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Polynomials with no zeros on a face of the bidisk

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POLYNOMIALS WITH NO ZEROS ON A FACE OF THE BIDISK

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Abstract. We present a Hilbert space geometric approach to the problem of characterizing the positive bivariate trigonometric polynomials that can be represented as the square of a two variable polynomial possessing a certain stability requirement, namely no zeros on a face of the bidisk. Two different characterizations are given using a Hilbert space structure naturally associated to the trigonometric polynomial; one is in terms of a certain orthogonal decomposition the Hilbert space must possess called the "split-shift orthogonality condition" and another is an operator theoretic or matrix condition closely related to an earlier characterization due to the first two authors. This approach allows several refinements of the characterization and it also allows us to prove a sums of squares decomposition which at once generalizes the Cole-Wermer sums of squares result for two variable stable polynomials as well as a sums of squares result related to the Schur-Cohn method for counting the roots of a univariate polynomial in the unit disk.

Contents

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1. Introduction

This article is concerned with harmonic analysis and moment problems as motivated by prediction theory and connections to analytic function theory and operator theory, continuing a tradition of classic works such as Helson-Lowdenslager [13], Helson-Szegő [14], and Wiener-Masani [25]. Most of these works are concerned with harmonic analysis on the unit circle and function theory on the unit disk. In this article, we work in the setting of the two-torus or bi-circle. Helson-Lowdenslager [13] was perhaps the first paper to pin down which aspects of harmonic analysis on the circle extend in a straightforward way to the bi-circle. Factorization of positive trigonometric polynomials is one area that most certainly does not extend in a straightforward way from one variable to two, and this topic serves as a good starting point to motivate the rest of the paper.

The classical Fejér-Riesz lemma states that a non-negative trigonometric polynomial $t(\theta)$ in one variable can be factored as

$$t(\theta) = |p(e^{i\theta})|^2$$

where $p \in \mathbb{C}[z]$ is a polynomial with no zeros in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. While this is one of the simplest factorization results it is useful in signal processing, trigonometric moment problems, and wavelets. It is also a prototype for more advanced and important factorization results, such as Szegő's theorem. A simple degrees of freedom argument shows that this result cannot be extended without conditions to two variables. In recent years, progress has been made in extending this result to two variables. First in Geronimo-Woerdeman [11] a characterization was given of positive bivariate trigonometric polynomials t that can be factored as

$$t(\theta, \phi) = |p(e^{i\theta}, e^{i\phi})|^2 \tag{1.1}$$

where $p \in \mathbb{C}[z, w]$ is stable, i.e. has no zeros in the closed bidisk $\overline{\mathbb{D}}^2 = \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. The characterization is in terms of trigonometric moments of the measure

$$\frac{d\theta d\phi}{(2\pi)^2 t(\theta, \phi)}$$

on $[0, 2\pi]^2$, and the necessary and sufficient conditions for the characterization come from studying measures on $\mathbb{T}^2 = (\partial\mathbb{D}) \times (\partial\mathbb{D})$ of the form

$$\frac{|dz||dw|}{(2\pi)^2 |p(z, w)|^2}$$

where p is a polynomial. These are called Bernstein-Szegő measures. In one variable, such measures play a natural role since they can be used to match a finite sequence of moments of a given positive Borel measure in an "entropy maximizing" way—see Landau [21] or Simon [24].

Surprisingly, the development of the above result passes through a sums of squares formula related to p which yields a famous inequality of Andô from multivariable operator theory and in turn yields Agler's Pick interpolation theorem for bounded analytic functions on the bidisk. This connection is described in Cole-Wermer [8] and Knese [17]. The Hilbert space geometry approach of [17] made it possible to extend the characterization to the setting where p has no zeros on the open bidisk $\mathbb{D}^2$ in [19].

In another direction, an investigation was begun in [12] of orthogonal polynomials associated with bivariate measures supported on the bicircle constructed using the lexicographical or reverse lexicographical ordering and the recurrence formulas associated with these polynomials were developed. As in the one variable case a spectral theory type result was proved relating the vanishing of certain coefficients in the recurrence formulas to the existence of a Fejér-Riesz type factorization and of a Bernstein-Szegő measure with p a stable polynomial. Recently in [9], this viewpoint yielded extensions of the above results to the problem of characterizing positive bivariate trigonometric polynomials that can be factored as in (1.1) where now $p \in \mathbb{C}[z,w]$ has no zeros on a *closed face of the bidisk.* A closed face of the bidisk refers to either $\mathbb{T} \times \overline{\mathbb{D}}$ or $\overline{\mathbb{D}} \times \mathbb{T}$. This result is significantly more difficult because much of the analyticity of $1/p$ is lost. However the moments can still be computed using the one variable residue theorem. Furthermore the factorization is in general not of the Helson-Lowdenslager type [13] which would give a rational function rather than a polynomial factorization. Special consideration was given when the trigonometric polynomial $t(\theta,\phi) = |p(z,w)|^2 = |q(z,w)|^2$ where $p(z,w) \neq 0$ for $|z| = 1,\ |w| \leq 1$ whereas $q(z,w) \neq 0$ for $|w| = 1,\ |z| \leq 1$, for which a spectral theory result analogous to the characterization of the Bernstein-Szegő measures on the circle was shown to hold.

In this article we refine, extend, and give a more complete picture of the results in [9]; the case where p has no zeros on a closed face of the bidisk mentioned above. In particular we emphasize that positive linear forms $\mathcal{T}$ on bivariate Laurent polynomials of bounded degree which can be represented as a Bernstein-Szegő measure as above with $p(z,w) \neq 0$ for $|z| = 1,\ |w| \leq 1$ can be characterized in two different ways by using $\mathcal{T}$ to define an inner product on polynomials: (1) a matrix condition involving certain natural truncated shift operators, and (2) the existence of a special orthogonal decomposition of spaces of polynomials we call *the split-shift orthogonality condition*. The "matrix condition" is easier to verify (and matches the condition presented in [9] when we choose an appropriate basis) while the "split-shift orthogonality condition" provides more information about the geometry of the spaces involved as well as the polynomial p. In particular, this latter condition is key to proving a generalization of the sums of squares formula alluded to above. A subtle fact is that the spaces involved in the split-shift condition are in general not unique but are in one-to-one correspondence with Fejér-Riesz type factorizations of the positive trigonometric polynomial t. Furthermore given p we present an explicit description of these spaces in terms of the decomposition of $p(z,0)$ as a product of stable and unstable factors. This makes it possible to characterize a whole stratification of factorizations of t as $|p|^2$ where p has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ and $p(z,0)$ has a specified number of zeros in $\mathbb{D}$. The case where $p(z,0)$ has no zeros in $\mathbb{D}$ recovers the Geronimo-Woerdeman characterization result, and the case where $p(z,0)$ has all zeros in $\mathbb{D}$ results in a related characterization of when $t = |p|^2$ where p has no zeros in $(\mathbb{C} \setminus \mathbb{D}) \times \overline{\mathbb{D}}$. In between these two extremes we can characterize when $t = |p|^2$ where the zero set of p in $\mathbb{D} \times \mathbb{C}$ has a specified number of sheets over $z \in \mathbb{D}$ sitting in $\mathbb{D} \times \mathbb{D}$ (and a complementary number of sheets sitting in $\mathbb{D} \times (\mathbb{C} \setminus \overline{\mathbb{D}})$).

We proceed as follows. In Section 2 we introduce the notation used throughout the paper and state the main theorems. In Section 3 we derive basic orthogonality relations associated with Bernstein-Szegő measures. In Section 4 we show that a Bernstein-Szegő measure with $p(z,w) \neq 0$ for $|z| = 1,\ |w| \leq 1$ implies the split-shift condition using a decomposition of $p(z,0)$ into stable and unstable factors. In Section 5 we show that the split-shift condition implies the existence of a Bernstein-Szegő measure of the type given above. Next in Section 6 the matrix condition mentioned above is shown to be equivalent to the split-shift condition. In Section 7 we describe all p that give rise to the same positive bivariate trigonometric polynomial. In Section 8 we show how to construct p from the moments associated with the positive linear form. In Section 9 we apply the previous results to solve the problem when an extended bivariate autoregressive model has a causal or acausal solution. Also in this section we give necessary and sufficient conditions in terms of moments when a bivariate Borel measure supported on the bicircle is a Bernstein-Szegő measure with p nonzero for $|z| = 1,\ |w| \leq 1$. Finally in Section 10 we adapt ideas from [20] to give a second proof of our generalized sum of squares formula which should be of independent interest, while we also consider "generalized distinguished varieties" and apply an argument of adapted from [18] to obtain a sum of squares formula for polynomials associated with these varieties. This allows us to obtain a determinantal representation of the polynomial giving rise to the variety. *Distinguished varieties* were introduced in [2] and play an important role in multivariable operator theory and function theory on the bidisk. Our determinantal representation generalizes one of the main theorems of [2].

2. Notation and statement of results

We denote spaces of Laurent polynomials by

$$\mathcal{L}_{j,k} = \vee\{z^s w^t : -j \leq s \leq j, -k \leq t \leq k\},$$

where $\vee$ denotes the complex linear span of a set, and we denote spaces of polynomials by

$$\mathcal{P}_{j,k} := \vee\{z^s w^t : 0 \leq s \leq j, 0 \leq t \leq k\},$$

where $j, k \in \mathbb{Z}_+$. In some parts of the paper, the spaces $\mathcal{L}_{j,k}$ and $\mathcal{P}_{j,k}$ appear naturally within the context of the Hilbert space $L^2(\mathbb{T}^2, \mu)$, where μ is a positive Borel measure on $\mathbb{T}^2$, in which case we may use also $j, k = \infty$ by considering the closed linear spans above.

A linear form $\mathcal{T} : \mathcal{L}_{n,m} \to \mathbb{C}$ is said to be *positive* if

$$\mathcal{T}(f(z,w)\bar{f}(1/z, 1/w)) > 0$$

for every nonzero $f \in \mathcal{P}_{n,m}$, where $\bar{f}(z,w) = \overline{f(\bar{z}, \bar{w})}$. With $\mathcal{T}$ we define an inner product on the space $\mathcal{P}_{n,m}$, via

$$\langle f, g\rangle_{\mathcal{T}} = \mathcal{T}(f(z,w)\bar{g}(1/z, 1/w)) \qquad f, g \in \mathcal{P}_{n,m}.$$

Let us write $\mathcal{H}_{\mathcal{T}}$ for the finite dimensional Hilbert space $(\mathcal{P}_{n,m}, \langle\cdot,\cdot\rangle_{\mathcal{T}})$.

For $(k,l) \in \mathbb{Z}_+^2$ where the inner product above is defined we denote the following orthogonal complements:

$$\mathcal{E}^1_{k,l} = \mathcal{P}_{k,l} \ominus w\mathcal{P}_{k,l-1}, \tag{2.1}$$

$$\mathcal{F}^1_{k,l} = \mathcal{P}_{k,l} \ominus \mathcal{P}_{k,l-1}, \tag{2.2}$$

$$\mathcal{E}^2_{k,l} = \mathcal{P}_{k,l} \ominus z\mathcal{P}_{k-1,l}, \tag{2.3}$$

$$\mathcal{F}^2_{k,l} = \mathcal{P}_{k,l} \ominus \mathcal{P}_{k-1,l}. \tag{2.4}$$

We will often employ the anti-unitary reflection operator $\overleftarrow{\cdot}$

$$g(z,w) \mapsto \overleftarrow{g}(z,w) := z^k w^l \bar{g}(1/z, 1/w)$$

which in this case we say is applied at the degree (k,l). This degree will usually be clear from context or explicitly stated. For example, applying this operator at degree (k,l) to the spaces $\mathcal{F}^1_{k,l}$ and $\mathcal{F}^2_{k,l}$ we see that

$$\mathcal{E}^1_{k,l} = \overleftarrow{\mathcal{F}}^1_{k,l} \text{ and } \mathcal{E}^2_{k,l} = \overleftarrow{\mathcal{F}}^2_{k,l},$$

since the operator is an anti-unitary in $\mathcal{H}_{\mathcal{T}}$.

Definition 2.1. A positive linear form $\mathcal{T}$ on $\mathcal{L}_{n,m}$ satisfies the *split-shift orthogonality condition* if there exist subspaces of polynomials $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}_{\mathcal{T}}$ such that

(1) $\mathcal{E}^1_{n-1,m} = \mathcal{K}_1 \oplus \mathcal{K}_2$
(2) $\mathcal{K}_1 \perp z\mathcal{K}_2$ and
(3) $\mathcal{K}_1, z\mathcal{K}_2 \subset \mathcal{E}^1_{n,m}$.

The point of conditions (2) and (3) is that they imply $\mathcal{K}_1 \oplus z\mathcal{K}_2 \subset \mathcal{E}^1_{n,m}$. This condition actually characterizes positive linear forms coming from a Bernstein-Szegő measure. What is interesting is that this condition can also be expressed using a simple matrix condition.

To present the matrix condition let us define three operators

$$\begin{aligned} A &= P_{w\mathcal{E}^2_{n,m-1}} M_z : \mathcal{E}^1_{n-1,m} \to w\mathcal{E}^2_{n,m-1} \\ B &= P_{\mathcal{E}^1_{n-1,m}} : w\mathcal{F}^2_{n,m-1} \to \mathcal{E}^1_{n-1,m} \\ T &= P_{\mathcal{E}^1_{n-1,m}} M_z : \mathcal{E}^1_{n-1,m} \to \mathcal{E}^1_{n-1,m} \end{aligned}$$

where M_z is multiplication by z and $P_{\mathcal{H}}$ represents orthogonal projection onto a subspace $\mathcal{H} \subset \mathcal{H}_{\mathcal{T}}$. Notice that T is just truncation of multiplication by z to $\mathcal{E}^1_{n-1,m}$.

Theorem 2.2. *Let $\mathcal{T}$ be a positive linear form on $\mathcal{L}_{n,m}$. The following are equivalent.*

(1) *(Bernstein-Szegő condition) There exists $p \in \mathbb{C}[z,w]$ with no zeros on $\mathbb{T}\times\overline{\mathbb{D}}$ and degree at most (n,m) such that*

$$\mathcal{T}(z^j w^k) = \int_{\mathbb{T}^2} z^j w^k \frac{|dz||dw|}{(2\pi)^2 |p(z,w)|^2} \qquad |j| \le n, |k| \le m. \tag{2.5}$$

(2) *(Split-shift condition) $\mathcal{T}$ satisfies the split-shift orthogonality condition.*
(3) *(Matrix condition) The invariant subspace of T generated by the range of B is contained in the kernel of A. More concretely,*

$$AT^j B = 0 \text{ for } j = 0, 1, \ldots, n-1. \tag{2.6}$$

This theorem is a more geometric formulation of the results in Geronimo-Iliev [9]. In particular, the coordinate free formulation of condition (3) makes it possible to give a straightforward proof of the equivalence of (2) and (3) in Propositions 6.1 and 6.2—the original proof in [9] involves some non-trivial linear algebra. Of greater significance, however, is *our emphasis on the split-shift condition and the rather complete knowledge it provides of the geometry of Bernstein-Szegő measures of the above type.* A version of the split-shift condition was recognized as an important stepping stone in [9], but at that time it was not clear how to construct the spaces involved directly with Hilbert space geometry—this question was explicitly raised as [9, Remark 5.3]. The approach developed here resolves this.

Theorem 2.3. *Let $\mathcal{T}$ be a positive linear form on $\mathcal{L}_{n,m}$ satisfying the Bernstein-Szegő condition of Theorem 2.2 with polynomial $p(z,w)$ having no zeros on $\mathbb{T}\times\overline{\mathbb{D}}$ and degree at most (n,m). Then*

$$\mathcal{K}_1 = P_{\mathcal{E}^1_{n-1,m}}\{g(z)a(z) : g \in \mathbb{C}[z], \deg g < \deg b\}$$
$$\mathcal{K}_2 = P_{\mathcal{E}^1_{n-1,m}}\{g(z)b(z) : g \in \mathbb{C}[z], \deg g < n - \deg b\},$$

satisfy the split-shift condition. Here $p(z,0) = a(z)b(z)$ where $a \in \mathbb{C}[z]$ has no zeros in $\overline{\mathbb{D}}$ and $b \in \mathbb{C}[z]$ has all zeros in $\mathbb{D}$.

More explicitly, if we form the span of the following projections of one variable polynomials

$$z^i a(z) - P_{w\mathcal{P}_{n-1,m}} z^i a(z) \text{ for } 0 \le i < \deg b$$

and

$$z^i b(z) - P_{w\mathcal{P}_{n-1,m}} z^i b(z) \text{ for } 0 \le i < n - \deg b$$

then the resulting subspaces satisfy all the orthogonality conditions in the split-shift definition.

Why should we emphasize the abstract looking split-shift condition in the first place? One answer to this is that the spaces in the split-shift condition appear naturally in the following sum of (hermitian) squares result that ends up being an important by-product of our work.

Theorem 2.4. *Suppose $p \in \mathbb{C}[z,w]$ has no zeros on $\mathbb{T}\times\overline{\mathbb{D}}$ and $\deg p = (n,m)$. Define $\overleftarrow{p}(z,w) = z^n w^m \overline{p(1/\bar{z}, 1/\bar{w})}$. Then, there exist polynomials $A_1,\dots,A_m$, $B_1,\dots,B_{n_1}$, $C_1,\dots,C_{n_2} \in \mathbb{C}[z,w]$ such that*

$$\begin{aligned}&|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2\\ &= (1-|w|^2)\sum_{j=1}^{m}|A_j(z,w)|^2 + (1-|z|^2)\left(\sum_{j=1}^{n_1}|B_j(z,w)|^2 - \sum_{j=1}^{n_2}|C_j(z,w)|^2\right)\end{aligned}$$

where n_2 is the number of zeros of $p(z,0)$ in $\mathbb{D}$ and $n_1 = n - n_2$. The same result holds if p has no zeros in $\mathbb{T}\times\mathbb{D}$ and no factors in common with $\overleftarrow{p}$.

The different sums of squares terms can be constructed from important subspaces of $L^2(\frac{|dz||dw|}{|p|^2})$: the A_j form an orthonormal basis of $\mathcal{E}^2_{n,m-1}$, the B_j form an orthonormal basis of $\overleftarrow{\mathcal{K}}_2$ (the reflection of $\mathcal{K}_2$), and the C_j form an orthonormal basis of $\mathcal{K}_1$. (See Theorem 5.5.) This formula illustrates how natural are the spaces in the split-shift condition, and it also reproves some important formulas as special cases.

