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LOCALIZATION AND COMPACTNESS OF OPERATORS ON FOCK SPACES

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ABSTRACT. For $0 < p \leq \infty$, let F^p_φ be the Fock space induced by a weight function φ satisfying $dd^c\varphi \simeq \omega_0$. In this paper, given $p \in (0,1]$ we introduce the concept of weakly localized operators on F^p_φ, we characterize the compact operators in the algebra generated by weakly localized operators. As an application, for $0 < p < \infty$ we prove that an operator T in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on F^p_φ if and only if its Berezin transform satisfies certain vanishing property at ∞. In the classical Fock space, we extend the Axler-Zheng condition on linear operators T, which ensures T is compact on F^p_α for all possible $0 < p < \infty$.

1. Introduction

Let $H(\mathbb{C}^n)$ be the collection of all entire functions on $\mathbb{C}^n$, and let $\omega_0 = dd^c|z|^2$ be the Euclidean Kähler form on $\mathbb{C}^n$, where $d^c = \frac{\sqrt{-1}}{4}(\overline{\partial} - \partial)$. Set $B(z,r)$ to be the Euclidean ball in $\mathbb{C}^n$ with center z and radius r, and $B(z,r)^c = \mathbb{C}^n \backslash B(z,r)$. Throughout the paper, we assume that $\varphi \in C^2(\mathbb{C}^n)$ is real-valued and there are two positive numbers M_1, M_2 such that

$$M_1\omega_0 \leq dd^c\varphi \leq M_2\omega_0 \tag{1.1}$$

in the sense of currents. The expression (1.1) will be denoted as $dd^c\varphi \simeq \omega_0$. Given $0 < p < \infty$ and a positive Borel measure μ on $\mathbb{C}^n$, let $L^p_\varphi(\mu)$ be the space defined by

$$L^p_\varphi(\mu) = \left\{ f \text{ is } \mu\text{-measurable on } \mathbb{C}^n : f(\cdot)e^{-\varphi(\cdot)} \in L^p(\mathbb{C}^n, d\mu) \right\}.$$

When $d\mu = dV$, the Lebesgue measure on $\mathbb{C}^n$, we write L^p_φ for $L^p_\varphi(\mu)$ and set

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{C}^n} \left| f(z)e^{-\varphi(z)} \right|^p dV(z) \right)^{\frac{1}{p}}.$$

For $0 < p < \infty$ the Fock space F^p_φ is defined as $F^p_\varphi = L^p_\varphi \cap H(\mathbb{C}^n)$, and

$$F^\infty_\varphi = \left\{ f \in H(\mathbb{C}^n) : \|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}^n} |f(z)|\, e^{-\varphi(z)} < \infty \right\}.$$

F^p_φ is a Banach space with norm $\|\cdot\|_{p,\varphi}$ when $1 \leq p \leq \infty$ and F^p_φ is a Fréchet space with distance $\rho(f,g) = \|f-g\|^p_{p,\varphi}$ if $0 < p < 1$. The typical model of φ is $\varphi(z) = \frac{\alpha}{2}|z|^2$ with $\alpha > 0$, which induces the classical Fock space. For this particular special weight φ, F^p_φ and $\|\cdot\|_{p,\varphi}$ will be written as F^p_α and $\|\cdot\|_{p,\alpha}$, respectively. The space F^p_α has been studied by

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many authors, see [2, 5, 7, 18–21] and the references therein. Another special case is with $\varphi(z) = \frac{\alpha}{2}|z|^2 - \frac{m}{2}\ln(A + |z|^2)$ with suitable $A > 0$, and then F^p_φ is the Fock-Sobolev space $F^{p,m}_\alpha$ studied in [3, 4].

It is well-known that F^2_φ is a Hilbert space with inner product

$$\langle f, g\rangle_{F^2_\varphi} = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-2\varphi(z)}dV(z).$$

Given $z, w \in \mathbb{C}^n$, the reproducing kernel of F^2_φ will be denoted by $K_z(w) = K(w, z)$. We write $k_z = \frac{K_z}{\|K_z\|_{2,\varphi}}$ to denote the normalized reproducing kernel. Given some bounded linear operator T on F^p_φ, the Berezin transform of T is well defined as

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle_{F^2_\varphi},$$

since $Tk_z \in F^p_\varphi \subset F^\infty_\varphi$ and $k_z \in F^1_\varphi$. Set P to be the projection from L^2_φ to F^2_φ, that is

$$Pf(z) = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-2\varphi(w)}dV(w) \quad \text{for } f \in L^2_\varphi.$$

For a complex Borel measure μ on $\mathbb{C}^n$ and $f \in F^p_\varphi$, we define the Toeplitz operator T_μ to be

$$T_\mu f(z) = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-2\varphi(w)}d\mu(w).$$

If $d\mu = gdV$, for short, we will use T_g to stand for the induced Toeplitz operator and will use that $\widetilde{g} = \widetilde{T}_g$.

In the case of Fock spaces F^2_α, fixed g bounded on $\mathbb{C}^n$, $|\langle T_g k_z, k_w\rangle|$ as a function of (z, w) decays very fast off the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$, see [20, Proposition 4.1]. From this point of view, Xia and Zheng in [20] introduced the notion of "sufficiently localized" operators on F^2_α which include the algebra generated by Toeplitz operators with bounded symbols, and they proved that, if T is in the C^*-algebra generated by the class of sufficiently localized operators, T is compact on F^2_α if and only if its Berezin transform tends to zero when z goes to infinity. In [10], Isralowitz extended [20] to the generalized Fock space F^2_φ with $dd^c\varphi \simeq \omega_0$. Isralowitz, Mitkovski and the third author extended Xia and Zheng's idea further in [11] to what they called "weakly localized" operators on F^p_φ with $1 < p < \infty$. They showed that, if T is in the C^*-algebra generated by the class of weakly localized operators, T is compact on F^p_φ if and only if its Berezin transform shares certain vanishing property near infinity. We would like to emphasize that the prior results in the area, for example [1, 2, 5, 8–10, 14, 15, 17, 19–22], depend strongly on two points. The first is the use of Weyl unitary operators induced by holomorphic self mappings of the domain; and the second is the restriction on the range of the exponent p, for example $p = 2$ or $1 < p < \infty$, so that Banach space techniques are applicable. But on F^p_φ with $0 < p < 1$ and $dd^c\varphi \simeq \omega_0$ these two points are not available.

The main purpose of this work is, on F^p_φ with $0 < p < 1$ and $dd^c\varphi \simeq \omega_0$, to study the so called "weakly localized" operators WL^φ_p and to characterize those compact operators $T \in \mathrm{WL}^\varphi_p$. The paper is divided into four sections. In Section 2, we introduce the concept of weakly localized operators WL^φ_p for $0 < p \le 1$, we will characterize the compact operators in WL^φ_p, and furthermore give a quantity equivalent to the essential norm of an operator in WL^φ_p. Section 3 is devoted to the compactness of Toeplitz operators induced by

BMO symbols acting on F^p_φ for all $0 < p < \infty$, our theorem shows an operator T in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on F^p_φ if and only if its Berezin transform satisfies a certain vanishing property at ∞ (more precisely, $\lim_{z\to\infty} \widetilde{T}(z) = 0$ when $\varphi(z) = \frac{\alpha}{2}|z|^2$). In Section 5, we extend Axler-Zheng's condition on linear operators T, which insures T are bounded (or compact) on F^p_α for all possible $0 < p < \infty$. In the final section, we provide some remarks and point to some open problems.

In what follows, C will denote a positive constant whose value may change from one occasion to another but does not depend on the functions or operators in consideration. For two positive quantities A and B, the expression $A \simeq B$ means there is some $C > 0$ such that $\frac{1}{C}B \le A \le CB$.

2. The operator class WL^φ_p with $0 < p \le 1$

As a generalization of the "strongly localized" operators of Xia and Zheng in [20], Isralowitz, Mitkovski and the third author introduced "weakly localized" operators on F^p_φ with $1 < p < \infty$, see [11]. In this section, we first give the definition of weakly localized operators on F^p_φ when $0 < p \le 1$. We use $\mathcal{D}$ to stand for the linear span of all normalized reproducing kernel functions $k_z(\cdot)$. It is obvious that $\mathcal{D}$ is dense in F^p_φ. As in [11], we will assume that the domain of every linear operator T appearing in this paper contains $\mathcal{D}$, and that the function $z \mapsto TK_z$ is conjugate holomorphic. We also assume the range of T is in F^∞_φ. Then $\langle Tk_z, k_w\rangle_{F^2_\varphi}$ can make sense.

Definition 2.1. *Let $0 < p < \infty$, set $s = \min\{1, p\}$. A linear operator T from $\mathcal{D}$ to F^∞_φ is called weakly localized for F^p_φ if*

$$\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^s dV(w) < \infty, \qquad \sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\langle k_z, Tk_w\rangle_{F^2_\varphi}\right|^s dV(w) < \infty; \tag{2.1}$$

and

$$\lim_{r\to\infty}\sup_{z\in\mathbb{C}^n}\int_{B(z,r)^c}\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^s dV(w) = 0, \tag{2.2}$$

$$\lim_{r\to\infty}\sup_{z\in\mathbb{C}^n}\int_{B(z,r)^c}\left|\langle k_z, Tk_w\rangle_{F^2_\varphi}\right|^s dV(w) = 0. \tag{2.3}$$

The algebra generated by weakly localized operators for F^p_φ will be denoted by WL^φ_p. For $\varphi(z) = \frac{\alpha}{2}|z|^2$, we write $WL^\varphi_p = WL^\alpha_p$ for convenience.

