x-intercepts. Similarly, students assume from using the vertex formula that two parabolas have the same vertex without checking that the y-coordinates match. This habit is likely formed when students learn to factor, since there is no y-coordinate involved until factoring is used in the context of functions.

The greatest difficulty students have when taking on the functions perspective is transforming a graphical representation into algebraic form. This bears great importance to curriculum design, since the Common Core emphasizes algebraic *functions*. Plus, graphical to algebraic transformations are an important means of analyzing real-world data. Students who cannot meet this objective will be unprepared for successful work in math-related careers.

Finally, students are uncomfortable with the nonstandard algebraic forms of a quadratic, despite their utility. Most students will actually expand a quadratic in factored or vertex form at the onset of attempting a problem. This preference might also stem from students' familiarity with the standard form from its exposure during initial factoring and expansion exercises.

Eraslan (2005, 2007, 2007b) has contributed in depth case studies to the research of teaching quadratics. Aspinwall and Eraslan (2007) analyzed the work of a tenth grade honors student to gain insight into student understanding of quadratic functions during four essential translation tasks.

While attempting to translate from a given graph to an algebraic representation, the student assumed the leading coefficient, a, was 1. The student also assumed that the coordinates of the vertex translated directly to the coefficients of b and c, respectively. When translating from an algebraic to a graphical representation, the student simply reversed this process. Next, the student was alternatively given the quadratic function in vertex form, and the immediate response was to translate the problem into standard form. Next the student applied the same strategy in graphing it as he had in the previous task. Consistency in this student's strategies is important with respect to instructional implications that will be discussed later in this paper.

For the final task, the student was prompted to translate the standard form of a quadratic function into vertex-form. The student translated the function into the form $y = x(ax + b) + c,$ rather than the appropriate vertex-form $y = a(x - h)^2 + k.$ In representation, the student conceived (b, c) to be the vertex. Here, the student illustrates again consistent evidence of linear-quadratic confusion.

Aspinwall and Eraslan (2007) attribute the student's misconceptions to "his tendency to make an unfamiliar idea more familiar." (p. 237) It is clear that the student was trying to utilize techniques that pertain to linear functions when faced with quadratic translations. This helps to explain the key misconception of linear-quadratic confusion found by Zaslavsky (1997). In fact, many of the key misconceptions are illustrated by this student's work.

Eraslan (2007) later released a more detailed study of interviews with this student to investigate the phenomenon of compartmentalization in his conception of quadratic functions. Compartmentalization is defined as a conflict of two different cognitive schemes representing a single concept. For example, a conflict is clearly evident between situation-specific procedures involved in working with linear and quadratic functions when this student tries to generalize those procedures to any function. Erasalan (2007) found that this phenomenon was especially present in the student's difficulty in making connections between algebraic and graphical representations of a single function.

In some instances, the student did display the correct intuitive reasoning in the graphical representation of a problem, but could not translate this reasoning algebraically. The student also focused only on the x-coordinate of the vertex in comparing the vertices of two parabolas (by graphing them), after correctly writing the formula for finding both coordinates of each vertex. This shows difficulty in moving from a correct algebraic representation to a graphical one. Hence, the student has difficulty moving back and forth between graphical and algebraic thinking, in either direction, adding a significant detail to Zaslavsky's (1997) findings.

In an earlier case study of another student, Eraslan (2005) found that the student tried to translate a function from graphical form by using the factored form, $y = a(x - x_1)(x - x_2)$. However, the student used the wrong sign for the roots and did not know how to expand the result into standard form. Not knowing to substitute another point in to find a, the student resorted to finding the slope, consistent with the linear-quadratic compartmentalization found in Eraslan's other case studies. The student later applied linear techniques in attempting to find the vertex, axis of symmetry, and intercepts.

In graphically representing another problem, this student envisioned a vertical asymptote to the parabola. This is consistent with the first key misconception described by Zaslavsky (1997). Another consistency was present in this student's avoidance of using the vertex-form, even to find the vertex. The student did successfully use factoring and the quadratic formula to solve quadratics, but made no attempt to solve real world application problems using any method.

As opposed to Zaslavsky's (1997) focus on content-related misconceptions, Zakaria (2010) categorizes errors into different levels of general problem solving.

Zakaria (2010) conducted interviews and diagnostic tests to 30 algebra students in a secondary school in Indonesia to gain insight into the types of errors students make specifically during tasks involving factorization, completing the square, and the quadratic formula.

Zakaria (2010) found that most students were unable to apply the method of completing the square. Students produced errors at the transformation level, meaning they failed to initiate the necessary procedures. Errors also involved vocabulary, prerequisites skills, and comprehension of the problems. Errors in applying the quadratic formula were purely computational. Students made errors multiplying, dividing, replacing signs, and adding and subtracting negative numbers. These were categorized as process errors.

Errors in factorization were primarily rooted in transformation and computational fluency (process). Computation errors involved multiplication, especially that which involved negative numbers; and failure to copy negative signs from step to step. These errors compounded with student difficulties in expanding polynomials into quadratic form. Zakaria (2010) also noted student difficulties in comprehending the meaning of the roots of an equation.

The study found that most errors were at the transformation and process levels, with the greatest number of transformation errors occurring during factorization. This can be attributed to the fact that there are three different types of factoring problems, as well as the cognitive demand placed on students by the intuitive approach required during the factorization process.

Kotsopoulos (2007) describes problems students in her classroom have had with learning quadratics as they relate to cognitive theory. She notes that students have difficulty retrieving multiplication facts necessary for factoring quadratics. Through the lens of cognitive science, she suspects this difficulty relates to a lack of *procedural* facts in the long-term semantic memory.

For students to successfully store multiplication facts into long-term memory, Kotsopoulos (2007) argues that order matters. For example, a teacher cannot assume that teaching the commutative property of multiplication will automatically result in students achieving proficiency with the other half of their multiplication tables. Rather, students need to learn multiplication facts explicitly in order to build up to the concept of the commutative property.

Similarly, students have difficulty understanding the symbolic relationships *within* and between the factored form, the standard form, and the vertex form of quadratic expressions. Order matters. Take the quadratic equation, $x^2 + 2x - 3 = 0$. A student may have no problem with solving it. When encountered with the equivalent expression, $x^2 + 3x + 1 = x + 4$, the same student will likely not know what to do. Students have particular difficulty finding the roots of the function $y = (x - 1)(2 - x)$, because it is out of standard order (Kotsopoulos, 2007).

These difficulties cannot justifiably be attributed to a lack of conceptual knowledge if the situation is novel to the student and the underlying procedures haven't been developed enough to justify the concept. Further, Kotsopoulos (2007) proposes that the order (sequencing) in which the types of representations are presented to students may affect their storage into long-term semantic memory. These are critically important assertions with respect to the instructional implications to be discussed later in this paper.

In sum, students face many challenges in learning to interpret quadratic functions. The first major challenge is the prerequisite arithmetic skills needed to make necessary algebraic manipulations. Second, students lack a conception of what quadratic functions are and how their different representations relate to one another. Third, students present a lack of understanding regarding functions in general.

These are, in my interpretation of the research, the three major roadblocks that deny students access. Without fluency in arithmetic, students lack the necessary tools to work on quadratics. Without an elementary understanding of functions, students cannot speak the language necessary to interpret quadratics. Without a conception of what quadratic relationships are, students lack a clear task.

These three roadblocks need to be overcome before students are expected to become proficient working in the abstraction of algebraic forms; and certainly before students are expected to retain the many procedures related to the abstract forms. These manipulations alone have long been the shortsighted focus of traditional instruction. It is no wonder that students are running into problems.

Techniques

This section gathers information from the research on the three primary techniques for solving quadratic equations: factoring, completing the square, and the quadratic formula. The research will highlight the effectiveness of each technique, the relative importance and practicality of each technique, and ways to better utilize each technique. Let’s begin with an honest portrayal of factoring from a teacher blog:

> “I think there is value to factoring on some level, but what we do in beginning algebra is overkill. When skills are a means to a greater end that students will see and can appreciate, that can go a long way into making sense of the skill and improving fluency with it. And when they’re taught in a vacuum, learning and retention don’t usually follow. I’ve taught factoring for 15 years and regardless of the level, the same outcome always happens: teach it for a couple of weeks, practice myriads of problems, they take (and most pass) the test, and the day after the test it’s like a lobotomy happens. I see it worse with that topic more so than any other. It’s literally like their memories are wiped clean. We use it again in rational expressions and they don’t recall it nor see the value. It’s just moving letters around to them so the skills don’t stick. I’d rather do less of it and really use it for something other than problem recognition. Show them when it can make sense and can be quite useful, like the GCF when rewriting business or science formulas. If our goal is brain calisthenics, there are so many more beneficial ways to get there. To me, factoring is right up there with square roots by hand. Sure, we can do it. But why?” (Almy, 2011, Webpage)

Factoring of quadratic equations has been treated as perhaps the most essential skill in algebra. It takes students a great deal of time to acquire proficiency in factoring. So a substantial amount of class time is spent on it. Typically, factoring is practiced during three different units in the algebra curriculum: factoring quadratic expressions, factoring to solve quadratic equations, and factoring to simplify rational expressions. This usually accounts for a month or more of class time. Many students still struggle with factoring during their third encounter with it.

Bossé and Nandakumar (2005) argue that factoring is overemphasized in the algebra curriculum in relation to its utility in real world mathematics. As Table 1 shows, the probability of a quadratic expression with integer coefficients actually being factorable is very small, once the range of possible coefficients exceeds ± 10. In a real world context, the case for factoring is far less optimistic than this. Integer coefficients rarely occur in quadratic models of real world data, so in nearly all situations factoring is out the window.

Table 1. Probability that a randomly generated quadratic is factorable.

Range $a \neq 0$	Number of possible quadratics formed	Number of factorable quadratics	Percent of quadratics that are factorable
$-2 \leq r \leq 2$	100	36	36.00
$-5 \leq r \leq 5$	1210	276	32.81
$-10 \leq r \leq 10$	8820	1348	15.20
$-100 \leq r \leq 100$	8 080 200	226 912	2.81
$-200 \leq r \leq 200$	64 320 400	1 028 860	1.60
$-300 \leq r \leq 300$	216 720 600	2 481 820	1.14
$-400 \leq r \leq 400$	513 280 800	4 619012	0.90
$-500 \leq r \leq 500$	1 002 001 000	7 468 652	0.74
$-1000 \leq r \leq 1000$	8 008 002 000	33 052 872	0.41

Bossé and Nandakumar, 2005, p.146

Despite its lack of real-world relevance, factoring remains a staple of the algebra curriculum. In order to accommodate factoring exercises, quadratics are grossly misrepresented in math textbooks. Bossé and Nandakumar (2005) conducted a survey of 27 algebra textbooks and found that 94% of all quadratic expressions represented in the texts *were* factorable. In some texts, all quadratics were factorable. This representation of quadratics certainly does not match the statistics in Table 1, nor is it consistent with the vision of the Common Core Standards that "high school standards set a rigorous definition of college and career readiness, by helping students develop a depth of understanding and ability to apply mathematics to novel situations, as college students and employees regularly do." (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage)

Bossé and Nandakumar (2005) argue that the alternative approaches of completing the square and the quadratic formula should be emphasized more. First, these approaches can be used to find the roots of any quadratic equation. Second, they are often more efficient in solving even those quadratic equations that *are* factorable $(when\ |a| > 1)$. Third, both of the alternative forms reveal more information about a quadratic function than the factored form does. The vertex-form derived from completing the square gives the coordinates for the vertex. The axis of symmetry of a quadratic function is embedded within the quadratic formula.

The alternative approaches do, however, involve much more intensive arithmetic including fractions, radicals, and exponents. Bossé and Nandakumar (2005) stress that students will need extra practice to gain comfort and proficiency with these techniques. This advice aligns well with the inclusion of completing the square and the quadratic formula in the Model Course Pathway for Algebra 1 recommended by the Common Core (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010). Traditionally, these refined approaches are introduced in Algebra 2. Earlier exposure places more emphasis on these techniques, allows for additional practice, and could result in increased comfort with using them.

Kandall (2003) exposes an awesome algorithm that can be used to factor any quadratic with rational roots. This approach treats the quadratic formula as a catalyst to factoring problems with large coefficients. When students conquer such problems, they feel great about it and it becomes a memorable experience. Hence, factoring becomes engrained into the memory. Before moving on, I challenge the reader to factor this problem: $124x^2 - 2747x - 6305.$

Kandall's (2003) algorithm can be adapted to suit an Algebra 1 class as follows. Start by applying the quadratic formula. If the discriminant is non-negative, the quadratic is indeed factorable. If it is not, the quadratic *cannot* be factored but the roots can be found by continuing. For those problems that can be factored, leave the roots in fraction form. If using a calculator, students can either calculate the numerator and denominator separately, or they can use the ”Frac” command under the MATH menu to get a decimal answer back into fraction form. Then take the first root and reduce it into simplest form. Students can use the “gcd” command under the MATH, NUM menus to facilitate simplification, especially if the numbers are big.

Now we are seeking *p* and *q* such that $ax^2 + bx + c = (px + q)(rx + s) = 0$. So we need $px + q = 0$. Suppose the first root is 6/7. We would then have $p\left(\frac{6}{7}\right) + q = 0$. Letting *p* equal the denominator of the root, in this case 7, and *q* equal to the numerator times -1, in this case -6, satisfies the equation. Suppose the second root is -5/3 once it is reduced into simplest form. Here we need to find *r* and *s* such that $r\left(\frac{-5}{3}\right) + s = 0$. So r = 3 and s = 5. The factored form of our quadratic is then $a(7x - 6)(3x + 5)$. Note that *a* can be found by distributing and comparing the resulting coefficients to those of the original function.

This method can help students to understand the relationships between factoring, the quadratic formula, roots, and the discriminant. This is a great way to provide meaning to factoring as Almy (2011) desires for her students. It also allows for students to factor quadratics with a much greater range of coefficients, while acclimating them to the quadratic formula—a great solution to the dilemma posed by Bossé and Nandakumar (2005). Ultimately, this can be a fun and motivating activity for students, because it works even on problems with huge coefficients.

Li (2011) observed a teacher use the process of completing the square in order to derive the quadratic formula from the standard form. In order to facilitate this proof, the teacher demonstrated it with a specific quadratic equation (with numeric coefficients) and the general form of the quadratic equation (with variable coefficients) side by side. This is probably the best approach to clarifying the derivation for students, but it does take 14 steps. That is 28 steps accounting for both representations! Li (2011) notes that the students had not achieved proficiency applying the quadratic formula at the conclusion of the unit. Picciotto (2008) describes using the same instructional approach to no avail. Even though it may be the best approach to teaching the derivation, the derivation itself may simply be inappropriate for 9th grade students. The Common Core Model Course Pathway for Algebra 1 *does* recommend this derivation (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010). It may be better to focus on operational proficiency in using completing the square and the quadratic formula, before expecting such a leap.

In either event, to make more room for the new techniques time needs to be allocated away from factoring. But how can students achieve proficiency in an area of such difficulty in less time? Hutchings and McCuaig (2008) illustrate an ingenious factoring technique, much more efficient than either completing the square or the quadratic formula. The following three examples show how using a simple array of numbers can help students to factor and find the roots of a quadratic equation.

Begin in the first column and write two factors that multiply to *a*. Then write two factors that multiply to *c* in the third column.

Figure 1.1. Matrix factoring technique step 1.

1 8	3 9	5 3
1 -3	1 -2	7 -2
$x^2 + 5x - 24$	$3x^2 + 25x - 18$	$35x^2 + x - 6$

Next, multiply the numbers in each row and write the product in the middle. The middle terms must add to *b*. If they do not, try different factor combinations. Once the middle terms add to b, the quadratic is factored and can be written by grouping diagonally opposed factors. For example, $35x^2 + x - 6 = (5x - 2)(7x + 3)$.

Figure 1.2. Matrix factoring technique step 2.

1 8 8	3 27 9	5 15 3
1 -3 -3	1 -2 -2	7 -14 -2
$x^2 + 5x - 24$	$3x^2 + 25x - 18$	$35x^2 + x - 6$

To finish, divide each of the factors of c by the diagonally opposed factor of a. Multiply the results by -1. Write these final entries in the fourth column. These are the roots of the equation!

Figure 1.3. Matrix factoring technique step 3.

1 8 8 -8	3 27 9 -9	5 15 3 -3 / 7
1 -3 -3 3	1 -2 -2 2 / 3	7 -14 -2 2 / 5
$x^2 + 5x - 24$	$3x^2 + 25x - 18$	$35x^2 + x - 6$

Hutchings and McCuaig, 2008, p.164

Students can begin using this technique selecting the initial factors by trial and error. After some practice and class discussion, students should begin to intuitively select factors that will lead to success more quickly. This approach might work even better if students wait until the second step to determine the necessary signs (positive or negative), and then transfer those signs to the first and third columns as needed.

The power of this matrix-factoring technique is in its structure. Students can keep track of steps visually rather than mentally. This greatly reduces the cognitive load of the traditional process, which is purely intuitive and requires multi-steps to be performed mentally. As students write and visually track the numbers, they will be able to recognize patterns in them and constructively build intuition.

This technique is also very efficient in the sense that students can practice many problems without writing much down. Hutchings and McCuaig (2008) found that students only had to write 19 to 25 symbols per problem using it. In doing the same problems, 102 to 119 symbols were required in executing the quadratic formula, and 92 to 155 symbols were required to complete the square! Naturally, the likelihood of error increases in direct proportion to these figures.

To truly appreciate this technique, just try it out on a problem set. It is like playing a game. It is particularly surprising how much easier it is to factor equations with leading coefficients greater than one. Hutchings and McCuaig (2008) prove that factoring can be accessible and effective, when approached from an innovative angle.

The real purpose of researching the different techniques is to increase student proficiency and understanding in modeling and solving quadratics. Different techniques suit different situations and must be viewed in light of students' evolving perspectives. Sensitivity in matching the problem situation, the technique, and student conceptions is the key to developing quality curriculum.

Worth noting is the possibility of developing the matrix-factoring technique into an applet. This could eliminate any scribbling and erasing, while providing immediate corrective feedback to students. We have already seen the calculator allow students to tackle "cool" problems with Kandall's (2003) algorithm. The next section will expand on further possibilities that technology brings to the study of quadratics.