When $n_2 = 0$, p is stable and we get the Cole-Wermer type of sum of squares formula [8] which can be used to prove Agler's Pick interpolation theorem on the bidisk; see also [11], [17], [10], [20], [7]. The exact numbers of squares involved in this case turned out to be important in recent work on extending Löwner's theory of matrix monotone functions to two variables in Agler-McCarthy-Young [5]. When $m = 0$ (i.e. p does not depend on w) we get a decomposition which readily implies part of the Schur-Cohn method for counting the roots of a polynomial inside and outside the unit circle.

The case of p with merely no zeros on $\mathbb{T}\times\mathbb{D}$ can be derived from a limiting argument as in [17]. We give a second proof of the sum of squares formula using ideas of Kummert [20] in Section 10.1. This proof should be of independent interest and has the advantage of working directly for all cases. See Section 10.2 for an application of the formula to proving a determinantal representation for a class of curves generalizing the *distinguished varieties* of Agler-McCarthy [2].

Now that we see that the split-shift condition is natural, we get into a deeper discussion of Theorem 2.2, and its extensions. From the maximum entropy principle [6] (see also the proof of Theorem 2.8) there are at most finitely many $p(z,w)$ for which the Bernstein-Szegő condition holds. Theorem 2.3 therefore gives one particular way to construct $\mathcal{K}_1, \mathcal{K}_2$ in the definition of split-shift and this way is uniquely determined by the choice of p. However, since a trigonometric polynomial factored as $|p|^2$ can potentially be factored in more than one such way—roughly speaking these polynomials can be obtained from one another by permuting the factors in $|p(z,w)|^2$ which depend only on z—each such factorization will yield spaces as in the split-shift condition via the above theorem. We prove that these are all the possible split-shift decompositions corresponding to $|p(z,w)|^2$. See Proposition 7.3.

While each choice of p in the factorization of $t = |p|^2$ yields a canonically associated pair of spaces $\mathcal{K}_1, \mathcal{K}_2$ in the split-shift condition, the matrix condition naturally gives rise to two canonical choices for such pairs.

Theorem 2.5. *Let $\mathcal{T}$ be a positive linear form on $\mathcal{L}_{n,m}$ satisfying the matrix condition of Theorem 2.2. Then, $(\mathcal{K}_1, \mathcal{K}_2) = (\mathcal{E}^1_{n-1,m} \ominus \mathcal{B}, \mathcal{B})$ satisfies the split-shift condition where*

$$\mathcal{B} = \vee\{T^j B f : f \in w\mathcal{F}^2_{n,m-1}, j = 0, 1, \dots\}.$$

Similarly, $(\mathcal{K}_1, \mathcal{K}_2) = (\mathcal{A}, \mathcal{E}^1_{n-1,m} \ominus \mathcal{A})$ satisfies the split-shift condition where

$$\mathcal{A} = \vee\{(T^*)^j A^* f : f \in w\mathcal{E}^2_{n,m-1}, j = 0, 1, \dots\}.$$

If $(\mathcal{K}_1', \mathcal{K}_2')$ is any other pair satisfying the split-shift condition, then $\mathcal{A} \subset \mathcal{K}_1'$ and $\mathcal{B} \subset \mathcal{K}_2'$.

To be clear, $T^* : \mathcal{E}^1_{n-1,m} \to \mathcal{E}^1_{n-1,m}$ is given by $P_{\mathcal{E}^1_{n-1,m}} M_{1/z}$ and $A^* : w\mathcal{E}^2_{n,m-1} \to \mathcal{E}^1_{n-1,m}$ is given by $P_{\mathcal{E}^1_{n-1,m}} M_{1/z}$. See Theorems 7.4 and 7.5 where we also show how the spaces in Theorem 2.5 relate to those in Theorem 2.3.

Theorems 2.3 and 2.5 directly show how the Bernstein-Szegő condition and the matrix condition yield the split-shift condition. On the other hand, if the split-shift condition holds, $\mathcal{K}_1 \oplus z\mathcal{K}_2$ has co-dimension one in $\mathcal{E}^1_{n,m}$ and we shall show that the Bernstein-Szegő condition holds using any unit norm element p in the one dimensional space $\mathcal{E}^1_{n,m} \ominus (\mathcal{K}_1 \oplus z\mathcal{K}_2)$. A sum of squares result related to Theorem 2.4 ends up being crucial here. In Section 8, we describe a simple procedure for constructing p from the moments $\mathcal{T}(z^j w^k)$ once we know the split-shift condition holds.

Our emphasis on the split-shift condition permits several interesting refinements that were not evident before. Notice that if p does not vanish on $\mathbb{T} \times \overline{\mathbb{D}}$, then the argument principle shows that the number of zeros of $p(\cdot, w)$ in $\mathbb{D}$ will be constant as w varies in $\overline{\mathbb{D}}$. Thus it is possible to prove a "stratified" version of Theorem 2.2, where we characterize factorizations involving p with no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$ such that $p(z,0)$ has a specified number of zeros in $\mathbb{D}$. See the end of Section 7 for the proof of the following corollary.

Corollary 2.6. *Let $\mathcal{T}$ be a positive linear form on $\mathcal{L}_{n,m}$ and let $0 \le d \le n$. The following are equivalent.*

(1) *(Bernstein-Szegő condition) There exists* $p \in \mathbb{C}[z,w]$ *with no zeros on* $\mathbb{T}\times\overline{\mathbb{D}}$, *degree at most* (n,m), *and where* $p(z,0)$ *has* d *zeros in* $\mathbb{D}$ *such that*

$$\mathcal{T}(z^j w^k) = \int_{\mathbb{T}^2} z^j w^k \frac{|dz||dw|}{(2\pi)^2|p(z,w)|^2} \qquad |j|\leq n, |k|\leq m.$$

(2) *(Split-shift condition)* $\mathcal{T}$ *satisfies the split-shift orthogonality condition where* $\mathcal{K}_1$ *has dimension* d.

(3) *(Matrix condition) The invariant subspace of* T *generated by the range of* B *is contained in the kernel of* A, *and*

$$\dim \mathcal{A} \leq d \leq n - \dim \mathcal{B}.$$

Note $\mathcal{A}$ *and* $\mathcal{B}$ *are as in Theorem 2.5.*

In particular, the case $d=0$ yields the Geronimo-Woerdeman result (as well as much simpler looking conditions). In this case, the split-shift condition merely says

$$z\mathcal{E}^1_{n-1,m} \subset \mathcal{E}^1_{n,m}. \tag{2.7}$$

The matrix condition in this case implies $\mathcal{A}=\{0\}$ which implies $A=0$. Since the range of A is $P_{w\mathcal{E}^2_{n,m-1}} z\mathcal{E}^1_{n-1,m}$, this means $w\mathcal{E}^2_{n,m-1} \perp z\mathcal{E}^1_{n-1,m}$, which is equivalent to (2.7) because of the orthogonal decomposition

$$\mathcal{E}^1_{n,m} \oplus w\mathcal{E}^2_{n,m-1} = z\mathcal{E}^1_{n-1,m} \oplus \mathcal{E}^2_{n,m}.$$

By performing the reflection operation, $w\mathcal{E}^2_{n,m-1} \perp z\mathcal{E}^1_{n-1,m}$ is equivalent to $\mathcal{F}^1_{n-1,m} \perp \mathcal{F}^2_{n,m-1}$.

Corollary 2.7 (Geronimo-Woerdeman [11]). *Let* $\mathcal{T}$ *be a positive linear form on* $\mathcal{L}_{n,m}$. *There exists* $p \in \mathbb{C}[z,w]$ *with no zeros on* $\overline{\mathbb{D}}^2$ *and degree at most* (n,m) *such that*

$$\mathcal{T}(z^j w^k) = \int_{\mathbb{T}^2} z^j w^k \frac{|dz||dw|}{(2\pi)^2|p(z,w)|^2} \qquad |j|\leq n, |k|\leq m,$$

if and only if

$$\mathcal{F}^1_{n-1,m} \perp \mathcal{F}^2_{n,m-1}.$$

To use the language of [22], the last condition can be neatly phrased as saying $\mathcal{P}_{n-1,m}$ and $\mathcal{P}_{n,m-1}$ intersect at right angles.

As in [9], Theorem 2.2 allows us to characterize when a positive two variable trigonometric polynomial can be factored as $|p(z,w)|^2$ on $\mathbb{T}^2$ where p has no zeros in $\mathbb{T}\times\overline{\mathbb{D}}$.

Theorem 2.8. *Suppose* $t(z,w) = \sum_{j=-n}^{n}\sum_{k=-m}^{m} t_{jk} z^j w^k > 0$ *for* $(z,w) \in \mathbb{T}^2$. *Then, there exists* $p \in \mathbb{C}[z,w]$ *of degree at most* (n,m) *with no zeros on* $\mathbb{T}\times\overline{\mathbb{D}}$ *such that* $t=|p|^2$ *on* $\mathbb{T}^2$ *if and only if the positive linear form* $\mathcal{T}$ *on* $\mathcal{L}_{n,m}$

$$\mathcal{T}(z^j w^k) = \int_{\mathbb{T}^2} z^j w^k \frac{|dz||dw|}{(2\pi)^2 t(z,w)} \qquad |j|\leq n, |k|\leq m$$

satisfies the split-shift condition.

See the end of Section 5 for a proof of this theorem.

We say a finite, positive Borel measure μ on $\mathbb{T}^2$ is *non-degenerate* if

$$\int_{\mathbb{T}^2} |f|^2 d\mu > 0,$$

for every nonzero polynomial $f \in \mathbb{C}[z,w]$. We next turn to the problem of characterizing which such measures μ on $\mathbb{T}^2$ are of the form

$$\frac{1}{|p|^2}d\sigma \tag{2.8}$$

where $p \in \mathbb{C}[z,w]$ has no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$ and degree at most (n,m); $d\sigma$ denotes normalized Lebesgue measure on $\mathbb{T}^2$.

Some necessary conditions turn out to be

$$\mathcal{E}^2_{n,M} = \mathcal{E}^2_{n+j,M} \tag{2.9}$$

for $M \geq m-1$ and $j \geq 0$. These conditions are most likely not sufficient though.

Surprisingly, in [9], it was noticed that conditions (2.9) combined with the analogous conditions obtained by interchanging the roles of z and w characterize when μ has the form

$$\frac{1}{|p(z,w)q(1/z,w)|^2}d\sigma$$

where $p,q \in \mathbb{C}[z,w]$ have no zeros in $\overline{\mathbb{D}}^2$.

We now provide the following necessary and sufficient conditions for μ to have the form (2.8). Define the following one dimensional spaces

$$\mathcal{H}_M := \mathcal{P}_{2n,M} \ominus \vee\{z^j w^k : 0 \leq j \leq 2n, 0 \leq k \leq M, (j,k) \neq (n,0)\}. \tag{2.10}$$

Theorem 2.9. *Let $d\mu$ be a non-degenerate, finite, positive Borel measure on $\mathbb{T}^2$. There exists $p \in \mathbb{C}[z,w]$ of degree at most (n,m) with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ such that*

$$d\mu = \frac{d\sigma}{|p|^2}$$

if and only if

$$\mathcal{E}^2_{n,M} = \mathcal{E}^2_{n+j,M} \text{ and } \mathcal{H}_m = \mathcal{H}_{m+j}$$

for $M \geq m-1$ and $j \geq 0$.

This theorem is proved in Section 9.2, and in 9.3 it is expressed concretely in terms of the moments of μ. In Section 9.1, we discuss the close connection of our main theorem, Theorem 2.2, to autoregressive filters as was done in [11].

3. Basic orthogonalities of Bernstein-Szegő measures

The next two sections are occupied with proving that the Bernstein-Szegő condition implies the split-shift condition in Theorem 2.2, which is the content of Theorem 4.8. The approach is an extension of [10].

Let $p \in \mathbb{C}[z,w]$ and assume $p(z,w) \neq 0$ for $(z,w) \in \mathbb{T} \times \overline{\mathbb{D}}$. Let $\deg p \leq (n,m)$, $\overleftarrow{p}(z,w) = z^n w^m \overline{p(1/\bar{z}, 1/\bar{w})}$. Let $d\sigma$ denote normalized Lebesgue measure on $\mathbb{T}^2$. We use $d\sigma_1(z) = |dz|/(2\pi)$ or $d\sigma_1(w) = |dw|/(2\pi)$ to denote normalized Lebesgue measure on $\mathbb{T}$ using the variable z or w. We use $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(1/|p|^2 d\sigma, \mathbb{T}^2)$ and $\vee$ to denote closed linear span in both $L^2(\mathbb{T}^2)$ and $L^2(1/|p|^2 d\sigma)$. This is legitimate because $|p|$ is bounded above and below on $\mathbb{T}^2$ so the identity map on $\mathbb{C}[z,w]$ extends to a homeomorphism of $L^2(1/|p|^2 d\sigma)$ to $L^2(\mathbb{T}^2)$. The next lemma shows that p and $\overleftarrow{p}$ are orthogonal to all monomials in half planes.

Lemma 3.1. *In* $L^2(\frac{1}{|p|^2}d\sigma)$,

$$p \perp z^j w^k \text{ for } j \in \mathbb{Z}, k \geq 1 \tag{3.1}$$

and

$$\overleftarrow{p} \perp z^j w^k \text{ for } j \in \mathbb{Z}, k < m. \tag{3.2}$$

Also,

$$\vee\{z^j p : j \in \mathbb{Z}\} = \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k \leq m\} \ominus \vee\{z^j w^k : j \in \mathbb{Z}, 1 \leq k \leq m\}.$$

Proof.

$$\langle z^j w^k, p\rangle = \int_{\mathbb{T}} z^j \int_{\mathbb{T}} \frac{w^k}{p(z,w)} d\sigma_1(w) d\sigma_1(z) = 0$$

for $k \geq 1$ since $1/p(z,w)$ is holomorphic in $w \in \overline{\mathbb{D}}$ when $z \in \mathbb{T}$. The proof for $\overleftarrow{p}$ is similar.

For the final part, we have just shown the inclusion $\subset$. On the other hand, if $f \in \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k \leq m\} \ominus \vee\{z^j w^k : j \in \mathbb{Z}, 1 \leq k \leq m\}$ and $f \perp z^j p$ for all $j \in \mathbb{Z}$, then

$$0 = \int_{\mathbb{T}^2} \frac{f(z,w)z^{-j}}{p(z,w)} d\sigma = \int_{\mathbb{T}} \frac{f(z,0)z^{-j}}{p(z,0)} d\sigma_1(z)$$

for all $j \in \mathbb{Z}$ implies $f(z,0)/p(z,0) = 0$ for a.e. $z \in \mathbb{T}$. (Note that $f(z,0)$ should be interpreted as $\sum_{j\in\mathbb{Z}} \hat{f}(j,0)z^j$ in L^2.) Therefore, $f(z,0) = 0$ which implies $f \in \vee\{z^j w^k : j \in \mathbb{Z}, 1 \leq k \leq m\}$ making f orthogonal to itself. So, $f = 0$. $\square$

Define

$$J_\eta(z,w) = z^n \frac{p(z,w)\overline{p(1/\bar{z},\eta)}}{1 - w\bar{\eta}}$$

$$H_\eta(z,w) = z^n \frac{\overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar{z},\eta)}}{1 - w\bar{\eta}}.$$

By (3.2) and expanding the denominator in H_η, we see that $H_\eta \perp \vee\{z^j w^k : j \in \mathbb{Z}, k < m\}$ for $\eta \in \mathbb{D}$. Similarly $J_\eta \perp \vee\{z^j w^k : j \in \mathbb{Z}, k \geq 0\}$ for $|\eta| > 1$ since for $(z,w) \in \mathbb{T}^2$

$$J_\eta(z,w) = \frac{-\bar{w}}{\bar{\eta}} z^n \frac{p(z,w)\overline{p(1/\bar{z},\eta)}}{1 - \bar{w}/\bar{\eta}}.$$

Define

$$L_\eta(z,w) = L(z,w;\eta) = z^n \frac{p(z,w)\overline{p(1/\bar{z},\eta)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar{z},\eta)}}{1 - w\bar{\eta}} = J_\eta(z,w) - H_\eta(z,w)$$

which is a polynomial in $(z,w,\bar{\eta})$ of degree $(2n, m-1, m-1)$. Notice that

$$z^{2n}(w\bar{\eta})^{m-1}\overline{L(1/\bar{z}, 1/\bar{w}; 1/\bar{\eta})} = \bar{\eta}^{m-1}\overleftarrow{L}_{1/\bar{\eta}}(z,w) = L_\eta(z,w). \tag{3.3}$$

Similarly we define

$$G_\eta(z,w) = G(z,w;\eta) = J_\eta(z,w) - w\bar{\eta}H_\eta(z,w)$$

which is a polynomial in $(z,w,\bar{\eta})$ of degree $(2n, m, m)$. Note that the reflection symmetry for $L_\eta(z,w)$ implies the following symmetry for $G_\eta(z,w)$

$$z^{2n}(w\bar{\eta})^m \overline{G_{1/\bar{\eta}}(1/\bar{z}, 1/\bar{w})} = G_\eta(z,w). \tag{3.4}$$

Up to factors of z^n, $L_\eta(z,w)$ and $G_\eta(z,w)$ are parametrized one-variable Christoffel-Darboux kernels. In the next few lemmas, the orthogonality properties of p and $\overleftarrow{p}$ are used to obtain orthogonality properties on pieces of these kernels.

Lemma 3.2. *If $f \in L^2$ and supp$(\hat{f}) \subset \mathbb{Z}\times\mathbb{Z}_+$, then in $L^2(1/|p|^2 d\sigma)$*

$$\langle f, J_\eta\rangle = \sum_{k\geq 0} \hat{f}(n,k)\eta^k$$

for $\eta \in \mathbb{D}$. In particular, $J_\eta \perp \vee\{z^j w^k : k \geq 0, j \neq n\}$ for $\eta \in \mathbb{D}$.