When $1 \le p < \infty$ $\mathrm{WL}^\varphi_p = \mathrm{WL}^\varphi_1$ by definition, and then Definition 2.1 was first introduced in [11]. Let $\mathcal{T}^\varphi_p$ denote the Toeplitz algebra on F^φ_p generated by L^∞ symbols, and let $\mathcal{K}(F^p_\varphi)$ be the set of all compact operators on F^φ_p. We use $\|T\|_{e,F^p_\varphi}$ to stand for the essential norm of a given operator T on F^p_φ

$$\|T\|_{e,F^p_\varphi} = \inf\left\{\|T - A\|_{F^p_\varphi\to F^p_\varphi} : A \in \mathcal{K}(F^p_\varphi)\right\}.$$

The purpose of this section is to characterize compact operators in WL^φ_p, $0 < p \le 1$. To carry out our analysis, we need some preliminary facts.

Lemma 2.2 ([16])**.** *Given φ as in the introduction, the Bergman kernel $K(\cdot,\cdot)$ for F^2_φ satisfies the following estimates:*

(1) *There exists C and $\theta>0$ such that*

$$|K(z,w)|e^{-\varphi(z)}e^{-\varphi(w)} \le Ce^{-\theta|z-w|} \quad \text{for } z,w\in\mathbb{C}^n.$$

(2) *There exists some $r>0$ such that*

$$|K(z,w)|\, e^{-\varphi(z)}e^{-\varphi(w)} \simeq 1 \quad \text{whenever } w\in B\,(z,r) \text{ and } z\in\mathbb{C}^n.$$

(3) *For $0<p\le\infty$ fixed,*

$$\|K(\cdot,z)\|_{p,\varphi} \simeq e^{\varphi(z)} \simeq \sqrt{K(z,z)}, \quad z\in\mathbb{C}^n.$$

Lemma 2.3 ([8])**.** *Suppose $0<p<\infty$ and $r>0$. Then there exists C such that for $f\in H(\mathbb{C}^n)$ and $z\in\mathbb{C}^n$, we have*

$$\left|f(z)e^{-\varphi(z)}\right|^p \le C\int_{B(z,r)} \left|f(w)e^{-\varphi(w)}\right|^p dV(w)$$

and

$$\int_{\mathbb{C}^n} \left|f(z)e^{-\varphi(z)}\right|^p d\mu(z) \le C\int_{\mathbb{C}^n} \left|f(z)e^{-\varphi(z)}\right|^p \widehat{\mu}_r(z)dV(z)$$

where μ is some given positive Borel measure and $\widehat{\mu}_r(\cdot)=\frac{\mu(B(\cdot,r))}{V(B(\cdot,r))}$.

Let $d(\cdot,\cdot)$ be the Euclidean distance on $\mathbb{C}^n$. Given some domain $\Omega\subseteq\mathbb{C}^n$, write $\Omega^+=\{z\in\mathbb{C}^n : d(z,\Omega)<1\}$, and Ω^+ is again a domain. Set $\mathcal{L}=\left\{a+bi : a,b\in\frac{1}{4}\mathbb{Z}^n\right\}$, $\mathcal{L}$ is countable so that we may write $\mathcal{L}=\{z_1,z_2,\cdots,z_j,\cdots\}$. It is obvious that $\mathcal{L}$ forms a 1/4-lattice in $\mathbb{C}^n$ (see [21] for the definition). For $E\subset\mathbb{C}^n$, let χ_E be the characteristic function of E. We have some absolute constant $N>0$ such that

$$\sum_{z_j\in\mathcal{L}} \chi_{B(z_j,\frac{1}{2})}(w) \le N \qquad \text{for } w\in\mathbb{C}^n. \tag{2.4}$$

Lemma 2.4. *For $0<p\le1$ there is some constant C (depending only on p and n) such that for any domain $\Omega\subset\mathbb{C}^n$ and $f\in H(\mathbb{C}^n)$,*

$$\left(\int_\Omega \left|f(w)e^{-\varphi(w)}\right| dV(w)\right)^p \le C\int_{\Omega^+} \left|f(w)e^{-\varphi(w)}\right|^p dV(w).$$

Proof. It is trivial to see that $(u+v)^p \le u^p + v^p$ for positive u, v and $0 < p \le 1$. Applying Lemma 2.3 and (2.4), for $f \in H(\mathbb{C}^n)$ we have

$$\begin{aligned}\left(\int_\Omega \left|f(w)e^{-\varphi(w)}\right| dV(w)\right)^p &\le \left(\sum_{z_j\in\mathcal{L}} \int_{\Omega\cap B(z_j,1/4)} \left|f(w)e^{-\varphi(w)}\right| dV(w)\right)^p \\ &\le C \sum_{z_j\in\mathcal{L}, d(z_j,\Omega)<1/4} \max_{|w-z_j|\le 1/4} \left|f(w)e^{-\varphi(w)}\right|^p \\ &\le C \sum_{z_j\in\mathcal{L}, d(z_j,\Omega)<1/4} \int_{|w-z_j|<1/2} \left|f(w)e^{-\varphi(w)}\right|^p dV(w) \\ &\le C \int_{\Omega^+} \sum_{z_j\in\mathcal{L}, d(z_j,\Omega)<1/4} \chi_{B(z_j,1/2)}(w) \left|f(w)e^{-\varphi(w)}\right|^p dV(w) \\ &\le C \int_{\Omega^+} \left|f(w)e^{-\varphi(w)}\right|^p dV(w).\end{aligned}$$

It is easy to check that the constants C above depend only on p and n. □

With the assumption that $w \mapsto TK_w$ is conjugate holomorphic, we know $\langle TK_w, K_z\rangle$ is conjugate holomorphic with w. And also, $\langle TK_z, K_w\rangle_{F^2_\varphi}$ is holomorphic with w. For $0 < p < 1$, apply Lemma 2.4 to get

$$\begin{aligned}\int_\Omega |\langle Tk_z, k_w\rangle_{F^2_\varphi}| dV(w) &= \int_\Omega |\langle Tk_z, K_w\rangle_{F^2_\varphi} e^{-\varphi(w)}| dV(w) \\ &\le C \left(\int_{\Omega^+} |\langle Tk_z, K_w\rangle_{F^2_\varphi} e^{-\varphi(w)}|^p dV(w)\right)^{\frac{1}{p}} \\ &= C \left(\int_{\Omega^+} |\langle Tk_z, k_w\rangle_{F^2_\varphi}|^p dV(w)\right)^{\frac{1}{p}}.\end{aligned}$$

And, similarly

$$\int_\Omega |\langle k_z, Tk_w\rangle_{F^2_\varphi}| dV(w) \le C \left(\int_{\Omega^+} |\langle k_z, Tk_w\rangle_{F^2_\varphi}|^p dV(w)\right)^{\frac{1}{p}}.$$

These two inequalities tell us $\mathrm{WL}^\varphi_p \subset \mathrm{WL}^\varphi_1$ with $0 < p \le 1$. With the relation $\langle TK_z, K_w\rangle_{F^2_\varphi} = \langle K_z, T^*K_w\rangle_{F^2_\varphi}$, we know T^* is well defined on $\mathcal{D}$. In [11] it is pointed out that WL^φ_1 is contained in the set of all bounded operators on F^p_φ for all $1 \le p < \infty$. When $0 < p < 1$, we have the two following lemmas.

Lemma 2.5. *For $0 < p \le 1$, if $T \in WL^\varphi_p$ then T is bounded on F^p_φ.*

Proof. Set

$$G(T) = \max\left\{\sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w), \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} \left|\langle k_z, Tk_w\rangle_{F^2_\varphi}\right|^p dV(w)\right\}.$$

Then, $G(T) < \infty$. For $f \in \mathcal{D}$, we have

$$Tf(z) = \langle Tf, K_z\rangle_{F^2_\varphi} = \langle f, T^*K_z\rangle_{F^2_\varphi} = \int_{\mathbb{C}^n} f(w)\langle K_w, T^*K_z\rangle_{F^2_\varphi} e^{-2\varphi(w)} dV(w). \tag{2.5}$$

Applying (2.5), Lemma 2.2 (estimate (3)) and Lemma 2.4 with $\Omega = \mathbb{C}^n$ to have

$$\begin{aligned}\left|Tf(z)e^{-\varphi(z)}\right|^p &\le C\left(\int_{\mathbb{C}^n}\left|f(w)\langle K_w, T^*k_z\rangle_{F^2_\varphi}\right|e^{-2\varphi(w)}dV(w)\right)^p\\ &\le C\left(\int_{\mathbb{C}^n}|f(w)\langle Tk_w, k_z\rangle_{F^2_\varphi}|e^{-\varphi(w)}dV(w)\right)^p\\ &\le C\int_{\mathbb{C}^n}\left|f(w)\langle Tk_w, k_z\rangle_{F^2_\varphi}e^{-\varphi(w)}\right|^p dV(w).\end{aligned}$$

Now, integrate both sides over $\mathbb{C}^n$, and apply Fubini's Theorem to obtain

$$\|Tf\|^p_{p,\varphi} \le C\int_{\mathbb{C}^n}\left|f(w)e^{-\varphi(w)}\right|^p dV(w)\int_{\mathbb{C}^n}\left|\langle Tk_w, k_z\rangle_{F^2_\varphi}\right|^p dV(z) = CG(T)\|f\|^p_{p,\varphi}.$$

When $0 < p < 1$, although F^p_φ is only a Fréchet space, with $P|_{F^p_\varphi} = \mathrm{Id}$ we know that $\mathcal{D}$ is dense in F^p_φ. Therefore, T is bounded on F^p_φ. □

Isralowitz, Mitkovski and the third author demonstrated in [11] that WL^φ_1 is a $*$-algebra. Lemma 2.6 tells us WL^φ_p is closed under the F^p_φ operator norm while $0 < p \le 1$.