Technology

The most important and practical technology medium used in secondary math classrooms is the graphing calculator. The calculator is used during daily exercises, explorations, and assessments—including end-of-year state assessments. Optimal use of the calculator is a vital part of instruction. The modeling capabilities of the calculator are especially useful in the study of quadratics.

Mittag and Taylor (2001) provide an overview of approaches that students can use to solve quadratic equations using the TI-83/84 graphing calculator. The authors refer to these approaches as the four “modern wonders” of the quadratic world, as opposed to the “ancient wonders” traditionally used to solve quadratics: factoring, completing the square, and the quadratic formula.

The traditional solution methods are definitely essential parts of the algebra curriculum. They are necessary processes that carry over into other topics in algebra and into later courses in mathematics. Unfortunately, students have a great deal of difficulty mastering these techniques. Further, students experience difficulty reasoning about quadratic functions and applying them to the real world, by using “ancient” methods alone. This is because the techniques, being purely symbolic, force students to work entirely at the abstract level.

The “modern wonders” of technology provide students with interactive and concrete representations of quadratics. They provide a point of access for students to reason about quadratic relationships. They also help to bring meaning to the symbolic representations composed by the traditional methods. So when the “ancient” and “modern” wonders are used together, students learn to model quadratics graphically and symbolically, and to make connections between the two. They get to see the full picture.

Since this paper is intended for a teacher-audience, the benefits of each method will be explained along with enough detail to navigate through the necessary calculator functions that are involved. This will help teachers who are a little uncomfortable with technology to get started. The most efficient way to find additional information is to simply “Google” the specific feature. Doing so will lead quickly to the desired information. A teacher may find it inefficient to dig through pages and indexes of any one particular site.

The first of the modern methods is the calculator’s CALC feature. After graphing a quadratic function, the student can use the “zero” command under the CALC menu to display the roots, or *zeros*, of the function. The CALC function is more reliable than the calculator table, since it will even locate roots with multiple decimal places. The authors suggest that students visually inspect the graph to determine the number of roots and to estimate their values, before using the CALC feature for verification. Likewise, the CALC feature could be used to verify algebraic solutions.

The second modern method is to *program* the quadratic formula into the calculator. It is fairly easy to do. The resulting program prompts the student for the coefficients, then outputs the roots at the press of a button. The program will even display complex solutions when they arise! This is an effective way of teaching students that solutions *do* exist outside of the real number system without requiring them to calculate those solutions—as specified in the Common Core standards for Algebra I.

This program can be used to quickly verify solutions. Students can get instant feedback and self-correct arithmetic errors that often occur during traditional use of the formula. The program is also ideal for real world modeling situations that would otherwise be too laborious. In fact, students can also do the work of real engineers by determining how to enter the formulas during the programming phase.

The third recommended method is the use of the calculator TABLE to locate roots of a function. The TABLE can also be used to evaluate the function at any point. Again, this is a quick and easy way to "perform" many calculations in the context of a real world problem. Ultimately, the TABLE utility might be the best tool to help students overcome each of the major "road blocks" mentioned earlier in this paper. Essentially, the calculator TABLE is a dynamic version of a T-table that a student would create by hand in order to draw a graph. This will be discussed in detail as a key instructional implication.

The final method described by the author is the SOLVER feature. This method serves the same purpose as the "zero" command under the CALC menu. SOLVER is found under the MATH menu. The quadratic equation must be entered and then ALPHA, ENTER (SOLVE). In order to find multiple solutions, the student must TRACE the graph close to the desired root before solving. SOLVER can actually be used on any equation, even those that are impossible to solve with algebraic methods! However, an equation that is not in "= 0" form must be entered as a difference of the two sides of the equation. Also, SOLVER (and any of these methods aside from the quadratic formula program) will not yield complex roots.

The other technology medium that can benefit student understanding of quadratics is the applet. Applets are topic-specific interactive programs that represent concepts visually and dynamically. Essentially, applets allow students to play with a math concept and see the patterns that emerge. Applets also typically provide the student with constant corrective feedback, easing the facilitation of instruction from an assessment perspective.

Daher (2009) found that most mathematics teachers are unfamiliar with the use of applets. Teachers benefit most from exploring applets directly, playing with them and solving problems with them. Then the teacher can decide which applets would be truly beneficial and what difficulties might arise during implementation. Daher (2009) allowed five preservice math teachers the opportunity to explore applets and to evaluate their usefulness.

"Though most of the participants thought that mathematical problems could be solved without applets, they emphasized the role of applets as fostering, facilitating and clarifying mathematical problems' statement and solution. The participants pointed at applets as tools which learners *enjoy* working with, so they will be encouraged to solve mathematical problems using them." (Daher, 2009, p. 394)

Many applets are available for free online. NCTM Illuminations Resources for Teaching Math provides a great selection of applets along with instructions for using them. Others can be found, again, by simply "Googling" the desired concept along with the keyword "applet." Be aware that many of the applets require a Java-enabled browser to run. The Java download is free. Web addresses for some excellent quadratics applets are provided in the Activities section of the Appendix. These recommended applets also align directly to the Common Core Standards.

Students can explore translations of quadratic equations and the effects of changing coefficients using two of the recommended applets. These explorations allow students to make direct connections between the graphical and algebraic representations of quadratic functions. Another applet provides a geometric representation of completing the square. It illustrates why $(\frac{b}{2a})^2$ is added to both sides of the equation. Knowing this eliminates some abstraction from the process. The visual also provides a great way for students to remember the procedure. The "Pan Balance" applet allows students to solve a system of one quadratic and one linear equation dynamically. The NCTM "Algebra Tiles" applet uses a visual representation of variables and numbers consistent with physical manipulatives that are often used during algebra instruction.

Before integrating any of the technology resources discussed, the first step a teacher should take is to sit down and play with the calculator or applet. This will bring an increased comfort level and reveal any potential difficulties. Then the teacher can determine if the technology resource is an appropriate instrument to meet the desired learning objective.

Instructional Implications

This section will discuss how the information gathered thus far informs classroom practice. Instructional implications will be presented as they relate to the three major roadblocks that students face in achieving procedural and conceptual understanding in quadratics, to the standards of the Common Core, in Algebra 1.

The first major roadblock is the prerequisite arithmetic skills needed to make necessary algebraic manipulations. Without fluency in arithmetic, students lack the necessary tools to work on quadratics. Kotsopoulos' (2007) findings are very difficult for a teacher to address. When students have difficulty multiplying numbers, it is even more difficult to generalize those same procedures to increasingly abstract symbols.

In time, this situation should be improved by the rigorous standards that are being set forth by the Common Core (in arithmetic and number sense) in the early grades. Within the scope of this paper, the research provides some ways to immediately benefit those students who have difficulties with prerequisite arithmetic.

Factoring may seem out of reach for students who have difficulty retrieving multiplication facts. The process of factoring often does not "stick" even for those students who are adept in arithmetic (Almy, 2011). Within this already confusing process, students must perform both multiplication and addition mentally. Negative numbers further complicate those computations.

Hutchings' and McCuaig's (2008) matrix-factoring technique is a promising solution. The inherent structure of matrix-factoring allows students to keep track of steps visually rather than mentally. This greatly reduces the cognitive load of the traditional process, which is purely intuitive and requires multi-steps to be performed mentally.

Once students have this concrete strategy to use, the process as well as its computations can be addressed incrementally. This can be done visually (rather than mentally), with the aid of multiplication tables, or with a calculator. To further scaffold the procedure, students can wait until the second step of the process to determine the necessary signs (positive or negative) and then transfer those signs to the first and third columns as needed. This should be taught explicitly and discussed as a problem solving strategy.

To make factoring more fun and memorable for students, it can be treated as a game. Students can begin playing by selecting the initial factors by trial and error. After some practice, students should begin to intuitively select factors that will lead to success more quickly. Class discussion can allow students to share strategies for playing the game, improving accuracy and speed.

Matrix-factoring is very efficient, so students will be able to practice many more problems. In doing so, they will have a much greater opportunity to recognize patterns in them and to build intuition. Again, class discussion can be used to discuss these patterns and lead to a deeper conceptual understanding of factoring. The teacher should be prepared with prompts to guide students toward target understandings.

Since this factoring technique is universal, it eliminates the need for learning three different types of factoring. This not only makes factoring easier, but it also frees up time. That time can be used to practice contextualized problems, as Almy (2011) wished she could have done with her students. Time is also freed up to introduce other solution approaches beyond factoring.

The alternative approaches do, however, involve much more intensive arithmetic—including fractions, radicals, and exponents. Bossé and Nandakumar (2005) stress that students will need extra practice to gain comfort and proficiency with these techniques. This advice aligns well with the inclusion of completing the square and the quadratic formula in the Model Course Pathway for Algebra 1 recommended by the Common Core (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010). Traditionally, these refined approaches are introduced in Algebra 2. Earlier exposure places more emphasis on these techniques, allows for additional practice, and could result in increased student comfort with using them.

The calculator is crucial for providing all students with access to these techniques. Students can use the quadratic formula program to solve equations and explore applied problems, without the intensive arithmetic creating a roadblock. The teacher must be sure, however, to spend some time strictly on identifying coefficients so that students can enter them correctly into the program. This should include examples that are not in standard form, where determining the coefficients is not as straightforward. Ideally, students can practice the arithmetic portions and use the program to verify answers or to get corrective feedback. The CALC function can be used to achieve the same purpose, while also reinforcing the graphical representation of solutions.

Similarly, applets provide students with corrective feedback and access to exploring concepts. Daher (2009) found that students actually enjoy working with applets and are encouraged to solve problems with them. Students can observe patterns dynamically and without the limitation nor the stress of static computations. Consequently, all students can build the essential concepts. The matrix-factoring game would be an especially powerful tool if programmed into an applet.

The calculator TABLE is perhaps the most powerful tool for working around computational difficulties and it is the basis of overcoming the second conceptual roadblock as well. That is, students present a lack of understanding regarding functions in general. Without an elementary understanding of functions, students cannot speak the language necessary to interpret quadratics.

It is often challenging, as a math teacher, to remember that the fundamental procedures and concepts that seem easy enough to rush through are not so easy for someone who is seeing them for the first time. The rigor and multidimensional nature of many of the Common Core Algebra Standards may tempt a teacher to rush through the “easy stuff.” Be aware that it is extremely important to take time, especially at the beginning of a unit, to lay a solid foundation for later concepts. Do not rush into generalizations, abstract techniques, and concepts before students have learned to analyze functions at a basic level.

Emphasizing the evaluation and interpretation of points along a function, particularly in the context of real world problems, will greatly reduce many potential misconceptions. Students are much more likely to remember the y-coordinates of roots and vertices. Students can also realize the true range of a parabola by evaluating it dynamically. Further, students can better evaluate the meaning of any abstract representation, since they can trace it backwards to a concrete source.

Recall the most alarming of Zaslavsky's (1997) observed misconceptions. Students fail to understand the graphical representation of a quadratic function. Also, students fail to realize each point on a parabola is determined by a specific input and output from the expression that defines the function. These are defining aspects of quadratic functions. Each needs to be addressed explicitly and thoroughly at the beginning of the unit and reinforced all throughout the unit.

The most basic way to understand a function is by simply plugging values into it to evaluate it. Ideally, this exercise takes place in a problem context. For example, a student could be asked to determine the height, h, of a falling object from the ground 4 seconds after it is released from the top of a 275 foot tall building, given the motion of the object is defined by the function, $h(t) = -16t^2 + 275$. (This is actually an accurate model based on acceleration due to gravity!)

Students plug 4 in for t and do the arithmetic to find $h(4) = -16(4)^2 + 275 = 19$. Students can interpret, verbally or in writing, that after 4 seconds of falling, the object is 19 feet above the ground. Here, students are using a procedure that applies to *all* functions. It should already be friendly and familiar. So students will be standing on solid ground. They can also visualize from the context, what the function means.

Next students can create a T-table, in which they evaluate and record several points. Students should discuss and interpret these points as the heights of the object at different times as the object is falling. Students can then use the points in the T-table to sketch a graph of the function. The graph represents all times and the respective heights of the object.

During this process, students are constructing a graphical representation of the function. They are also learning how the function behaves. A great discussion could ensue about how an object falls faster and faster and how that is represented by the shape of the graph. This is an ideal opportunity to also contrast the function with a linear one, say a graph representing an object moving at a constant velocity.

Once students have mastered these techniques manually, the calculator GRAPH and TABLE can be introduced as extensions of the tools that students have already used to evaluate functions. Essentially, the calculator TABLE is a dynamic version of the T-table that a student would create by hand in order to draw a graph. The calculator GRAPH is a precise and unbounded version of what a student would draw.

Do not assume that students will make this connection on their own! Make sure they understand the calculator representations as being the same as the representations they have constructed themselves. Then students will have a deeper understanding of the utility of the calculator in evaluating and making sense of functions. Be sure to provide students with opportunities throughout the unit to model problems both by hand and with technology.

Make sure students are adept at using the TABLE so they can take full advantage of its capabilities. To set up a table of values, press 2^{nd}, TblSet (above WINDOW). Enter the value where you would like the table to begin by the TblStart prompt. Enter desired increments of x values in your table by the ΔTbl prompt. Demonstrate explicitly that vertices sometimes fall on decimal values of x, so they may not show up in a table without the proper increment setting. If the following menu option is changed from “Auto” to “Ask,” then you can enter specific values to pinpoint a “hiding” vertex. Press 2^{nd} TABLE (above GRAPH) to display the table. Use the arrow keys to scroll further up or down the table.

Remember that the calculator GRAPH and TABLE also provide access for students who have difficulties with arithmetic to evaluate a function at several points and observe the behavior and key features of the function. The Table provides corrective feedback to students who desire to carry out arithmetic by hand. And since students can model functions much faster with the calculator, they have greater opportunities to explore and interpret real world problems. This provides an excellent conceptual foundation for studying quadratic functions and should be used to support all of the techniques discussed in the following section.

The third roadblock is that students lack a conception of what quadratic functions are and how their different representations relate to one another. Without a conception of what quadratic relationships are, students lack a clear task. An essential mathematical skill that helps students to conceptualize what quadratic functions are is *comparison*. The Common Core Standards do well in emphasizing the comparison of different families of functions. This is the key to addressing the most pervasive of the student misconceptions, linear-quadratic confusion.

Recall that Aspinwall and Eraslan (2007) attribute the student's misconceptions to his tendency to make an unfamiliar idea more familiar. If instruction begins with what students already know about linear functions and compare that to the characteristics of quadratics, it can help students to feel more familiar with the new function. It is important to discuss the differences between the symbolic and graphic representations explicitly. A great place to start is by evaluating the different functions and comparing their resulting graphs. The gravity versus constant velocity problem would be an ideal context.

Next, focus on the meaning of the coefficients. This is crucial, since both families of functions use the coefficient symbols *a* and *b*, but the meanings of the coefficients differ per family. Two of the recommended applets from NCTM Illuminations are ideal for this purpose. Students can explore translations of quadratic functions and the effects of changing their coefficients. These explorations allow students to make direct connections between the graphical and algebraic representations, a major conceptual obstacle that has been identified.

A fundamental understanding that will emerge is the fact that a quadratic function changes to a linear one when the coefficient *a* is set to zero. It is also important to explore situations where the other coefficients are equal to zero. Then students are much better prepared to practice problems finding the vertex, the y-coordinate, and the axis of symmetry. Students will have a greater familiarity with the meaning of the coefficients. This will alleviate problems with linear-quadratic confusion.

I will stress again that real life problems also bring meaning to the features of quadratic representations. Here, we can revisit the previous context that students are already familiar with. In the falling object problem, students can discuss the maximum height and the point of impacting the ground, and what part of the graph is relevant to the context of the problem. This illustrates the meanings of the vertex, the roots, an extraneous root, and the range of the function in a concrete way before these concepts are introduced abstractly.

Students will then be able to picture what these terms mean when they are finding them symbolically. It will help them to distinguish and check the algebraic procedures involved. Real world meaning along with the meaning gained from the previous explorations of coefficients will result in algebraic procedures making more sense to students.

The modeling capabilities of the calculator are especially useful in real world applications of quadratics. The TABLE, GRAPH, and quadratic formula program have already been discussed. Another useful tool is the CALC function. The CALC function can be used to find the roots of a quadratic function. Mittag and Taylor (2001) suggest that students visually inspect the graph to determine the number of roots and to estimate their values, before using the CALC feature for verification. Likewise, the CALC feature could be used to verify algebraic solutions. The MAX/MIN commands under the CALC menu can be used to find the vertex. This would be used to verify results after applying the vertex formula. These approaches allow students to make connections between the graphical and algebraic representations of the problem they are modeling.

The case of factoring in real world applications should be discussed with students. A variety of problems that suit the different solution techniques should be presented. These should include a fair number of problems with irrational solutions and non-integer coefficients. These are representative of real world problems that are not factorable. Contextualizing these problems would make them even more realistic. Ultimately, it is *not* realistic for factoring to be the sole technique for solving quadratics in Algebra 1. This real world consideration aligns well with the inclusion of completing the square and the quadratic formula in the Model Course Pathway for Algebra 1 recommended by the Common Core (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The challenge of transforming a graphical representation into algebraic form recognized by Zaslavsky's (1997) and Zakaria (2010) is the ideal situation to orient students to completing the square conceptually. But doing so would be jumping the gun. Due to the complexity of this approach, students first need to learn the process procedurally. So the initial motivator should simply be a quadratic equation that cannot be factored. The process should be demonstrated directly. After doing enough practice problems to become somewhat comfortable with the process, students should be prompted to explain each step of solving a problem in writing.

Then students can be oriented to the greater utility of completing the square. That is, in translating from graphical representations of real world data to algebraic form. The maximums or minimums of graphs (vertices) can be translated directly into the vertex form of a quadratic function. The vertex form is the result of completing the square. Students should experience an application where they are given many graphs of data and must determine the function for each one. Then students can solve each function and interpret the results in the context of the problem situation, ideally making a decision based on their findings.

Despite the Common Core recommendation of using completing the square to derive the quadratic equation in Algebra 1, it may not be practical. Students may not have matured enough in mathematical reasoning at the 9th grade level to benefit from it. More time should be devoted to having students gain operational proficiency with it, and with applying it to solve real world problems. A much more practical alternative to build conceptual understanding is the use of the geometric representation of completing the square in the NCTM applet. The applet illustrates why $(\frac{b}{2a})^2$ is added to both sides of the equation. Knowing this eliminates some abstraction from the process of completing the square. The visual representation also provides a great way for students to remember the procedure, which is a primary concern voiced in the research.