Proof.

$$\begin{aligned}\langle f, J_\eta\rangle &= \iint_{\mathbb{T}^2} \frac{f(z,w)}{p(z,w)} \frac{\bar{z}^n p(z,\eta)}{1-\bar{w}\eta} d\sigma_1(w) d\sigma_1(z)\\ &= \int_{\mathbb{T}} \frac{f(z,\eta)}{p(z,\eta)} p(z,\eta)\bar{z}^n d\sigma_1(z)\\ &= \sum_{k\geq 0}\hat{f}(n,k)\eta^k.\end{aligned}$$

□

Lemma 3.3. *In $L^2(1/|p|^2 d\sigma)$, for all $\eta \in \mathbb{C}$*

$$L_\eta \perp \vee\{z^j w^k : j \neq n, 0 \leq k < m\}$$

and for $f \in \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k < m\}$

$$\langle f, L_\eta\rangle = \sum_{k=0}^{m-1} \hat{f}(n,k)\eta^k.$$

Proof. Note $L_\eta = J_\eta - H_\eta$. As $f \perp H_\eta$ and $\langle f, J_\eta\rangle = \sum_{k=0}^{m-1}\hat{f}(n,k)\eta^k$, we see that the desired formula holds for $\eta \in \mathbb{D}$. Since both sides are polynomials in η, the formula holds for all $\eta \in \mathbb{C}$. □

Corollary 3.4. *In $L^2(1/|p|^2 d\sigma)$*

$$\begin{aligned}\vee\{L_\eta : \eta \in \mathbb{D}\} =& \mathcal{P}_{2n,m-1} \ominus (\mathcal{P}_{n-1,m-1} \vee z^{n+1}\mathcal{P}_{n-1,m-1})\\ &= \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k < m\}\\ &\quad \ominus \vee\{z^j w^k : j \neq n, 0 \leq k < m\}.\end{aligned}$$

Proof. We have already shown the L_η's are in the orthogonal complements on the right. On the other hand, if any f (in either orthogonal complement space) is orthogonal to L_η for all η, then $\hat{f}(n,k) = 0$ for $0 \leq k < m$, implying $f = 0$. □

The next Proposition (see also Corollary 3.7) shows that certain orthogonal subspaces are mapped into each other by multiplication by z.

Proposition 3.5. *In $L^2(1/|p|^2 d\sigma)$*

$$\begin{aligned}&\mathcal{P}_{\infty,m-1} \ominus \mathcal{P}_{n-1,m-1} = \vee\{z^j L_\eta : j \geq 0, \eta \in \mathbb{D}\}\\ &= \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k < m\} \ominus \vee\{z^j w^k : j < n, 0 \leq k < m\}\end{aligned}$$

$$\begin{aligned}&\mathcal{P}_{\infty,m} \ominus \mathcal{P}_{n-1,m} = \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\}\\ &= \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k \leq m\} \ominus \vee\{z^j w^k : j < n, 0 \leq k \leq m\}.\end{aligned}$$

Proof. By the Corollary, $z^j L_\eta$ is in the orthogonal complement spaces for all $j \geq 0, \eta \in \mathbb{D}$. On the other hand, if anything in these orthogonal complements is orthogonal to $z^j L_\eta$ for all $j \geq 0, \eta \in \mathbb{D}$, then such an element will have no Fourier support in the set $\{(j,k) : j \geq n, 0 \leq k < m\}$ and will be orthogonal to itself.

We get similar decompositions when we use G_η instead of L_η and allow $k = m$. $\square$

If we apply the anti-unitary reflection operation $\overleftarrow{\cdot}$ at degree $(n-1, m-1)$ we get other useful decompositions

$$
\begin{aligned}
(3.5) \qquad \vee\{z^j w^k : j < n, 0 \leq k < m\} \ominus \mathcal{P}_{n-1,m-1} &= \vee\{z^{j-n} L_\eta : j < 0, \eta \in \mathbb{D}\} \\
&= \vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k < m\} \ominus \vee\{z^j w^k : j \geq 0, 0 \leq k < m\}.
\end{aligned}
$$

Multiplication of the above equation by z gives the following important consequence which provides necessary conditions for the full measure characterization in Section 9.2.

Corollary 3.6. *In* $L^2(1/|p|^2 d\sigma)$,

$$\mathcal{E}^2_{n,M} \perp z\mathcal{P}_{\infty,M}$$

for all $M \geq m-1$.

We get this for all $M \geq m-1$ simply because we can view p as a polynomial of degree at most $(n, M+1)$ for any $M \geq m-1$. Similarly, we obtain the following corollary.

Corollary 3.7. *In* $L^2(1/|p|^2 d\sigma)$,

$$\mathcal{P}_{\infty,M} \ominus \mathcal{P}_{n,M} = z(\mathcal{P}_{\infty,M} \ominus \mathcal{P}_{n-1,M}),$$

for all $M \geq m-1$.

4. Bernstein-Szegő condition implies split-shift condition

Using the same setup as the previous section, we now delve into the more refined orthogonalities necessary to prove that the Bernstein-Szegő condition implies the split-shift condition in Theorem 2.2. Write $p(z,0) = a(z)b(z)$ where a has no zeros in $\overline{\mathbb{D}}$ and b has all zeros in $\mathbb{D}$. Let $\beta := \deg b$ and $\overleftarrow{b}(z) = z^\beta \overline{b(1/\bar{z})}$.

Lemma 4.1. *In* $L^2(1/|p|^2 d\sigma)$, *for* $\eta \in \mathbb{D}$, *we have*

$$az^j \perp wJ_\eta$$

for all $j < \beta$.

For $|\eta| > 1$,

$$w^m \overleftarrow{b} z^j \perp H_\eta$$

for all $j < n - \beta$.

Proof. Observe that for $j < \beta$ and $0 < |\eta| < 1$, $\langle az^j, wJ_\eta\rangle$ equals

$$\begin{aligned}
\iint_{\mathbb{T}^2} \frac{a(z)z^j}{p(z,w)}\frac{\bar{w}\bar{z}^n p(z,\eta)}{1-\bar{w}\eta}\frac{dw}{2\pi i w}d\sigma_1(z) &= \int_{\mathbb{T}} z^{j-n}a(z)p(z,\eta)\int_{\mathbb{T}}\frac{dw}{2\pi i p(z,w)(w-\eta)w}d\sigma_1(z)\\
&= \int_{\mathbb{T}} z^{j-n}a(z)p(z,\eta)\left(\frac{1}{\eta p(z,\eta)} - \frac{1}{\eta p(z,0)}\right)d\sigma_1(z)\\
&= \frac{1}{\eta}\left(\int_{\mathbb{T}}\bar{z}^{n-j}a(z)d\sigma_1(z) - \int_{\mathbb{T}}\frac{z^{j-n}p(z,\eta)}{b(z)}d\sigma_1(z)\right)\\
&= -\frac{1}{\eta}\overline{\int_{\mathbb{T}}\frac{z^{n+\beta-j}\overline{p(z,\eta)}}{\overleftarrow{b}(z)}d\sigma_1(z)} = 0
\end{aligned}$$

since $n - j > \deg a$. The proof is easier when $\eta = 0$.

For $|\eta| > 1$, $\overline{\langle w^m\overleftarrow{b}z^j, H_\eta\rangle}$ equals

$$\begin{aligned}
\langle w^m\overleftarrow{b}z^j, z^n\frac{\overleftarrow{p}\,\overline{\overleftarrow{p}(z,\eta)}}{1-w\bar{\eta}}\rangle^* &= \int_{\mathbb{T}} z^{n-\beta-j}b\bar{\eta}^m p(z,1/\bar{\eta})\int_{\mathbb{T}}\frac{1}{p(z,w)\bar{\eta}(1/\bar{\eta}-w)}\frac{dw}{2\pi i w}d\sigma_1(z)\\
&= \int_{\mathbb{T}} z^{n-\beta-j}b\bar{\eta}^m p(z,1/\bar{\eta})\left(-\frac{1}{p(z,1/\bar{\eta})} + \frac{1}{p(z,0)}\right)d\sigma_1(z)\\
&= -\int_{\mathbb{T}} z^{n-\beta-j}b\bar{\eta}^m d\sigma_1(z) + \int_{\mathbb{T}}\bar{\eta}^m z^{n-\beta-j}\frac{p(z,1/\bar{\eta})}{a(z)}d\sigma_1(z)\\
&= 0
\end{aligned}$$

for $n - \beta > j$. $\square$

Lemma 4.2. *In $L^2(1/|p|^2 d\sigma)$, if $f \in \vee\{z^j : j \geq 0\}$, then $f \perp z^k p$ for all $k \geq 0$ if and only if $f(z) = a(z)q(z)$ where $q \in \mathbb{C}[z]$ has degree less than β.*

If $f \in \vee\{w^m z^j : j \geq 0\}$, then $f \perp z^k\overleftarrow{p}$ for all $k \geq 0$ if and only if $f(z,w) = w^m\overleftarrow{b}(z)q(z)$ where $q \in \mathbb{C}[z]$ has degree less than $n - \beta$.

Proof. If $z^k p \perp f \in \vee\{z^j : j \geq 0\}$ for all $k \geq 0$ then

$$0 = \iint_{\mathbb{T}^2}\bar{z}^k\frac{f(z)}{p(z,w)}d\sigma_1(w)d\sigma_1(z) = \int_{\mathbb{T}}\frac{\bar{z}^k f(z)}{p(z,0)}d\sigma_1(z)$$

for all $k \geq 0$ implies $f(z)/p(z,0) = \bar{z}\overline{g(z)}$ for $g \in H^2(\mathbb{T}) = \vee\{z^j : j \geq 0\}$. Then, $f(z) = p(z,0)\bar{z}\overline{g(z)}$ and so

$$z^{n-1}\overline{f(z)} = z^n\overline{p(z,0)}g(z) \in H^2$$

implies $f(z)$ is a polynomial of degree at most $n-1$. In addition, $\overleftarrow{f} = \overleftarrow{a}\overleftarrow{b}g$ implies $\overleftarrow{a}$ divides $\overleftarrow{f}$. (We reflect b at degree β and a at degree $n - \beta$.) So, $\overleftarrow{f} = \overleftarrow{a}h$ where $h \in \mathbb{C}[z]$ has degree less than β. Finally, $f = aq$ where $q \in \mathbb{C}[z]$ has degree less than β.

For the converse, let $f = aq$. Then,

$$\begin{aligned}
\iint_{\mathbb{T}^2}\frac{\bar{z}^k f(z)}{p(z,w)}d\sigma_1(w)d\sigma_1(z) &= \int_{\mathbb{T}}\frac{\bar{z}^k f(z)}{p(z,0)}d\sigma_1(z) = \int_{\mathbb{T}}\frac{\bar{z}^k q(z)}{b(z)}d\sigma_1(z)\\
&= \overline{\int_{\mathbb{T}}\frac{z^{k+\beta}\overline{q(z)}}{\overleftarrow{b}(z)}d\sigma_1(z)} = \overline{\int_{\mathbb{T}}\frac{z^{k+1}\overleftarrow{q}(z)}{\overleftarrow{b}(z)}d\sigma_1(z)} = 0
\end{aligned}$$

for all $k \geq 0$.

The proof of the second part is very similar. $\square$

Define

$$\mathcal{K} := \mathcal{P}_{\infty,m} \ominus w\mathcal{P}_{\infty,m-1}$$
$$\mathcal{L} := \mathcal{P}_{\infty,m} \ominus \mathcal{P}_{\infty,m-1}.$$

Notice

$$\vee\{z^j p : j \geq 0\} \subset \mathcal{K}$$

and

$$\vee\{z^j \overleftarrow{p} : j \geq 0\} \subset \mathcal{L}.$$

Let P_0 denote orthogonal projection onto $\mathcal{P}_{\infty,m-1}$ and $P_0^\perp = I - P_0$.

Let P_1 denote orthogonal projection onto $w\mathcal{P}_{\infty,m-1}$ and $P_1^\perp = I - P_1$.

The next two lemmas and corollary construct the spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ in the split-shift condition in Definition 2.1.

Lemma 4.3. *In* $L^2(1/|p|^2 d\sigma)$

$$\mathcal{K} \ominus \vee\{z^j p : j \geq 0\} = P_1^\perp(\vee\{az^j : 0 \leq j < \beta\})$$
$$\mathcal{L} \ominus \vee\{z^j \overleftarrow{p} : j \geq 0\} = P_0^\perp(\vee\{w^m \overleftarrow{b} z^j : 0 \leq j < n - \beta\}).$$

Proof. Let $f \in \mathcal{K}$ and $f \perp z^j p$ for all $j \geq 0$. Write $f(z,w) = f(z,0) - wg(z,w)$ where $g \in \mathcal{P}_{\infty,m-1}$ and notice that $P_1 f = 0 = P_1(f(z,0)) - wg(z,w)$ so that $f = f(z,0) - P_1 f(z,0) = P_1^\perp f(z,0)$. Since $P_1 f(z,0) \perp z^j p$ for all $j \geq 0$, we see that $f(z,0) \perp z^j p$ for all $j \geq 0$. By Lemma 4.2, $f(z,0) = a(z)q(z)$ where $\deg q < \beta$. This shows the inclusion $\subset$.

On the other hand, $P_1^\perp(az^j) = az^j - P_1 az^j \in \mathcal{K}$, $P_1(az^j) \perp z^k p$ for all $k \geq 0$, and $az^j \perp z^k p$ for all $k \geq 0$ by Lemma 4.2.

The second equation has a similar proof.

□

Lemma 4.4. *In* $L^2(1/|p|^2 d\sigma)$

$$P_1^\perp(\vee\{az^j : 0 \leq j < \beta\}) \subset \mathcal{P}_{n-1,m}$$
$$P_0^\perp(\vee\{w^m \overleftarrow{b} z^j : 0 \leq j < n - \beta\}) \subset \mathcal{P}_{n-1,m}.$$

Proof. For $0 \leq j < \beta$, $P_1^\perp(az^j) = az^j - P_1(az^j)$. Clearly, $az^j \in \mathcal{P}_{n-1,m}$, so the main thing to show is that $P_1(az^j) \in \mathcal{P}_{n-1,m}$.

For $k \geq 0$, $\eta \in \mathbb{D}$

$$\begin{aligned}
\langle P_1(az^j), wz^k L_\eta \rangle &= \langle az^j, wz^k L_\eta \rangle && \text{since } wz^k L_\eta \in w\mathcal{P}_{\infty,m-1} \\
&= \langle az^{j-k}, wL_\eta \rangle \\
&= \langle az^{j-k}, wJ_\eta \rangle && \text{since } az^{j-k} \perp wH_\eta \text{ when } |\eta| < 1 \\
&= 0
\end{aligned}$$

by Lemma 4.1. On the other hand, since $f = \bar{w} P_1(az^j)$ is an element of $\mathcal{P}_{\infty,m-1}$

$$\langle P_1(az^j), wz^k L_\eta \rangle = \langle f \bar{z}^k, L_\eta \rangle = \sum_{t=0}^{m-1} \hat{f}(n+k,t)\eta^t \equiv 0$$

by Lemma 3.3. So, $\hat{f}(j,k) = 0$ for $j \geq n$ and $k \in \mathbb{Z}$. This shows $P_1(az^j) = wf \in \mathcal{P}_{n-1,m}$.

For the second inclusion, the main thing to show is $f = P_0(w^m \overleftarrow{b} z^j) \in \mathcal{P}_{n-1,m}$ for $0 \leq j < n-\beta$. For $\eta \in \mathbb{C}$,

$$\langle f, z^k L_\eta \rangle = \sum_{t=0}^{m-1} \hat{f}(n+k,t)\eta^t$$

on one hand, while for $|\eta| > 1$

$$\begin{aligned}\langle f, z^k L_\eta \rangle &= \langle w^m \overleftarrow{b} z^j, z^k L_\eta \rangle \\ &= \langle w^m z^{j-k} \overleftarrow{b}, -H_\eta \rangle \text{ since } J_\eta \perp w^m z^{j-k} \overleftarrow{b} \text{ when } |\eta| > 1 \\ &= 0\end{aligned}$$

by Lemma 4.1 since $j - k < n - \beta$. This implies $\hat{f}(n+k,t) = 0$ for $k \geq 0$ and $0 \leq t < m$ as desired.

□

Set

$$\mathcal{K}_1 = \mathcal{K} \ominus \vee\{z^j p : j \geq 0\}.$$
$$\mathcal{L}_1 = \mathcal{L} \ominus \vee\{z^j \overleftarrow{p} : j \geq 0\}.$$

Corollary 4.5. *In* $L^2(1/|p|^2 d\sigma)$

$$\begin{aligned}\mathcal{K}_1 &= P_{\mathcal{E}^1_{n-1,m}}(\vee\{az^j : 0 \leq j < \beta\}) \subset \mathcal{E}^1_{n,m}, \\ \mathcal{L}_1 &= P_{\mathcal{F}^1_{n-1,m}}(\vee\{w^m \overleftarrow{b} z^j : 0 \leq j < n-\beta\}) \subset \mathcal{F}^1_{n,m}, \\ \overleftarrow{\mathcal{L}}_1 &= P_{\mathcal{E}^1_{n-1,m}}(\vee\{bz^j : 0 \leq j < n-\beta\}) \subset \bar{z}\mathcal{E}^1_{n,m}.\end{aligned}$$

Proof. By Lemmas 4.3 and 4.4, $\mathcal{K}_1$ is contained in both $\mathcal{E}^1_{n-1,m}$ and $\mathcal{E}^1_{n,m}$. Let $P^1_{n-1,m-1}$ denote orthogonal projection onto $w\mathcal{P}_{n-1,m-1}$. For any $f = az^j - P_1(az^j)$ we know $f \in \mathcal{P}_{n-1,m}$ for $0 \leq j < \beta$, and so we see that $P_1(az^j) = P^1_{n-1,m-1}P_1(az^j) = P^1_{n-1,m-1}(az^j)$. Therefore, $f = az^j - P^1_{n-1,m-1}(az^j) = P_{\mathcal{E}^1_{n-1,m}}(az^j)$, which proves $\mathcal{K}_1 = P_{\mathcal{E}^1_{n-1,m}}(\vee\{az^j : 0 \leq j < \beta\})$.

The second set of equations has a similar proof. The last set of equations follows from the second set by taking the reflection operation $\overleftarrow{\cdot}$ at the degree $(n-1,m)$. □

Let $\mathcal{K}_2 := \mathcal{E}^1_{n-1,m} \ominus \mathcal{K}_1$ so that

$$\mathcal{E}^1_{n-1,m} = \mathcal{K}_1 \oplus \mathcal{K}_2.$$

Similarly, define $\mathcal{L}_2$ so that

$$\mathcal{F}^1_{n-1,m} = \mathcal{L}_1 \oplus \mathcal{L}_2.$$

The next lemma gives a different characterization of the spaces $\mathcal{K}_2$ and $\mathcal{L}_2$.