Lemma 2.6. *For $0 < p \le 1$, WL^φ_p is closed under the operator norm on F^p_φ.*

Proof. We only need to prove $\overline{\mathrm{WL}}^\varphi_p = \mathrm{WL}^\varphi_p$. For $T \in \overline{\mathrm{WL}}^\varphi_p$ we show

$$\lim_{r\to\infty}\sup_{z\in\mathbb{C}^n}\int_{B(z,r)^c}\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w) = 0.$$

In fact, for any $\varepsilon > 0$ we have some $A_\varepsilon \in \mathrm{WL}^\varphi_p$ such that $\|T - A_\varepsilon\|_{F^p_\varphi\to F^p_\varphi} < \varepsilon$. For this A_ε we have some r such that

$$\sup_{z\in\mathbb{C}^n}\int_{B(z,r)^c}\left|\langle A_\varepsilon k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w) < \varepsilon.$$

This implies

$$\begin{aligned}&\int_{B(z,r)^c}\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &\le \int_{B(z,r)^c}\left|\langle (T - A_\varepsilon)k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w) + \int_{B(z,r)^c}\left|\langle A_\varepsilon k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &\le \int_{\mathbb{C}^n}\left|\langle (T - A_\varepsilon)k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w) + \int_{B(z,r)^c}\left|\langle A_\varepsilon k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &= \|(T - A_\varepsilon)\,k_z\|^p_{p,\varphi} + \int_{B(z,r)^c}\left|\langle A_\varepsilon k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &\le \|T - A_\varepsilon\|^p_{F^p_\varphi\to F^p_\varphi}\|k_z\|^p_{p,\varphi} + \int_{B(z,r)^c}\left|\langle A_\varepsilon k_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &< C\varepsilon,\end{aligned}$$

where the constant C does not depend on ε. □

To characterize the compactness of those $T \in \mathrm{WL}^\varphi_p$ in the case $0 < p \le 1$, we will borrow ideas from [17] and will be approximating a given operator $T \in \mathrm{WL}^\varphi_p$ by infinite sums of well localized pieces. To get this approximation we need the following covering lemma from [11]. See also [2, Lemma 3.1].

Lemma 2.7. *There exists a positive integer N such that for each $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}_{j=1}^{\infty}$ of $\mathbb{C}^n$ by disjoint Borel sets satisfying:*

(1) *every point of $\mathbb{C}^n$ belongs to at most N of the sets $G_j = \{z : d(z, F_j) \le r\}$;*
(2) *$\operatorname{diam} F_j \le 2r$ for every j.*

Notice that, if $r > 1$, we have some absolute constant $N > 0$ such that

$$\sum_{j=1}^{\infty} \chi_{F_j^+}(w) \le \sum_{j=1}^{\infty} \chi_{G_j}(w) \le \sum_{j=1}^{\infty} \chi_{G_j^+}(w) \le N \quad \forall w \in \mathbb{C}^n. \tag{2.6}$$

This covering $\mathcal{F}_r$ can also be chosen in a simple way. For example, let $\{a_j\}$ be an enumeration of the lattice $\frac{2r}{\sqrt{n}}\mathbb{Z}^{2n}$. And take F_j to be the cube with centers a_j, side-length $\frac{2r}{\sqrt{n}}$ and half of the boundary so that $\cup_{j=1}^{\infty} F_j = \mathbb{C}^n$, $F_j \cap F_k = \emptyset$ if $j \ne k$.

Proposition 2.8. *Let $0 < p \le 1$ and $T \in WL_p^{\varphi}$. Then for every $\varepsilon > 0$, there is some $r > 0$ sufficiently large such that, for the covering $\{F_j\}_{j=1}^{\infty}$ and $\{G_j\}_{j=1}^{\infty}$ (associated to r) from Lemma 2.7, we have*

$$\left\| T - P\left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} T P M_{\chi_{G_j}}\right) \right\|_{F_\varphi^p \to F_\varphi^p} < \varepsilon. \tag{2.7}$$

Proof. Let $T \in \mathrm{WL}_p^{\varphi}$ be given. For $\varepsilon > 0$, we have some $r > 0$ sufficiently large (we may assume $r > 10$) such that

$$\int_{B(z,r-1)^c} |\langle Tk_z, k_w\rangle_{F_\varphi^2}|^p dV(w) < \varepsilon \quad \text{and} \quad \int_{B(z,r-1)^c} |\langle k_z, Tk_w\rangle_{F_\varphi^2}|^p dV(w) < \varepsilon.$$

Take $\{F_j\}_{j=1}^{\infty}$ and $\{G_j\}_{j=1}^{\infty}$ to be as in Lemma 2.7 with r. For $w \in F_j^+$ and $u \in G_j^c$ we have $|u - w| > r - 1$, then $u \in B(w, r-1)^c$. That is $G_j^c \subset B(w, r-1)^c$ whenever $w \in F_j^+$. Hence, for $w \in F_j^+$,

$$\begin{aligned}
\left|\left(TPM_{\chi_{G_j^c}} f\right)(w)\right| &= \left|\langle PM_{\chi_{G_j^c}} f, T^* K_w\rangle_{F_\varphi^2}\right| \\
&= \left|\langle M_{\chi_{G_j^c}} f, T^* K_w\rangle_{F_\varphi^2}\right| \\
&\le \int_{G_j^c} |f(u)| \left|\langle K_u, T^* K_w\rangle_{F_\varphi^2}\right| e^{-2\varphi(u)} dV(u) \\
&\le \int_{B(w,r-1)^c} |f(u)| \left|\langle K_u, T^* K_w\rangle_{F_\varphi^2}\right| e^{-2\varphi(u)} dV(u).
\end{aligned}$$

Set $S = TP - \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. Then

$$
\begin{aligned}
&|PSf(z)|^p \\
&\le \left(\int_{\mathbb{C}^n} \left| Sf(w) K(z,w) e^{-2\varphi(w)} \right| dV(w) \right)^p \\
&= \left(\int_{\mathbb{C}^n} \left| \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j^c}} f(w) \right| |K(z,w)| e^{-2\varphi(w)} dV(w) \right)^p \\
&= \left(\sum_{j=1}^{\infty} \int_{\mathbb{C}^n} \left| M_{\chi_{F_j}} TPM_{\chi_{G_j^c}} f(w) \right| |K(z,w)| e^{-2\varphi(w)} dV(w) \right)^p \\
&\le \sum_{j=1}^{\infty} \left(\int_{F_j} \left| TPM_{\chi_{G_j^c}} f(w) \right| |K(z,w)| e^{-2\varphi(w)} dV(w) \right)^p .
\end{aligned}
$$

Notice that $|K(z,w)| = |K(w,z)|$, applying Lemma 2.4 twice to above, we get

$$
\begin{aligned}
&|PSf(z)|^p \\
&\le C \sum_{j=1}^{\infty} \int_{F_j^+} \left| TPM_{\chi_{G_j^c}} f(w) \right|^p |K(w,z)|^p e^{-2p\varphi(w)} dV(w) \\
&\le C \sum_{j=1}^{\infty} \int_{F_j^+} |K(w,z)|^p e^{-p\varphi(w)} \left(\int_{B(w,r-1)^c} \left| f(u) e^{-\varphi(u)} \langle Tk_u, k_w \rangle_{F^2_\varphi} \right| dV(u) \right)^p dV(w) \\
&\le C \sum_{j=1}^{\infty} \int_{F_j^+} |K(w,z)|^p e^{-p\varphi(w)} \left(\int_{B(w,r-2)^c} \left| f(u) e^{-\varphi(u)} \langle Tk_u, k_w \rangle_{F^2_\varphi} \right|^p dV(u) \right) dV(w).
\end{aligned}
$$

By Fubini's Theorem and (2.6), we get $\|PSf\|^p_{p,\varphi}$ is no more than

$$
\begin{aligned}
& C \sum_{j=1}^{\infty} \int_{\mathbb{C}^n} \left| f(u) e^{-\varphi(u)} \right|^p \int_{F_j^+} \chi_{B(u,r-1)^c}(w) \left| \langle Tk_u, k_w \rangle_{F^2_\varphi} \right|^p e^{-p\varphi(w)} \\
& \quad \times \int_{\mathbb{C}^n} |K(w,z)|^p e^{-p\varphi(z)} dV(z) dV(w) dV(u) \\
& \le CN \int_{\mathbb{C}^n} \left| f(u) e^{-\varphi(u)} \right|^p \left(\int_{B(u,r-1)^c} \left| \langle Tk_u, k_w \rangle_{F^2_\varphi} \right|^p dV(w) \right) dV(u) \\
& \le C\varepsilon \|f\|^p_{p,\varphi}.
\end{aligned}
$$

The constants C above are independent of ε. Notice that $PTP = T$ on F^p_φ, so $PS = T - P\left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right)$ is well defined on F^p_φ and the estimate (2.7) is proved under the restriction that $T \in \mathrm{WL}^\varphi_p$. □

Lemma 2.9. *Given $0 < p \le 1$, there is some constant C such that for all bounded linear operator T on F^p_φ and $\{F_j\}_{j=1}^\infty$, $\{G_j\}_{j=1}^\infty$ associated to $r > 1$ as in Lemma 2.7 and each* *positive integer m, we have*

$$\limsup_{m\to\infty} \|PT_m\|_{F^p_\varphi\to F^p_\varphi} \le C \limsup_{m\to\infty} \sup_{w\in\cup_{j>m}G_j^+} \|Tk_w\|_{p,\varphi}, \tag{2.8}$$

where $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$.