Despite having difficulties with teaching the derivation of the quadratic formula, the teacher observed by Li (2011) has some excellent methods for introducing students to the formula itself. She begins by writing the formula on the board symbolically. Students read the formula, then sing it as a rhythm: "*x* equals negative *b*, plus or minus the square root, b squared minus 4*ac*, all over 2*a*." (Li, 2011, p. 6) Next, the students are given a phonetic cue in the form of a story, which the students rehearse several times:

> The bee is sad (negative), and he is feeling wishy-washy, maybe he will go or maybe he won't (plus or minus). It's about going to the radical party. He's feeling a little squared, about the four awesome cheerleaders. The entire party was over, however, by 2 AM (Li, 2011, p. 6).

This is a great organizational scheme that helps students remember the formula. The story could be more powerful than deriving the formula, if the end objective is that the students will actually be able to successfully use the formula to solve problems.

The teacher follows this introduction with examples of using the formula, making sure to include examples with a=1, a>1, b=0, and b>0, consistent with Zaslavsky's (1997) recommendations. It would also be wise to include example problems that are not in standard form, with respect to order. The teacher used a variety of these examples to show students how to identify and plug in the coefficients, then carry out the necessary computations.

It might be even better to isolate each of these steps for practice at first. Try setting up several practice problems by just plugging in the coefficients. Then compute several of them by hand. Finally, practice entering several problems into the calculator. It is very important for students to master the use of the quadratic formula at an operational level. This is easier if the process is broken down into steps. As discussed earlier, programing the quadratic formula into the calculator can help facilitate mastery of the computations involved.

Kandall's (2003) algorithm would be a great extension of the quadratic formula to challenge students and show the relationship between the formula and the factored form of a quadratic equation. After doing practice problems, it could be especially effective to have students make their own problems and solve them. This is ideal for a project or performance-based assessment. In order to create problems, students would need to start with the factored form and choose whatever prime coefficients (p,q,r,s). Then expand to the standard form. Students would likely choose huge coefficients and make the project a fun and memorable experience.

As a final note on the techniques of factoring, completing the square, and the quadratic formula; students should practice applying every technique to a single problem. At the end of the unit, students could complete a problem set in this manner. Then the class can discuss which techniques are optimal in certain situations. Having discussed how the content should be developed procedurally and conceptually for students based upon the research, now the Common Core Standards can be presented and mapped accordingly.

Chapter Three: Curriculum Design

Preliminary Analysis of the Common Core Algebra 1 Trajectory

The platform for this curriculum design is the Common Core State Standards for Mathematics Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010). It can be exceedingly difficult to design curriculum based on the Common Core standards without using the appendix. The main body of the standards document does not delineate the standards between Algebra 1 and Algebra 2 courses. Appendix A does provide a pathway specifying the recommended inclusion and sequence of standards for Algebra 1 (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The recommended Algebra 1 trajectory is organized into five units with quadratics appearing in the fourth and fifth units. Unit four focuses on algebraic representations and processes associated with quadratic expressions and equations. The standards call for the creation of quadratic equations and implementation of solution processes. The fifth unit shifts focus to graphical representations of quadratic functions. Functions are used to model real world phenomena. Solutions are extended to non-integral roots using the Quadratic Formula. The solution technique of completing the square is also incorporated (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The first step in analyzing the recommended trajectory for curriculum design is to map the existing state standards (NCTM, 2000) to each section of new standards in the recommended sequence. See Figure 3 for the existing state standards. Mapping the old standards to the new ones reveals specific changes in the concepts and procedures students are expected to learn. Notes are provided at the end of each section to highlight necessary changes as well as concerns regarding the implementation of such changes. See Table 2 for this mapping.

The first section calls for the interpretation of algebraic representations of quadratics, such as coefficients and factors of expressions. Students are expected to use the structure of expressions to manipulate them algebraically, into forms that highlight a desired relationship or quantity. This includes manipulation of nonstandard forms. The old standard of identifying an expression that is the difference of perfect squares is extended to identifying the product of such expressions and decomposing it (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

These changes warrant major concerns in light of the existing research on student conceptions of quadratics. Mittag and Taylor (2001) and Kotsopoulos (2007) provide evidence that students benefit from concrete representations of quadratics before manipulating algebraic representations. The algebraic representations are otherwise abstract or even meaningless to students. Even if students learn an algebraic procedure without an appropriately supporting context, they are unlikely to retain the procedure. Interpreting and manipulating nonstandard forms should be modeled explicitly and practiced nearer the end of a unit after students are familiar with standard forms. Here it is expected at the beginning of the unit and with greater generality than might be practical.

The following section requires further manipulation of equations. Students are expected to factor and solve quadratic equations and to complete the square to determine the maximum or minimum value of a quadratic function. Students are also expected to manipulate exponential functions (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

These are complicated procedures that must be practiced in isolation. As evidenced by Kotsopoulos (2007), students cannot be expected to grasp concepts without explicit and focused procedural instruction. Completing the square is a particularly abstract procedure that needs to be motivated and supported graphically (Zaslavsky, 1997; Zakaria, 2010). Students have difficulty completing the square without extensive focused practice (Zaslavsky, 1997; Bossé and Nandakumar, 2005; Eraslan, 2005; Hutchings and McCuaig, 2008; Zakaria, 2010). Exponential functions is an unrelated topic and should not be a part of this sequence. Here, it is evident that the pathway lacks coherence and practicality.

The next section extends the property of closure to operations on polynomials. This is helpful in the conceptual understanding of algebraic manipulations to polynomial functions such as quadratics. However, polynomials are covered in a preceding unit. It would be more appropriate to address this standard in that unit to maintain coherence (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The following section is consistent with the old standards in the creation of equations and graphs. Rational and exponential functions are, however, included. These should be moved to another unit to maintain coherence. The standards do not call for creating an equation from a given graph, which would properly incorporate the vertex form of quadratic functions, as recommended by Zaslavsky (1997) and Zakaria (2010).

This section includes the standard exhibited in the Common Core section of this paper. Students are expected to solve equations by taking square roots, factoring, completing the square, and applying the Quadratic Formula. Quadratic Formula problems are to include equations with complex solutions.

Each of these techniques takes considerable time to practice (Bossé and Nandakumar, 2005; Hutchings and McCuaig, 2008). Exposure to these techniques might not have adequately prepared students to accomplish this standard, since concrete and graphical supports were absent (Zaslavsky, 1997; Mittag and Taylor, 2001; Kotsopoulos, 2007; Zakaria, 2010). This standard also calls for the derivation of the Quadratic Formula, which was already discussed in this paper (Picciotto, 2008; Li, 2011). It would be more appropriate to allow time for students to become comfortable with applying the alternative techniques (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The concluding section of this unit requires solving a system of one linear and one quadratic equation. The new standard extends the solution process to the algebraic method, in addition to the graphical method. Solutions are extended from integers to rationals. This standard also extends the quadratic equation to a circle. It might be better to address this standard in a separate culminating unit, since a circle is of a much different nature that the quadratic equations addressed in this unit. The unit is already cumbersome without this standard (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The functions and modeling unit begins with properties of operations on rational numbers, which would be more appropriate for the preceding unit on polynomials. The following section calls for students to interpret the key characteristics of graphical and tabular representations within problem constraints. Students also must calculate and interpret average rates of change (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010). This section would be excellent for comparing linear and quadratic functions to alleviate confusion between associated procedures, as stressed by Zaslavsky (1997) and Aspinwall and Eraslan (2007).

The following section compares different representations of functions as well as different families of functions. As stated earlier, the research supports comparison of linear and quadratic equations. The other families of functions should be compared in a separate culminating unit. In this section, graphical representations do support the solution techniques of factoring and completing the square, as supported throughout this paper (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The following section calls for students to create quadratic functions to model real world problems and to explain solution steps. This motivates and helps students to visualize the meaning of the algebra procedures, as supported by Mittag and Taylor (2001) and Kotsopoulos (2007). The standard also calls for composition of functions to model problems. An ideal application to fulfill this standard is the constant function of initial position added to a quadratic model of physical motion (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The next section calls for students to investigate physical transformations of quadratics using technology. This standard is strongly supported by Mittag and Taylor (2001) in providing students with interactive and concrete representations of quadratics. Technology provides a point of access for students to reason about quadratic relationships and also helps bring meaning to symbolic representations. This section also expects students to find inverse functions. The ideal application of this standard is to solve the parent function of quadratics, $y = x^2$, by taking its square root (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

The concluding section involves exponential functions, which should be addressed in a subsequent unit. This is one of the major changes that needs to be made in the curriculum design. Function families outside of linear and quadratic functions should be addressed in separate units. For sake of comparison, these other functions could be covered in a culminating unit (National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010).

It is also evident that the addition of new topics, namely the Quadratic Formula and completing the square, necessitates the movement of factoring expressions to the preceding unit on polynomials. Hutchings' and McCuaig's (2008) matrix-factoring technique can be used in the polynomials unit to save time and can then can be carried into the quadratics curriculum. Solving quadratic equations by factoring will be the focus of factoring in the quadratics curriculum, not factoring itself. This maintains coherence and distributes appropriate emphasis to the different solution techniques.

The biggest change that needs to be made to the recommended pathway is the inclusion of graphical representations to support algebraic procedures. The research, as discussed earlier, stresses visual representations precede symbolic ones. These two recommended units would align much more closely to the research if their order was switched.

In the same vain, technology should be capitalized as much as possible, especially early in the units. Technology-supported representations help students, especially those who have difficulty with prerequisite arithmetic, to understand the behavior of quadratic functions. Technology even allows students to model and solve problems before the algebraic techniques are mastered (Mittag and Taylor, 2001). Here, two of the major conceptual roadblocks discussed earlier in this paper are addressed. The reorganization of the recommended pathway in the next section will be conceptually based around the major conceptual roadblocks identified in the research.

Table 2. NCTM 2000 New York State Standards for Quadratics Units

Content standards are indicated in bold.

A.A.8	Analyze and solve verbal problems that involve quadratic equations
A.A.11	Solve a system of one linear and one quadratic equation in two variables, where only factoring is required. Note: The quadratic equation should represent a parabola and the solution(s) should be integers.
A.A.13	Add, subtract, and multiply monomials and polynomials
A.A.14	Divide a polynomial by a monomial or binomial, where the quotient has no remainder
A.A.19	Identify and factor the difference of two perfect squares
A.A.20	Factor algebraic expressions completely, including trinomials with a lead coefficient of one (after factoring a GCF)
A.A.26	Solve algebraic proportions in one variable which result in linear or quadratic equations
A.A.27	Understand and apply the multiplication property of zero to solve quadratic equations with integral coefficients and integral roots
A.A.28	Understand the difference and connection between roots of a quadratic equation and factors of a quadratic expression
A.A.41	Determine the vertex and axis of symmetry of a parabola, given its equation (See A.G.10)
A.PS.2	Recognize and understand equivalent representations of a problem situation or a mathematical concept
A.PS.4	Use multiple representations to represent and explain problem situations (e.g., verbally, numerically, algebraically, graphically)
A.PS.5	Choose an effective approach to solve a problem from a variety of strategies (numeric, graphic, algebraic)
A.PS.8	Determine information required to solve a problem, choose methods for obtaining the information, and define parameters for acceptable solutions
A.PS.9	Interpret solutions within the given constraints of a problem
A.PS.10	Evaluate the relative efficiency of different representations and solution methods of a problem
A.RP.4	Develop, verify, and explain an argument, using appropriate mathematical ideas and language

A.CM.2	Use mathematical representations to communicate with appropriate accuracy, including numerical tables, formulas, functions, equations, charts, graphs, Venn diagrams, and other diagrams
A.CM.4	Explain relationships among different representations of a problem
A.CM.11	Represent word problems using standard mathematical notation
A.CN.1	Understand and make connections among multiple representations of the same mathematical idea
A.CN.2	Understand the corresponding procedures for similar problems or mathematical concepts
A.CN.5	Understand how quantitative models connect to various physical models and representations
A.CN.6	Recognize and apply mathematics to situations in the outside world
A.R.1	Use physical objects, diagrams, charts, tables, graphs, symbols, equations, or objects created using technology as representations of mathematical concepts
A.R.2	Recognize, compare, and use an array of representational forms
A.R.3	Use representation as a tool for exploring and understanding mathematical ideas
A.R.5	Investigate relationships between different representations and their impact on a given problem
A.R.6	Use mathematics to show and understand physical phenomena (e.g., find the height of a building if a ladder of a given length forms a given angle of elevation with the ground)
A.R.8	Use mathematics to show and understand mathematical phenomena (e.g., compare the graphs of the functions represented by the equations $y = x^2$ and $y = -x^2$)
A.N.1	Identify and apply the properties of real numbers (closure, commutative, associative, distributive, identity, inverse). Note: Students do not need to identify groups and fields, but students should be engaged in the ideas.
A.N.2	Simplify radical terms (no variable in the radicand)
A.G.4	Identify and graph linear, quadratic (parabolic), absolute value, and exponential functions
A.G.5	Investigate and generalize how changing the coefficients of a function affects its graph
A.G.8	Find the roots of a parabolic function graphically. Note: Only quadratic equations with integral solutions.
A.G.9	Solve systems of linear and quadratic equations graphically. Note: Only use systems of linear and quadratic equations that lead to solutions whose coordinates are integers.
A.G.10	Determine the vertex and axis of symmetry of a parabola, given its graph (See A.A.41). Note: The vertex will have an ordered pair of integers and the axis of symmetry will have an integral value.

(NCTM, 2000, Webpage)

Table 3. Mapping 2000 Standards to Common Core Trajectory for Algebra 1

2000 Standards	Common Core Pathway
Expressions and Equations	
A.A.8 Analyze and solve verbal problems that involve quadratic equations **A.CM.11** Represent word problems using standard mathematical notation **A.CN.6** Recognize and apply mathematics to situations in the outside world **A.G.5** Investigate and generalize how changing the coefficients of a function affects its graph **A.A.19** Identify and factor the difference of two perfect squares	**A.SSE.1** Interpret expressions that represent a quantity in terms of its context.★ a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret P(1+r)n as the product of P and a factor not depending on P.* **A.SSE.2** Use the structure of an expression to identify ways to rewrite it. *For example, see x4 – y4 as (x2)2 – (y2)2, thus recognizing it as a difference of squares that can be factored as (x2 – y2)(x2 + y2).* **Notes: Extension of DOTS. Interpreting and manipulating nonstandard forms. This should be modeled explicitly and practiced nearer the end of a unit after students are familiar with standard forms. Here it is expected at the beginning of the unit and with greater generality than might be practical.**
A.A.20 Factor algebraic expressions completely, including trinomials with a lead coefficient of one (after factoring a GCF) **A.A.27** Understand and apply the multiplication property of zero to solve quadratic equations with integral coefficients and integral roots	**A.SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ a. Factor a quadratic expression to reveal the zeros of the function it defines.

A.A.28 Understand the difference and connection between roots of a quadratic equation and factors of a quadratic expression	b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15t can be rewritten as (1.151/12)12t ≈ 1.01212t to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* **Notes: Is it assumed that students will be able to factor without focused practice? Completing the square is new and is an abstract idea that should be supported graphically / geometrically. Exponents is an unrelated topic.**
A.A.13 Add, subtract, and multiply monomials and polynomials **A.A.14** Divide a polynomial by a monomial or binomial, where the quotient has no remainder **A.N.1** Identify and apply the properties of real numbers (closure, commutative, associative, distributive, identity, inverse)	**A.APR.1** Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. **Note: Extends property of closure to polynomials**
A.A.8 Analyze and solve verbal problems that involve quadratic equations **A.CM.11** Represent word problems using standard mathematical notation **A.PS.4** Use multiple representations to represent and explain problem situations (e.g., verbally, numerically, algebraically,	**A.CED.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*. **A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **A.CED.4** Rearrange formulas to highlight a quantity of interest, using the

graphically) **A.G.8** Find the roots of a parabolic function graphically Note: Only quadratic equations with integral solutions. **A.A.41** Determine the vertex and axis of symmetry of a parabola, given its equation (See A.G.10)	same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* **Note: Same except rearranging would include vertex form. What about creating an equation, given its graph?**
A.A.20 Factor algebraic expressions completely, including trinomials with a lead coefficient of one (after factoring a GCF) **A.A.28** Understand the difference and connection between roots of a quadratic equation and factors of a quadratic expression	**A.REI.4** Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form *(x – p)2 = q* that has the same solutions. Derive the quadratic formula from this form. b. Solve quadratic equations by inspection (e.g., for *x2* = 49), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as *a ± bi* for real numbers *a* and *b*. **Note: Completing the square and the quadratic equation are moved from Algebra II to Algebra I. These will take considerable time to practice. Derivation of the quadratic formula may be inappropriate.**
A.A.11 Solve a system of one linear and one quadratic equation in two variables, where only factoring is required Note: The quadratic equation should represent a parabola and the solution(s) should be integers.	**A.REI.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line y = –3x and the circle x2 + y2 = 3*.