Lemma 4.6. *In* $L^2(1/|p|^2 d\sigma)$

$$\begin{aligned}\vee\{z^j p : j \geq 0\} \oplus \vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\} \\ = \mathcal{K}_2 \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\}\end{aligned} \tag{4.1}$$

$$\begin{aligned}\vee\{z^j \overleftarrow{p} : j \geq 0\} \oplus \vee\{z^j L_\eta : j \geq 0, \eta \in \mathbb{D}\} \\ = \mathcal{L}_2 \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\}.\end{aligned} \tag{4.2}$$

Proof. Now,

$$\begin{aligned}
&\mathcal{P}_{\infty,m} \ominus w\mathcal{P}_{n-1,m-1} = (\mathcal{P}_{\infty,m} \ominus w\mathcal{P}_{\infty,m-1}) \oplus w(\mathcal{P}_{\infty,m-1} \ominus \mathcal{P}_{n-1,m-1}) \\
=&\mathcal{K} \oplus \vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\} \\
=&\mathcal{K}_1 \oplus \vee\{z^j p : j \geq 0\} \oplus \vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\}
\end{aligned}$$

by Proposition 3.5. The same set is equal to

$$\mathcal{E}^1_{n-1,m} \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\}$$

and after canceling $\mathcal{K}_1$ we obtain (4.1). The proof for $\mathcal{L}_2$ follows along the same lines by considering $\mathcal{P}_{\infty,m} \ominus \mathcal{P}_{n-1,m-1}$. □

Lemma 4.7. *In $L^2(1/|p|^2 d\sigma)$*

$$\begin{aligned}
\vee \{z^j p : j < 0\} &\oplus \vee\{wz^{j-n} L_\eta : j < 0, \eta \in \mathbb{D}\} \\
&= \overleftarrow{\mathcal{L}}_2 \oplus \vee\{z^{j-n} G_\eta : j < 0, \eta \in \mathbb{D}\}
\end{aligned} \tag{4.3}$$

where $\overleftarrow{\mathcal{L}}_2$ is obtained by reflecting $\mathcal{L}_2$ at degree $(n-1, m)$ and

$$\begin{aligned}
\vee \{z^j p : j \in \mathbb{Z}\} &\oplus \vee\{wz^{j-n} L_\eta : j < 0, \eta \in \mathbb{D}\} \oplus \vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\} \\
&= \vee\{z^{j-n} G_\eta : j < 0, \eta \in \mathbb{D}\} \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\} \oplus \mathcal{E}^1_{n-1,m}.
\end{aligned} \tag{4.4}$$

Proof. The first part follows from applying the reverse operation $\overleftarrow{\cdot}$ at the degree $(n-1, m)$ in (4.2) and by using (3.3) and (3.4).

The second part comes from decomposing

$$\vee\{z^j w^k : j \in \mathbb{Z}, 0 \leq k \leq m\} \ominus w\mathcal{P}_{n-1,m-1}$$

in two different ways. By Proposition 3.5, equation (3.5) and Lemma 3.1 it equals

$$\begin{aligned}
&\vee \{z^j w^k : j \in \mathbb{Z}, 0 \leq k \leq m\} \ominus \vee\{z^j w^k : j \in \mathbb{Z}, 1 \leq k \leq m\} \\
&\oplus \vee\{wz^{n-j} L_\eta : j < 0, \eta \in \mathbb{D}\} \oplus \vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\} \\
= &\vee \{z^j p : j \in \mathbb{Z}\} \oplus \vee\{wz^{j-n} L_\eta : j < 0, \eta \in \mathbb{D}\} \oplus \vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\},
\end{aligned}$$

while it also equals the right hand side of (4.4). □

Theorem 4.8. *Assume $p \in \mathbb{C}[z, w]$ has no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$ and degree at most (n, m). In $L^2(1/|p|^2 d\sigma)$, for*

$$\mathcal{K}_1 = P_{\mathcal{E}^1_{n-1,m}}(\vee\{az^j : 0 \leq j < \beta\})$$

$$\overleftarrow{\mathcal{L}}_1 = P_{\mathcal{E}^1_{n-1,m}}(\vee\{bz^j : 0 \leq j < n - \beta\})$$

we have

$$\mathcal{E}^1_{n-1,m} = \mathcal{K}_1 \oplus \overleftarrow{\mathcal{L}}_1$$

and

$$\mathcal{E}^1_{n,m} = \mathcal{K}_1 \oplus z\overleftarrow{\mathcal{L}}_1 \oplus \mathbb{C}p.$$

Consequently, the split-shift orthogonality condition holds for a positive linear form associated to a Bernstein-Szegő measure with p having no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$.

Proof. By Corollary 4.5, it is enough to prove

$$\overleftarrow{\mathcal{L}}_1 = \mathcal{K}_2$$

and

$$\mathcal{E}^1_{n,m} = \mathcal{K}_1 \oplus z\mathcal{K}_2 \oplus \mathbb{C}p.$$

The direct sum of the left sides of (4.1) and (4.3) yields the left side of (4.4). So, the direct sum of the corresponding right hand sides are equal which means

$$\begin{aligned} &\mathcal{K}_2 \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\} \oplus \overleftarrow{\mathcal{L}}_2 \oplus \vee\{z^{j-n} G_\eta : j < 0, \eta \in \mathbb{D}\} \\ = &\vee \{z^{j-n} G_\eta : j < 0, \eta \in \mathbb{D}\} \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\} \oplus \mathcal{E}^1_{n-1,m}. \end{aligned}$$

Therefore, $\mathcal{E}^1_{n-1,m} = \mathcal{K}_2 \oplus \overleftarrow{\mathcal{L}}_2$. But, $\mathcal{E}^1_{n-1,m} = \overleftarrow{\mathcal{F}}^1_{n-1,m} = \overleftarrow{\mathcal{L}}_1 \oplus \overleftarrow{\mathcal{L}}_2$ and so $\mathcal{K}_2 = \overleftarrow{\mathcal{L}}_1$.

We know $\mathbb{C}p, \mathcal{K}_1 \subset \mathcal{E}^1_{n,m}$ by Lemma 3.1 and Corollary 4.5. By definition of $\mathcal{K}_1$ we know $\mathcal{K}_1 \perp p$.

Now, using Corollary 3.7, we see that

$$\vee\{z^j w^k : j \geq 0, 0 \leq k \leq m\} \ominus \vee\{z^j w^k : 0 \leq j \leq n, 1 \leq k \leq m\}$$

decomposes into

$$\begin{aligned} &\mathcal{K} \oplus zw(\mathcal{P}_{\infty,m-1} \ominus \mathcal{P}_{n-1,m-1}) \\ = &\vee \{z^j p : j \geq 0\} \oplus \mathcal{K}_1 \oplus \vee\{zwz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\} \\ = &\mathbb{C}p \oplus \mathcal{K}_1 \oplus z(\vee\{z^j p : j \geq 0\}) \oplus z(\vee\{wz^j L_\eta : j \geq 0, \eta \in \mathbb{D}\}) \\ = &\mathbb{C}p \oplus \mathcal{K}_1 \oplus z(\mathcal{K}_2 \oplus \vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\}) \text{ by (4.1)} \\ = &\mathbb{C}p \oplus \mathcal{K}_1 \oplus z\mathcal{K}_2 \oplus z(\vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\}) \end{aligned}$$

but it also decomposes into

$$\mathcal{E}^1_{n,m} \oplus z(\vee\{z^j G_\eta : j \geq 0, \eta \in \mathbb{D}\})$$

and therefore

$$\mathcal{E}^1_{n,m} = \mathbb{C}p \oplus \mathcal{K}_1 \oplus z\mathcal{K}_2.$$

□

5. Split-shift condition implies Bernstein-Szegő condition

The goal now is to prove that the split-shift condition (see Definition 2.1) implies that $\mathcal{T}$ can be represented using a Bernstein-Szegő measure whose associated polynomial has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$.

We call the pair $(\mathcal{K}_1, \mathcal{K}_2)$ a *shift-split* of $\mathcal{E}^1_{n,m}$. By dimensional considerations $\mathcal{E}^1_{n,m} \ominus (\mathcal{K}_1 \oplus z\mathcal{K}_2)$ will be one dimensional, and therefore of the form $\mathbb{C}p$ for some unit norm p. We shall call p a *split-poly* associated to the shift-split. The point now will be to prove that a split-poly p has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ and along the way we will prove some interesting formulas for p (which will also give formulas for an arbitrary p with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ since we can apply our formulas to $1/|p|^2 d\sigma$).

There may be more than one shift-split of $\mathcal{E}^1_{n,m}$, but we shall see that each split-poly is associated to one shift-split. We will provide a description of all split-polys (and hence all shift-splits via the previous section) in Section 7.

Let $K_{j,k}$ be the reproducing kernel for $\mathcal{P}_{j,k}$ in $\mathcal{H}_\mathcal{T}$. Namely, for $(\zeta, \eta) \in \mathbb{C}^2$, $(K_{j,k})_{(\zeta,\eta)}(\cdot, \cdot) = K_{j,k}(\cdot, \cdot; \zeta, \eta)$ is the unique element of $\mathcal{P}_{j,k}$ such that

$$\langle f, (K_{j,k})_{(\zeta,\eta)} \rangle_\mathcal{T} = f(\zeta, \eta)$$

for all $f \in \mathcal{P}_{j,k}$.

Remark 5.1. We shall use some standard facts about reproducing kernels of polynomials on $\mathbb{T}^2$. See Section 3 of [17].

(1) The reproducing kernel of an orthogonal direct sum is the sum of the reproducing kernels.
(2) Shifting a subspace by z (resp. w) multiplies the reproducing kernel by $z\bar{\zeta}$ (resp. $w\bar{\eta}$).
(3) The "reflection" $\overleftarrow{\cdot}$ of a subspace "reflects" the reproducing kernel.

On this last point, if $\mathcal{H}$ is a subspace of polynomials of degree at most (j,k) and H is its reproducing kernel, the subspace

$$\overleftarrow{\mathcal{H}} := \{z^j w^k \bar{f}(1/z, 1/w) : f \in \mathcal{H}\}$$

has reproducing kernel

$$\overleftarrow{H}(z,w;\zeta,\eta) := (z\bar{\zeta})^j (w\bar{\eta})^k H(1/\bar{\zeta}, 1/\bar{\eta}; 1/\bar{z}, 1/\bar{w}).$$

The degree (j,k) at which we reflect will either be mentioned explicitly or will be the maximal degree of the elements of the subspace.

Using these manipulations we get the following formulas.

$$\begin{aligned}
E_j^1 &= K_{j,m} - w\bar{\eta}K_{j,m-1} = \text{ the reproducing kernel for } \mathcal{E}_{j,m}^1 \\
F_j^1 &= \overleftarrow{E}_j^1 = K_{j,m} - K_{j,m-1} = \text{ the reproducing kernel for } \mathcal{F}_{j,m}^1 \\
E_k^2 &= K_{n,k} - z\bar{\zeta}K_{n-1,k} = \text{ the reproducing kernel for } \mathcal{E}_{n,k}^2 \\
F_k^2 &= \overleftarrow{E}_k^2 = K_{n,k} - K_{n-1,k} = \text{ the reproducing kernel for } \mathcal{F}_{n,k}^2.
\end{aligned}$$

For example, the first formula follows from the orthogonal decomposition

$$\mathcal{P}_{j,m} = w\mathcal{P}_{j,m-1} \oplus \mathcal{E}_{j,m}^1.$$

We record some basic formulas which do not require any special orthogonality conditions. In fact, they are just the result of manipulating the equations above.

Lemma 5.2. *We have*

$$\begin{aligned}
E_j^1(z,w;\zeta,\eta) - F_j^1(z,w;\zeta,\eta) &= (1 - w\bar{\eta})K_{j,m-1}(z,w;\zeta,\eta) \\
E_k^2(z,w;\zeta,\eta) - F_k^2(z,w;\zeta,\eta) &= (1 - z\bar{\zeta})K_{n-1,k}(z,w;\zeta,\eta).
\end{aligned}$$

If $\{E_0(z,w), \dots, E_m(z,w)\}$ is an orthonormal basis for $\mathcal{E}_{n,m}^2$ then we write

$$\vec{E}_m^2(z,w) = (E_0(z,w), \dots, E_m(z,w)) = (1, w, \dots, w^m)E_m^2(z)$$

for an appropriate $(m+1) \times (m+1)$ matrix polynomial $E_m^2(z)$. Then,

$$E_m^2(z,w;\zeta,\eta) = \vec{E}_m^2(z,w)\vec{E}_m^2(\zeta,\eta)^* = (1, w, \dots, w^m)E_m^2(z)E_m^2(\zeta)^*(1, \eta, \dots, \eta^m)^*.$$

Lemma 5.3. *The matrix polynomial $E_m^2(z)$ is invertible for all $z \in \overline{\mathbb{D}}$.*

Proof. Suppose $E_m^2(z_0)$ is singular for some $z_0 \in \mathbb{C}$ and choose nonzero $v \in \mathbb{C}^{m+1}$ such that $E_m^2(z_0)v = 0$. Then,

$$f(z,w) = \vec{E}_m^2(z,w)v = (1, w, \dots, w^m)E_m^2(z)v$$

is in $\mathcal{E}_{n,m}^2$, and $f(z_0, w) = 0$ for all w. So, $f(z,w) = (z - z_0)g(z,w)$ for some $g \in \mathcal{P}_{n-1,m}$. Since $f \perp zg$ we have

$$\|f - zg\|^2 = \|f\|^2 + \|g\|^2 = |z_0|^2\|g\|^2.$$

Then, $\|f\|^2 = \|v\|^2 = (|z_0|^2 - 1)\|g\|^2$ which implies $|z_0| > 1$. $\square$

Let K_1 be the reproducing kernel for $\mathcal{K}_1$, and let K_2 be the reproducing kernel for $\mathcal{K}_2$. When p has unit norm, the reproducing kernel for $\mathbb{C}p$ is $p(z,w)\overline{p(\zeta,\eta)}$ but we will simply write $p\bar{p}$.

Lemma 5.4. *If (K_1, K_2) is a shift-split of $\mathcal{E}^1_{n,m}$ with split-poly p then*

$$E^1_{n-1} = K_1 + K_2$$

$$E^1_n = K_1 + z\bar{\zeta}K_2 + p\bar{p}$$

$$F^1_{n-1} = \overleftarrow{K}_1 + \overleftarrow{K}_2$$

$$F^1_n = z\bar{\zeta}\overleftarrow{K}_1 + \overleftarrow{K}_2 + \overleftarrow{p}\overline{\overleftarrow{p}}$$

where the kernels $\overleftarrow{K}_1$ and $\overleftarrow{K}_2$ are reflected at the degree $(n-1, m)$.

Proof. These all follow from Remark 5.1 and the definition of shift-split. $\square$

Theorem 5.5. *If (K_1, K_2) is a shift-split of $\mathcal{E}^1_{n,m}$ with split-poly p then*

$$\begin{aligned} p\bar{p} - \overleftarrow{p}\overline{\overleftarrow{p}} &= (1 - w\bar{\eta})E^2_{m-1} + (1 - z\bar{\zeta})(\overleftarrow{K}_2 - K_1) \\ &= (1 - w\bar{\eta})F^2_{m-1} + (1 - z\bar{\zeta})(K_2 - \overleftarrow{K}_1) \\ &= (1 - w\bar{\eta})F^2_{m-1} + (1 - z\bar{\zeta})(\overleftarrow{K}_2 - K_1) + (1 - z\bar{\zeta})(1 - w\bar{\eta})K_{n-1,m-1} \end{aligned}$$

and

$$\begin{aligned} p\bar{p} - w\bar{\eta}\overleftarrow{p}\overline{\overleftarrow{p}} &= (1 - w\bar{\eta})E^2_m + (1 - z\bar{\zeta})(w\bar{\eta}\overleftarrow{K}_2 - K_1) \\ &= (1 - w\bar{\eta})F^2_m + (1 - z\bar{\zeta})(K_2 - w\bar{\eta}\overleftarrow{K}_1) \\ &= (1 - w\bar{\eta})F^2_m + (1 - z\bar{\zeta})(w\bar{\eta}\overleftarrow{K}_2 - K_1) + (1 - z\bar{\zeta})(1 - w\bar{\eta})K_{n-1,m}. \end{aligned}$$

Proof. Combining Lemmas 5.2 and 5.4, we get

$$z\bar{\zeta}(K_1 + K_2 - (\overleftarrow{K}_1 + \overleftarrow{K}_2)) = (1 - w\bar{\eta})z\bar{\zeta}K_{n-1,m-1}$$

and

$$K_1 + z\bar{\zeta}K_2 + p\bar{p} - (z\bar{\zeta}\overleftarrow{K}_1 + \overleftarrow{K}_2 + \overleftarrow{p}\overline{\overleftarrow{p}}) = (1 - w\bar{\eta})K_{n,m-1}.$$

Subtract these two formulas to get

$$(1 - z\bar{\zeta})(K_1 - \overleftarrow{K}_2) + p\bar{p} - \overleftarrow{p}\overline{\overleftarrow{p}} = (1 - w\bar{\eta})E^2_{m-1}$$

which rearranges to get the first desired formula. Similar arguments give the remaining formulas. $\square$

If $\mathcal{T}$ comes from a Bernstein-Szegő measure $1/|p|^2 d\sigma$ where p has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$, then Theorem 4.8 implies p is a split-poly and then the above theorem immediately implies Theorem 2.4, the sum of squares theorem from the introduction since reproducing kernels can be written as a sum of squares of an orthonormal basis. In general, the sum of squares formula implies a split-poly has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$.

Corollary 5.6. *If p is a split-poly, then p has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$.*

Proof. We use the second set of formulas in Theorem 5.5. Suppose $p(z,w) = 0$ for some $(z,w) \in \mathbb{T} \times \overline{\mathbb{D}}$. Setting $z = \zeta \in \mathbb{T}$ and $w = \eta \in \overline{\mathbb{D}}$ we have

$$|p(z,w)|^2 - |w|^2 |\overleftarrow{p}(z,w)|^2 = -|w\overleftarrow{p}(z,w)|^2 = (1-|w|^2) E^2_m(z,w;z,w) \geq 0$$

which shows $w\overleftarrow{p}(z,w) = 0$. Then, for arbitrary $\eta \in \mathbb{C}$ we have

$$0 = E^2_m(z,w;z,\eta) = E^2_m(z,w)E^2_m(z,\eta)^* = (1,w,\dots,w^m)E^2_m(z)E^2_m(z)^*(1,\eta,\dots,\eta^m)^*$$

which implies

$$0 = (1,w,\dots,w^m)E^2_m(z)E^2_m(z)^*$$

contradicting the fact that $E^2_m(z)$ is invertible from Lemma 5.3. □

We can now prove the split-shift orthogonality condition implies the Bernstein-Szegő condition in Theorem 2.2.