Proof. First, we are going to show

$$\sup_{f\in F^p_\varphi\setminus\{0\}} \left\| TP\left(\frac{\chi_{G_j} f}{\|\chi_{G_j^+} f\|_{p,\varphi}}\right)\right\|_{p,\varphi} \le C \sup_{w\in G_j^+} \|Tk_w\|_{p,\varphi}. \tag{2.9}$$

In fact, given $f \in F^p_\varphi$ not identically zero, set

$$g_j = P\left(\frac{\chi_{G_j} f}{\|\chi_{G_j^+} f\|_{p,\varphi}}\right).$$

Then

$$g_j(z) = \int_{G_j} \frac{f(w)K(z,w)e^{-2\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w).$$

It is trivial to see that $g_j \in F^p_\varphi$ because of the compactness of $\overline{G_j}$. Since T is bounded on F^p_φ, then

$$|T(g_j)(z)| \le \int_{G_j} \frac{|f(w)||TK_w(z)|e^{-2\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w)$$

Note that TK_w is conjugate holomorphic respecting to w. From Lemma 2.4 we have

$$\begin{aligned}
\|T(g_j)\|^p_{p,\varphi} &\le \int_{\mathbb{C}^n}\left(\int_{G_j} \frac{|f(w)|\left|\overline{TK_w(z)}\right| e^{-2\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w)\right)^p e^{-p\varphi(z)} dV(z)\\
&\le C\int_{\mathbb{C}^n}\left(\int_{G_j^+} \frac{|f(w)|^p\left|\overline{TK_w(z)}\right|^p e^{-2p\varphi(w)}}{\|\chi_{G_j^+} f\|^p_{p,\varphi}} dV(w)\right) e^{-p\varphi(z)} dV(z)\\
&\le C\int_{G_j^+} \frac{\left|f(w)e^{-\varphi(w)}\right|^p}{\|\chi_{G_j^+} f\|^p_{p,\varphi}}\left(\int_{\mathbb{C}^n} \left|Tk_w(z)e^{-\varphi(z)}\right|^p dV(z)\right) dV(w)\\
&\le C \sup_{w\in G_j^+} \|Tk_w\|^p_{p,\varphi} \int_{G_j^+} \frac{\left|f(w)e^{-\varphi(w)}\right|^p dV(w)}{\|\chi_{G_j^+} f\|^p_{p,\varphi}}\\
&= C \sup_{w\in G_j^+} \|Tk_w\|^p_{p,\varphi}.
\end{aligned}$$

This gives (2.9). To prove (2.8), we have from Lemma 2.4 that

$$\begin{aligned}
&\left|P\left(\chi_{F_j}(\cdot)\int_{G_j}\frac{\langle f,k_w\rangle_{F^2_\varphi}(Tk_w)(\cdot)}{\|\chi_{G_j^+}f\|_{p,\varphi}}dV(w)\right)(z)\right|^p\\
&\le\left|\int_{F_j}K(z,u)e^{-2\varphi(u)}\int_{G_j}\frac{\langle f,k_w\rangle_{F^2_\varphi}(Tk_w)(u)}{\|\chi_{G_j^+}f\|_{p,\varphi}}dV(w)dV(u)\right|^p\\
&\le C\int_{F_j^+}|K(z,u)|^pe^{-2p\varphi(u)}\left|\int_{G_j}\frac{\langle f,k_w\rangle_{F^2_\varphi}(Tk_w)(u)}{\|\chi_{G_j^+}f\|^p_{p,\varphi}}dV(w)\right|^p dV(u)\\
&\le C\int_{F_j^+}|K(z,u)|^pe^{-2p\varphi(u)}\left(\int_{G_j^+}\frac{|\langle f,k_w\rangle_{F^2_\varphi}|^p|(Tk_w)(u)|^p}{\|\chi_{G_j^+}f\|^p_{p,\varphi}}dV(w)\right)dV(u).
\end{aligned}$$

Hence, integrating both sides and interchanging the order of integrations we obtain

$$\begin{aligned}
&\left\|P\left(M_{\chi_{F_j}}Tg_j\right)\right\|^p_{p,\varphi}\\
&\le C\int_{G_j^+}\frac{|\langle f,k_w\rangle_{F^2_\varphi}|^p}{\|\chi_{G_j^+}f\|^p_{p,\varphi}}\int_{F_j^+}|(Tk_w)(u)|^pe^{-2p\varphi(u)}\int_{\mathbb{C}^n}|K(z,u)|^pe^{-p\varphi(z)}dV(z)dV(u)dV(w)\\
&\le C\int_{G_j^+}\frac{|\langle f,k_w\rangle_{F^2_\varphi}|^p}{\|\chi_{G_j^+}f\|^p_{p,\varphi}}\left(\int_{F_j^+}|(Tk_w)(u)|^pe^{-p\varphi(u)}dV(u)\right)dV(w).
\end{aligned}$$

This gives

$$\left\|P\left(M_{\chi_{F_j}}Tg_j\right)\right\|^p_{p,\varphi}\le C\left(\sup_{w\in G_j^+}\|Tk_w\|^p_{p,\varphi}\right)\int_{G_j^+}\frac{|\langle f,k_w\rangle_{F^2_\varphi}|^p}{\|\chi_{G_j^+}f\|^p_{p,\varphi}}dV(w)=C\sup_{w\in G_j^+}\|Tk_w\|^p_{p,\varphi}.$$

Therefore, (2.6) yields

$$\begin{aligned}
\|PT_mf\|^p_{p,\varphi}&\le\sum_{j>m}\|PM_{\chi_{F_j}}TPM_{\chi_{G_j}}f\|^p_{p,\varphi}\\
&=\sum_{j>m}\|P\left(M_{\chi_{F_j}}Tg_j\right)\|^p_{p,\varphi}\,\|\chi_{G_j^+}f\|^p_{p,\varphi}\\
&\le C\sum_{j>m}\sup_{w\in G_j^+}\|Tk_w\|^p_{p,\varphi}\|\chi_{G_j^+}f\|^p_{p,\varphi}\\
&\le CN\left(\sup_{w\in\cup_{j>m}G_j^+}\|Tk_w\|^p_{p,\varphi}\right)\|f\|^p_{p,\varphi}.
\end{aligned}$$

From this, (2.8) follows. □

In the case of $1\le p<\infty$, the projection P is bounded from L^p_φ to F^p_φ, and so is PM_{χ_E} when $E\subset\mathbb{C}^n$ is measurable. But P is not bounded on L^p_φ if $0<p<1$. The following lemma, Lemma 2.10, says PM_{χ_E} is still bounded on F^p_φ.

Lemma 2.10. *Suppose* $0<p\le 1$. *There exists some constant* C *such that for any domain* E *in* $\mathbb{C}^n$ *we have* $\|PM_{\chi_E}\|_{F^p_\varphi\to F^p_\varphi}\le C$.

Proof. Suppose $E \in \mathbb{C}^n$ is a domain. For $f \in F^p_\varphi$, we have $|f(w)K(z,w)| = |f(w)K(w,z)|$. Lemma 2.4 and Lemma 2.2, estimate (3) gives

$$\begin{aligned}
\|PM_{\chi_E} f\|^p_{p,\varphi} &= \int_{\mathbb{C}^n} \left| \int_E f(w)K(z,w)e^{-2\varphi(w)} dV(w) \right|^p e^{-p\varphi(z)} dV(z) \\
&\leq C \int_{\mathbb{C}^n} \left(\int_{E^+} \left| f(w)K(w,z)e^{-2\varphi(w)} \right|^p dV(w) \right) e^{-p\varphi(z)} dV(z) \\
&= C \int_{E^+} |f(w)|^p e^{-2p\varphi(w)} \left(\int_{\mathbb{C}^n} \left| K(w,z)e^{-\varphi(z)} \right|^p dV(z) \right) dV(w). \\
&\leq C \|f\|^p_{p,\varphi}.
\end{aligned}$$

□

Lemma 2.11. *Suppose* $0 < p \leq 1$ *and* $T \in \mathcal{K}(F^p_\varphi)$*. Then*

$$\lim_{R\to\infty} \|PM_{\chi_{B(0,R)}} T - T\|_{F^p_\varphi \to F^p_\varphi} = 0.$$

Proof. Notice that, $PT = T$ on F^p_φ. For $f \in F^p_\varphi$ with $\|f\|_{p,\varphi} \leq 1$, we get

$$\begin{aligned}
\left\| \left(PM_{\chi_{B(0,R)}} T - T \right) f \right\|^p_{p,\varphi} &= \left\| \left(PM_{\chi_{B(0,R)}} T - PT \right) f \right\|^p_{p,\varphi} \\
&= \int_{\mathbb{C}^n} \left| \int_{|w|\geq R} Tf(w)K(z,w)e^{-2\varphi(w)} dV(w) \right|^p e^{-p\varphi(z)} dV(z).
\end{aligned}$$

Then by Lemma 2.4,

$$\begin{aligned}
\left\| \left(PM_{\chi_{B(0,R)}} T - T \right) f \right\|^p_{p,\varphi} &\leq C \int_{\mathbb{C}^n} \left(\int_{|w|\geq R-1} \left| Tf(w)K(w,z)e^{-2\varphi(w)} \right|^p dV(w) \right) e^{-p\varphi(z)} dV(z) \\
&= \int_{|w|\geq R-1} \left| Tf(w)e^{-2\varphi(w)} \right|^p \left(\int_{\mathbb{C}^n} |K(w,z)|^p e^{-p\varphi(z)} dV(z) \right) dV(w) \\
&\leq C \int_{|w|\geq R-1} \left| Tf(w)e^{-\varphi(w)} \right|^p dV(w).
\end{aligned}$$