	Note: Extends quadratic equations to circles and solutions from integers to rationals.
2000 Standards	**Common Core Pathway**
Quadratic Functions and Modeling	
A.N.1 Identify and apply the properties of real numbers (closure, commutative, associative, distributive, identity, inverse) Note: Students do not need to identify groups and fields, but students should be engaged in the ideas. **A.N.2** Simplify radical terms (no variable in the radicand)	**N.RN.3** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. **Note: Extends properties of numbers to quadratic equation.**
A.A.41 Determine the vertex and axis of symmetry of a parabola, given its equation (See A.G.10) **A.PS.8** Determine information required to solve a problem, choose methods for obtaining the information, and define parameters for acceptable solutions **A.PS.9** Interpret solutions within the given constraints of a problem **A.R.8** Use mathematics to show and understand mathematical phenomena (e.g., compare the graphs of the functions represented by the equations) **A.G.4** Identify and graph linear, quadratic (parabolic), absolute value, and exponential functions **A.G.5** Investigate and generalize how changing the coefficients of a function affects its graph	**F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★ **F.IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ **F.IF.6** Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ **Note: More attention placed on features of graphs. Average rate of change is a great way to compare linear and quadratic functions.**

A.G.8 Find the roots of a parabolic function graphically. Note: Only quadratic equations with integral solutions.	
A.G.4 Identify and graph linear, quadratic (parabolic), absolute value, and exponential functions **A.G.5** Investigate and generalize how changing the coefficients of a function affects its graph **A.A.41** Determine the vertex and axis of symmetry of a parabola, given its equation (See A.G.10) **A.A.20** Factor algebraic expressions completely, including trinomials with a lead coefficient of one (after factoring a GCF) **A.A.28** Understand the difference and connection between roots of a quadratic equation and factors of a quadratic expression **A.PS.9** Interpret solutions within the given constraints of a problem **A.R.5** Investigate relationships between different representations and their impact on a given problem	**F.IF.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. **F.IF.8** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay.* **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

	Note: Comparing the different types of functions might be more appropriate for a separate culminating unit, with the exception of comparing linear and quadratic functions.
A.CN.5 Understand how quantitative models connect to various physical models and representations **A.CN.6** Recognize and apply mathematics to situations in the outside world	**F.BF.1** Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* **Note: Add a constant function to quadratic function to represent initial position in motion problems.**
A.G.5 Investigate and generalize how changing the coefficients of a function affects its graph	**F.BF.3** Identify the effect on the graph of replacing *f(x)* by *f(x) + k, k f(x), f(kx),* and *f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.* **F.BF.4** Find inverse functions. a. Solve an equation of the form *f(x) = c* for a simple function *f* that has an inverse and write an expression for the inverse. *For example, f(x) = 2 x3 or f(x) = (x+1)/(x-1) for x ≠ 1.* **Note: Investigate shifts and dilations of quadratics using technology.**

	Solve quadratic $y = x^2$ by taking square root, at beginning of unit.
N/A	**F.LE.3** Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. **Note: Would be appropriate for a culminating unit comparing the different types of functions.**

(National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage; NCTM, 2010, Webpage)

Algebra 1 Pathway Revised and Unpacked into Objectives

The next step in designing the quadratics curriculum is to unpack the Common Core standards into instructional objectives and to organize the resulting objectives in a conceptual progression that aligns with the research on students' conceptualizations of quadratics. Unpacking the standards means that they must be broken into focused objectives that can typically be accomplished during a single lesson. Once students have met the objectives that support a given standard, they have met the standard itself. In cases where students compare different types of functions or techniques, the standard is approached holistically once the supporting objectives have been met.

The lesson objectives that accompany each standard are listed in Table 3 next to the respective standard. In many cases, a standard applies to more than one unit. In such cases, supporting objectives change according to the conceptual focus of the unit. The curriculum is organized into six units based upon the conceptual sequencing supported by the research.

The conceptual roadblocks determined in the synthesis of the literature on student misconceptions guide the conceptual framework of the curriculum revision. Recall the first major roadblock is the prerequisite arithmetic skills needed to make necessary algebraic manipulations. The second roadblock is that students lack a conception of what quadratic functions are and how their different representations relate to one another. The third roadblock is that students present a lack of understanding regarding functions in general. These roadblocks are not necessarily numbered by conceptual sequence. Also, more than one roadblock can be addressed at a time.

The first unit addresses roadblock 3 directly by viewing quadratics as functions. The unit emphasizes the evaluation and interpretation of points along a function in the context of real world problems, and representing the function graphically by using these points (Zaslavsky, 1997). Students evaluate points to construct T-tables and draw graphs based upon the tables. Once students have mastered these techniques manually, the calculator GRAPH and TABLE are taught as dynamic extensions of the tools that students have already used to evaluate functions.

Remember that the calculator GRAPH and TABLE also provide access for students who have difficulties with arithmetic (roadblock 1) to evaluate a function at several points and observe the behavior and key features of the function (roadblock 2). The TABLE provides corrective feedback to students who desire to carry out arithmetic by hand. And since students can model functions much faster with the calculator, they have greater opportunities to explore and interpret real world problems. This provides a solid conceptual foundation for studying quadratic functions.

The second unit is entitled Characteristics of Quadratic Graphs. This unit is designed to address roadblock 2. Students continue to produce graphical representations of functions, but the graphing calculator for most applications so that concentration can be placed on the important features of graphs. Students can analyze and compare more graphs using technology than would be possible otherwise. Students also will use technology to perform translations to graphs.

These objectives give students visual representations to build concepts about the behavior of quadratic functions, and also support later algebraic representations of quadratics. Technology also allows students who have difficulties with arithmetic to understand the behavior and features of functions. This is the rationale for introducing graphical and technology based units first in the curriculum.

Unit 3 aims to alleviate students' confusion of procedures corresponding to linear and quadratic functions by comparing the two families of functions directly (Zaslavsky, 1997; Aspinwall and Eraslan, 2007). The unit objectives address the specific points of confusion evidenced in the research, such as the concept of slope and the meaning of coefficients. Technology is used to illuminate these comparisons. Students will also learn how to use regression to create equations that model real world data and to distinguish whether a linear or quadratic function is a more accurate model of a data set.

In unit 4, students begin to apply solution techniques to quadratics. Graphing calculators are used to solve quadratics graphically. Graphical and algebraic solution methods are used side by side to make the symbolic techniques more concrete for students. Students will also be able to verify each solution to see if corrections need to be made. As discussed in the previous section, instruction on factoring will have already been covered in the polynomials unit. So focus in this unit can be placed on using factoring as a solution technique.

The final two units on completing the square and the quadratic formula allow students to practice each technique in isolation, since each technique involves intensive arithmetic and many steps. For this reason, students will also explain the steps of each process in writing. At the end of the last unit, students will apply all three algebraic methods to problems and compare the efficiency of the different techniques. Graphical solutions will be carried through both units to verify solutions and support each of the techniques. As discussed throughout this paper, derivation of the Quadratic Formula will not be included in this curriculum design for Algebra 1.

Table 4. Algebra 1 Pathway Revised and Unpacked into Objectives

Unit	Standards	Objectives
Quadratics as Functions	**A.REI.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). **F.IF.1** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x. The graph of f is the graph of the equation y = f(x). **F.IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.	• Evaluate a quadratic function at a point • Interpret a point on the graph of a quadratic function in the context of a real world problem • Represent a quadratic function in a T-table • Graph quadratic functions by hand • Graph quadratic functions using technology • Use calculator Table and Trace features to evaluate quadratic functions • Describe domain and range of a quadratic function
Characteristics of Quadratic Graphs	**F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. **F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. **F.BF.3** Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and	• Graph quadratic functions by hand • Graph quadratic functions using technology • Analyze the characteristics of the graphs of quadratic functions: domain, range, orientation, symmetry, axis of symmetry, maximum/minimum, vertex, roots, intercepts • Describe the characteristics of the graph of a quadratic function in the context of a real

	negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. **A.SSE.1a** Interpret parts of an expression, such as terms, factors, and coefficients.	world problem • Investigate the effects of changing the coefficients of a quadratic function *using* technology • Apply translations to quadratic functions
Comparing Quadratic and Linear Functions	**F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. **A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. **A.SSE.1a** Interpret parts of an expression, such as terms, factors, and coefficients. **A.REI.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.	• Use a given quadratic function to investigate the rate of change of a quadratic function • Compare linear and quadratic functions • Compare the meaning of coefficients in linear and quadratic functions • Identify whether a function is linear or quadratic from given data • Write equations that model data • Use a graphing calculator to find an appropriate regression equation for a set of data • Use a graphing calculator to solve a system of one linear and one quadratic equation
Solving Quadratic Equations Graphically and by Factoring	**F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.	• Solve quadratic equations by graphing by hand and with technology • Estimate solutions of quadratic equations by graphing • Interpret the solutions of a quadratic function in the context of a real world problem • Use multiplicative property of zero to determine solutions to the factored form of

	F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. **A.SSE.3a** Factor a quadratic expression to reveal the zeros of the function it defines.	quadratic equations • Solve equations of the form $x^2 + bx + c = 0$ • Check solutions to quadratic equations by substitution • Solve quadratics of the form $ax^2 + bx + c = 0$
Applying Completing the Square to Quadratics	**A.REI.4a** Use the method of completing the square to transform any quadratic equation in x into an equation of the form (x – p)2 = q that has the same solutions. Derive the quadratic formula from this form. **Without derivation.** **F.BF.1a** Determine an explicit expression, a recursive process, or steps for calculation from a context. **A.SSE.2** Use the structure of an expression to identify ways to rewrite it. For example, see x4 – y4 as (x2)2 – (y2)2, thus recognizing it as a difference of squares that can be factored as (x2 – y2)(x2 + y2). **F.IF.8a** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.	• Complete the square to write perfect square trinomials • Describe the steps of completing the square • Complete the square to determine the vertex of a quadratic function • Solve quadratic equations by completing the square • Identify coefficients to be used in the quadratic formula
Using the Quadratic Formula to Solve Quadratic Equations	**A.REI.4b** Solve quadratic equations by inspection (e.g., for x2 = 49), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b. **F.BF.1a** Determine an explicit expression, a recursive process, or steps for calculation from a context.	• Solve quadratic equations by using the quadratic formula • Use the discriminant to determine the number and nature of solutions to a quadratic equation • Describe the steps in applying the quadratic formula • Use the quadratic formula to solve real world

		problems and determine if solutions are acceptable in the context of the problem • Compare the efficiency of factoring, completing the square, and the quadratic formula
Address in Previous Unit	**F.LE.2** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). **F.LE.5** Interpret the parameters in a linear or exponential function in terms of a context.	

(National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage; NCTM, 2010, Webpage)

Final Revision for Optimal Sequencing and Conceptual Integration of Completing the Square

The final step in revising the quadratics curriculum capitalizes on the power of completing the square and its resulting vertex form to explore the characteristics of quadratic graphs. The true utility of the vertex form becomes evident through the technology applets that allow students to explore the effects of changing coefficients. The vertex form allows students to model physical situations dynamically, since the vertex is imbedded into its algebraic form. This makes translations between graphical and algebraic forms much more straightforward and intuitive (Zaslavsky, 1997)

Completing the square is integrated into the technology investigations of the characteristics of graphs, first in using the vertex form to shift graphs. This motivates the subsequent instruction of the algebraic steps of completing the square to achieve the vertex form (Zaslavsky, 1997; Zakaria, 2010). At this point students will have some familiarity, comfort, and motivation with this solution technique.

Since the research urges intensive practice for student mastery of completing the square, the technique is carried into the subsequent unit of solution techniques. Not only does this allow for extra practice, but students are also afforded the opportunity to make earlier comparisons of the different solution techniques: graphical, completing the square, and factoring (Zaslavsky, 1997; Bossé and Nandakumar, 2005; Eraslan, 2005; Hutchings and McCuaig, 2008; Zakaria, 2010).

This shift works beautifully to synthesize the curriculum in more manageable amounts conceptually (Kotsopoulos, 2007). Factoring and completing the square are normally used when quadratics have integer coefficients. The Quadratic Formula is the technique of choice otherwise. The concluding unit can then focus more directly on applying the formula to problems with irrational and complex solutions. Synthesis of the techniques will also be easier for students having completed the comparison objectives in the previous unit (Kotsopoulos, 2007). The grouping of objectives becomes optimal in regards to application and the underlying number concepts.

To facilitate continuity between the two units mentioned above, the unit comparing linear and quadratic functions is taught earlier. This is advantageous in dispelling confusion as well as capitalizing early on the prior knowledge that students have about linear functions (Zaslavsky, 1997). Students also get to see the meaning of the coefficients in the standard form of quadratics right before seeing the meaning and advantages of the variables in the vertex form (Kotsopoulos, 2007). Shifting this unit results in more comparisons and continuity in building both the graphical and symbolic concepts of quadratics.

Table 5. Pathway Revised for Improved Sequencing and Conceptual Integration of Completing the Square

Unit	Standards	Objectives
Evaluating and Graphing Quadratic Functions	**A.REI.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). **F.IF.1** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x. The graph of f is the graph of the equation y = f(x). **F.IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.	• Evaluate a quadratic function at a point • Interpret a point on the graph of a quadratic function in the context of a real world problem • Represent a quadratic function in a T-table • Graph quadratic functions by hand • Graph quadratic functions using technology • Use calculator table and trace features to evaluate quadratic functions • Describe domain and range of a quadratic function
Comparing Quadratic and Linear Functions	**F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. **A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the	• Use a given quadratic function to investigate the rate of change of a quadratic function • Compare linear and quadratic functions • Compare the meaning of coefficients in linear and quadratic functions • Identify whether a function is linear or quadratic from given data • Write equations that model data

	relationship. **A.SSE.1a** Interpret parts of an expression, such as terms, factors, and coefficients. **A.REI.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.	• Use a graphing calculator to find an appropriate regression equation for a set of data • Use a graphing calculator to solve a system of one linear and one quadratic equation
Characteristics of Quadratic Graphs and Completing the Square	**F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. **F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. **F.BF.3** Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. **A.SSE.1a** Interpret parts of an expression, such as terms, factors, and coefficients. **A.REI.4a** Use the method of completing the square to transform any quadratic equation in x into an equation of the form (x – p)2 = q that has the same solutions. Derive the quadratic formula from this form. **Without derivation.** **F.BF.1a** Determine an explicit expression, a recursive process, or steps for calculation from a context. **A.SSE.2** Use the structure of an expression to identify ways to rewrite it.	• Graph quadratic functions using technology • Complete the square to write perfect square trinomials • Describe the steps of completing the square • *Translate between the standard and vertex forms of the quadratic equation* • Complete the square to determine the vertex of a quadratic function • *Translate between graphical and algebraic representations using the vertex form of the quadratic equation* • Analyze the characteristics of the graphs of quadratic functions: domain, range, orientation, symmetry, axis of symmetry, maximum/minimum, vertex, roots, intercepts • Investigate the effects of changing the coefficients of a quadratic function using technology • *Apply translations to quadratic functions* • Describe the characteristics of the graph

	For example, see x4 – y4 as (x2)2 – (y2)2, thus recognizing it as a difference of squares that can be factored as (x2 – y2)(x2 + y2). **F.IF.8a** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.	of a quadratic function in the context of a real world problem
Solving Quadratic Equations Graphically and Algebraically	**F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum. **F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. **A.SSE.3a** Factor a quadratic expression to reveal the zeros of the function it defines.	• Solve quadratic equations by graphing by hand and with technology • Estimate solutions of quadratic equations by graphing • Interpret the solutions of a quadratic function in the context of a real world problem • Use multiplicative property of zero to determine solutions to the factored form of quadratic equations • Solve equations of the form $x^2+bx+c=0$ • Check solutions to quadratic equations by substitution • Solve quadratics of the form $ax^2+bx+c=0$ • Solve quadratic equations by completing the square • *Identify graphs by inspection given standard, factored, and vertex forms of quadratics*

Using the Quadratic Formula to Solve Quadratic Equations	**A.REI.4b** Solve quadratic equations by inspection (e.g., for x2 = 49), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b. **F.BF.1a** Determine an explicit expression, a recursive process, or steps for calculation from a context.	• Identify coefficients to be used in the quadratic formula • Solve quadratic equations by using the quadratic formula • Use the discriminant to determine the number and nature of solutions to a quadratic equation • Describe the steps in applying the quadratic formula • Use the quadratic formula to solve real world problems and determine if solutions are acceptable in the context of the problem • Compare the efficiency of factoring, completing the square, and the quadratic formula. • Use a graphing calculator to find an appropriate quadratic regression equation for a set of data. Solve, graph, and interpret in the context of the problem.
Address in Previous Unit	**F.LE.2** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). **F.LE.5** Interpret the parameters in a linear or exponential function in terms of a context.	

(National Governors Association Center for Best Practices; Council of Chief State School Officers, 2010, Webpage; NCTM, 2010, Webpage)

Table 6. Instructional Calendar for Quadratics in Algebra 1

	Monday	Tuesday	Wednesday	Thursday	Friday
Unit 1 **Evaluating and Graphing Quadratic Functions**	Investigating Trajectory Motion Cannonball Applet, Evaluate Points Construct T-Tables and Graph Data by Hand, Discuss and Interpret Important Features	Connect T-tables and Graphs by Hand to Calculator Table and Graphs with Trace and Zoom Features, Discuss Domain and Range of the Function and in the Context of the Cannonball Problem	Investigate Variables of Trajectory Motion: Wind Resistance and Initial Heights, Model and Compare with Calculator Graphs, Interpret Effects of Variables on Motion	**Unit 2** Investigate Average Rate of Change: Falling Book Problem, Quadratic Regression Features of Calculator to Model Problem, Calculate Slope Over Intervals	Falling Book Problem, Graph and Draw Features of Calculator to Compare Linear and Quadratic Functions, Discuss Differences in the Graph and in the Context of the Problem
Unit 2 **Comparing Quadratic and Linear Functions**	Identify Data Sets as Having Linear or Quadratic using Regression and R-value Features of Calculator, Create Equations to Model Data	Compare Meanings of Coefficients in Linear and Quadratic Functions Using Dynamic Applets, Discuss and Write about Similarities and Differences	**Unit 3** Compare Standard and Vertex Forms Using Dynamic Applets, Advantages of Vertex Form in Translating Graphs, Explore Vocabulary: Vertex, Max, Min, y-intercept	Dynamic Problem Solving: Will It Make the Hoop? Applet, Translating Graphs and Determining if a Point is in the Function, Explore Vocabulary: Axis of Symmetry, Roots, Domain, Range	**Test Units 1, 2**
Unit 3 **Characteristics of Quadratic Graphs and Completing the Square**	Vocab Practice Clickers, Applet Activity 1: How to Write Vertex Form of Quadratic Functions, Dynamic Translations of Graphs	Applet Activity 2: Determining the Vertex Form of Quadratic Functions Given the Vertex and a Point	Applet Activity 2, Determining the Vertex Form, Graph Functions by Hand and with Calculator Given Vertex Form	Complete the Square to Write Perfect Square Trinomials, Identify Vertex and Max/Min, Explain Steps Verbally and in Writing	Translate from Vertex Form to Standard Form by Distribution, Translating Back and Forth, Given Graphs Find Vertex Form and Convert to Standard Form