Corollary 5.7. *Suppose two positive linear forms $\mathcal{T}_1$ and $\mathcal{T}_2$ both satisfy the split-shift condition with the same split-poly p. Then, $\mathcal{T}_1 = \mathcal{T}_2$ and the linear forms agree with the linear form associated with the measure $1/|p|^2 d\sigma$.*

Proof. It is enough to show the reproducing kernels $K_{n,m}$ are the same for both forms. We can form a matrix polynomial $E^2_m(z)$ corresponding to each form $\mathcal{T}_1$ and $\mathcal{T}_2$, say $E_1(z)$ and $E_2(z)$ (just in this proof; we will not use this notation elsewhere). Using Theorem 5.5 for $z = \zeta \in \mathbb{T}$ and arbitrary $w, \eta \in \mathbb{C}$ we get $E_1(z)E_1(z)^* = E_2(z)E_2(z)^*$ for $z \in \mathbb{T}$ using arguments similar to the previous proof. Then,

$$E_2^{-1}(z)E_1(z) = \bar{E}_2(1/z)^t \bar{E}_1^{-1}(1/z)^t.$$

By Lemma 5.3, the left hand side is analytic for $|z| \leq 1$, while the right hand side is analytic for $|z| \geq 1$. By Liouville's theorem $E_2^{-1}(z)E_1(z) = V$ is a constant unitary matrix. This in turn implies the reproducing kernels $E^2_m(z,w;\zeta,\eta)$ for $\mathcal{T}_1$, $\mathcal{T}_2$ are the same. By the next lemma, we may conclude that $\mathcal{T}_1 = \mathcal{T}_2$. □

Lemma 5.8. *The inner product in $\mathcal{H}_{\mathcal{T}}$ is determined by the reproducing kernel E^2_m.*

Proof. Notice $F^2_m(z,w;\zeta,\eta) = (z\bar{\zeta})^n (w\bar{\eta})^m E^2_m(1/\bar{\zeta}, 1/\bar{\eta}; 1/\bar{z}, 1/\bar{w})$. So, E^2_m determines F^2_m. By Lemma 5.2, E^2_m determines $K_{n-1,m}$ and since $K_{n,m} = E^2_m + z\bar{\zeta}K_{n-1,m}$ we see that E^2_m determines $K_{n,m}$ as well. □

We can now prove Theorem 2.8.

Proof of Theorem 2.8. We already know that if $t = |p|^2$ then the split-shift condition holds. On the other hand, if the split-shift condition holds with split-poly p, then p has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ and the form corresponding to $1/|p|^2 d\sigma$ agrees with the form $\mathcal{T}$. Then, by Cauchy-Schwarz

$$1 = \left(\int_{\mathbb{T}^2} \frac{\sqrt{t}}{|p|} \frac{|p|}{\sqrt{t}} d\sigma \right)^2 \leq \int_{\mathbb{T}^2} \frac{t}{|p|^2} d\sigma \int_{\mathbb{T}^2} \frac{|p|^2}{t} d\sigma = \int_{\mathbb{T}^2} \frac{t}{t} d\sigma \int_{\mathbb{T}^2} \frac{|p|^2}{|p|^2} d\sigma = 1$$

since the forms agree. Since we have equality in our application of Cauchy-Schwarz, it is not hard to see $t = |p|^2$. □

6. The matrix condition

The abstract flavor of the split-shift orthogonality condition makes it difficult to check. This section is devoted to showing it is equivalent to checking that a number of natural operators vanish.

The "matrix condition" (2.6) from Theorem 2.2 can be viewed as saying the smallest invariant subspace of T containing the range of B is contained in the kernel of A.

Proposition 6.1. *If a positive linear form $\mathcal{T}$ satisfies the split-shift condition with shift-split $(\mathcal{K}_1, \mathcal{K}_2)$, then the matrix condition* (2.6) *holds and*

$$\vee\{(T^*)^j A^* f : f \in w\mathcal{E}^2_{n,m-1}, j = 0, 1, \dots\} \subset \mathcal{K}_1 \tag{6.1}$$

$$\vee\{T^j B f : f \in w\mathcal{F}^2_{n,m-1}, j = 0, 1, \dots\} \subset \mathcal{K}_2. \tag{6.2}$$

Proof. Let $(\mathcal{K}_1, \mathcal{K}_2)$ be a shift-split of $\mathcal{E}^1_{n,m}$. Then, $\mathcal{E}^1_{n-1,m} = \mathcal{K}_1 \oplus \mathcal{K}_2$ and $\mathcal{E}^1_{n,m} = \mathcal{K}_1 \oplus z\mathcal{K}_2 \oplus \mathbb{C}p$ where p is the associated split-poly.

The strategy is to prove (1) the range of B is contained in $\mathcal{K}_2$, (2) $\mathcal{K}_2$ is an invariant subspace of T ($T\mathcal{K}_2 \subset \mathcal{K}_2$), and (3) $A\mathcal{K}_2 = 0$. This will imply that (2.6) holds as well as (6.2).

Since $\mathcal{K}_1 \subset \mathcal{E}^1_{n,m}$, $\mathcal{K}_1 \perp w\mathcal{F}^2_{n,m-1}$ and therefore for $g \in \mathcal{K}_1, f \in w\mathcal{F}^2_{n,m-1}$

$$\langle Bf, g\rangle = \langle P_{\mathcal{E}^1_{n-1,m}} P_{w\mathcal{F}^2_{n,m-1}} f, g\rangle = \langle f, P_{w\mathcal{F}^2_{n,m-1}} g\rangle = 0.$$

So, the range of B is orthogonal to $\mathcal{K}_1$ and therefore must be contained in $\mathcal{K}_2$.

To show $T\mathcal{K}_2 \subset \mathcal{K}_2$, let $f_2 \in \mathcal{K}_2$ and $g_1 \in \mathcal{K}_1$. Since $z\mathcal{K}_2 \perp \mathcal{K}_1$, we know $g_1 \perp zf_2$ and therefore

$$\langle Tf_2, g_1\rangle = \langle M_z f_2, g_1\rangle = 0.$$

So, $T\mathcal{K}_2 \perp \mathcal{K}_1$ and thus $T\mathcal{K}_2 \subset \mathcal{K}_2$.

Finally, since $z\mathcal{K}_2 \subset \mathcal{E}^1_{n,m} \perp w\mathcal{E}^2_{n,m-1}$, we must have

$$A\mathcal{K}_2 = P_{w\mathcal{E}^2_{n,m-1}} M_z \mathcal{K}_2 = 0.$$

The proof of (6.1) is similar if we work with adjoints of our operators. □

Proposition 6.2. *Suppose $\mathcal{T}$ is a positive linear form on $\mathcal{L}_{n,m}$ satisfying the matrix condition* (2.6)*. Set*

$$\mathcal{K}_2 = \vee\{T^j B f : f \in w\mathcal{F}^2_{n,m-1}, j = 0, 1, \dots\}$$

and $\mathcal{K}_1 = \mathcal{E}^1_{n-1,m} \ominus \mathcal{K}_2$. Then, $(\mathcal{K}_1, \mathcal{K}_2)$ is a shift-split of $\mathcal{E}^1_{n-1,m}$ and hence $\mathcal{T}$ satisfies the split-shift orthogonality condition.

Proof. Notice that by the Cayley-Hamilton theorem we do not need to consider all powers of T in the definition of $\mathcal{K}_2$, so that

$$\mathcal{K}_2 = \vee\{T^j B f : j = 0, 1, \dots, n-1, f \in w\mathcal{F}^2_{n,m-1}\}$$

and also $AT^jB = 0$ for $j = 0, 1, 2, \dots$.

We need to show $z\mathcal{K}_2 \perp \mathcal{K}_1$ and $z\mathcal{K}_2, \mathcal{K}_1 \subset \mathcal{E}^1_{n,m}$.

To prove $z\mathcal{K}_2 \perp \mathcal{K}_1$, simply note that for $f \in w\mathcal{F}^2_{n,m-1}$, $g \in \mathcal{K}_1$, $j = 0, 1, 2, \dots$, we have

$$\langle zT^j Bf, g\rangle = \langle T^{j+1} Bf, g\rangle = 0.$$

The proves $z\mathcal{K}_2 \perp \mathcal{K}_1$ since $z\mathcal{K}_2$ is spanned by elements of the form zT^jBf.

To show $z\mathcal{K}_2 \subset \mathcal{E}^1_{n,m}$, note that

$$z\mathcal{K}_2 \subset \mathcal{P}_{n,m} \ominus zw\mathcal{P}_{n-1,m-1} = \mathcal{E}^1_{n,m} \oplus w\mathcal{E}^2_{n,m-1}$$

and therefore it is enough to show $z\mathcal{K}_2 \perp w\mathcal{E}^2_{n,m-1}$. So, for $f \in w\mathcal{F}^2_{n,m-1}$ and $g \in w\mathcal{E}^2_{n,m-1}$ we have

$$\langle zT^j Bf, g\rangle = \langle AT^j Bf, g\rangle = 0 \quad j = 0, 1, 2 \dots$$

since $A = P_{w\mathcal{E}^2_{n,m-1}} M_z$ and $AT^j B = 0$. This proves $z\mathcal{K}_2 \subset \mathcal{E}^1_{n,m}$.

Similarly, to show $\mathcal{K}_1 \subset \mathcal{E}^1_{n,m}$ it is enough to show $\mathcal{K}_1 \perp w\mathcal{F}^2_{n,m-1}$, since $\mathcal{K}_1 \subset \mathcal{P}_{n,m} \ominus w\mathcal{P}_{n-1,m-1}$. Observe that for $f \in w\mathcal{F}^2_{n,m-1}$ and $g \in \mathcal{K}_1$, $Bf \in \mathcal{K}_2$ and so

$$0 = \langle Bf, g\rangle = \langle f, g\rangle$$

and therefore $w\mathcal{F}^2_{n,m-1} \perp \mathcal{K}_1$. $\square$

7. Description of shift-splits and split-polys

If the split-shift condition holds for $\mathcal{T}$, then we have seen that $\mathcal{T}$ can be represented using moments of a measure $1/|p|^2 d\sigma$ where $p \in \mathbb{C}[z, w]$ has no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$ and p has degree at most (n, m). The description of all such p is essentially an algebra problem.

Lemma 7.1. *Let $t(z, w)$ be a two variable trigonometric polynomial which can be factored as $|p(z, w)|^2$ where p has degree at most (n, m) and no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$.*

Then, there exists a $g \in \mathbb{C}[z, w]$ with no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$ none of whose irreducible factors involve z alone, and there exists a stable polynomial $q \in \mathbb{C}[z]$ (no zeros on $\overline{\mathbb{D}}$) such that

$$t(z, w) = |q(z)g(z, w)|^2 \text{ for } (z, w) \in \mathbb{T}^2$$

Moreover, if $t(z, w) = |p_1(z, w)|^2$ where p_1 has degree at most (n, m) and no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$, then there exist $q_1, q_2 \in \mathbb{C}[z]$ such that $q = q_1 \overleftarrow{q_2}$ and

$$p_1(z, w) = q_1(z)q_2(z)g(z, w)$$

Proof. Suppose $t = |p|^2$ as above. We may factor $p(z, w) = h(z)g(z, w)$ where g has no irreducible factors involving z alone. By the one variable Fejér-Riesz lemma we can factor $|h|^2 = |q|^2$ with q, a stable one variable polynomial. Then, $t(z, w) = |q(z)g(z, w)|^2$ on $\mathbb{T}^2$.

Now, if $t = |p_1|^2$ as above, then again $p_1(z, w) = h_1(z)g_1(z, w)$ where g_1 has no irreducible factors involving z alone. Now,

$$h_1(z)g_1(z, w)\overline{h_1(z)g_1(z, w)} = h(z)g(z, w)\overline{h(z)g(z, w)}$$

on $\mathbb{T}^2$ which implies

$$h_1(z)g_1(z, w)\overleftarrow{h_1}(z)\overleftarrow{g_1}(z, w) = q(z)g(z, w)\overleftarrow{q}(z)\overleftarrow{g}(z, w)$$

on all of $\mathbb{C}^2$, when we reflect at appropriate degrees. Then, for $z \in \mathbb{T}$

$$\frac{h_1(z)g_1(z, w)}{q(z)g(z, w)} = \frac{\overleftarrow{q}(z)\overleftarrow{g}(z, w)}{\overleftarrow{h_1}(z)\overleftarrow{g_1}(z, w)}$$

and the left side is holomorphic for all $w \in \overline{\mathbb{D}}$ and the right side is holomorphic for $|w| \geq 1$ making the function entire and rational in w. The same can be said for the reciprocal and this forces the function to be constant in w. So, for $z \in \mathbb{T}$

$$\frac{h_1(z)g_1(z,w)}{q(z)g(z,w)} = \frac{h_1(z)g_1(z,0)}{q(z)g(z,0)}$$

and we see

$$g_1(z,w)g(z,0) = g(z,w)g_1(z,0).$$

This extends to all $z \in \mathbb{C}$ and since g and g_1 have no irreducible factors involving z alone, we may conclude they are constant multiples of one another. The constant can be absorbed into the definition of h_1 so that $p_1(z,w) = h_1(z)g(z,w)$. Then, $|p_1|^2 = |p|^2$ on $\mathbb{T}^2$ implies that $|h_1|^2 = |q|^2$ on $\mathbb{T}$. It is then elementary to show h_1 is obtained by flipping some of the roots of q to inside $\mathbb{D}$. □

Lemma 7.2. *If the split-shift orthogonality condition holds with a given split-poly p, then the spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ are uniquely determined by p.*

Proof. Looking at the formulas in Theorem 5.5, we see that since all of the E or F kernels are uniquely determined, the kernels $\overleftarrow{K}_2 - K_1$ and $w\bar{\eta}\overleftarrow{K}_2 - K_1$ are uniquely determined. We see that $(1 - w\bar{\eta})K_1$ is uniquely determined and so K_1 is uniquely determined. A similar argument shows K_2 is uniquely determined. □

We can now give a description of all possible shift-splits.

Proposition 7.3. *If the split-shift condition holds for a positive linear form $\mathcal{T}$ on $\mathcal{L}_{n,m}$, then there exists $g \in \mathbb{C}[z,w]$ with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ and no irreducible factors involving z alone and stable $q \in \mathbb{C}[z]$, such that $q(z)g(z,w)$ is a split-poly. Write $n_1 := \deg_z g$ and $n_0 := n - n_1$. Every other split-poly is of the form*

$$q_1(z)q_2(z)g(z,w)$$

where $\deg q_1 q_2 \leq n_0$, $q(z) = q_1(z)\overleftarrow{q}_2(z)$, and $\overleftarrow{q}_2$ is reflected at the degree of q_2. The associated shift-split $(\mathcal{K}_1, \mathcal{K}_2)$ is given by

$$\mathcal{K}_1 = \vee P_{\mathcal{E}^1_{n-1,m}}\{z^j q_1(z)g_1(z) : 0 \leq j < \deg q_2 + \deg g_2\}$$

$$\mathcal{K}_2 = \vee P_{\mathcal{E}^1_{n-1,m}}\{z^j q_2(z)g_2(z) : 0 \leq j < n - \deg q_2 - \deg g_2\}$$

where $g(z,0) = g_1(z)g_2(z)$ with g_1 having no zeros in $\overline{\mathbb{D}}$ and g_2 having all zeros in $\mathbb{D}$.

Proposition 6.2 singles out the shift-split with minimal $\mathcal{K}_2$ which would correspond to the split-poly $\overleftarrow{q}(z)g(z,w)$, where $\overleftarrow{q}(z)$ is reflected at degree n_0. The shift-split with minimal $\mathcal{K}_1$ corresponds to split-poly $q(z)g(z,w)$. This leads to a canonical decomposition of $\mathcal{E}^1_{n-1,m}$ which does not depend on a choice of shift-split. Let $\deg g = (n_1, m)$ and $n_0 := n - n_1$.

Define

$$\mathcal{K}_0 = \vee\{z^j g(z,w) : 0 \leq j < n_0\}$$

$$\mathcal{A} = \vee P_{\mathcal{E}^1_{n-1,m}}\{z^j q(z)g_1(z) : 0 \leq j < \deg g_2\}$$

$$\mathcal{B} = \vee P_{\mathcal{E}^1_{n-1,m}}\{z^j \overleftarrow{q}(z)g_2(z) : 0 \leq j < n_1 - \deg g_2\}.$$

Theorem 7.4. *Let $\mathcal{T}$ be a positive linear form on $\mathcal{L}_{n,m}$ satisfying the split-shift condition. Then,*

$$\mathcal{E}^1_{n-1,m} = \mathcal{K}_0 \oplus \mathcal{A} \oplus \mathcal{B}.$$

Both $(\mathcal{K}_0 \oplus \mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{K}_0 \oplus \mathcal{B})$ are shift-splits. If $(\mathcal{K}_1, \mathcal{K}_2)$ is any shift-split, then $\mathcal{A} \subset \mathcal{K}_1$ and $\mathcal{B} \subset \mathcal{K}_2$.

Proof. It follows by inspection of definitions that $\mathcal{A} \subset \mathcal{K}_1$ and $\mathcal{B} \subset \mathcal{K}_2$ using $\mathcal{K}_1$ and $\mathcal{K}_2$ from Proposition 7.3.

Let $g \in \mathbb{C}[z, w]$ and $q \in \mathbb{C}[z]$ be as in the previous proposition. In $L^2(1/|qg|^2 d\sigma)$, $g \perp z^j w^{k+1}$ for $j \in \mathbb{Z}$ and $k \geq 0$ because

$$\langle z^j w^{k+1}, g\rangle = \int_{\mathbb{T}} \frac{z^j}{|q(z)|^2} \int_{\mathbb{T}} \frac{w^{k+1}}{g(z,w)} \frac{|dw||dz|}{(2\pi)^2} = 0$$

since $1/g(z, \cdot)$ is holomorphic. Therefore, $z^j g(z, w) \in \mathcal{E}^1_{n-1,m}$ for $0 \leq j < n_0$ and we see

$$\mathcal{K}_0 = \vee P_{\mathcal{E}^1_{n-1,m}} \{z^j g_1(z) g_2(z) : 0 \leq j < n_0\}$$

is contained in

$$\vee P_{\mathcal{E}^1_{n-1,m}} \{z^j g_1(z) : 0 \leq j < n_0 + \deg g_2\}$$

but this corresponds to $\mathcal{K}_1$ in the shift-split coming from the split-poly $\overline{q}(z) g(z, w)$. Hence, this space and $\mathcal{K}_0$ must be orthogonal to the associated $\mathcal{K}_2$ which happens to be $\mathcal{B}$. Notice also that

$$\mathcal{A} \subset \vee P_{\mathcal{E}^1_{n-1,m}} \{z^j g_1(z) : 0 \leq j < n_0 + \deg g_2\}.$$

A similar argument shows $\mathcal{K}_0$ is orthogonal to $\mathcal{A}$, and by dimension considerations

$$\mathcal{K}_0 \oplus \mathcal{A} = \vee P_{\mathcal{E}^1_{n-1,m}} \{z^j g_1(z) : 0 \leq j < n_0 + \deg g_2\}$$

and again by dimension considerations

$$\mathcal{K}_0 \oplus \mathcal{A} \oplus \mathcal{B} = \mathcal{E}^1_{n-1,m}.$$

We already noted that $\mathcal{K}_0 \oplus \mathcal{A}$ corresponds to "$\mathcal{K}_1$" in some shift-split. Therefore, $(\mathcal{K}_0 \oplus \mathcal{A}, \mathcal{B})$ is a shift-split. By a similar argument, $(\mathcal{A}, \mathcal{K}_0 \oplus \mathcal{B})$ is a shift-split. □

Propositions 6.1 and 6.2 together show that the invariant subspace of T generated by the range of B is the minimal possible "$\mathcal{K}_2$" occurring in a shift-split. We have already computed the minimal $\mathcal{K}_2$, which is $\mathcal{B}$. A similar argument can be used for the minimal $\mathcal{K}_1$ which is $\mathcal{A}$. This implies the following.