Since $T \in \mathcal{K}(F^p_\varphi)$, $\{Tf : \ f \in F^p_\varphi \text{ with } \|f\|_{p,\varphi} \leq 1\} \subset F^p_\varphi$ is relatively compact. By [8, Lemma 3.2], for each $\varepsilon > 0$ there is some $R > 0$ such that

$$\sup_{f\in F^p_\varphi, \|f\|_{p,\varphi}\leq 1} \int_{|w|>R-1} \left| Tf(w)e^{-\varphi(w)} \right|^p dV(w) < \varepsilon^p.$$

Therefore,

$$\|PM_{\chi_{B(0,R)}} T - T\|_{F^p_\varphi \to F^p_\varphi} = \sup_{f\in F^p_\varphi, \|f\|_{p,\varphi}\leq 1} \left\| \left(PM_{\chi_{B(0,R)}} T - T \right) f \right\|_{p,\varphi} < C\varepsilon,$$

where C is independent of ε. □

Lemma 2.12. *Suppose* $0 < p \leq 1$*. Then for* T *bounded on* F^p_φ *we have*

$$\|T\|_{e,F^p_\varphi} \simeq \limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}} T\|_{F^p_\varphi \to F^p_\varphi}.$$

Proof. For any $R > 0$, $PM_{\chi_{B(0,R)}}$ is a Toeplitz operator induced by $\chi_{B(0,R)}$, Lemma 2.9 from [8] tells us it is compact on F^p_φ. Given T bounded on F^p_φ, $PM_{\chi_{B(0,R)}}T$ is compact. Thus,

$$\|T\|_{e,F^p_\varphi} \le \|T - PM_{\chi_{B(0,R)}}T\|_{F^p_\varphi \to F^p_\varphi}.$$

This yields

$$\|T\|_{e,F^p_\varphi} \le \limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}}T\|_{F^p_\varphi \to F^p_\varphi}.$$

On the other hand, for any $A \in \mathcal{K}(F^p_\varphi)$, Lemma 2.11 shows

$$\limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}}A\|_{F^p_\varphi \to F^p_\varphi} = 0.$$

From Lemma 2.10, we know

$$\begin{aligned}\limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}}T\|_{F^p_\varphi \to F^p_\varphi} &= \limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}}(T-A)\|_{F^p_\varphi \to F^p_\varphi} \\ &\le \limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}}\|_{F^p_\varphi \to F^p_\varphi}\|T-A\|_{F^p_\varphi \to F^p_\varphi} \\ &\le C\|T-A\|_{F^p_\varphi \to F^p_\varphi}.\end{aligned}$$

Hence,

$$\limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}}T\|_{F^p_\varphi \to F^p_\varphi} \le C\|T\|_{e,F^p_\varphi}.$$

□

Now we are in the position to characterize those compact operators in WL^φ_p with $0 < p \le 1$, which extends the main results in [10, 11, 20] to the small exponential case.

Theorem 2.13. *Let $0 < p \le 1$ and $T \in WL^\varphi_p$. The following statements are equivalent:*

(A) $T \in \mathcal{K}(F^p_\varphi)$;

(B) $\lim_{z\to\infty} \sup_{w\in B(z,r)} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| = 0$ *for any $r > 0$;*

(C) $\lim_{z\to\infty} \sup_{w\in\mathbb{C}^n} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| = 0$;

(D) $\lim_{z\to\infty} \|Tk_z\|_{p,\varphi} = 0$.

Proof. It is trivial that (C)⇒(B). We will show the implication (B)⇒(D) under the hypothesis $T \in \mathrm{WL}^\varphi_p$. In fact, for any $\varepsilon > 0$, by (2.2) we have some $r > 0$ such that

$$\sup_{z\in\mathbb{C}^n} \int_{B(z,r)^c} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w) < \varepsilon.$$

Combining the above inequality with (B), we get

$$\begin{aligned}\|Tk_z\|_{p,\varphi}^p &= \int_{\mathbb{C}^n} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &= \left(\int_{B(z,r)^c} + \int_{B(z,r)}\right) \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p dV(w)\\ &\le \varepsilon + A(B(z,r)) \sup_{w\in B(z,r)} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|^p\\ &\le \varepsilon + Cr^{2n}\left(\sup_{w\in B(z,r)} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right|\right)^p\\ &< 2\varepsilon\end{aligned}$$

whenever $|z|$ is sufficiently large. Therefore, (B) implies (D).

Suppose T satisfies (D). By Lemma 2.3 we know

(2.10)
$$\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| = \left|Tk_z(w)e^{-\varphi(w)}\right| \le C\left(\int_{B(w,1)} \left|Tk_z(u)e^{-\varphi(u)}\right|^p dV(u)\right)^{\frac{1}{p}} \le C\|Tk_z\|_{p,\varphi}.$$

Then,

$$\sup_{w\in\mathbb{C}^n} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| \le C\|Tk_z\|_{p,\varphi}$$

which gives the implication (D)⇒(C).

To prove (D)⇒(A), given $\varepsilon > 0$ we pick some $r > 10$ with sets $\{F_j\}_j$ and $\{G_j\}_j$ as in Proposition 2.8 so that

$$\left\|T - P\left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}}\right)\right\|_{F^p_\varphi\to F^p_\varphi} < \varepsilon.$$

For each positive integer m, set $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. Since $P\left(\sum_{j=1}^{m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}\right)$ is compact on F^p_φ, we get

$$\|T\|_{e,F^p_\varphi}^p \le \left\|T - P\left(\sum_{j=1}^{m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}\right)\right\|_{F^p_\varphi\to F^p_\varphi}^p < \varepsilon^p + \|PT_m\|^p_{F^p_\varphi\to F^p_\varphi}. \tag{2.11}$$

Suppose T satisfies (D), then there exists $t > 0$ such that $\|Tk_z\|_{p,\varphi} < \varepsilon$ for $|z| \ge t$. Notice that, $\cup_{j>m} G_j^+ \subset B(0,t)^c$ whenever m is large enough. So, (2.8) in Lemma 2.9 and (2.11) imply $\|T\|_{e,F^p_\varphi} = 0$ which gives the compactness of T.

To finish our proof, we only need to prove the implication (A)⇒(B). Given $T \in \mathcal{K}(F^p_\varphi)$, Lemma 2.11 tells us

$$\lim_{R\to\infty} \|PM_{\chi_{B(0,R)}}T - T\|_{F^p_\varphi\to F^p_\varphi} = 0. \tag{2.12}$$

First, we claim that

$$\lim_{z\to\infty} \sup_{w\in B(z,r)} \left|\langle PM_{\chi_{B(0,R)}}Tk_z, k_w\rangle_{F^2_\varphi}\right| = 0. \tag{2.13}$$

In fact, Lemma 2.10 shows $PM_{\chi_{B(0,R)}}Tk_z \in F^p_\varphi \subset F^2_\varphi$, we obtain

$$\begin{aligned}
\left|\langle PM_{\chi_{B(0,R)}}Tk_z, k_w\rangle_{F^2_\varphi}\right| &= \left|\langle M_{\chi_{B(0,R)}}Tk_z, k_w\rangle_{F^2_\varphi}\right| \\
&\le \int_{B(0,R)} \left|Tk_z(u)\overline{k_w(u)}\right| e^{-2\varphi(u)}dV(u) \\
&\le \|Tk_z\|_{\infty,\varphi} \int_{B(0,R)} |k_w(u)|\, e^{-\varphi(u)}dV(u) \\
&\le C\|Tk_z\|_{p,\varphi} \sup_{u\in B(0,R)} |k_w(u)|\, e^{-\varphi(u)} \\
&\le C\|T\|_{F^p_\varphi\to F^p_\varphi}\|k_z\|_{p,\varphi}e^{-\theta|w|} \\
&\le Ce^{-\theta|w|},
\end{aligned}$$

where the constants C are independent of z and w. Hence, (2.13) is true. Using (3) in Lemma 2.2 and (2.12) to get that

$$\begin{aligned}
\left|\left\langle \left(T - PM_{\chi_{B(0,R)}}T\right) k_z, k_w\right\rangle_{F^2_\varphi}\right| &\le \left\|\left(T - PM_{\chi_{B(0,R)}}T\right) k_z\right\|_{\infty,\varphi} \|k_w\|_{1,\varphi} \\
&\le C\left\|\left(T - PM_{\chi_{B(0,R)}}T\right) k_z\right\|_{p,\varphi} \\
&\le C\|T - PM_{\chi_{B(0,R)}}T\|_{F^p_\varphi\to F^p_\varphi}\|k_z\|_{p,\varphi} \\
&\le C\|T - PM_{\chi_{B(0,R)}}T\|_{F^p_\varphi\to F^p_\varphi} \to 0
\end{aligned}$$

as $R\to\infty$. Combining the above with (2.13), we obtain

$$\sup_{w\in B(z,r)} \left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| \le \sup_{w\in B(z,r)} \left|\left\langle PM_{\chi_{B(0,R)}}Tk_z, k_w\right\rangle_{F^2_\varphi}\right| + \sup_{w\in B(z,r)} \left|\left\langle \left(T - PM_{\chi_{B(0,R)}}T\right) k_z, k_w\right\rangle_{F^2_\varphi}\right|$$

goes to 0 as $z\to\infty$. □

If $1<p<\infty$, $k_z\to 0$ weakly on F^p_φ, which implies $\lim_{z\to\infty}\|T(k_z)\|_{p,\varphi} = 0$ for $T\in\mathcal{K}(F^p_\varphi)$. Theorem 1.2 in [11] tells us that the equivalence from (A) to (D) remains true for $T\in \mathrm{WL}^\varphi_p$ if $1<p<\infty$. For our later applications, we exhibit the following result.