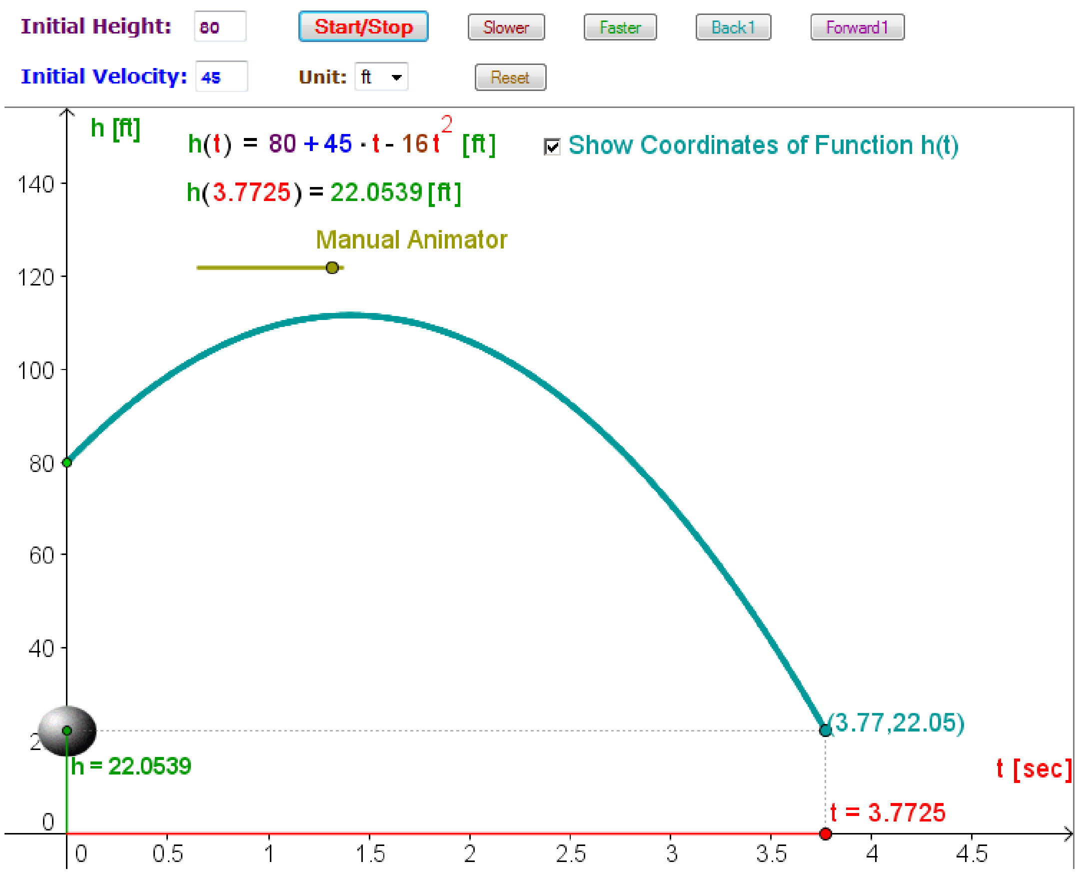
Unit 4 **Solving Quadratic Equations Graphically and Algebraically**	**Go Over Unit 1,2 Test** **Review**	**Unit 3 Test**	**Unit 4** Review Factoring, Factor to Solve Equations, Check by Hand and Verify Graphically with Calculator	Whiteboards Activity: Factor to Solve Equations, Check by Hand and Verify Graphically with Calculator Group Practice	Solve Equations by Completing the Square, Check by Hand and Verify Graphically with Calculator Group Practice
Unit 5 **Using the Quadratic Formula to Solve Quadratic Equations**	Station Activity: Solve Equations by Completing the Square, Factoring, and Graphically	Matching Game: Given the Three Forms of Each Equation Match the Graph, Independent Practice with 1-on-1 Help	**Unit 4 Quiz** **Go Over Quiz** **Go Over Unit 3 Test** **Review**	**Unit 5** Setting Up Quadratic Formula to Solve Problems with Integral Roots, Solve and Verify Graphically with Calculator, Independent Practice	Groups Solve Problems with Integral Roots, Verify Graphically with Calculator, Describe Steps to Solution, Groups Present
	Review Simplification of Radicals and Rounding to Decimal Form, Groups Practice Problems with Non-integral Roots and Verify Solutions Graphically	Groups Practice Problems with Non-integral Roots and Verify Solutions Graphically, Groups Present	Complex Numbers, Discuss Factorability in the Real World, Discriminant and Nature of Roots, Problems with Complex Solutions and/or Non-integral Coefficients	Discuss Cannonball Problem and Rejecting Roots in Context, Bungee Jump Activity Using Regression and Applying and Comparing All Solution Techniques	Cannonball Problem Given Graph, Applying and Comparing All Solution Techniques, Review for **Unit 5 Test on Monday**

Activities and Problem Sets

Unit 1: Monday-Tuesday

Figure 2. Cannonball Trajectory Motion Applet

http://www.mathcasts.org/mtwiki/InterA/MotionV



Unit 1: Monday

Evaluating Functions

Evaluate the following functions at the specified times.

1. $h(t) = -16t^2 + 45t + 80$

t = 0

t = 2

t = 4

2. $h(t) = -16t^2 + 60t + 110$

t = 0

t = 1

t = 3

3. $h(t) = -16t^2 - 6t + 64$

t = 0

t = 0.5

t = 0.9

Evaluate the following functions at the specified values of x.

1. $y = 2x^2 + 5 - 11$

x = 0

x = 8

x = - 3

2. $y = -x^2 + 5 - 11$

x = 24

x = -17

x = 5

Construct T-tables and graph the following functions.

1. $h(t) = -16t^2 + 45t + 80$

2. $h(t) = -16t^2 + 60t + 110$

3. $2x^2 + 5x - 11$

Unit 1: Tuesday

Calculator Tables and Graphs

Analyze the following quadratic functions using the Table and Graphing features.

1. $y = x^2$
2. $y = 2x^2$
3. $y = 2x^2 + 5$
4. $y = 2x^2 + 5x - 11$
5. $y = -x^2$
6. $y = -2x^2$
7. $y = x^2 + 4x - 3$
8. $y = 23x^2 - 12x - 151$
9. $y = 0.65x^2 + 0.95x - 14.10$
10. $y = 230x^2 + 5064x - 611000$

Unit 1: Wednesday

Investigating Variables of Trajectory Motion

The **height of a falling object** at time t is can be found using the function,

$$h(t) = -16t^2 + v_0 t + h_0$$

where h_0 is the *initial height*,

v_0 is the *initial velocity*,

and the -16 is the *acceleration constant* due to gravity.

We already used this formula to model the motion of a cannonball trajectory. Now we are going to model the motion of a falling sheet of plywood.

Guiding Questions

1. Do you think we should use the same formula? Why or why not.
2. Would a sheet of plywood fall the same way a cannonball would?
3. Why would the plywood fall slower?
4. How can we change our formula to make it more accurate?
5. How do we write the function if the wind resistance changes the acceleration coefficient to -8 and the sheet of plywood falls from the top of a building that is 120 feet tall?
6. If a cannonball was also falling from the roof, how could we find out what height the plywood is at when the cannonball hits the ground?

Test the following conditions for $(\boldsymbol{v_0}, \boldsymbol{h_0})$.

(-8, 120), (-4, 120), (0, 120), (-32, 120), (-60, 120), (-120, 120)

(-16, 100), (-16, 70), (-16, 30), (-16, 500), (-16, 0), (-16, 16)

Unit 2: Thursday-Friday

Investigating Average Rate of Change: Falling Book Problem

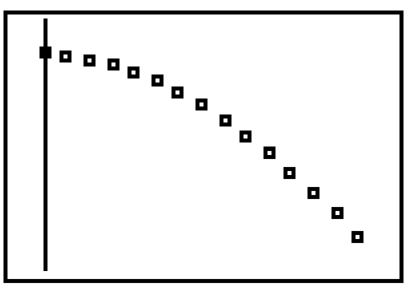
A book was dropped from a height of 0.865 meters and its height above ground was monitored with a Calculator-Based Laboratory (CBL). The following table contains data of the book's height in meters versus time in seconds.

Time	0.00	0.02	0.04	0.06	0.08	0.10	0.12
Height	0.865	0.858	0.848	0.836	0.819	0.799	0.775

Time	0.14	0.16	0.18	0.20	0.22	0.24	0.26	0.28
Height	0.748	0.716	0.681	0.642	0.600	0.554	0.506	0.454

NCTM, 2007.

Create a Scatter Plot of the given data with time x and height y.



NCTM, 2007.

Group Problem Solving: Find the ***average velocity*** of the falling book.

Assign each group two random points as well as the two end points in the table. Have students calculate average velocities and compare results on board.

Discussion:

1. How do you interpret the meaning of the slopes you found?
2. Why do our results differ?
3. How does the velocity change as the book falls?
4. Describe the motion of the falling book.
5. Is it linear? What does it look like if you zoom in on the graph?

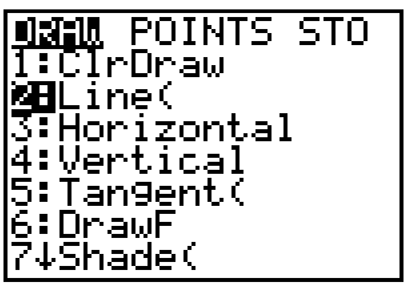
Practice: Find the average velocities for the following time intervals.

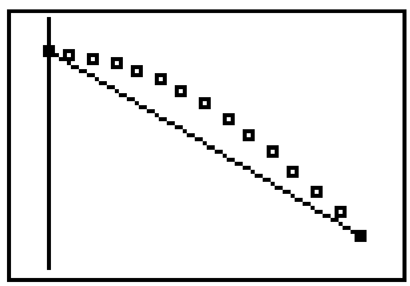
1. t = 0 and t = 0.16

2. t = 0.16 and t = 0.28

3. Compare the motion of the book during these intervals.

Draw Feature of Calculator: 2nd Draw → 2: Line → Line(x_1, y_1, x_2, y_2)





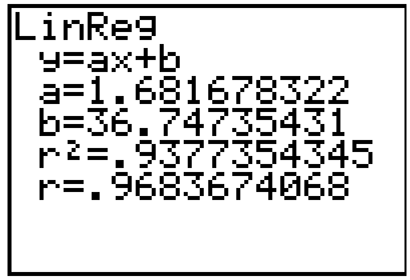
NCTM, 2007.

Unit 2: Monday

Identifying Data Sets as Linear or Quadratic Using Regression and R-values

Make sure that the R-value is enabled on the calculator by entering:

2nd CATALOG → Diagnostic On → Enter → Enter.



Texas Instruments, 2012.

Data Sets

Time	5	6	7	8	9	10	11
Height	-6.3	-4.3	-2.3	0.1	2.4	4.3	6.2

Time	0	1	2	3	4	5	6
Height	24.5	19.3	14.1	9.3	4.2	-1.4	-6.2

Time	0	1	2	3	4	5	6
Height	45.3	36.1	27.4	18.2	9.5	0.1	-9.2

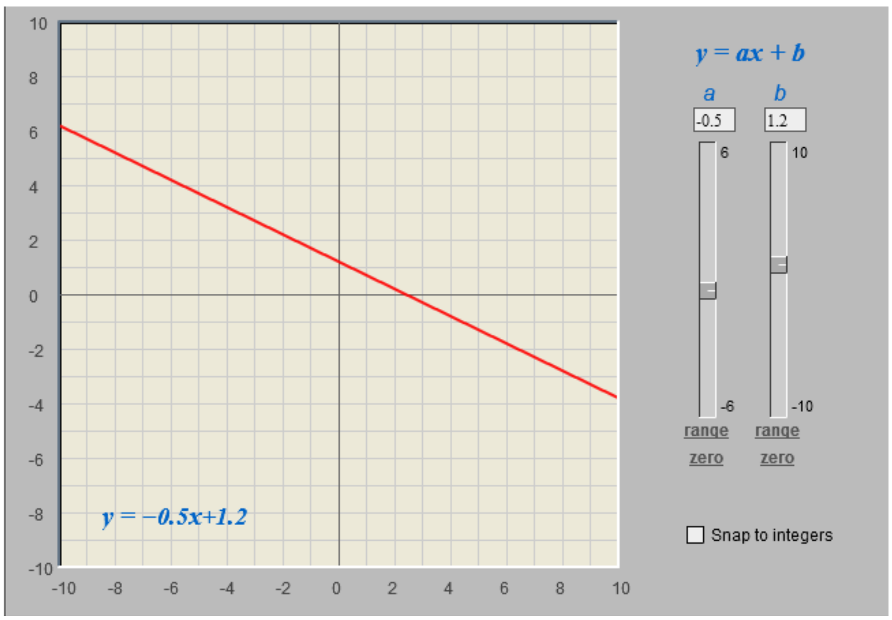
Time	2	3	4	5	6	7	8
Height	-12.1	-8.1	-4.0	0.2	4.3	8.0	12.1

Time	3	4	5	6	7	8	9
Height	-146	-118	-82	-38	14	74	142

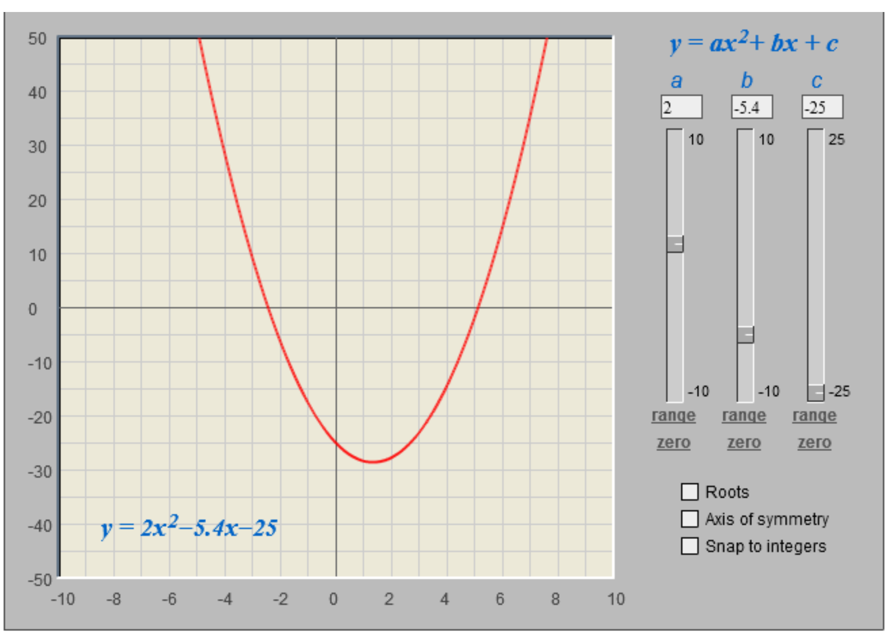
Unit 2: Tuesday

Figure 5. Applets to Compare Meanings of Coefficients of Linear and Quadratic Functions

http://www.mathopenref.com/linearexplorer.html



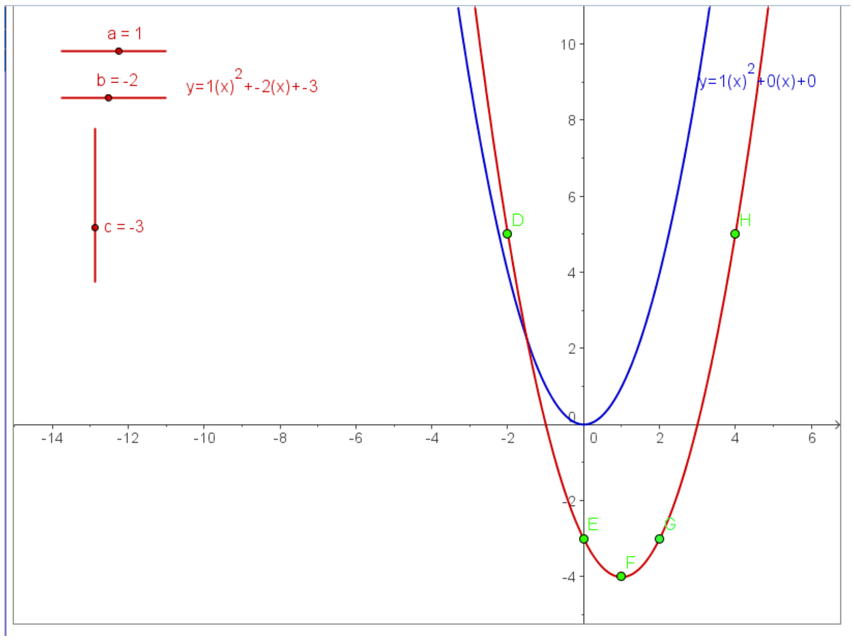
http://www.mathopenref.com/quadraticexplorer.html



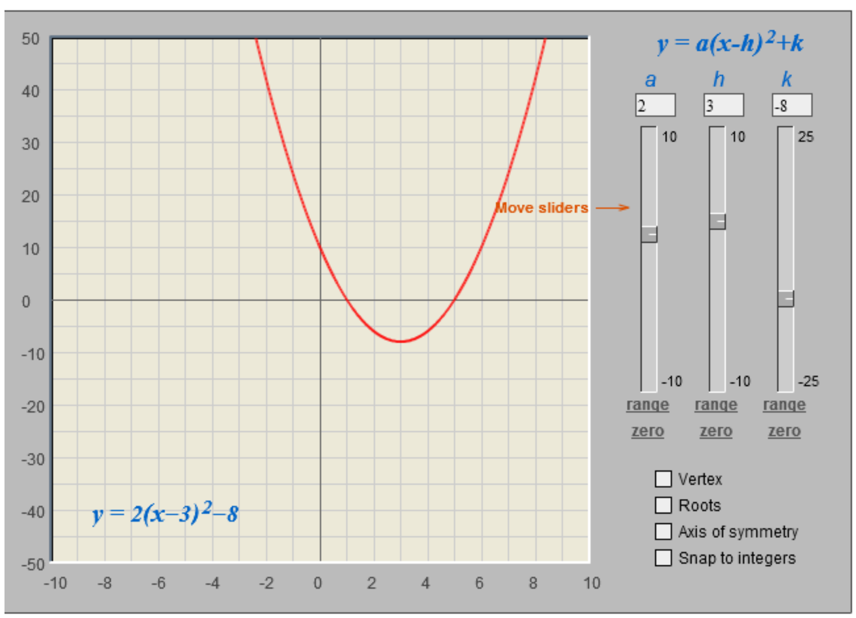
Unit 3: Wednesday

Figure 6. Applets to Compare Standard and Vertex Forms

http://www.andrews.edu/~adamst/quadgraph.html



http://www.mathopenref.com/quadvertexexplorer.html

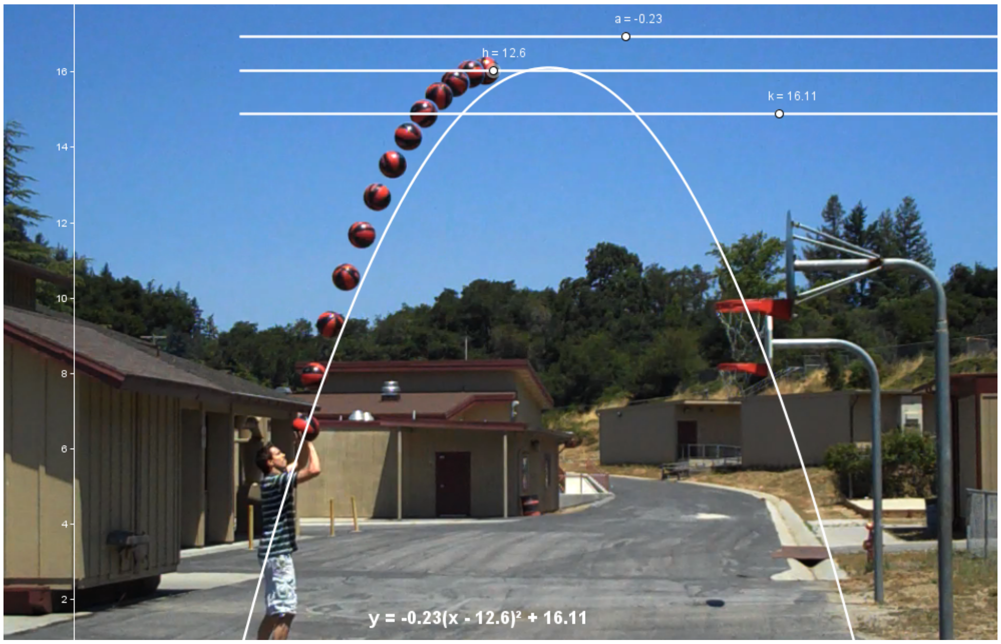


Unit 3: Thursday

Figure 7. Translations Applet: Will It Make the Hoop?

http://www.livebinders.com/play/play_or_edit?id=330740

Will it go in the hoop? Adjust the sliders to model the shot with a quadratic function.