Theorem 7.5. *If the split-shift condition holds,*

$$\mathcal{B} = \vee\{T^j B f : f \in w\mathcal{F}^2_{n,m-1}, j = 0, 1, \ldots, n-1\}$$

and

$$\mathcal{A} = \vee\{(T^*)^j A^* f : f \in w\mathcal{E}^2_{n,m-1}, j = 0, 1, \ldots, n-1\}$$

where $A^* = P_{\mathcal{E}^1_{n-1,m}} M_{1/z} : w\mathcal{E}^2_{n,m-1} \to \mathcal{E}^1_{n-1,m}$ and $T^* = P_{\mathcal{E}^1_{n-1,m}} M_{1/z} : \mathcal{E}^1_{n,m-1} \to \mathcal{E}^1_{n-1,m}$.

We can now prove the stratified characterization of Bernstein-Szegő measures.

Proof of Corollary 2.6. Suppose $\mathcal{T}$ is a positive linear form on $\mathcal{L}_{n,m}$ given by

$$\mathcal{T}(z^j w^k) = \int_{\mathbb{T}^2} z^j w^k \frac{|dz||dw|}{(2\pi)^2|p(z,w)|^2}, \qquad |j| \leq n, |k| \leq m,$$

where $p \in \mathbb{C}[z,w]$ has no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$, degree at most (n,m) and $p(z,0)$ has d zeros in $\mathbb{D}$. We write $p(z,0) = a(z)b(z)$ where b has all zeros in $\mathbb{D}$ and a has no zeros in $\overline{\mathbb{D}}$. By Theorem 4.8, $\mathcal{T}$ possesses a shift-split $(\mathcal{K}_1, \mathcal{K}_2)$ where $\mathcal{K}_1$ has dimension $d = \deg b(z)$.

Next, supposing $\mathcal{T}$ possesses a split-shift $(\mathcal{K}_1, \mathcal{K}_2)$ where $\mathcal{K}_1$ has dimension d, by Theorems 7.4 and 7.5 we have $\mathcal{A} \subset \mathcal{K}_1 \subset \mathcal{A} \oplus \mathcal{K}_0$. Therefore,

$$\dim \mathcal{A} \leq d \leq n - \dim \mathcal{B}. \tag{7.1}$$

Finally, if $\mathcal{T}$ satisfies the matrix condition and (7.1), then we see from Proposition 7.3 that it is possible to choose $\mathcal{K}_1$ with dimension d and the corresponding split-poly p has the desired property that $p(z,0)$ has d roots in $\mathbb{D}$. □

8. Construction of p from Fourier coefficients

In this section, it is useful to write $z = (z_1, z_2)$ for an element of $\mathbb{C}^2$ as opposed to (z,w), so that we can use multi-index notation $z^u = z_1^{u_1} z_2^{u_2}$.

Supposing the split-shift condition does hold, how do we construct p directly from the Fourier coefficients

$$\mathcal{T}(z^{-u}) = c_u \ ?$$

In principle, one could construct $(\mathcal{K}_1, \mathcal{K}_2)$ and then produce p as an element of $\mathcal{E}_{n,m} \ominus (\mathcal{K}_1 \oplus z\mathcal{K}_2)$; however, this is quite involved. In this section we describe a simpler procedure assuming we already know that the shift-split condition holds.

First, we construct an orthonormal basis for $\mathcal{E}^2_{n,m}$. It helps to use interval notation for subsets of integers as in $[0,n] = \{0,\ldots,n\}$. Let $S_j = [0,n] \times [0,m] \setminus \{(0,0),\ldots,(0,j-1)\}$, $S_0 = [0,n] \times [0,m]$. Let

$$(\gamma^{(j)}_{u,v})_{u,v \in S_j} = (c_{v-u})^{-1}_{u,v \in S_j}$$

and define

$$\phi_j(z) = \sum_{v \in S_j} \gamma^{(j)}_{(0,j),v} z^v \Big/ \sqrt{\gamma^{(j)}_{(0,j),(0,j)}}.$$

Then, $\phi_0, \phi_1, \ldots, \phi_m$ form an orthonormal basis for $\mathcal{E}^2_{n,m}$. To see this let $u \in S_{j+1}$ and $\gamma = \sqrt{\gamma^{(j)}_{(0,j),(0,j)}}$. We compute

$$\langle \phi_j, z^u \rangle = \sum_{v \in S_j} \gamma^{(j)}_{(0,j),v} c_{u-v}/\gamma = \delta_{(0,j),u}/\gamma = 0$$

since $(0,j) \notin S_{j+1}$ and $S_{j+1} \subset S_j$. For $k > j$, ϕ_k is a combination of z^u with $u \in S_k \subset S_{j+1}$ and therefore $\phi_k \perp \phi_j$ for $k > j$. Also,

$$\langle \phi_j, \phi_j \rangle = \sum_{v,u \in S_j} \gamma^{(j)}_{(0,j),v} c_{u-v} \bar{\gamma}^{(j)}_{(0,j),u}/\gamma^2 = \sum_{u \in S_j} \delta_{(0,j),u} \bar{\gamma}^{(j)}_{(0,j),u}/\gamma^2 = \gamma^{(j)}_{(0,j),(0,j)}/\gamma^2 = 1.$$

The reproducing kernel for $\mathcal{E}^2_{n,m}$ is therefore $E^2_m(z;\zeta) = \sum_{j=0}^m \phi_j(z)\overline{\phi_j(\zeta)}$.

We assume $p(z_1, z_2) = q(z_1)g(z_1, z_2)$ with q stable and g has no factors with z_1 alone and then show how to construct g and q using only the moments c_u. Theorem 5.5 proves

$$z_1^n E_m^2(z; 1/\bar{z}_1, 0) = p(z)z_1^n \bar{p}(1/z_1, 0) = q(z_1)g(z)z_1^n \bar{q}(1/z_1)\bar{g}(1/z_1, 0).$$

From this we can calculate g up to a constant multiple. The key point is that the product of all factors of the above polynomial that involve z_1 alone will be the greatest common divisor of the coefficients of powers of z_2.

Let us write

$$z_1^n E_m^2(z; 1/\bar{z}_1, 0) = \sum_{j=0}^{m} E_j(z_1)z_2^j,$$

and then compute $Q = \gcd\{E_0, E_1, \ldots, E_m\}$ using the Euclidean algorithm. Then, $Q(z_1) = Cq(z_1)z_1^n \bar{q}(1/z_1)\bar{g}(1/z_1, 0)$ for some constant C. This gives

$$z_1^n E_m^2(z; 1/\bar{z}_1, 0)/Q(z_1) = g(z)$$

possibly with a constant. At this stage we look at the one variable moment problem

$$c_j = \mathcal{T}(z_1^{-j} g(z)\bar{g}(1/z_1, 1/z_2)) = \int_{\mathbb{T}} z_1^{-j} \frac{|dz_1|}{2\pi|q(z_1)|^2}$$

for $|j| \leq n_0 := n - \deg_z g$. Set

$$\gamma_{j,k} = (c_{k-j})^{-1}_{j,k\in[0,n_0]}$$

and then we can construct

$$q(z_1) = \sum_{j=0}^{n_0} \gamma_{0,j} z_1^j / \sqrt{\gamma_{0,0}}$$

(up to a unimodular multiple) by one variable theory.

Hence, we have constructed p as $p(z) = q(z_1)g(z)$.

9. Applications

9.1. **Autoregressive filters.** A direct application of the above work is to two variable autoregressive models [23].

We consider (wide sense) stationary processes $X = (X_u)_{u\in\mathbb{Z}^2}$ depending on two discrete variables defined on a fixed probability space $(\Omega, \mathcal{A}, P)$. We shall assume that X is a *zero mean* process, i.e. the means $E(X_u)$ are equal to zero. Recall that the space $L^2(\Omega, \mathcal{A}, P)$ of square integrable random variables endowed with the inner product

$$\langle X, Y\rangle := E(XY^*)$$

is a Hilbert space. A sequence $X = (X_m)_{m\in\mathbb{Z}^2}$ is called a *stationary process* on $\mathbb{Z}^2$ if for $m, n \in \mathbb{Z}^2$ we have that

$$E(X_m X_n^*) = E(X_{m+u}X_{n+u}^*) =: R_X(m - n), \text{ for all } u \in \mathbb{Z}^2.$$

It is known that the function R_X, termed the *covariance function* of X, defines a *positive semi-definite function* on $\mathbb{Z}^2$, i.e.

$$\sum_{i,j=1}^{k} \alpha_i \bar{\alpha}_j R_X(r_i - r_j) \geq 0,$$

for all $k \in \mathbb{N}$, $\alpha_1, \dots, \alpha_k \in \mathbb{C}, r_1, \dots, r_k \in \mathbb{Z}^2$ and Bochner's Theorem states that for such a function R_X there is a positive regular bounded measure μ_X defined for Borel sets on the torus $[0, 2\pi]^2$ such that

$$R_X(u) = \int e^{-i\langle u,t\rangle} d\mu_X(t),$$

for all two tuples of integers u. The measure μ_X is referred to as the *spectral distribution measure* of the process X. The *spectral density* $f_X(t)$ of the process X is the spectral density of the absolutely continuous part of μ_X, i.e. the absolutely continuous part of μ_X equals

$$f_X(t_1, t_2)\frac{dt_1 dt_2}{(2\pi)^2}.$$

Let $\tilde{H} = \{(k,l) : -\infty < k < \infty, l > 0\} \cup \{(k,0), k > 0\}$ and let $\Lambda_{n,m} = \{(k,l) : 0 \le k \le n, 0 \le l \le m\} \subset \tilde{H} \cup \{(0,0)\}$ be a finite set. A zero-mean stationary stochastic process $X = (X_u)_{u\in\mathbb{Z}^2}$ is said to be *extended autoregressive* or eAR(n,m), if there exist complex numbers $a_k, k \in \Lambda_{n,m}$ with $a_{(i,0)} \neq 0$ for some $0 \le i \le n$, so that for every u

$$\sum_{k\in\Lambda_{n,m}} a_k X_{u-k} = \mathcal{E}_u, \qquad u \in \mathbb{Z}^2, \tag{9.1}$$

where $\{\mathcal{E}_u \ : u \in \mathbb{Z}^2\}$ is a white noise zero mean process with variance 1. The eAR(n,m) process is said to be *acausal (in z)* if there is a solution to equations (9.1) of the form

$$X_u = \sum_{k\in\tilde{H}\cup\{(0,0),(-1,0),\dots\}} \phi_k \mathcal{E}_{u-k}, u \in \mathbb{Z}^2,$$

with $\sum_{k\in\tilde{H}\cup\{(0,0),(-1,0)\dots\}} |\phi_k| < \infty$ and it is said to be *causal* if there is a solution of the form

$$X_u = \sum_{k\in\tilde{H}\cup\{(0,0)\}} \phi_k \mathcal{E}_{u-k}, u \in \mathbb{Z}^2,$$

with $\sum_{k\in\tilde{H}\cup\{(0,0)\}} |\phi_k| < \infty$. From the general theory of autoregressive models it follows that if (9.1) has a causal (acausal (in z)) solution then

$$p(z,w) = \sum_{v\in\Lambda_{n,m}} a_v z^{v_1} w^{v_2}. \tag{9.2}$$

is stable on $\bar{\mathbb{D}}^2$ ($\mathbb{T} \times \overline{\mathbb{D}}$).

The bivariate extended autoregressive (eAR) model problem concerns the following. Given autocorrelation elements

$$c_k = E(X_0 X_k^*), \ k \in \Lambda_{n,m} - \Lambda_{n,m}$$

determine, if possible, the coefficients $a_l, l \in \Lambda_{n,m}$ of an acausal autoregressive filter representation. In [11] necessary and sufficient conditions were given for the autocorrelation coefficients in order for the eAR(n,m) to have a causal solution. Here we give necessary and sufficient conditions in order for an eAR(n,m) model to have an acausal solution.

If we begin with a polynomial that is nonzero for $(z,w) \in \mathbb{T}\times\overline{\mathbb{D}}$ then choosing the autoregressive filter coefficients as in equation (9.2) give an eAR(n,m) model whose

Fourier coefficients give a linear form that is positive and satisfies conditions in Theorem 2.2. Conversely, we can use the conditions in Theorem 2.2 to characterize the existence of acausal (in z) solution.

Theorem 9.1. *Given autocorrelation elements $c_{k,l}$, $(k,l) \in \Lambda_{n,m} - \Lambda_{n,m}$ there exists an acausal (in z) solution to the* $\mathrm{eAR}(n,m)$ *problem if and only if the linear form $\mathcal{T}$ determined by the Fourier coefficients c_k is positive and satisfies one of the equivalent conditions of Theorem 2.2*

Corollary 9.2. *With the hypotheses of the above Theorem there exists a casual solution to the* $\mathrm{eAR}(n,m)$ *problem if and if the linear form $\mathcal{T}$ determined by the Fourier coefficients c_k is positive and $A = 0$.*

9.2. **Full measure characterization.** We now identify which measures $d\mu$ are of the form

$$\frac{1}{|p(z,w)|^2} d\sigma$$

where $p \in \mathbb{C}[z,w]$ has no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$.

Corollary 3.6 provides necessary conditions which can be encoded as

$$\mathcal{E}^2_{n,M} = \mathcal{E}^2_{n+j,M} \tag{9.3}$$

for all $j \geq 0$ and $M \geq m-1$. By performing the reflection operation, it follows that

$$z^j \mathcal{F}^2_{n,M} = \mathcal{F}^2_{n+j,M}$$

for all $j \geq 0, M \geq m-1$.

It turns out that conditions (9.3) for $M = m-1, m$ are sufficient to show that the moments $\int z^j w^k d\mu$ agree with the moments of a Bernstein-Szegő measure when $j \in \mathbb{Z}$ and $|k| \leq m$ (i.e. on a strip). It is then another issue to prove that the Bernstein-Szegő measure obtained with a particular m agrees with other choices.

Fix m and define $A_N = P_{w\mathcal{E}^2_{N,m-1}} M_z P_{\mathcal{E}^1_{N-1,m}}$, $T_N = P_{\mathcal{E}^1_{N-1,m}} M_z P_{\mathcal{E}^1_{N-1,m}}$, and $B_N = P_{\mathcal{E}^1_{N-1,m}} P_{w\mathcal{F}^2_{N,m-1}}$.

Lemma 9.3. *Assume* (9.3) *holds for $j \geq 0$ and for $M = m-1, m$. Then, $A_N T_N^k B_N = 0$ for $N \geq n$, $k \geq 0$.*

Proof. Let $P = P_1 + P_2$ be the projection onto the space

$$\mathcal{P}_{N+k,m} \ominus w\mathcal{P}_{N-1,m-1} = \mathcal{E}^1_{N-1,m} \bigoplus_{j=0}^{k} \mathcal{F}^2_{N+j,m}$$

where P_1, P_2 are the projections onto $\mathcal{E}^1_{N-1,m}, \bigoplus_{j=0}^k \mathcal{F}^2_{N+j,m}$ respectively.

Noting that $z \bigoplus_{j=0}^k \mathcal{F}^2_{N+j,m} = \bigoplus_{j=0}^k \mathcal{F}^2_{N+j+1,m}$ by (9.3), we have $P_1 M_z P_2 = 0$. Then, $T_N = P_1 M_z P_1 = P_1 M_z (P_1 + P_2) = P_1 M_z P$. Therefore, for $k \geq 1$ we have

$$T_N^k = P_1 M_z (P M_z P)^{k-1} P.$$

Similarly, $A_N = P_{w\mathcal{E}^2_{N,m-1}} M_z P_1 = P_{w\mathcal{E}^2_{N,m-1}} M_z P$, so that

$$A_N T_N^k = P_{w\mathcal{E}^2_{N,m-1}} M_z (P M_z P)^k P.$$

Next, $P P_{w\mathcal{F}^2_{N,m-1}} = P_{w\mathcal{F}^2_{N,m-1}}$ while

$$P M_z P_{w\mathcal{F}^2_{N+j,m-1}} = P_{w\mathcal{F}^2_{N+j+1,m}} M_z P_{w\mathcal{F}^2_{N+j,m-1}}$$

since $zw\mathcal{F}^2_{N+j,m-1} = w\mathcal{F}^2_{N+j+1,m-1}$ by (9.3) so that inductively we have

$$A_N T_N^k B_N = P_{w\mathcal{E}^2_{N,m-1}} M_z P_{w\mathcal{F}^2_{N+k,m-1}} \prod_{j=0}^{k-1} \left(M_z P_{w\mathcal{F}^2_{N+j,m-1}} \right)$$

where the product is multiplied from right to left as j goes from 0 to $k-1$ (if $k=0$, the product is I). But, $P_{w\mathcal{E}^2_{N,m-1}} M_z P_{w\mathcal{F}^2_{N+k,m-1}} = 0$ as $zw\mathcal{F}^2_{N+k,m-1} = w\mathcal{F}^2_{N+k+1,m-1} \perp w\mathcal{E}^2_{N,m-1}$. $\square$

Therefore, assuming (9.3) for $M = m, m-1$ and $j \geq 0$, the matrix condition holds for the positive linear form on $\mathcal{L}_{N,m}$ for $N \geq n$. So for each N, there is a $p_N \in \mathbb{C}[z,w]$ of degree at most (N,m) with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ such that the Bernstein-Szegő measure for p_N matches the moments of $d\mu$ on $\mathcal{L}_{N,m}$. We can further assume that each p_N has been normalized so that $p_N(z,w) = q_N(z) g_N(z,w)$ where q_N is stable in z and g_N has no factors involving z alone. By Theorem 5.5 and (9.3), if we set $z = \zeta \in \mathbb{T}$, $w \in \mathbb{C}, \eta = 0$, we get

$$p_n(z,w)\overline{p_n(z,0)} = p_{n+j}(z,w)\overline{p_{n+j}(z,0)}$$

for $j \geq 0$. This implies $g_n = g_{n+j}$ for each $j \geq 0$ (after absorbing constants into q's if necessary) and then by stability of each q_N, $q_n = q_{n+j}$ for each $j \geq 0$. Therefore, the moments of $d\mu$ on the strip $\{z^j w^k : j \in \mathbb{Z}, |k| \leq m\}$ are matched by those of $1/|p_n|^2 d\sigma$.