Theorem 2.14. *Let $0<p<\infty$ and $T\in WL^\varphi_p$. The following statements are equivalent:*

(A) $T\in\mathcal{K}(F^p_\varphi)$;

(B) $\lim_{z\to\infty}\sup_{w\in B(z,r)}\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| = 0$ *for any* $r>0$;

(C) $\lim_{z\to\infty}\sup_{w\in\mathbb{C}^n}\left|\langle Tk_z, k_w\rangle_{F^2_\varphi}\right| = 0$;

(D) $\lim_{z\to\infty}\|Tk_z\|_{p,\varphi} = 0$.

From Theorem 2.13, it is natural to ask whether the essential norm of $T\in \mathrm{WL}^\varphi_p$ can be dominated by its behavior on normalized reproducing kernel k_z? This problem has attracted much interest, see [10, Section 6] for example. Our Theorem 2.15 says the answer is affirmative when $0<p\le 1$.

Theorem 2.15. *Suppose $0<p\le 1$. Then for $T\in WL^\varphi_p$ we have*

$$\|T\|_{e,F^p_\varphi} \simeq \limsup_{z\to\infty}\|Tk_z\|_{p,\varphi}. \tag{2.14}$$

Proof. Suppose $T \in \mathrm{WL}_p^\varphi$. From Lemma 2.5 we know T is bounded on F_φ^p which implies $\limsup_{z\to\infty} \|Tk_z\|_{p,\varphi} < \infty$. By Theorem 2.13, $\|T\|_{e,F_\varphi^p} = 0$ if $\limsup_{z\to\infty} \|Tk_z\|_{p,\varphi} = 0$. So, we may assume $\limsup_{z\to\infty} \|Tk_z\|_{p,\varphi} = \varepsilon_1 > 0$. From Proposition 2.8, we have two sequences of sets $\{F_j\}_j$ and $\{G_j\}_j$ so that

$$\left\| T - P\left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}}\right)\right\|_{F_\varphi^p \to F_\varphi^p} < \varepsilon_1.$$

Then, for $m = 1, 2, \cdots$, from (2.8) and (2.11) we have

$$\|T\|_{e,F_\varphi^p} \le \varepsilon_1 + \|PT_m\|_{F_\varphi^p\to F_\varphi^p} \le \varepsilon_1 + C \sup_{z\in \cup_{j>m} G_j^+} \|Tk_z\|_{p,\varphi}.$$

Since $0 < p \le 1$, Lemma 2.9 tells us that the constants C above do not depend on the precise choice of $\{F_j\}_j$ and $\{G_j\}_j$, and hence do not depend on T. Let $m \to \infty$, we have the desired estimate

$$\|T\|_{e,F_\varphi^p}^p \le \varepsilon_1^p + C \limsup_{z\to\infty} \|Tk_z\|_{p,\varphi}^p = C \limsup_{z\to\infty} \|Tk_z\|_{p,\varphi}^p.$$

On the other hand, fixed $R > 0$, notice that $PM_{\chi_{B(0,R)}}$ is a Toeplitz operator induced by $\chi_{B(0,R)}$, which is a bounded function. So $PM_{\chi_{B(0,R)}} \in \mathrm{WL}_p^\varphi$ and $PM_{\chi_{B(0,R)}}T \in \mathrm{WL}_p^\varphi$, because WL_p^φ is a algebra. Since $PM_{\chi_{B(0,R)}}$ is compact and T is bounded on F_φ^p (see Lemma 2.5), we get that $PM_{\chi_{B(0,R)}}T$ is compact on F_φ^p. Theorem 2.13 tells us

$$\lim_{z\to\infty} \left\| PM_{\chi_{B(0,R)}} Tk_z \right\|_{p,\varphi} = 0. \tag{2.15}$$

Therefore, Lemma 2.12, (2.15) and the fact that $PT = T$ yield

$$\begin{aligned}
\|T\|_{e,F_\varphi^p}^p &\simeq \limsup_{R\to\infty} \|PM_{\chi_{B(0,R)^c}} T\|_{F_\varphi^p\to F_\varphi^p}^p \\
&\ge \limsup_{R\to\infty} \limsup_{z\to\infty} \left\| PM_{\chi_{B(0,R)^c}} Tk_z \right\|_{p,\varphi}^p \\
&\ge \limsup_{R\to\infty} \limsup_{z\to\infty} \left(\|Tk_z\|_{p,\varphi}^p - \left\| PM_{\chi_{B(0,R)}} Tk_z \right\|_{p,\varphi}^p \right) \\
&\ge \limsup_{z\to\infty} \|Tk_z\|_{p,\varphi}^p.
\end{aligned}$$

□

3. Toeplitz Operators with BMO Symbols

In this section, we are going to discuss the characterizations on Toeplitz operators with BMO symbols. First, we will characterize the boundedness (and the compactness) of Toeplitz operators T_f on F_φ^p with BMO symbols f. Furthermore, we will characterize those compact operators on F_φ^p which are in the algebra generated by bounded Toeplitz operators with BMO symbols. For this purpose, we need some more auxiliary function spaces.

Fixed $r > 0$, recall that $B(\cdot, r) = \{w \in \mathbb{C}^n : |w - \cdot| < r\}$. Given a locally Lebesgue integrable function f on $\mathbb{C}^n$ (written as $f \in L_{loc}^1(\mathbb{C}^n)$), write

$$\omega_r(f)(\cdot) = \sup \{|f(w) - f(\cdot)| : w \in B(\cdot, r)\}$$

and

$$MO_r(f)(\cdot) = \frac{1}{V\left(B(\cdot, r)\right)} \int_{B(\cdot,r)} \left| f - \widehat{f}_r(\cdot) \right| dV$$

where

$$\widehat{f}_r(\cdot) = \frac{1}{V\left(B(\cdot, r)\right)} \int_{B(\cdot,r)} f dV.$$

For f on $\mathbb{C}^n$ with $f(\cdot)|k_z(\cdot)|^2 \in L^1_\varphi$ for all $z \in \mathbb{C}^n$, the Berezin transform of f is defined as

$$\widetilde{f}(z) = \int_{\mathbb{C}^n} f(w) \left|k_z(w)\right|^2 e^{-2\varphi(w)} dV(w).$$

Let BO_r be the collection of all continuous functions f on $\mathbb{C}^n$ such that $\omega_r(f)$ is bounded. We use BA_r and BMO_r to denote respectively the set of all $f \in L^1_{loc}(\mathbb{C}^n)$ such that $\widehat{|f|}_r$ and $MO_r(f)$ are bounded on $\mathbb{C}^n$. The space BMO is the family of all measurable function f on $\mathbb{C}^n$ satisfying $f(\cdot)|k_z(\cdot)|^2 \in L^1_\varphi$ for $z \in \mathbb{C}^n$ and

$$\|f\|_{\mathrm{BMO}} = \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} \left| f(w) - \widetilde{f}(z) \right| |k_z(w)|^2 e^{-2\varphi(w)} dV(w) < \infty.$$

By Lemma 3.33 in [21], we obtain that the spaces BO_r and BA_r are independent of r, they will be denoted as BO and BA below. The next lemma says BMO_r is independent of r as well.

Lemma 3.1. *Suppose $f \in L^1_{loc}(\mathbb{C}^n)$. The following three statements are equivalent:*
(A) *$f \in BMO_r$ for some (or any) $r > 0$;*
(B) *$f \in BMO$;*
(C) *$f = f_1 + f_2$, where $f_1 \in BA$ and $f_2 \in BO$.*

Proof. For $n = 1$ and $\varphi(z) = \frac{\alpha}{2}|z|^2$, this is Theorem 3.34 from [21]. For general n and φ satisfying $dd^c\varphi \simeq \omega_0$, the proof can be carried out as that of [21] with a little modification. The details will be omitted here. □

For $f \in \mathrm{BMO}$, say $f = f_1 + f_2$ with $f_1 \in \mathrm{BA}$ and $f_2 \in \mathrm{BO}$, similar to [7, Lemma 4.1], we know the Toeplitz operator T_{f_1} is well defined on F^p_φ. From [21], $|f_2(z)| \le a|z| + b$ with constants $a, b > 0$, T_{f_2} is also well defined on F^p_φ. Thus, T_f is well defined on F^p_φ, where $0 < p < \infty$. Moreover,

$$\langle T_f k_z, k_w \rangle = \int_{\mathbb{C}^n} k_z(u)\overline{k_w(u)} f(u) e^{-2\varphi(u)} dV(u). \tag{3.1}$$

Coburn, Isralowitz and Li [5] proved that T_f ($f \in \mathrm{BMO}$) is compact on the classical Fock space $F^2_{1/2}$ if and only if the Berezin transform $\widetilde{f}$ vanished at the infinity. The first two authors extended this result to the setting of $F^p_{1/2}$ with $0 < p < \infty$ in [8]. Under the assumption that S is a linear combination of operators of form $T_{f_1} \cdots T_{f_m}$ with each function f_j satisfying $\widetilde{|f_j|}$ bounded, Isralowitz proved that S is compact on $F^2_{1/2}$ if and only if $\widetilde{S}$ vanishes at the infinity, see [9] for details. In all these references, the Weyl unitary operators acting on F^2_α by $W_z f(\cdot) = k_z f(\cdot - z)$ and the involutive unitary operators $U_z f(\cdot) = k_z f(z - \cdot)$ play as a very crucial role. Unfortunately, there are not these kinds of unitary operators on our generalized Fock space F^p_φ.

We will use $\mathcal{B}_p^\varphi$ to denote the collection of all linear combination of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$, where each function $f_j \in \mathrm{BMO}$ and $\widetilde{f_j}$ is bounded on $\mathbb{C}^n$.