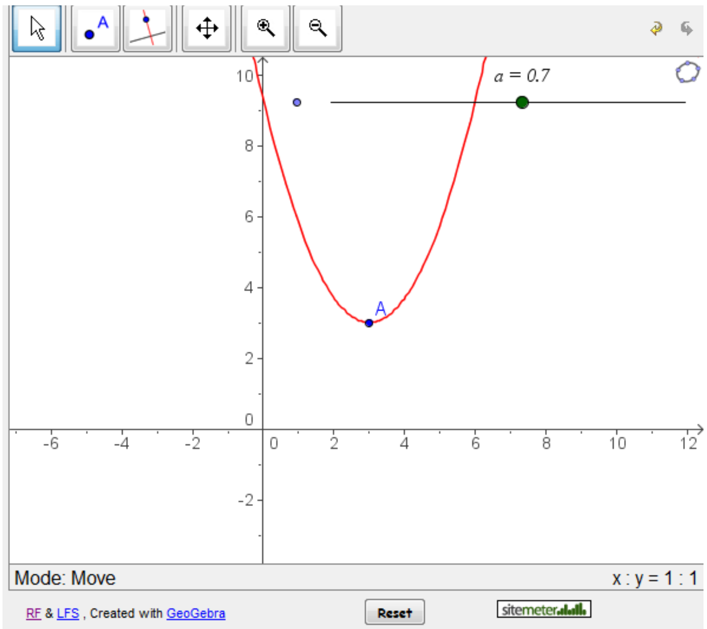
If you just know the equation, how do you figure out if the ball goes in?

Unit 3: Monday

Figure 8. How to Write the Vertex Form Applet

http://www.mathcasts.org/gg/student/quadratics/quad_fun1/index.html

Activity 1: Vertex and Coefficient a



Guiding Questions

1. What is the vertex? The turning point (h, k) of the parabola.

2. How does the coefficient *a* affect the graph of a parabola? Width and vertical orientation.

3. What happens to the graph when a = 0? Line y = k.

4. What is the axis of symmetry? X = h.

5. What affects the axis of symmetry? Vertex.

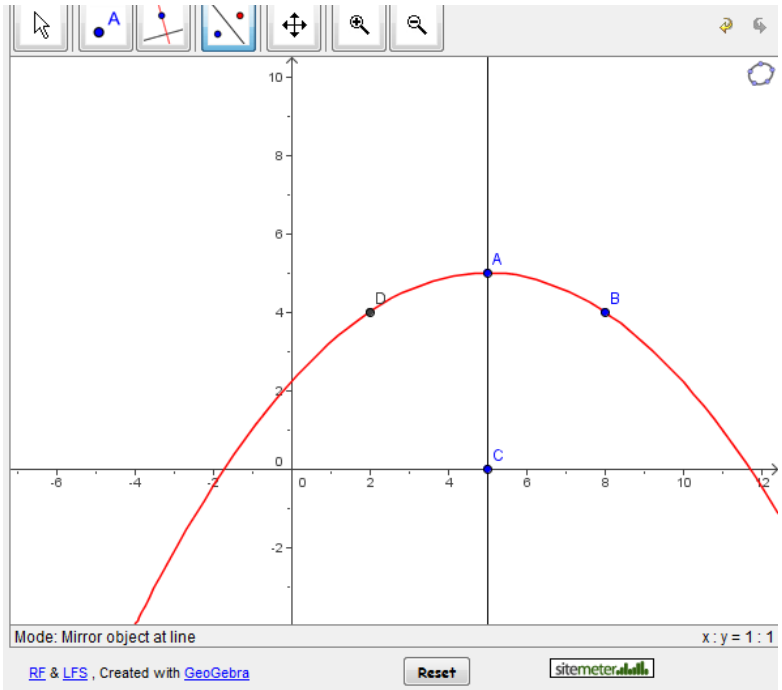
6. How do I find the equation for the quadratic function given the graph? Substitute a, h, k into the vertex form of the equation.

Unit 3: Tuesday-Wednesday

Figure 9. Applet Determining the Vertex Form Given the Vertex and a Point

http://www.mathcasts.org/gg/student/quadratics/quad_fun2/index.html

Activity 2: Vertex and a Point



Guiding Questions

1. What do I know about a point reflected across the axis of symmetry? It is on the parabola.

2. How do I draw the axis of symmetry? Perpendicular line through the vertex.

3. How do I calculate the coefficient a given a graph of a quadratic? Substitute in any point (x, f(x)) and the vertex (h, k) into the vertex form of the quadratic equation. Then solve for a.

4. What did I do wrong? Remember to square (x – h). Check signs. Applet provides feedback algebraically and graphically.

Unit 3: Wednesday

Determining the Vertex Form Given Vertex and a Point

Determine the vertex form of a quadratic function for each of the following.

1. vertex (2, 6), point (3, 2)
2. vertex (0, 0), point (2, 4)
3. vertex (-1, 2), point (-3, 6)
4. vertex (3, -2), point (4, 3)
5. vertex (-4, -1), point (-8, -10)

Entering Vertex Form into Calculator

Practice entering the vertex form of quadratic functions for each of these examples.

1. $y = 2(x - 3)^2 + 5$
2. $y = (x + 5)^2 + 9$
3. $y = -3(x + 1)^2 - 4$
4. $y = -(x - 3)^2 + 5$
5. $y = 4(x - 1)^2 - 6$

Unit 3: Thursday

Completing the Square to Write Perfect Square Trinomials

Complete the Square to get the following quadratic equations into vertex form.

1. $x^2 + 7x - 11 = 0$
2. $x^2 + 6x + 2 = 0$
3. $2x^2 - 12x + 4 = 0$
4. $10x^2 - 20x - 80 = 0$
5. $3x^2 - 18x - 48 = 0$
6. $x^2 - 16x + 48 = 0$
7. $10x^2 - 20x - 86 = 0$

Unit 3: Friday

Translating Between Vertex Form and Standard Form

Use distribution to translate the following equations into standard form.

1. $(x + 5)^2 + 9 = 0$
2. $2(x - 3)^2 + 5 = 0$
3. $-3(x + 1)^2 - 4 = 0$
4. $-(x - 3)^2 + 5 = 0$
5. $4(x - 1)^2 - 6 = 0$

Translate the following equations from standard form into vertex form.

1.

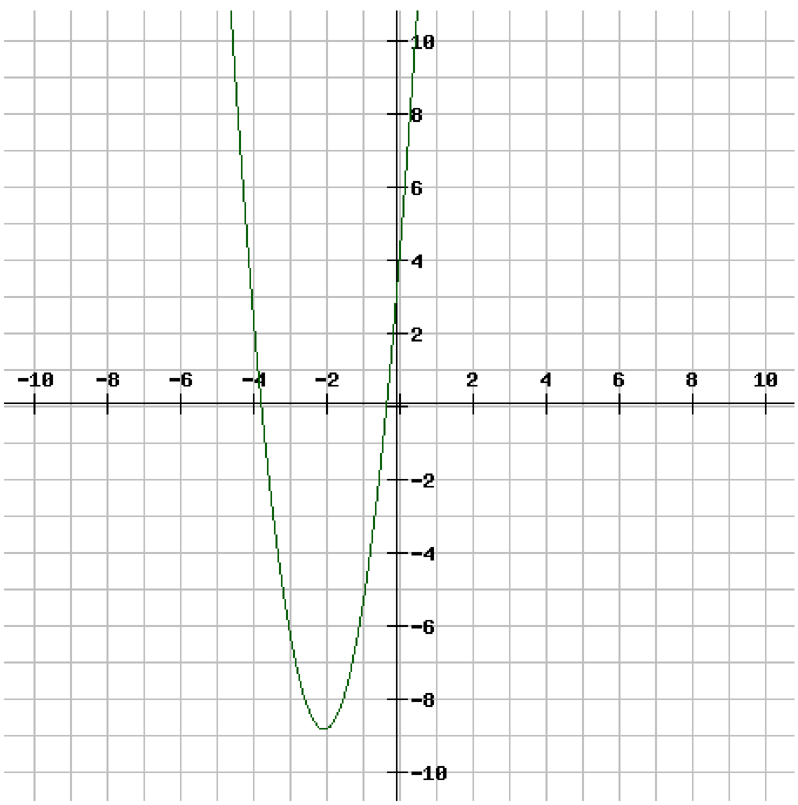
2.

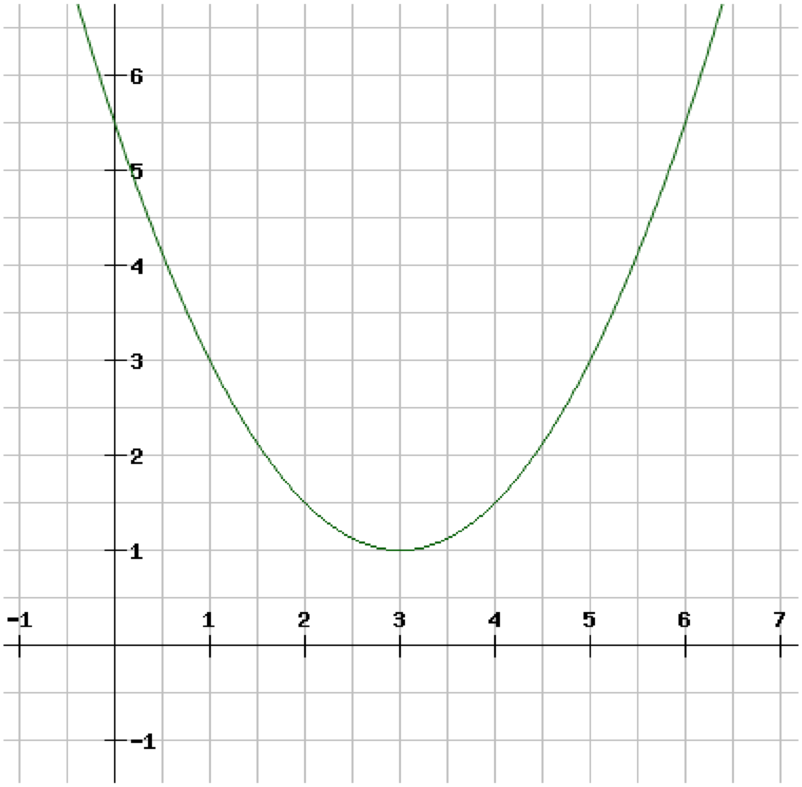
3.

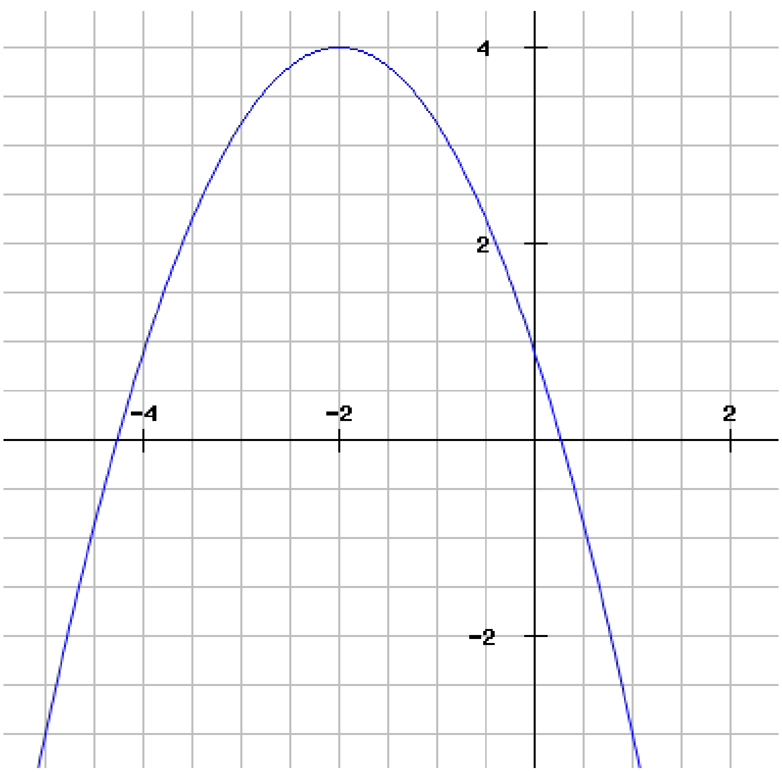
4.

5.

Determine the vertex form of the function for each of the following graphs and translate into standard form.




6
5
4
3
2
1
-1
1
2
3
4
5
6
7
-1


4
2
-4
-2
2
-2

Unit 4: Wednesday

Factor to Solve Quadratics

Solve the following equations by factoring and verify graphically with your calculator.

1. $3x^2 = 48$
2. $x^2 - 5x = 0$
3. $x^2 - 6x = 0$
4. $x^2 - 2x - 15 = 0$
5. $m^2 - 3m - 10 = 0$
6. $x^2 - 5x - 24 = 0$
7. $x^2 - 5x + 6 = 0$
8. $x^2 + 11x + 28 = 0$
9. $x^2 - 4x - 12 = 0$
10. $x^2 - 5x = 6$

Unit 4: Thursday

Whiteboard Problems

Solve the following equations by factoring.

1. $x^2 - 10x + 21 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 - 7x + 6 = 0$
4. $3x^2 - 27x = 0$
5. $x^2 + 3x - 18 = 0$
6. $(x+4)(x-3) = 0$
7. $x^2 - x = 6$
8. $x^2 = 30 - 13x$
9. $4x^2 - 36 = 0$
10. $x^2 + 2x - 24 = 0$
11. $x^2 + 3x - 40 = 0$
12. $x^2 + 3x - 28 = 0$

Unit 4: Friday

Solving Equations by Completing the Square

Solve the following quadratic equations by completing the square and verify graphically with your calculator.

1. $x^2 - 5x - 24 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 + 11x + 28 = 0$
4. $x^2 - 4x - 12 = 0$
5. $x^2 - 5x = 6$
6. $x^2 - x = 6$
7. $x^2 = 30 - 13x$
8. $4x^2 - 36 = 0$
9. $x^2 + 2x - 24 = 0$
10. $x^2 + 3x - 40 = 0$

Unit 4: Monday

Stations Activity

Station One: Find the roots of the following functions graphically.

1. $y = 15x^2 - 75x + 265$
2. $y = 5x^2 + 20x - 90$
3. $y = -3x^2 - 27x + 72$
4. $y = -3/5(x-8)^2 + 6$
5. $y = 4(x + 2)^2 - 3$
6. $y = .52x^2 + 2.97x + 8.16$

Station Two: Solve the following equations by completing the square.

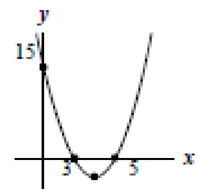
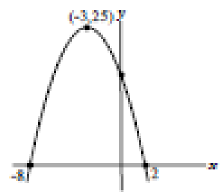
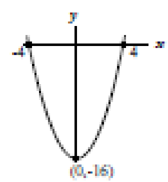
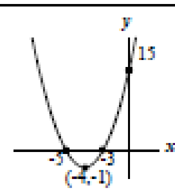
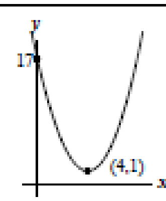
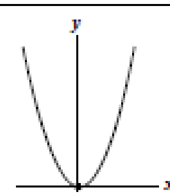
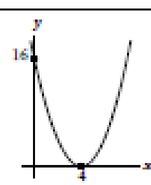
1. $x^2 - 6x + 18 = 0$
2. $5x^2 + 10x + 7 = 0$
3. $-\frac{1}{3x^2} + 4x - 15 = 0$
4. $2x^2 + 36x + 163 = 0$
5. $-4x^2 - 40x - 115 = 0$

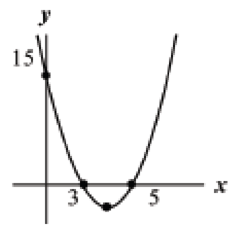
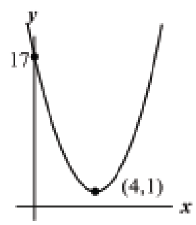
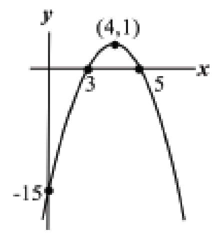
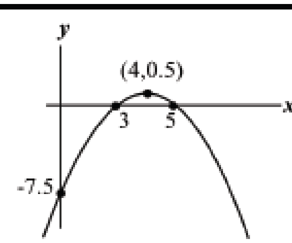
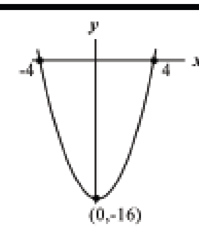
Station Three: Solve by factoring.