Theorem 9.4. *Let $d\mu$ be a positive Borel measure. If*

$$\mathcal{E}^2_{n,m} = \mathcal{E}^2_{n+j,m} \quad \mathcal{E}^2_{n,m-1} = \mathcal{E}^2_{n+j,m-1}$$

for $j \geq 0$, then there exists $p \in \mathbb{C}[z,w]$ of degree at most (n,m) with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ such that

$$\int z^j w^k d\mu = \int \frac{z^j w^k}{|p(z,w)|^2} d\sigma$$

for $j \in \mathbb{Z}$ and $|k| \leq m$.

To get the full measure characterization, we note that for a Bernstein-Szegő measure (and recalling G_η from Section 3)

$$G_0(z,w) = p(z,w) z^n \bar{p}(1/z, 0)$$

is an element of the one dimensional space $\mathcal{H}_m$ defined in (2.10), but by all of the orthogonality relations for Bernstein-Szegő measures it is also in $\mathcal{H}_M$ for $M \geq m$. Therefore, a set of necessary conditions is

$$\mathcal{H}_m = \mathcal{H}_{m+j} \text{ for } j \geq 0.$$

Theorem 9.5. *Let $d\mu$ be a positive Borel measure on $\mathbb{T}^2$ satisfying*

$$\mathcal{E}^2_{n,M} = \mathcal{E}^2_{n+j,M} \text{ and } \mathcal{H}_m = \mathcal{H}_{m+j}$$

for $M \geq m-1$ and $j \geq 0$. Then, there exists $p \in \mathbb{C}[z,w]$ of degree at most (n,m) with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ such that

$$d\mu = \frac{d\sigma}{|p(z,w)|^2}$$

Proof. The conditions on $\mathcal{E}^2_{\cdot,\cdot}$ imply that for each $M \geq m$, there exists $p_M \in \mathbb{C}[z,w]$ of degree at most (n, M) with no zeros on $\mathbb{T}\times\overline{\mathbb{D}}$ such that the moments of $d\sigma/|p_M|^2$ match those of $d\mu$ on the strip $\{z^j w^k : j \in \mathbb{Z}, |k| \leq M\}$. We normalize $p_M(z,w) = q_M(z)g_M(z,w)$ where q_M is stable and g_M has no factors involving z alone.

Using the assumption $\mathcal{H}_m = \mathcal{H}_M$ for $m \geq M$, it follows that for each $m \geq M$

$$p_m(z,w)z^n\bar{p}_m(1/z,0) = Cp_M(z,w)z^n\bar{p}_M(1/z,0)$$

for some constant C. We then must have that g_m and g_M are constant multiples and then since q_m, q_M are stable, they too must be constant multiples of one another. Therefore, p_m and p_M must be constant multiples. The constant must be unimodular since p_m and p_M have unit norm. Therefore, the measures $1/|p_m|^2 d\sigma = 1/|p_M|^2 d\sigma$ match all of the moments of $d\mu$. Hence, $1/|p_m|^2 d\sigma = d\mu$. □

9.3. **Concrete expression for the full measure characterization.** The conditions

$$\mathcal{E}^2_{n,M} = \mathcal{E}^2_{n+j,M} \text{ and } \mathcal{H}_m = \mathcal{H}_{m+j}$$

for $M \geq m-1$ and $j \geq 0$ given above can be written directly in terms of the Fourier coefficients of μ

$$c_u = \int z^{-u} d\mu \qquad u = (u_1, u_2) \in \mathbb{Z}^2$$

as follows. Similar to Section 8 it is useful to write $z = (z_1, z_2)$ for an element of $\mathbb{C}^2$ as opposed to (z,w) and we will use multi-index notation $z^u = z_1^{u_1} z_2^{u_2}$. Also, $[0, N] = \{0, 1, \ldots, N\}$; there should be no confusing this with a closed interval of real numbers.

Let

$$(\gamma^{N,M}_{u,v})_{u,v\in[0,N]\times[0,M]} = (c_{v-u})^{-1}_{u,v\in[0,N]\times[0,M]}.$$

A basis for $\mathcal{E}^2_{N,M}$ consists of

$$f^{N,M}_j(z) = \sum_{v\in[0,N]\times[0,M]} \gamma^{N,M}_{(0,j),v} z^v$$

for $j = 0, 1, \ldots, M$. In order for $\mathcal{E}^2_{N+1,M} = \mathcal{E}^2_{N,M}$ to hold we need the coefficients of $z_1^{N+1} z_2^k$ for $k = 0, 1, \ldots, M$ to vanish in $\mathcal{E}^2_{N+1,M}$. Looking at $f^{N+1,M}_j$ this amounts to

$$\gamma^{N+1,M}_{(0,j),(N+1,k)} = 0$$

for $j, k = 0, 1, \ldots, M$.

Therefore, the conditions $\mathcal{E}^2_{n,M} = \mathcal{E}^2_{n+j,M}$ for $j \geq 0$ and $M \geq m - 1$ can be expressed as

$$\gamma^{N+1,M}_{(0,j),(N+1,k)} = 0$$

for $N \geq n$, $M \geq m - 1$, $j, k = 0, 1, \ldots, M$.

Next we turn to the conditions

$$\mathcal{H}_m = \mathcal{H}_{m+j} \text{ for } j \geq 0.$$

Recall

$$\mathcal{H}_M = \mathcal{P}_{2n,M} \ominus \vee\{z_1^j z_2^k : 0 \leq j \leq 2n, 0 \leq k \leq M, (j,k) \neq (n,0)\}$$

so that a nonzero element of the one dimensional space $\mathcal{H}_M$ is given by

$$g_M(z) = \sum_{u\in[0,2n]\times[0,M]} \xi^M_{(n,0),u} z^u$$

where we define

$$\xi^M_{u,v} = (c_{v-u})^{-1}_{u,v\in[0,2n]\times[0,M]}.$$

The condition $\mathcal{H}_M = \mathcal{H}_{M-1}$ can then be expressed as

$$\xi^M_{(n,0),(j,M)} = 0 \qquad j = 0,1,\dots,2n.$$

Let us summarize everything.

Theorem 9.6. *Let μ be a positive, finite measure on $\mathbb{T}^2$ with moments c_u for $u \in \mathbb{Z}^2$. There exists a polynomial $p \in \mathbb{C}[z,w]$ of degree at most (n,m) with no zeros on $\mathbb{T}\times\overline{\mathbb{D}}$ such that*

$$d\mu = \frac{1}{|p(z,w)|^2} d\sigma$$

if and only if

(1) *for all $N, M \geq 0$,*

$$\det(c_{v-u})_{u,v\in[0,N]\times[0,M]} \neq 0$$

(2) *for $N \geq n$, $M \geq m-1$, $j,k = 0,1,\dots,M$*

$$\gamma^{N+1,M}_{(0,j),(N+1,k)} = 0$$

and

(3) *for $M \geq m$, $j = 0,1,\dots,2n$*

$$\xi^M_{(n,0),(j,M)} = 0$$

where

$$(\gamma^{N,M}_{u,v})_{u,v\in[0,N]\times[0,M]} = (c_{v-u})^{-1}_{u,v\in[0,N]\times[0,M]}$$

$$\xi^M_{u,v} = (c_{v-u})^{-1}_{u,v\in[0,2n]\times[0,M]}.$$

10. Generalized distinguished varieties

10.1. **Construction of the sums of squares formula.** Here we use Kummert's approach as in [20] to give a different proof of the sums of squares formula Theorem 2.4. One advantage of this approach is that it works for p with no zeros on $\mathbb{T}\times\mathbb{D}$ and no factors in common with $\overleftarrow{p}$ (rather than assuming no zeros on $\mathbb{T}\times\overline{\mathbb{D}}$). This approach is also useful because it shows how to compute the reproducing kernels in the decomposition of p using only one variable theory.

Theorem 10.1. *Suppose $p \in \mathbb{C}[z,w]$ has degree (n,m), no zeros on $\mathbb{T}\times\mathbb{D}$ and no factors in common with $\overleftarrow{p}$. Let n_2 be the number of zeros of $p(z,0)$ in $\mathbb{D}$ and $n_1 = n - n_2$. Then, there exist vector polynomials $E \in \mathbb{C}^m[z,w], A \in \mathbb{C}^{n_1}[z,w], B \in \mathbb{C}^{n_2}[z,w]$ such that*

- $$|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2 = (1-|w|^2)|E(z,w)|^2 + (1-|z|^2)(|A(z,w)|^2 - |B(z,w)|^2).$$
- *E has degree at most $(n,m-1)$ and A and B have degree at most $(n-1,m)$, and*
- *the entries of A and B form a linearly independent set of polynomials.*

The last two details are needed in Section 10.2.

Consider for $z \in \mathbb{T}$

$$\frac{p(z,w)\overline{p(z,\eta)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(z,\eta)}}{1-w\bar{\eta}} = (1,\bar{\eta},\dots,\bar{\eta}^{m-1})T(z)(1,w,\dots,w^{m-1})^t$$

where $T(z)$ is an $m\times m$ matrix valued trigonometric polynomial which is positive definite for all but finitely many values of $z \in \mathbb{T}$. This is because $T(z)$ is positive definite for each value of z such that $p(z,\cdot)$ has no zeros in $\mathbb{T}$. There can only be finitely many zeros on $\mathbb{T}^2$ or else p and $\overleftarrow{p}$ would have a common factor. Let $S = \{z \in \mathbb{T} : \det T(z) = 0\}$.

By the matrix Fejér-Riesz theorem in one variable, we may factor

$$T(z) = E(z)^*E(z)$$

where E is an invertible matrix polynomial on $\mathbb{D}$ of degree at most n. Let

$$E(z,w) = E(z)(1,w,\dots,w^{m-1})^t$$

so that

$$p(z,w)\overline{p(z,\eta)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(z,\eta)} = (1-w\bar{\eta})E(z,\eta)^*E(z,w)$$

for $z \in \mathbb{T}$. We are using both the notations $E(z)$ and $E(z,w)$, but no confusion should arise.

Then, for fixed $z \in \mathbb{T}\setminus S$, the map which maps

$$\begin{pmatrix} p(z,w) \\ wE(z,w)\end{pmatrix} \mapsto \begin{pmatrix} \overleftarrow{p}(z,w) \\ E(z,w)\end{pmatrix}$$

extends to a unitary $U(z)$ which we can explicitly solve for. Write

$$p(z,w) = \sum_{j=0}^{m} p_j(z)w^j, \qquad \overleftarrow{p}(z,w) = \sum_{j=0}^{m} \overleftarrow{p}_{m-j}(z)w^j.$$

Then,

$$U(z) = \begin{pmatrix} \overleftarrow{p}_m(z) & \cdots & \overleftarrow{p}_1(z) & \overleftarrow{p}_0(z) \\ & E(z) & & 0 \end{pmatrix} \begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_m(z) \\ 0 & & E(z) & \end{pmatrix}^{-1}$$

which is unitary by construction for $z \in \mathbb{T}\setminus S$, but clearly extends to a matrix rational function with poles in $\mathbb{D}$ at the zeros in $\mathbb{D}$ of $p_0(z)$. Moreover, any singularities on $\mathbb{T}$ must be removable because U is bounded on a punctured neighborhood in $\mathbb{T}$ of each singularity.

Set n_2 to be the number of zeros of $p(z,0)$ in $\mathbb{D}$ and $n_1 = n - n_2$. Theorem 10.4 and Section 10.1.2 below prove that

$$\frac{I - U(\zeta)^*U(z)}{1-\bar{\zeta}z} = F(\zeta)^*F(z) - G(\zeta)^*G(z) \tag{10.1}$$

where F is $n_1 \times (m+1)$ and G is $n_2 \times (m+1)$, and the rows of F and G are linearly independent as vector functions; meaning there is no non-zero solution $(v_1, v_2) \in \mathbb{C}^n$ to

$$v_1F(z) + v_2G(z) \equiv 0. \tag{10.2}$$

Accepting all of this for now, we rearrange (10.1) to get

$$I + \bar{\zeta}F(\zeta)^*zF(z) + G(\zeta)^*G(z) = U(\zeta)^*U(z) + F(\zeta)^*F(z) + \bar{\zeta}G(\zeta)^*zG(z)$$

and so there exists an $(m+1+n)\times(m+1+n)$ unitary (with two indicated block decompositions)

$$V = \begin{array}{c c} & \begin{array}{cc}\mathbb{C}^{m+1} & \mathbb{C}^{n}\end{array} \\ \begin{array}{c}\mathbb{C}^{m+1}\\ \mathbb{C}^{n}\end{array} & \begin{pmatrix} V_1' & V_2' \\ V_3' & V_4' \end{pmatrix}\end{array} = \begin{array}{c c} & \begin{array}{cc}\mathbb{C}^{1} & \mathbb{C}^{m+n}\end{array} \\ \begin{array}{c}\mathbb{C}^{1}\\ \mathbb{C}^{m+n}\end{array} & \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}\end{array} \tag{10.3}$$

such that

$$V\begin{pmatrix} I \\ zF(z) \\ G(z)\end{pmatrix} = \begin{pmatrix} U(z) \\ F(z) \\ zG(z)\end{pmatrix}. \tag{10.4}$$

Multiplying both sides of this equation by $X(z,w) = \begin{pmatrix} p(z,w) \\ wE(z,w)\end{pmatrix}$ gives

$$V\begin{pmatrix} p(z,w) \\ wE(z,w) \\ zF(z)X(z,w) \\ G(z)X(z,w)\end{pmatrix} = \begin{pmatrix} \overleftarrow{p}(z,w) \\ E(z,w) \\ F(z)X(z,w) \\ zG(z)X(z,w)\end{pmatrix}. \tag{10.5}$$

Let $A(z,w) = F(z)X(z,w)$ and $B(z,w) = G(z)X(z,w)$. The entries of A and B are linearly independent, because if $v_1 \in \mathbb{C}^{n_1}, v_2 \in \mathbb{C}^{n_2}$ and

$$0 \equiv v_1A(z,w) + v_2B(z,w) = (v_1F(z) + v_2G(z))X(z,w)$$

then since

$$X(z,w) = \begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_m(z) \\ 0 & & E(z) & \end{pmatrix}\begin{pmatrix} 1 \\ w \\ \vdots \\ w^m\end{pmatrix}$$

we have

$$0 \equiv (v_1F(z) + v_2G(z))\begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_m(z) \\ 0 & & E(z) & \end{pmatrix}.$$

The matrix on the right is invertible in $\mathbb{D}$ except at possible zeros of p_0, so we get $v_1F(z) + v_2G(z) \equiv 0$ which implies $v_1 = 0$ and $v_2 = 0$.

Taking the norm squared of both sides of (10.5) gives the following formula since V is a unitary

$$\begin{aligned}&|p(z,w)|^2 + |w|^2|E(z,w)|^2 + |z|^2|A(z,w)|^2 + |B(z,w)|^2\\ &= |\overleftarrow{p}(z,w)|^2 + |E(z,w)|^2 + |A(z,w)|^2 + |z|^2|B(z,w)|^2.\end{aligned}$$

If we rearrange we get the desired sum of squares formula

$$|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2 = (1-|w|^2)|E(z,w)|^2 + (1-|z|^2)(|A(z,w)|^2 - |B(z,w)|^2).$$

One final technicality is that while E has entries that are polynomials, it is not clear that the same holds for A and B. To show they are polynomials we go through a longer process of proving the following "transfer function" representation which is interesting in its own right.

Set

$$\Delta(z,w) = \begin{pmatrix} wI_m & 0 & 0 \\ 0 & zI_{n_1} & 0 \\ 0 & 0 & I_{n_2} \end{pmatrix}$$
$$\Gamma(z,w) = \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & zI_{n_2} \end{pmatrix}. \tag{10.6}$$

Theorem 10.2. *Suppose $p \in \mathbb{C}[z,w]$ has no zeros in $\mathbb{T} \times \mathbb{D}$, degree (n,m), and no factors in common with $\overleftarrow{p}$. Then, there exists a $(1+m+n) \times (1+m+n)$ unitary matrix V such that*

$$\frac{\overleftarrow{p}(z,w)}{p(z,w)} = V_1 + V_2\Delta(z,w)(\Gamma(z,w) - V_4\Delta(z,w))^{-1}V_3. \tag{10.7}$$

Here we use the block form indicated in (10.3) *and again n_2 is the number of zeros of $p(z,0)$ in $\mathbb{D}$ and $n_1 = n - n_2$.*

A technicality we must address is whether the matrix we invert above is non-degenerate. The fact that the rows of F and G are linearly independent is used to show this.