Theorem 3.2. *Let $0<p<\infty$.*

(A) *If $f \in BMO$, then T_f is bounded on F_φ^p if and only if $\widetilde{f}$ is bounded on $\mathbb{C}^n$; T_f is compact on F_φ^p if and only if*

$$\lim_{z\to\infty}\sup_{w\in B(z,r)}\left|\langle T_f k_z, k_w\rangle_{F_\varphi^2}\right| = 0 \quad \forall r>0. \tag{3.2}$$

(B) *If $S\in\mathcal{B}_p^\varphi$, then S is compact on F_φ^p if and only if*

$$\lim_{z\to\infty}\sup_{w\in B(z,r)}\left|\langle S k_z, k_w\rangle_{F_\varphi^2}\right| = 0 \quad \forall r>0. \tag{3.3}$$

Proof. We claim $T_f \in \mathrm{WL}_p^\varphi$ if $f\in\mathrm{BMO}$ and $\widetilde{f}$ remains bounded. In fact, similar to [5, Lemma 1] it is trivial to verify

$$\sup_{z\in\mathbb{C}^n}\widetilde{|f|}(z) \le \|f\|_{\mathrm{BMO}} + \sup_{z\in\mathbb{C}^n}|\widetilde{f}(z)| < \infty. \tag{3.4}$$

By [8, Theorem 3.5], $|f|dV$ is a Fock-Carleson measure. Hence,

$$\begin{aligned}\left|\langle T_f k_z, k_w\rangle_{F_\varphi^2}\right| &\le \int_{\mathbb{C}^n}|k_z(u)k_w(u)|\,e^{-2\varphi(u)}|f(u)|dV(u)\\ &\le C\sup_{u\in\mathbb{C}^n}\widehat{|f|}_r(u)\int_{\mathbb{C}^n}|k_z(u)k_w(u)|\,e^{-2\varphi(u)}dV(u)\\ &\le C\sup_{u\in\mathbb{C}^n}\widetilde{|f|}(u)\int_{\mathbb{C}^n}e^{-\theta|z-u|-\theta|w-u|}dV(u)\\ &\le Ce^{-\frac{\theta}{2}|z-w|}.\end{aligned}$$

This implies $T_f\in\mathrm{WL}_p^\varphi$ for any $p\in(0,\infty)$ (and also, T_f is strongly localized in the sense of Xia and Zheng, see [20]).

(A). Suppose $f\in\mathrm{BMO}$. If $\widetilde{f}$ is bounded on $\mathbb{C}^n$, then $T_f\in\mathrm{WL}_p^\varphi$ which implies T_f is bounded on F_φ^p for any $p\in(0,\infty)$. Conversely, the condition that T_f is bounded implies $\widetilde{f}$ is bounded, which can be proved in a standard way with $\widetilde{f}(z)=\langle T_f k_z, k_z\rangle_{F_\varphi^2}$.

Now we deal with the compactness of T_f. If (3.2) holds, then $\widetilde{f}(z)=\langle T_f k_z,k_z\rangle_{F_\varphi^2}$ is bounded, hence $T_f\in\mathrm{WL}_p^\varphi$. Therefore, by (3.2) and Theorem 2.13, T_f is compact on F_φ^p for all $0<p<\infty$. Conversely, if T_f is compact on F_φ^p for some $0<p<\infty$. If $1<p<\infty$, we have $\lim\limits_{z\to\infty}\|T_f k_z\|_{p,\varphi}=0$ because k_z tends to zero weakly, from which (3.2) follows for any $r>0$. If $0<p\le 1$, we know $\widetilde{f}$ to be bounded. Then, $T_f\in\mathrm{WL}_p^\varphi$. Now the estimate (3.2) comes from Theorem 2.13.

(B) Since each $T_{f_j}\in\mathrm{WL}_p^\varphi$, we have $\mathcal{B}_p^\varphi\subset\mathrm{WL}_p^\varphi$ for $0<p<\infty$. Now the conclusion follows from Theorem 2.13. □

As shown by Isralowitz in [10, Proposition 1.5], on the classical Fock space F_α^p the estimate (3.3) is equivalent to $\lim\limits_{z\to\infty}\widetilde{S}(z)=0$. Therefore, Theorem 3.2 extends [5, 8].

As in [9], set BT to be the collection of all measurable functions f on $\mathbb{C}^n$ with $\widetilde{|f|}$ bounded. As shown in the proof of Theorem 4.2, $T_f \in \mathrm{WL}_p^\varphi$ if $f \in \mathrm{BT}$. We have Corollary 3.3 at once.

Corollary 3.3. *Let $0 < p < \infty$, and let S be in the family all linear combination of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$, where each function $f_j \in$BT. Then S is compact on F^p_φ if and only if one of the following three statements holds:*
(A) $\lim_{z\to\infty} \sup_{w\in B(z,r)} |\langle Tk_z, k_w\rangle| = 0$ *for any $r > 0$;*
(B) $\lim_{z\to\infty} \sup_{w\in\mathbb{C}^n} |\langle Tk_z, k_w\rangle| = 0$*;*
(C) $\lim_{z\to\infty} \|Tk_z\|_{p,\varphi} = 0.$

While $\varphi(z) = \frac{1}{4}|z|^2$, $p = 2$ and S is a linear combination of operators of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$ with each $f_j \in \mathrm{BT}$, Corollary 3.3 gives the main result of [9].

4. Operators Satisfying Axler and Zheng's Condition

In this section, we will restrict ourselves to the classical Fock space F^p_α, that is $\varphi(z) = \frac{\alpha}{2}|z|^2$ with $\alpha > 0$. We are going to characterize the boundedness and compactness of linear operators with the Axler-Zheng condition on F^p_α.

Let ϕ_z be the holomorphic self-map of $\mathbb{C}^n$, $\phi_z(\cdot) = z - \cdot$. U_z is the operator on F^p_α defined by $U_z f = (f \circ \phi_z)k_z$. Given some linear operator S on F^p_α, define

$$S_z = U_z S U_z^*.$$

In the context of Bergman space $A^2(\mathbb{D})$ on the unit disc $\mathbb{D}$, with $\phi_z(w) = \frac{w-z}{1-\bar{z}w}$ and $U_z f = (f \circ \phi_z)\phi'(z)$, Axler-Zheng introduced the condition

$$\sup_{z\in\mathbb{D}} \|S_z 1\|_{A^p} < \infty \text{ with some } p > 2$$

in [1]. The work in [5, 6, 13, 15, 22] also explored the condition $\|S_z 1\|_{A^p} \le C$. In the Fock space setting, Wang, Cao, and Zhu carried out related research in [19] to obtain that, if there exist some $p > 2$ such that

$$\sup_{z\in\mathbb{C}^n} \|S_z 1\|_{p,\frac{2\alpha}{p}} < \infty \quad (\text{or } \|S_z 1\|_{p,\frac{2\alpha}{p}} \to 0 \text{ as } z \to \infty),$$

the operator S is bounded (or compact) on F^2_α.

Theorem 4.1. *Suppose S is a linear operator defined on $\mathcal{D}$. If there are some $0 < \sigma < p < \infty$ such that*

$$M = \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) < \infty, \tag{4.1}$$

then

$$\left|\langle Sk_z, k_w\rangle_{F^2_\alpha}\right| \le CM^{\frac{1}{p}} e^{-\frac{\alpha(p-\sigma)}{2p}|z-w|^2},$$

so S is bounded on F^s_α for all $0 < s < \infty$. Furthermore, if both (4.1) *and*

$$\lim_{z\to\infty} \int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) = 0 \tag{4.2}$$

hold, then S is compact on F^s_α for $0 < s < \infty$.

Proof. Since $K(\cdot,\cdot) = e^{\alpha\langle\cdot,\cdot\rangle}$, it is easy to verify $k_z(z-u)k_z(u) = 1$ and

$$K(z-u, z-u) = K(z,z)K(u,u)|K(u,z)|^{-2}.$$

By the equality $S_z 1(u) = k_z(u)(Sk_z)(z-u)$ (see [21]) and Lemma 2.3 we have

$$\begin{aligned}
\int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) &= \int_{\mathbb{C}^n} |k_z(u)(Sk_z)(z-u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) \\
&= \int_{\mathbb{C}^n} |k_z(z-u)(Sk_z)(u)|^p e^{-\frac{\alpha\sigma}{2}|u-z|^2} dV(u) \\
&= \int_{\mathbb{C}^n} |(Sk_z)(u)|^p |k_z(u)|^{-p} e^{-\frac{\alpha\sigma}{2}|u-z|^2} dV(u) \\
&\geq \int_{B(w,1)} |(Sk_z)(u)|^p |k_z(u)|^{-(p-\sigma)} e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) \\
&\geq C|(Sk_z)(w)|^p |k_z(w)|^{-(p-\sigma)} e^{-\frac{\alpha\sigma}{2}|w|^2} \\
&= C|\langle Sk_z, k_w\rangle_{F^2_\alpha}|^p e^{\frac{\alpha(p-\sigma)}{2}|z-w|^2}.
\end{aligned}$$

From the above inequalities and (4.1), we get

$$|\langle Sk_z, k_w\rangle_{F^2_\alpha}| \leq CM^{\frac{1}{p}} e^{-\frac{\alpha(p-\sigma)}{2p}|z-w|^2}.$$

Since $p - \sigma > 0$, S is weakly localized for F^s_α, so S is bounded on F^s_α for all $0 < s < \infty$.

Furthermore, if both (4.1) and (4.2) are valid, from the proof above we have $S \in \mathrm{WL}^\alpha_s$. And also, for $p \in (0,\infty)$ there is some constant C_r such that

$$\sup_{w\in B(z,r)} \left|\langle Sk_z, k_z\rangle_{F^2_\alpha}\right| \leq C_r \left(\int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u)\right)^{\frac{1}{p}} \to 0$$

as $z \to \infty$. By Theorem 2.13 S is compact on F^s_α for all $0 < s < \infty$. □

Remark. If $p > \sigma = 2$ and $s = 2$, then Theorem 4.1 reduces to Theorems A and B in [19].