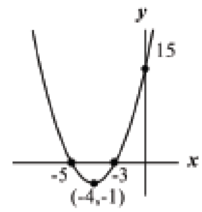
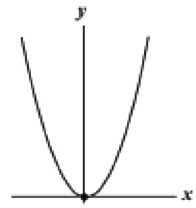
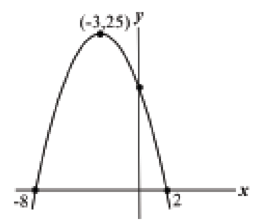
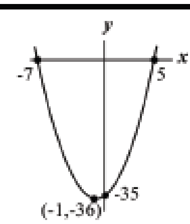
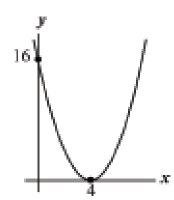
1. $x^2 - 4x + 3 = 0$
2. $3x^2 + 12x + 12 = 0$
3. $x^2 - 4x - 12 = 0$
4. $-2x^2 - 10x - 12 = 0$
5. $x^2 - 25 = 0$
6. $-x^2 - 2x + 24 = 0$
7. $5x^2 - 20 = 0$
8. $x^2 - 15x + 56 = 0$

Unit 4: Tuesday

Figure 10. Matching Game Standard, Vertex, and Factored Forms to Graph

A $y = x^2 + 2x - 35$ $y = (x-5)(x+7)$ $y = (x+1)^2 - 36$	
H $y = x^2 - 8x + 15$ $y = (x-3)(x-5)$ $y = (x-4)^2 - 1$	
E $y = -x^2 - 6x + 16$ $y = -(x+8)(x-2)$ $y = -(x+3)^2 + 25$	
F $y = x^2 - 16$ $y = (x-4)(x+4)$ $y = (x-0)^2 - 16$	
B $y = x^2 + 8x + 15$ $y = (x+5)(x+3)$ $y = (x+4)^2 - 1$	
G $y = x^2 - 8x + 17$ No roots $y = (x-4)^2 + 1$	
J $y = x^2$ $y = (x-0)(x-0)$ $y = (x-0)^2 + 0$	

A $y = x^2 + 2x - 35$ $y =$ .... $y =$	
B $y = x^2 + 8x$.......... $y =$ .... $y = (x+4)^2 - 1$	
C $y = x^2 - 8x$.......... $y = (x-4)(x-4)$ $y =$	
D $y = -x^2 + 8x$.......... $y =$ .... $y = -(x-4)^2 + 1$	
E $y = -x^2 - 6x + 16$ $y = -(x+8)(x-2)$ $y = -(x+3)^2 + 25$	

F $y = x^2$.................. $y = (x-4)(x+4)$ $y =$	
G $y = x^2 - 8x$........ No roots $y =$	
H $y = x^2 - 8x + 15$ $y = (x-3)(x-5)$ $y = (x-4)^2 - 1$	
I $y = -\frac{1}{2}x^2 + 4x$.......... $y = -\frac{(x-3)(x-5)}{2}$ $y =$	
J $y = x^2$ $y =$ .... $y =$	

MARS, 2012.

Which feature of the graph does each form of a quadratic function highlight? Use these features to match the function and graph cards. Then use the think-pair-share strategy with your partner to determine the missing information.

Unit 5: Thursday

Quadratic Formula Problems with Integral Roots

Solve using the quadratic formula and verify graphically using your calculator.

1. $-2x^2 - 10x - 12 = 0$
2. $x^2 - 4x + 3 = 0$
3. $x^2 - 15x + 16 = 0$
4. $-x^2 - 2x + 24 = 0$
5. $4x^2 - 36 = 0$
6. $5x^2 + 15x - 20 = 0$
7. $-3x^2 - 24x - 36 = 0$
8. $6x^2 - 6x - 12 = 0$
9. $-4x^2 + 24x - 36 = 0$
10. $9x^2 + 27x = 0$

Unit 5: Friday

Quadratic Formula Problems with Integral Roots for Group Problem Solving

1. $x^2 - 5x - 6 = 0$
2. $4x^2 - 20x + 24 = 0$
3. $-x^2 - 24x - 48 = 0$
4. $81 - x^2 = 0$
5. $2x^2 - 16x - 40 = 0$

6. $-5x^2 - 5x + 100 = 0$

7. $128 - 2x^2 = 0$

8. $2x^2 + 22x + 48 = 0$

9. $x^2 - 13x + 42 = 0$

10. $-3x^2 + 33x = 0$

11. $x^2 + x = 0$

12. $3x^2 - 33x + 54 = 0$

Unit 5: Monday

Discriminant Problems

1. The roots of the equation $x^2 - 2x - 2 = 0$ are
1) real, rational, and equal
2) real, rational, and unequal
3) real, irrational, and unequal
4) imaginary

2. The roots of the equation $2x^2 + 3x + 2 = 0$ are
1) irrational and unequal
2) imaginary
3) rational and equal
4) rational and unequal

3. The roots of the equation $2x^2 + 4x + 3 = 0$ are
1) real, rational, and unequal
2) real, irrational, and unequal
3) real, rational, and equal
4) imaginary

4. The roots of the equation $x^2 + 2x + 4 = 0$ are
1) real, rational, and unequal
2) imaginary and unequal
3) rational and equal
4) rational and unequal

5. The roots of the equation $x^2 + x + 1 = 0$ are
1) real, rational, and unequal
2) real, irrational, and unequal
3) real, rational, and equal
4) imaginary

6. The roots of the quadratic equation $5x^2 - 2x = -3$ are
1) imaginary
2) real and irrational
3) real, rational, and unequal
4) real, rational, and equal

7. If $b^2 - 4ac < 0$, the roots of the equation $ax^2 + bx + c = 0$ must be
1) real, irrational, and unequal
2) real, rational, and unequal
3) real, rational, and equal
4) imaginary

8. The roots of the equation $x^2 + 7x - 8 = 0$ are
1) real, rational, and equal
2) real, rational, and unequal
3) real, irrational, and equal
4) imaginary

9. The roots of the equation $-3x^2 = 5x + 4$ are
1) real, rational, and unequal
2) real, irrational, and unequal
3) real, irrational, and equal
4) imaginary

10. The roots of the quadratic equation $4x^2 = 2 + 7x$ are best described as
1) real, equal, and rational
2) real, unequal, and rational
3) real, unequal, and irrational
4) Imaginary

Quadratic Formula Problems with Non-integral Roots

Solve the following equations and reduce to simplest form.

1. $6x^2 + 15x - 9 = 0$
2. $3x^2 + 11x + 3 = 0$
3. $-5x^2 + 9x - 1 = 0$
4. $-x^2 + 13x - 9 = 0$
5. $2x^2 - 19x + 8 = 0$
6. $2x^2 - 10 = 0$
7. $-2x^2 + 24 = 0$
8. $x^2 + 6x + 2 = 0$
9. $-3x^2 + 2x + 2 = 0$
10. $-4x^2 + 3x + 5 = 0$
11. $4x^2 + 6x + 1 = 0$
12. $x^2 + 3x - 2 = 0$

Unit 5: Tuesday

Quadratic Formula Practice with Non-integral and Irrational Roots

Solve the following equations and reduce to simplest form.

1. $3x^2 + 2x - 2 = 0$
2. $-5x^2 + 12x - 6 = 0$
3. $3x^2 - 9x + 3 = 0$
4. $-x^2 - 10x + 20 = 0$
5. $8x^2 + 15x + 5 = 0$
6. $7x^2 + x - 3 = 0$
7. $-2x^2 - 4x + 9 = 0$
8. $13x - 9 - x^2 = 0$
9. $x^2 + 6x + 2 = 0$
10. $3x - 4x^2 + 5 = 0$
11. $1 + 4x^2 + 6x = 0$
12. $x^2 - 5 = 0$

Unit 5: Wednesday

Discriminant Problems

1. The roots of the equation $9x^2 + 3x - 4 = 0$ are
1) imaginary
2) real, rational, and equal
3) real, rational, and unequal
4) real, irrational, and unequal

2. The roots of the equation $x^2 - 10x + 25 = 0$ are
1) imaginary
2) real and irrational
3) real, rational, and equal
4) real, rational, and unequal

3. The roots of the equation $x^2 - 3x - 2 = 0$ are
1) real, rational, and equal
2) real, rational, and unequal
3) real, irrational, and unequal
4) imaginary

4. The roots of the equation $2x^2 - x = 4$ are
1) real and irrational
2) real, rational, and equal
3) real, rational, and unequal
4) Imaginary

5. The roots of the equation $2x^2 - 8x - 4 = 0$ are
1) imaginary
2) real, rational, and equal
3) real, irrational, and unequal
4) real, rational, and unequal

6. The roots of the equation $2x^2 - 5 = 0$ are
1) imaginary
2) real, rational, and equal
3) real, rational, and unequal
4) real and irrational

7. The roots of the equation $5x^2 - 2x + 1 = 0$ are
1) real, rational, and unequal
2) real, rational, and equal
3) real, irrational, and unequal
4) imaginary

8. The roots of $x^2 - 5x + 1 = 0$ are
1) real, rational, and unequal
2) real, rational, and equal
3) real, irrational, and unequal
4) Imaginary

Quadratic Formula Problems with Complex Roots

Solve the following equations and reduce to simplest form.

1. $x^2 + 4x + 8 = 0$
2. $-2x^2 - 5x - 10 = 0$
3. $3x^2 - 6x + 5 = 0$
4. $-3x^2 + 7x - 7 = 0$
5. $5x^2 - 2x + 5 = 0$
6. $-x^2 + 6x - 10 = 0$
7. $x^2 + 3x + 7 = 0$
8. $2x^2 - x + 6 = 0$
9. $-4x^2 - 3x - 3 = 0$
10. $4x^2 - 10x + 8 = 0$
11. $-5x^2 + 11x - 6 = 0$
12. $-x^2 + 4x - 9 = 0$

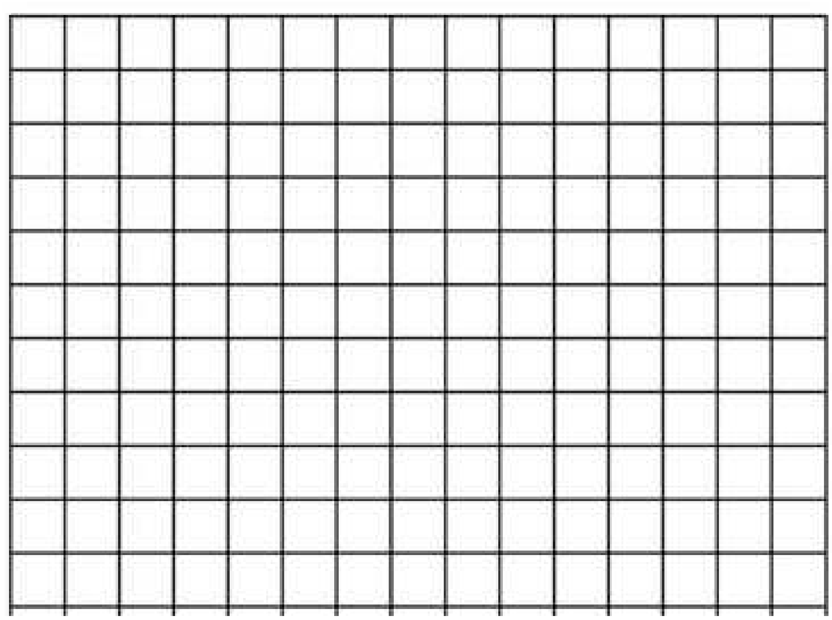
Unit 5: Thursday

Bungee Jump Activity

You have been hired by an amusement park to build a bungee jumping attraction. Your engineering team has built a jumping platform **144 feet** above a sparkling pool of water. You must now test the apparatus. You have outfitted a test jump dummy with a GPS device that can measure its height above the water at various times during the jump. Here is the resulting data:

Time (seconds)	Height (feet)
1	72
2	24
3	0
4	0
5	24
6	72

1. Use an appropriate **regression** model to determine a function that represents the data. **Write the function *here*.** (6 points)



2. **Graph** the function. (4 points)

   On the graph, *identify* and *label*:

3. The **roots** (4 points)

4. The **vertex** (1 point)

Write the **coordinates** of:

5. The **roots** (4 points)

6. The **vertex** (2 points)

7. **Explain** in a complete sentence how you used your calculator to find the **vertex**. (2points)

8. Verify the **roots** of your function by **factoring**. Show all work. (6 points)

9. Verify the **roots** of your function by using the **quadratic equation**. Show all work. (6 points)

10. Use the **vertex formula** and **substitution** to verify the **coordinates** for the vertex. (6 points)

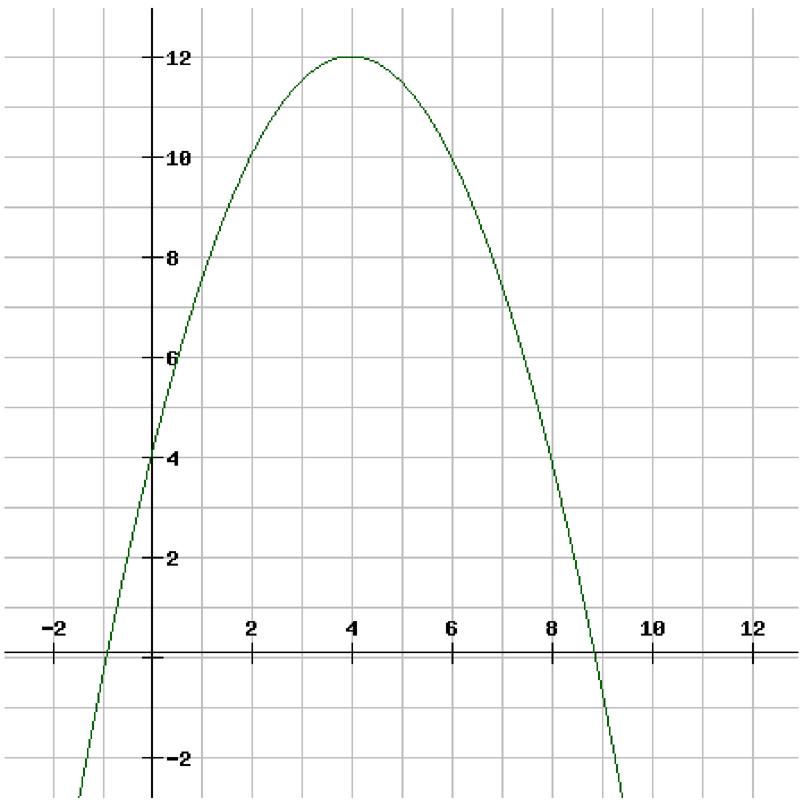
11. **Explain** in a complete sentence which method you prefer for finding the **vertex**. (3 points)

12. **Explain** in a clearly written paragraph what these **roots** and **vertex** mean in the context of the problem. Include details about **what happened** to Mr. Daily during the test jump, and **when it happened**. Include appropriate **units**. (8 points)

Unit 5: Friday

Basketball Problem Solving Activity

A basketball player makes a shot and the motion of the ball follows the function in this graph, where the x-axis represents horizontal position in feet and the y-axis represents vertical position in feet.



1. Determine a function that models the motion of the basketball and represent it in standard form. Show all work and be prepared to explain how you arrived at your answer.

2. Determine where the ball hits the court (floor). Show all work and be prepared to explain how you arrived at your answer.