5. Further Remarks

An important theme in analysis on function spaces is to characterize when a given operator is compact. In the setting of the Bergman space $A^p(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$, $1 < p < \infty$, in 2007 Suárez proved, see [17], that a bounded operator S is compact if and only if S is in the Toeplitz algebra and the Berezin transform of S vanishes on the boundary. Later on, Mitkovski, Suárez and the third author [14] extended [17] to the weighted Bergman space $A^p_\alpha(\mathbb{B}_n)$. On the classical Fock space F^p_α for $1 < p < \infty$, in [2] Bauer and Isralowitz showed that Suárez's characterization on compact operators is valid. For general φ with $dd^c\varphi \simeq \omega_0$ and $1 < p < \infty$, most recently in [10] Isralowitz obtained $\mathcal{K}(F^p_\varphi) = \mathcal{T}^p_\varphi(C^\infty_c(\mathbb{C}^n))$, which implies the results in [2].

For Toeplitz operators T_μ with positive Borel measures μ as symbols, the boundedness (or compactness) on F^p_φ with $0 < p \leq 1$ can be characterized with the same condition as that on F^q_φ with $q > 1$. Unfortunately, some differences appear when we talk about the structure of $\mathcal{K}(F^p_\varphi)$. For example, we find $\mathcal{K}(F^p_\varphi)\backslash\mathcal{T}^p_\varphi \neq \emptyset$ if $0 < p \leq 1$. To see this, from [21, Lemma

4.39] (or [12]) we take a separated sequence $\{z_j\}_{j=1}^{\infty}$ which is an interpolating sequence for F_α^∞. Hence, we have some $f \in F_\alpha^\infty$ such that

$$f(z_k)e^{-\frac{\alpha}{2}|z_k|^2} = 1, \qquad \forall k \in \mathbb{N}. \tag{5.1}$$

Although [21] is only concerned with one variable interpolation, take $\{z_j\} \subset \mathbb{C}$ and $f \in H(\mathbb{C})$ satisfying the interpolation above, extend f to $\mathbb{C}^n$ with the equation $f(z, z') = f(z)$ for $(z, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$, we will have f satisfying (5.1) in $\mathbb{C}^n$. Furthermore, for $0 < p \le 1$ take $g \in F_\alpha^p$ so that $g(0) \neq 0$. Define the operator T on F_α^p as

$$T(\cdot) = \langle \cdot, f\rangle_{F_\alpha^2} g. \tag{5.2}$$

T is bounded and of rank 1, so T is compact on F_α^p. Also,

$$\left|\langle Tk_{z_k}, k_w\rangle_{F_\alpha^2}\right| = \left|\overline{f(z_k)}e^{-\frac{\alpha}{2}|z_k|^2} g(w)e^{-\frac{\alpha}{2}|w|^2}\right|.$$

Because $\{z_j\}_{j=1}^{\infty}$ is separated, we have $\lim_{j\to\infty} z_j = \infty$. For each $r > 0$, as k is large enough we have from Lemma 2.3 that

$$\begin{aligned}\int_{B(z_k,r)^c} \left|\langle Tk_{z_k}, k_w\rangle_{F_\alpha^2}\right|^p dV(w) &= \int_{B(z_k,r)^c} \left|g(w)e^{-\frac{\alpha}{2}|w|^2}\right|^p dV(w)\\ &\ge \int_{B(0,1)} \left|g(w)e^{-\frac{\alpha}{2}|w|^2}\right|^p dV(w)\\ &\ge C\,|g(0)|^p .\end{aligned}$$

$T \notin \mathrm{WL}_p^\alpha$ by Definition 2.1. Hence, $\mathcal{K}(F_\alpha^p)\backslash\mathcal{T}_\alpha^p \neq \emptyset$ for $0 < p \le 1$. This tells us the characterization of compact operators T on F_φ^p with $0 < p \le 1$ is quite different from that with $1 < p < \infty$.

For $0 < p \le 1$, $\{k_z : z \in \mathbb{C}^n\}$ does not converge weakly to zero in F_α^p as z goes to ∞. In fact, take $f \in F_\alpha^\infty$ satisfying (5.1), since the dual space of F_α^p is F_α^∞ under the pairing $\langle g, f\rangle_{F_\alpha^2}$ (see [21]), we know that $F_f = \langle \cdot, f\rangle_{F_\alpha^2}$ is a bounded linear functional on F_α^p. However,

$$F_f(k_{z_k}) = \langle k_{z_k}, f\rangle_{F_\alpha^2} = 1$$

for all k.

The operator T defined as (5.2) also shows $\lim_{z\to\infty} \|Tk_z\|_{p,\alpha} \neq 0$, because $Tk_{z_j} = g$ for $j = 1, 2, \cdots$, which says Tk_z need not converge to 0 in F_α^p even if T is compact while $0 < p \le 1$. So, the hypothesis $T \in \mathrm{W}L_p^\varphi$ both in Theorem 2.13 and 2.14 can not be removed. But for the Berezin transform, we have the following Proposition 5.1.

Proposition 5.1. *Suppose $0 < p \le 1$ and $T \in \mathcal{K}(F_\varphi^p)$. Then $\widetilde{T}(z) \to 0$ as $z \to \infty$.*

Proof. For $R>0$ fixed, $PM_{\chi_{B(0,R)}}Tk_z \in F^p_\varphi \subset F^2_\varphi$. Lemma 2.2 estimate (1) and Lemma 2.2 estimate (3) give

$$\begin{aligned}\left|\langle PM_{\chi_{B(0,R)}}Tk_z, k_z\rangle_{F^2_\varphi}\right| \;=\; \left|\langle M_{\chi_{B(0,R)}}Tk_z, k_z\rangle_{F^2_\varphi}\right| &\le \int_{B(0,R)} \left|Tk_z(u)\overline{k_z(u)}\right| e^{-2\varphi(u)}dV(u)\\ &\le \|Tk_z\|_{\infty,\varphi}\int_{B(0,R)}|k_z(u)|\,e^{-\varphi(u)}dV(u)\\ &\le C\|Tk_z\|_{p,\varphi}\sup_{|u|\le R}|k_z(u)|\,e^{-\varphi(u)}\\ &\le \|T\|_{F^p_\varphi\to F^p_\varphi}\|k_z\|_{p,\varphi}e^{-\theta|z|}\\ &\le Ce^{-\theta|z|}\to 0\end{aligned}$$

as $z\to\infty$. Since $T\in\mathcal{K}(F^p_\varphi)$, Lemma 2.11 tells us

$$\begin{aligned}\left|\left\langle \left(T-PM_{\chi_{B(0,R)}}T\right)k_z,k_z\right\rangle_{F^2_\varphi}\right| &\le \left\|\left(T-PM_{\chi_{B(0,R)}}T\right)k_z\right\|_{\infty,\varphi}\|k_z\|_{1,\varphi}\\ &\le C\left\|\left(T-PM_{\chi_{B(0,R)}}T\right)k_z\right\|_{p,\varphi}\\ &\le C\|T-PM_{\chi_{B(0,R)}}T\|_{F^p_\varphi\to F^p_\varphi}\|k_z\|_{p,\varphi},\\ &\le C\|T-PM_{\chi_{B(0,R)}}T\|_{F^p_\varphi\to F^p_\varphi}\to 0\end{aligned}$$

as $R\to\infty$. Therefore, taking $z\to\infty$,

$$|\widetilde{T}(z)| = \left|\langle Tk_z,k_z\rangle_{F^2_\varphi}\right| \le \left|\left\langle PM_{\chi_{B(0,R)}}Tk_z,k_z\right\rangle_{F^2_\varphi}\right| + \left|\left\langle\left(T-PM_{\chi_{B(0,R)}}T\right)k_z,k_z\right\rangle_{F^2_\varphi}\right|\to 0.$$

□

Summarizing the discussion above we put forward the following problem.

Problem 5.2. *For $0<p\le 1$, what are the necessary and sufficient conditions to characterize the membership in $\mathcal{K}(F^p_\varphi)$?*

Under the restriction $0<p\le 1$, we dominate the essential norm of $T\in \mathrm{WL}^\varphi_p$ by its behavior on k_z, see Theorem 2.14. Our second problem is whether the estimate (2.14) still holds for $1<p<\infty$?.

Problem 5.3. *Suppose $1<p<\infty$. Does*

$$\|T_f\|_{e,F^p_\varphi} \simeq \limsup_{z\to\infty}\|T_fk_z\|_{p,\varphi}$$

hold for bounded f on $\mathbb{C}^n$?

In the previous section, with $f\in$ BMO we have obtained the compactness of Toeplitz operators T_f on F^p_φ. However, to consider the compactness of finite product $T_{f_1}T_{f_2}\cdots T_{f_m}$ of Toeplitz operators with BMO symbols we have assumed each symbol f_j has a bounded Berezin transform. Is this hypothesis necessary in the statement (B) of Theorem 3.2?

Problem 5.4. *Suppose $0<p<\infty$, and T is in the set of all linear combination of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$, where each function $f_j\in$ BMO. Can we conclude that T is compact on F^p_φ if and only if*

$$\lim_{z\to\infty}\sup_{w\in B(z,r)}\left|\langle Tk_z,k_w\rangle_{F^2_\varphi}\right| = 0$$

holds for each $r > 0$?

We also point to the general question of how the story is similar, or different, in the case of the Bergman space $A^p(\mathbb{D})$ when $0 < p < 1$.

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