Table 7. Modes of Instruction and Assessment

Unit 1: Evaluating and Graphing Quadratic Functions

3 Days	Monday	Tuesday	Wednesday
Modes of Instruction	Cannonball Applet Direct Instruction Investigation/Application Monitor Group Discussion Prompts Facilitate Whole Class Discussion	Calculator Display Direct Instruction Modeling/Application Call and Response Monitor Group Work Prompts Facilitate Whole Class Discussion	Calculator Display Investigation Modeling/Application Guided Practice Call and Response Monitor Group Work Prompts Facilitate Whole Class Discussion
Modes of Assessment	Group Discussion Worksheets Whole Class Discussion Homework Exercises	Call and Response Group Discussion Worksheets Whole Class Discussion Ticket Out the Door (TOTD) Homework Exercises	Call and Response Group Discussion Worksheets Group Quiz Homework Exercises Test Friday Week 2

Unit 2: Characteristics of Quadratic Graphs and Completing the Square

5 Days	Thursday	Friday	Monday	Tuesday
Modes of Instruction	Calculator Display Modeling/Application Monitor Group Discussion Prompts Facilitate Whole Class Discussion	Calculator Display Representation Guided Practice Call and Response Facilitate Whole Class Discussion	Calculator Display Model Guided Practice Call and Response Monitor Group Work Prompts	Applet Displays Exploration Model Monitor Group Discussion Prompts Whole Class Discussion

Modes of Assessment	Warm-up Slope Group Discussion Group Reports Worksheets Whole Class Discussion Homework Exercises	Call and Response Worksheets Whole Class Discussion Homework Exercises	Call and Response Group Work Worksheets Homework Exercises	Warm-up Linear Coefficients Worksheets Group Discussion Group Reports Whole Class Discussion Journal Entry Homework Exercises Test Friday

Unit 3: Comparing Quadratic and Linear Functions

8 Days	Wednesday	Thursday	Monday	Tuesday
Modes of Instruction	Applet Displays Exploration Vocabulary Notes Guiding Questions Monitor Group Discussion Prompts Facilitate Whole Class Discussion	Will It Make the Hoop? Applet Display Application Vocabulary Notes Guiding Questions Group Problem Solving Prompts Facilitate Whole Class Discussion	Applet Display Representation Guided Practice Call and Response Monitor Group Work Technology Support	Applet Display Representation Monitor Group Work Technology Support Guided Practice Algebra Steps Call and Response
Modes of Assessment	Group Discussion Worksheets Vocabulary in Own Words Whole Class Discussion Homework Exercises	Group Discussion Worksheets Vocabulary in Own Words Whole Class Discussion Homework Exercises	Vocab Practice Clickers Call and Response Group Work Worksheets TOTD Homework Exercises	Call and Response Group Work Worksheets TOTD Homework Exercises

Unit 3 Continued	Wednesday	Thursday	Friday	Monday
Modes of Instruction	Applet Display Representations Monitor Group Work Guided Practice Algebra Steps Calculator Display Call and Response	Direct Instruction Procedural Notes on Algebra Steps Guided Practice Call and Response Group Practice	Representations Procedural Connections Guided Practice Call and Response Monitor Group Work	Review Call and Response Independent Practice One-on-one Help
Modes of Assessment	Call and Response Group Work Worksheets / Graphs Group Reports TOTD Homework Exercises	Group Discussion Call and Response Worksheets Journal Entry Homework Exercises	Call and Response Group Work Worksheets TOTD Homework Exercises	Call and Response Monitor Independent Practice Questions Test Tuesday

Unit 4: Solving Quadratic Equations Graphically and Algebraically

6 Days	Wednesday	Thursday	Friday
Modes of Instruction	Direct Instruction Procedural Notes Call and Response Monitor Group Work Calculator Display Representations	Group Practice Monitor Group Work Procedural Representations Whiteboard Activity	Direct Instruction Procedural Notes Call and Response Monitor Group Work Calculator Display Representations
Modes of Assessment	Warm-up Factoring Call and Response Group Discussion Worksheets TOTD Homework Exercises	Group Discussion Worksheets Whiteboard Practice Homework Exercises	Warm-up Vertex Form Call and Response Group Discussion Worksheets Group Quiz Homework Exercises

Unit 4 Continued	Monday	Tuesday	Wednesday
Modes of Instruction	Stations Procedural Connections Monitor Group Work Whole Class Discussion	Matching Game Representations Monitor Group Work Whole Class Discussion Independent Practice One-on-one Help	Review Call and Response Independent Practice One-on-one Help
Modes of Assessment	Worksheets Group Work Whole Class Discussion Journal Entry Homework Exercises	Group Work Worksheets Whole Class Discussion Monitor Practice Problems Homework Exercises	Call and Response Worksheets Monitor Practice Problems Quiz

Unit 5: Using the Quadratic Formula to Solve Quadratic Equations

8 Days	Thursday	Friday	Monday
Modes of Instruction	Direct Instruction Procedural Notes Representations Call and Response Guided Practice Independent Practice One-on-one Help	Procedural Representations Monitor Group Work Groups Present Discuss Solution Steps	Direct Instruction Procedural Notes Call and Response Representations Monitor Group Work Go Over Problems
Modes of Assessment	Call and Response Worksheets Monitor Practice Problems TOTD Homework Exercises	Monitor Group Work Group Presentations Journal Entry Homework Exercises	Call and Response Group Work Worksheets TOTD Homework Exercises

Unit 5 Continued	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
Modes of Instruction	Procedural Representations Monitor Group Work Groups Present	Direct Instruction Procedural Notes Representations Call and Response Guided Practice Monitor Group Work Whole Class Discussion	Bungee Jump Activity Modeling/Application Procedural Connections Calculator Display Monitor Group Work Whole Class Discussion	Basketball Problem Solving Representations Procedural Connections Monitor Group Discussion Prompts Whole Class Discussion Review
Modes of Assessment	Warm-up Simplify Radicals Monitor Group Work Group Presentations Homework Exercises	Call and Response Worksheets Group Work Homework Exercises	Warm-up QF Group Work Worksheets Whole Class Discussion Homework Exercises	Group Work Worksheets Whole Class Discussion Homework Exercises Test Monday

Instructional Overview

Evaluating and Graphing Quadratic Functions

Students are introduced to quadratic functions in unit 1 by investigating trajectory motion. The Cannonball Applet shown on p. 81 illustrates for students that the motion of the cannonball is strictly vertical, whereas the graph of its position relative to time is parabolic. The speed of the cannonball icon falling demonstrates this relationship with a concrete visual alongside the graph. Students will be assigned different coefficients for height and initial velocity. The resulting functions will be evaluated with the applet giving meaning and feedback to students. Students will work in groups to construct T-tables and graphs by hand to model the problem. Groups will also discuss the shape of the graph and its meaning in the context of the problem. Groups will share their findings in a whole class discussion.

On day two, students extend tables and graphs by hand to the Table and Graph utilities of their graphing calculators. Students begin by modeling another cannonball function by hand. An overhead display is used to show students how to use the calculator to model the same function. Students compare the calculator Table to their constructed tables. Discussion will guide students to seeing the calculator Table as a dynamic version of the tables they created.

Both the Table and the Trace feature of the calculator will be used to evaluate functions. The Zoom feature of the calculator will be used to align the Graph to example functions and to interpret the domain and range of quadratics. The domain and range will also be discussed within the constraints of the cannonball problem.

Groups will practice modeling example functions. Then students will individually complete a Ticket Out The Door (TOTD) with one table and graph by hand and several evaluations using the calculator.

To conclude this introductory unit, students will extend the trajectory problem to consider the effects of aerodynamics and initial heights on the motion function. A whole class discussion will consider the difference between a falling sheet of plywood versus the falling cannonball. Guiding questions are given on p. 85. An example problem will be completed on the board. Then groups will model problems with their calculators in groups. Problems will involve varying initial heights and drag coefficients.

Students will practice evaluating and interpreting problems through this activity. Groups will share their findings in a whole class discussion. A group quiz will conclude the lesson. Investigating the meaning of coefficients in different contexts will prepare students for comparisons to be made in the following units.

Comparing Quadratic and Linear Functions

The second unit begins with a comparison of linear and quadratic functions through an investigation of average rate of change. The opening lesson is outlined on p. 86. The activity will take two days of instructional time. Students will model the falling book problem by using the quadratic regression utility of their graphing calculators. Students will already be familiar with using linear regression from a previous unit, so direct instruction will not be necessary. Once groups determine the quadratic function that models the problem, each group will be assigned two random points as well as the two end points in the table. Groups will calculate average velocities by using the slope formula.

Group results will be reported and compared on the board. Students will then have the opportunity to practice additional intervals to check for understanding. The class will discuss the difference between linear and quadratics as determined by slope. Local linearity will be illustrated on the overhead and discussed as well. On day two, students will further explore the relationship of the slope between intervals and the actual curve. To model this relationship, students will graph the function on their calculators and use the Draw feature to draw in the slope between intervals. Discussion will motivate the idea of instantaneous velocity in calculus.

Next students will extend the regression utility to distinguish between data sets that have linear or quadratic patterns. The sample data sets to be analyzed are displayed on p. 88 along with instructions for enabling the R-value display of the calculator. The regression coefficient, or R-value, tells how closely the function fits the data. Possible values range from 0 to 1, with 1 being a perfect fit.

In the concluding activity, students will use technology applets to compare meanings of coefficients of linear and quadratic functions. These applets are displayed on p. 89. One applet shows the effects of changing the coefficients in the graph of a linear function. The other applet shows the effects on graphs of quadratic functions in the standard form. Both applets use sliders which students can move to adjust a coefficient and watch the graph move.

Special observations that will be prompted are the relationship between the constant coefficients and the case of a quadratic with a leading coefficient of zero. Worksheets will prompt students to describe a variety of changes and comparisons. The teacher will monitor group discussions and give additional prompts as needed. The lesson will conclude with a journal entry comparing the meanings of the coefficients in the two families.

Characteristics of Quadratic Graphs and Completing the Square

Now that students have had some practice with the coefficients of the standard form, they will be introduced to the vertex form of the quadratic function. The vertex form will allow students to model situations more easily and to explore the relationships between the graphical and algebraic representations of functions more succinctly. They will begin by comparing graphical translations using the standard and vertex forms. As with the comparison of linear and quadratics in the previous unit, separate applets will be used to explore each form. These applets are illustrated on p. 90.

The standard form applet starts with coefficients equal to one and challenges students to move the parabola until it contains different points fixed on the graph. This is fairly difficult. Students will see the advantages of the vertex form in the following applet. Horizontal shifts are much easier to make, since the vertex is embedded into the variables they are changing. Students will take notes on the vocabulary terms: vertex, maximum, minimum, and y-intercept. During the applet activity, they will describe verbally and in writing how these features change as they make changes to the variables. Groups will share their observations in a whole class discussion.

Next students will apply translations using the vertex form in a fun application. The Will It Make the Hoop applet illustrated on p. 91 has students determine whether the basketball shot photographed in the applet will make it into the hoop. Students do this by constructing a quadratic function that contains the basketball player's hands and the hoop. Students can see whether the ball is following the appropriate path of their graph.

Students will take notes on more vocabulary terms: axis of symmetry, roots, domain, and range. Groups will discuss the meaning of these terms as they use the applet. Groups will then do a problem solving activity where they must determine if the ball will go in, given just the picture and the function modeling the shot. Groups should determine that the point of the location of the hoop needs to be a point in the function. Then they must evaluate the function at that point. The whole class will discuss solution methods to close the lesson.

The next three lessons transition students into translating from graphical to algebraic representations of quadratics. These lessons reinforce the vocabulary and how it translates along with these representations. The two applet activities illustrated on p. 92-93 scaffold this transition beautifully for students. Guiding questions are also given. The applets even provide hints, feedback, and solutions.

During the third lesson, determining the coefficient a can either be treated as a problem solving activity in which groups present, or it can be taught with direct instruction. In either case, it is imperative to include guided and independent practice. TOTD problems should be used to keep close track of student progress with algebra procedures. Students will also practice graphing by hand and with calculators from the vertex form.

During the next two lessons, students learn how to complete the square to determine the vertex form. This requires direct instruction of the required algebra steps. Guided practice with call and response is used and then students practice problems in groups. To reinforce these steps, students will explain them verbally and in writing as a journal entry.

During the second lesson in this sequence, students will use distribution to convert vertex form into standard form. They will practice moving back and forth between these forms. Ultimately, groups will practice problems that require producing the vertex form from a given graph, and then converting it into standard form. Students will be given an additional review day to practice problems with one-on-one help. Practice problem sets for this segment are given on p. 94-97.

Solving Quadratic Equations Graphically and Algebraically

In this unit, students will be solving quadratic equations and finding the roots of functions. The algebraic methods will be factoring and completing the square. Students will also find solutions graphically with graphing calculators. Calculators will be used to verify solutions to most exercises throughout the unit. Students will already know how to factor from the polynomials unit. They will also know how to complete the square from the previous unit. This unit will allow continued practice of these techniques with the additional steps of finding the solutions.

The first two days are spent on solving equations by factoring. Students begin with a factoring warm-up. The first lesson is taught with direct instruction, followed by guided practice with call and response. Then groups practice problems, and they are gone over at the end of class. Groups continue practice problems during the second day of the sequence.

Student learning is assessed on day one with a TOTD and on day two with a whiteboards activity. In the whiteboards activity, the problems are presented on the board. Students solve them individually, at their own pace, on mini dry erase boards. Once they have a solution, they raise the board. The teacher tells the student either to progress to the next problem or retry. Problem sets for this sequence are given on p. 98-99.

On the third day of the unit, completing the square is used to solve equations. This is taught in the same fashion as the previous sequence, culminating in group practice. Learning is assessed with a group quiz. During the next two days of the unit, the different solution techniques are practiced and compared.

The first of these lessons is done in a stations activity. Students move in groups to different stations where problem sets are categorized by type: factoring, completing the square, graphical. The teacher monitors the station work and takes note of common errors. These common errors are addressed in a whole class discussion, along with a comparison of the different techniques. See p. 101 for the problem sets for each station.

The second of these mixed-technique lessons is done as a matching game. See p. 102. Groups are given a group of cards on which each card has three representations of the same function: standard, factored, and vertex form. They are also given a group of cards, each having the graph of a quadratic function. The task is to match each graph card to its symbolic card, using the highlighting features of the different forms of each function. Some of the cards have missing information, which the groups must determine after matching the cards. The game is followed by a whole class discussion comparing student strategies. The remainder of the class is spent on independent practice with one-on-one help. The last day is spent on review with one-on-one help and a quiz.

Using the Quadratic Formula to Solve Quadratic Equations

In the concluding unit, students learn to solve quadratic equations using the Quadratic Formula. This is the most global of the solution techniques. It is applied to problems with non-integral coefficients and complex solutions, to which the other techniques do not apply. Students begin with a two day sequence of problems with integral solutions.

The first day consists of direct instruction, guided practice, and independent practice. Special attention is placed first on setting up the formula. Students must identify which coefficients to place correctly in the formula. Practice problems must include examples in nonstandard order and others with zero coefficients. Call and response is used to gauge student understanding. Then the arithmetic is practiced to reach solutions. Calculators are used to verify each solution graphically. Students practice problems independently with one-on-one help available. A TOTD is completed for assessment.

In the second lesson, groups practice additional problems with integral solutions. They also practice explaining each step in the solution process. Then groups are assigned problems to present to the whole class with an explanation of the solution steps. At the end of class, students explain the steps in writing as a journal entry.

The third lesson introduces problems with non-integral solutions. The lesson begins with a warm-up and review of simplifying radicals and rounding to decimal forms to express solutions. Guided practice is followed by group practice. After going over the problems, learning is assessed with a TOTD. The forth lesson provides additional practice of problems with non-integral solutions. The lesson is conducted in the same fashion as day two, with group practice and presentations.

The next lesson begins with a discussion of factorability in the real world and the possible nature of solutions to problems. Direct instruction and guided practice is used to teach the use of the discriminant for determining the nature and number of solutions. The calculator display on the overhead is used to show functions that will not have real solutions. Groups practice analyzing discriminants and solving mixed problems. These problem sets are provided on p. 106-111.

The following lesson begins with a discussion of when roots are rejected in the context of a real world problem. The Cannonball Problem is used as an example. Next, students apply all the solution techniques they have learned in the Bungee Jump Activity illustrated on p. 112. In this activity, students use regression to model data. They graph the resulting function by hand and identify the key features of the graph. The students use all of the techniques to verify the solutions. Then they interpret the solutions in the context of the problem. This assessment measures most of the major standards in the quadratics curriculum.

The final lesson includes a Basketball Problem Solving Activity illustrated on p. 115. In this problem, the motion of a basketball during a lay-up is given in a graph. Groups must find the algebraic form of this function and solve it to determine where the ball hits the court. This problem is left open-ended, so that groups can discuss their different solution techniques and why they chose them. This is done as a whole class discussion. The remainder of the class is used for review and one-on-one help.

Chapter Four: Conclusion

Development of this project shows positive steps toward effective implementation of the Common Core learning standards. A teacher might be initially overwhelmed with the task of designing curriculum from the new Algebra standards. The quadratics curriculum is also a particularly challenging area to practice implementation. The research indicates that students have many difficulties with quadratics (Zaslavsky, 1997; Bossé and Nandakumar, 2005; Eraslan, 2005; Hutchings and McCuaig, 2008; Zakaria, 2010). Implementing even more rigorous standards takes much consideration to meet both the increasing demands of the standards while meeting the student needs that already exist and may be compounded even more by any changes.

The systematic approach that emerged in this project can be used to develop curriculum that meets both of these objectives. The approach can also be generalized to other classes and topics in mathematics. The approach begins with the students and how they conceive the topic of study. The research highlights misconceptions students have (Zaslavsky, 1997; Eraslan, 2005; Aspinwall and Eraslan, 2007; Zakaria, 2010; Kotsopoulos, 2007). Alleviating each of the major misconceptions provides focal points for organizing the standards.

Breaking some of the standards into doable objectives reveals that a single standard might span weeks, as is the case with the prime example highlighted in this paper: students must be able to apply all of the solution techniques and be able to choose one that is appropriate for a given problem situation. Here, students must understand all of the techniques procedurally and conceptually. Hence, students must be able to make sense of the abstract symbols they are working with. How might students attach meaning to these symbols? Can they trace abstract concepts and procedures back to something concrete? Do they understand the situations that motivate these procedures and concepts?

Real world problems, technology, and graphical representations empower students in making meaning of the algebraic representations and procedures that instruction has traditionally focused on. Even the pathway recommended by the Common Core standards document begins by jumping straight into these abstract procedures. The literature on quadratics urges teachers to look into contexts and technologies that can help students make sense of quadratic relationships before implementing algebraic procedures (Mittag and Taylor, 2001; Daher, 2009; Kotsopoulos, 2007).

This project should help the reader become more familiar and comfortable with the technology that is available. The power of completing the square and the vertex form of quadratic functions is illustrated by the technology applets included in this project. This technology enhances a teacher's perspective on instruction of quadratics. The vertex form applets allow a conceptual transition from concrete representations into the symbolic procedures that accompany them. Integrating this technology allows a beautiful instructional sequence to emerge. The reader is urged to experiment with these applets. Verbal descriptions cannot quite convey what actual experience with the applets can.

The graphing calculator is another powerful medium for illuminating the meaning of quadratic functions. Even students who have difficulties with prerequisite arithmetic can model any problem, evaluate functions, represent functions graphically, and analyze the important features of functions. In light of the research, it is very important to have students make the connection between what the calculator is doing and the constructions of tables, graphs, and algebraic procedures that students perform by hand (Mittag and Taylor, 2001). It is imperative that the calculator features are taught as extensions of these basic constructions and not merely as short cuts for getting an answer.

This raises another important finding in doing this project. The research indicates that student misconceptions revolve around basic understandings (Zaslavsky, 1997; Eraslan, 2005; Aspinwall and Eraslan, 2007). It is important to consider the basic characteristics of quadratic functions and the basic concepts of functions in general as key elements of curriculum design. Difficulty with basic concepts of functions and prerequisite arithmetic also indicates the need for vertical planning across mathematics courses in order for implementation of the Common Core standards to be successful.

Developing the curriculum for this project was quite laborious and time consuming. For practical purposes, curriculum design would benefit greatly from collaborative efforts. Forthcoming resources supporting the Common Core standards in mathematics should help teachers to develop curriculum more efficiently, and to tailor instruction according to student needs. Professional development, especially in utilizing the best technology resources available, will also be a detrimental aspect of achieving success.

Like the Common Core standards themselves, the unit plans in this project should be treated as working and evolving documents. A teacher should make further revisions to this curriculum design based on available resources and feedback from students. Teachers should extend the type of research that was incorporated into this design. When implementing these units, a teacher can reflect on lessons with notes on what worked and did not work for students and any common errors students have made. These notes can then be organized as demonstrated in the preliminary analysis of the initial instructional trajectory used in this project. This will provide insight into further adjustments that will improve the design.

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