By (10.5), the map sending

$$\begin{pmatrix} p(z,w) \\ wE(z,w) \\ zA(z,w) \\ B(z,w) \end{pmatrix} \mapsto \begin{pmatrix} \overleftarrow{p}(z,w) \\ E(z,w) \\ A(z,w) \\ zB(z,w) \end{pmatrix}$$

extends to the unitary V. Then,

$$V_1 p + V_2\Delta(z,w)\begin{pmatrix} E \\ A \\ B \end{pmatrix} = \overleftarrow{p}$$
$$V_3 p + V_4\Delta(z,w)\begin{pmatrix} E \\ A \\ B \end{pmatrix} = \Gamma(z,w)\begin{pmatrix} E \\ A \\ B \end{pmatrix}$$

which implies

$$pV_3 = (\Gamma(z,w) - V_4\Delta(z,w))\begin{pmatrix} E \\ A \\ B \end{pmatrix}$$

We would like to invert the matrix on the right, so we need to make sure

$$\det\left(\Gamma(z,w) - V_4\Delta(z,w)\right) \tag{10.8}$$

is not identically zero. This is equivalent to

$$\det\left(\begin{pmatrix} wI_m & 0 & 0 \\ 0 & zI_{n_1} & 0 \\ 0 & 0 & z^{-1}I_{n_2} \end{pmatrix} - V_4\right)$$

being non-trivial by simple matrix manipulations. The coefficient of w^m will occur as

$$\det\left(\begin{pmatrix} zI_{n_1} & 0 \\ 0 & z^{-1}I_{n_2} \end{pmatrix} - V_4'\right)$$

where V_4' is the lower right $n \times n$ block of V as in (10.3). We shall show this determinant is non-vanishing for $z \in \mathbb{T}$. If it does vanish for some $z = \zeta \in \mathbb{T}$, then there exists a nonzero $v = (v_1, v_2) \in \mathbb{C}^n = \mathbb{C}^{n_1+n_2}$ such that

$$(v_1, v_2)\begin{pmatrix} \zeta I_{n_1} & 0 \\ 0 & \bar{\zeta} I_{n_2} \end{pmatrix} = (v_1, v_2)V_4'.$$

This implies that $\|v\| = \|vV_4'\|$ and since V is a unitary $vV_3' = 0$. Then, by (10.4)

$$(0, v_1, v_2)V\begin{pmatrix} I \\ zF(z) \\ G(z) \end{pmatrix} = (0, \zeta v_1, \bar{\zeta} v_2)\begin{pmatrix} I \\ zF(z) \\ G(z) \end{pmatrix} = (0, v_1, v_2)\begin{pmatrix} U(z) \\ F(z) \\ zG(z) \end{pmatrix}$$

so that

$$v_1 \zeta z F(z) + v_2 \bar{\zeta} G(z) = v_1 F(z) + v_2 z G(z)$$

and then

$$(z\zeta - 1)v_1 F(z) + (\bar{\zeta} - z)v_2 G(z) \equiv 0.$$

This implies

$$v_1 F(z) - \bar{\zeta} v_2 G(z) \equiv 0$$

contradicting (10.2). Therefore, the determinant in (10.8) is not identically zero, and

$$p(z,w)\left(\Gamma(z,w) - V_4\Delta(z,w)\right)^{-1} V_3 = \begin{pmatrix} E(z,w) \\ A(z,w) \\ B(z,w) \end{pmatrix} \tag{10.9}$$

which in turn yields (10.7). Examining (10.7) we see that since p has degree (n, m), since

$$\det\left(\begin{pmatrix} I_m & 0 & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & zI_{n_2} \end{pmatrix} - V_4\begin{pmatrix} wI_m & 0 & 0 \\ 0 & zI_{n_1} & 0 \\ 0 & 0 & I_{n_2} \end{pmatrix}\right)$$

has degree at most (n, m), and since p and $\overleftarrow{p}$ have no common factors, we must have that p is a constant multiple of the above determinant else (10.7) could be reduced further. This implies that the left hand side of (10.9) is a vector polynomial and finally we see that the entries of E, A, B are polynomials. By Cramer's rule the entries of E have degree at most $(n, m-1)$ and the entries of A and B have degree at most $(n-1, m)$.

10.1.1. *Unitary valued rational functions on the circle.* The following is undoubtedly well-known material from systems theory; however, we were unable to find a suitable reference so we include a detailed explanation.

Theorem 10.3 (Smith Normal form [15])**.** *Let R be a principal ideal domain and let A be an $N \times N$ matrix with entries in R. There exists a unique (up to units) diagonal matrix $D \in R^{N\times N}$, called the Smith Normal form, with entries $D_1|D_2|\dots|D_N$ such that*

$$A = SDT$$

where $S, T \in R^{N\times N}$ and $S^{-1}, T^{-1} \in R^{N\times N}$. The matrix S is formed through the row operations of (1) multiplying a row by an element of R and adding the result onto another row, (2) switching two rows, and (3) multiplying a row by a unit in R.

The entries D_j may also be computed as

$$D_j = \frac{\gcd_j(A)}{\gcd_{j-1}(A)}$$

where $\gcd_j(A)$ *represents the greatest common divisor of determinants of all $j \times j$ submatrices of A (*$\gcd_0 := 1$*).*

Let $U \in \mathbb{C}(z)^{N\times N}$ be a rational $N \times N$ matrix function of one variable which is unitary valued on the unit circle. Let R be the ring of fractions $\mathbb{C}[z]\mathcal{S}^{-1}$ where $\mathcal{S}$ is the multiplicative set $\mathcal{S} = \{q \in \mathbb{C}[z] : q(z) \neq 0 \text{ for } z \in \mathbb{D}\}$. We may write $U = \frac{1}{q}Q$ where $q \in \mathbb{C}[z]$ has all zeros in $\mathbb{D}$ and $Q \in R^{N\times N}$. Let D be the Smith Normal form of Q in R. Write

$$\frac{D_j}{q} = \frac{d_j}{q_j}$$

in lowest terms and define

$$N_1 = \sum_{j=1}^{N} \# \text{ zeros of } d_j \text{ in } \mathbb{D} \text{ counting multiplicity}$$

$$N_2 = \sum_{j=1}^{N} \# \text{ zeros of } q_j \text{ in } \mathbb{D} \text{ counting multiplicity.}$$

Theorem 10.4. *With U as above, there exist an $N_1 \times N$ matrix function F and an $N_2 \times N$ matrix function G such that*

$$\frac{I - U(\zeta)^*U(z)}{1 - z\bar{\zeta}} = F(\zeta)^*F(z) - G(\zeta)^*G(z).$$

The rows of F and G together form a linearly independent set of vector functions on $\mathbb{D}$.

Proof. We shall use $\vec{H}^2$ to denote the vector valued Hardy space $H^2(\mathbb{T}) \otimes \mathbb{C}^N$ for short. Now $U\vec{H}^2$ is a reproducing kernel Hilbert space with point evaluations in $\mathbb{D}$ except at the poles of U in $\mathbb{D}$. In $\vec{L}^2 = L^2(\mathbb{T}) \otimes \mathbb{C}^N$, if $f \in \vec{H}^2$ and $v \in \mathbb{C}^N$

$$\langle Uf, \frac{UU(\zeta)^*}{1 - \cdot\bar{\zeta}} v\rangle_{L^2} = \langle f, \frac{U(\zeta)^*v}{1 - \cdot\bar{\zeta}}\rangle_{L^2} = \langle U(\zeta)f(\zeta), v\rangle_{\mathbb{C}^N}$$

which shows $U\vec{H}^2$ has reproducing kernel

$$\frac{U(z)U(\zeta)^*}{1 - z\bar{\zeta}}.$$

Notice $U\vec{H}^2$ is not necessarily contained in $\vec{H}^2$. The space $U\vec{H}^2 \vee \vec{H}^2$ is therefore a reproducing kernel Hilbert space containing both spaces.

Consider the kernel

$$K(z;\zeta) = \frac{I - U(z)U(\zeta)^*}{1 - z\bar{\zeta}}$$

which is not necessarily positive definite but is rather the difference of two reproducing kernels

$$K = K_{\vec{H}^2} - K_{U\vec{H}^2}.$$

We shall in general use $K_{\mathcal{H}}$ to denote the reproducing kernel of a space $\mathcal{H}$ in $U\vec{H}^2 \vee \vec{H}^2$. We can decompose $\vec{H}^2 = (\vec{H}^2 \cap U\vec{H}^2) \oplus (\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2))$ so that

$$K_{\vec{H}^2} = K_{\vec{H}^2 \cap U\vec{H}^2} + K_{\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)}$$

and similarly

$$K_{U\vec{H}^2} = K_{\vec{H}^2 \cap U\vec{H}^2} + K_{U\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)}.$$

Therefore,

$$K = K_{\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)} - K_{U\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)}$$

The spaces $\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2), U\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)$ are actually finite dimensional. We can compute their dimensions as follows.

Write $U = \frac{1}{q}Q$ where q has all zeros in $\mathbb{D}$ and Q has entries in R. Note that since U is unitary on the circle, U has no poles on the circle (U is bounded near any potential singularities). Therefore, the entries of Q belong to the smaller ring $R_0 = \mathbb{C}[z]\mathcal{S}_0^{-1}$ where $\mathcal{S}_0$ is the multiplicative set $\mathcal{S}_0 = \{q \in \mathbb{C}[z] : q(z) \neq 0 \text{ for } z \in \overline{\mathbb{D}}\}$. By Theorem 10.3, we may write $Q = SDT$ where S, T are matrices with entries in R_0 whose inverses have the same property, and D is the Smith Normal form of Q in R_0. The elements $D_j \in R_0$ have no zeros on the unit circle since $\det U = \det Q/q^N = \det S \det T \prod (D_j/q)$ has no zeros on $\mathbb{T}$ and q has no zeros on $\mathbb{T}$. So, Q has the same Smith Normal form D in R by the gcd characterization of the Smith Normal form.

Now, $\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)$ is isomorphic as a vector space to the quotient $\vec{H}^2/(\vec{H}^2 \cap U\vec{H}^2)$, and since $T\vec{H}^2 = \vec{H}^2 = S\vec{H}^2$ we see that

$$\vec{H}^2/(\vec{H}^2 \cap U\vec{H}^2) \cong \vec{H}^2/(\vec{H}^2 \cap \frac{1}{q}D\vec{H}^2)$$

which breaks up into the algebraic direct sum of the spaces

$$H^2/(H^2 \cap \frac{D_j}{q}H^2).$$

Any zeros of D_j or q in $\mathbb{C} \setminus \mathbb{D}$ can be absorbed into H^2 so that $\frac{D_j}{q}H^2 = \frac{d_j}{q_j}H^2$ for some d_j and q_j with all zeros in $\mathbb{D}$ and no common zeros (after canceling). The space

$$H^2 \cap \frac{d_j}{q_j}H^2 = d_j H^2$$

and H^2/d_jH^2 has dimension equal to the number of zeros of d_j in $\mathbb{D}$. Therefore, $\vec{H}^2/(\vec{H}^2 \cap U\vec{H}^2)$ has dimension equal to

$$N_1 = \sum_{j=1}^{N} \# \text{ zeros of } d_j \text{ in } \mathbb{D} \text{ counting multiplicity.}$$

A similar analysis shows that $U\vec{H}^2/(\vec{H}^2 \cap U\vec{H}^2)$ has dimension equal to

$$N_2 = \sum_{j=1}^{N} \# \text{ zeros of } q_j \text{ in } \mathbb{D} \text{ counting multiplicity.}$$

If $\{f_1, \dots, f_{N_1}\}$ is an orthonormal basis for $\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)$ and $\{g_1, \dots, g_{N_2}\}$ is an orthonormal basis for $U\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)$ then

$$K(z;\zeta) = \sum f_j(z) f_j(\zeta)^* - \sum g_j(z) g_j(\zeta)^*$$

which if we form an $N \times N_1$ matrix $F = (f_1, \dots, f_{N_1})$ and an $N \times N_2$ matrix $G = (g_1, \dots, g_{N_2})$ can rewrite as

$$\frac{I - U(z)U(\zeta)^*}{1 - z\bar{\zeta}} = F(z)F(\zeta)^* - G(z)G(\zeta)^*.$$

Since $\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)$ and $U\vec{H}^2 \ominus (\vec{H}^2 \cap U\vec{H}^2)$ have trivial intersection, the columns of F and G form an independent set of vector functions.

Of course, applying the above work to U^t instead would yield a formula of the form

$$\frac{I - U(\zeta)^*U(z)}{1 - z\bar{\zeta}} = F(\zeta)^*F(z) - G(\zeta)^*G(z)$$

after switching z and ζ and taking conjugates, where now F and G are $N_1 \times N$ and $N_2 \times N$ valued respectively. Note we are not saying they are the same F and G as before, but the dimensions N_1 and N_2 are preserved because the transpose does not change the diagonal term in the Smith Normal form. The rows of F and G form an independent set of vector functions just as above. $\square$

10.1.2. *A particular choice of U.* Recall the matrix function U from Section 10.1

$$U(z) = \begin{pmatrix} \overleftarrow{p}_m(z) & \cdots & \overleftarrow{p}_1(z) & \overleftarrow{p}_0(z) \\ & E(z) & & 0 \end{pmatrix} \begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_m(z) \\ 0 & & E(z) & \end{pmatrix}^{-1}.$$

Note

$$U = \frac{1}{p_0} Q = \frac{1}{p_0} \begin{pmatrix} \overleftarrow{p}_m & \cdots & \overleftarrow{p}_1 & \overleftarrow{p}_0 \\ & E & & 0 \end{pmatrix} \begin{pmatrix} 1 & -\begin{pmatrix} p_1 & \cdots & p_m \end{pmatrix} E^{-1} \\ 0 & p_0 E^{-1} \end{pmatrix}$$

where Q has entries in R since $\det E$ may have zeros on $\mathbb{T}$ while $p_0(z) = p(z, 0)$ has no zeros on $\mathbb{T}$. We now show how to compute the Smith Normal form of Q in R.

Now,

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \overleftarrow{p}_m & \cdots & \overleftarrow{p}_1 & \overleftarrow{p}_0 \\ & I & & 0 \end{pmatrix} \begin{pmatrix} 1 & -\begin{pmatrix} p_1 & \cdots & p_m \end{pmatrix} \\ 0 & p_0 I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E^{-1} \end{pmatrix}.$$

It is not hard to see that the product of the inner two matrices can be converted to the diagonal matrix D with entries $1, p_0, \dots, p_0, p_0\overleftarrow{p}_0$ using row and column operations in R, which is the Smith Normal form of Q. The entries of $\frac{1}{p_0}D$ are then $1/p_0, 1, \dots, 1, \overleftarrow{p}_0$.

This proves that for n_2 equal to the number of zeros of p_0 in $\mathbb{D}$ and $n_1 = n - n_2$

$$\frac{I - U(\zeta)^*U(z)}{1 - \bar{\zeta}z} = F(\zeta)^*F(z) - G(\zeta)^*G(z)$$

where F is $n_1 \times (m+1)$ and G is $n_2 \times (m+1)$.

10.2. **Generalized distinguished varieties and determinantal representations.** Distinguished varieties are a class of curves introduced in Agler-McCarthy [2] because they play a natural role in multivariable operator theory and function theory on the bidisk (see [18], [16], [1]). The zero set Z_p of a polynomial $p \in \mathbb{C}[z, w]$ is a distinguished variety if

$$Z_p \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$$

where $\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}}$. Notice the curve $Z_p \cap \mathbb{D}^2$ exits the boundary of $\mathbb{D}^2$ through the distinguished boundary $\mathbb{T}^2$; hence the name *distinguished* variety. This area is part of a larger topic of understanding algebraic curves and their interaction with $\mathbb{T}^2$. See [3], [4].

The sums of squares theorem, Theorem 2.4 or Theorem 10.1, naturally leads to the study of a more general class of curves using the methods of [18]. We say that the zero set Z_p of $p \in \mathbb{C}[z,w]$ is a *generalized distinguished variety* if it satisfies

$$Z_p \subset (\mathbb{D}\times\mathbb{C}) \cup \mathbb{T}^2 \cup (\mathbb{E}\times\mathbb{C}) \text{ or} \tag{10.10}$$

$$Z_p \subset (\mathbb{C}\times\mathbb{D}) \cup \mathbb{T}^2 \cup (\mathbb{C}\times\mathbb{E}).$$

That is, Z_p does not intersect the (relatively small) set $(\mathbb{T}\times\mathbb{D}) \cup (\mathbb{T}\times\mathbb{E})$ in the former case above. We shall show that generalized distinguished varieties share much of the structure of distinguished varieties, and in particular they possess a determinantal representation generalizing one of the main theorems in [2]. We use the notation (10.6) below.

Theorem 10.5. *Suppose $p \in \mathbb{C}[z,w]$ has degree (n,m), has no factors involving z alone, and satisfies* (10.10)*. Then, there exists an $(m+n)\times(m+n)$ unitary matrix U such that p is a constant multiple of*

$$\det\left(U\Delta(z,w) - \Gamma(z,w)\right),$$

where n_2 is the number of zeros of $p(z,0)$ in $\mathbb{D}$ and $n_1 = n - n_2$.

If Z_p is a distinguished variety then $n_2 = n$ and we get the representation

$$Z_p = \left\{(z,w) : \det\left(U\begin{pmatrix} wI_m & 0 \\ 0 & I_n\end{pmatrix} - \begin{pmatrix} I_m & 0 \\ 0 & zI_n\end{pmatrix}\right) = 0\right\}.$$

If we write $U = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$, then the above zero set can be written as

$$\det(\Phi(w) - zI_n) = 0$$

in terms of the matrix rational inner function

$$\Phi(w) = D + wC(I - wA)^{-1}B$$

at least outside of the poles of Φ. This is how the characterization of distinguished varieties is stated in [2].

Lemma 10.6. *Suppose $p \in \mathbb{C}[z,w]$ has degree (n,m), has no factors involving z alone, and satisfies* (10.10)*. Then, $p = \mu\overleftarrow{p}$ for some $\mu \in \mathbb{T}$.*

Proof. For each $z \in \mathbb{T}$, $p(z,\cdot)$ has all zeros in $\mathbb{T}$—as does $\overleftarrow{p}(z,\cdot)$. Since $z \in \mathbb{T}$, we see that $p(z,\cdot)$ and $\overleftarrow{p}(z,\cdot)$ have the same roots (counting multiplicity, since they approach zero at the same rate near a root because $|p| = |\overleftarrow{p}|$ on $\mathbb{T}^2$). Therefore, if we write

$$p(z,w) = \sum_{j=0}^{m} p_j(z)w^j, \qquad \overleftarrow{p}(z,w) = \sum_{j=0}^{m} \overleftarrow{p}_{m-j}(z)w^j$$

then for all $z \in \mathbb{T}$, $\overleftarrow{p}_m(z)p(z,\cdot) = p_0(z)\overleftarrow{p}(z,\cdot)$ since these polynomials have the same roots and same leading coefficient. Hence, $\overleftarrow{p}_m(z)p(z,w) = p_0(z)\overleftarrow{p}(z,w)$ for all $(z,w) \in \mathbb{C}^2$. By assumption p has no factors involving z alone, and therefore p divides $\overleftarrow{p}$. Similarly $\overleftarrow{p}$ divides p, so that $p = C\overleftarrow{p}$ for some constant C. Since $|p| = |\overleftarrow{p}|$ on $\mathbb{T}^2$, C must be unimodular. $\square$

Because of this lemma we can assume $p = \overleftarrow{p}$ by replacing p with an appropriate constant multiple.

Lemma 10.7. *Suppose $p = \overleftarrow{p} \in \mathbb{C}[z,w]$ has degree (n,m) satisfies* (10.10) *and is irreducible. Then,*

- $mp = \frac{\partial \overleftarrow{p}}{\partial w} + w\frac{\partial p}{\partial w}$ *and*
- $\frac{\partial \overleftarrow{p}}{\partial w}$ *has no zeros in $\mathbb{T} \times \mathbb{D}$ and no factors in common with $\frac{\partial p}{\partial w}$.*

We reflect $\partial p/\partial w$ at the degree $(n, m-1)$.

Proof. The identity $mp = \frac{\partial \overleftarrow{p}}{\partial w} + w\frac{\partial p}{\partial w}$ is straightforward assuming $p = \overleftarrow{p}$.

For $t < 1$, let $p_t(z,w) = p(z,tw)$. Then, p_t has no zeros in $\mathbb{T} \times \overline{\mathbb{D}}$ and

$$|p_t(z,w)|^2 - |\overleftarrow{p_t}(z,w)|^2 \geq 0$$

for $(z,w) \in \mathbb{T} \times \overline{\mathbb{D}}$. Therefore,

$$\lim_{t \nearrow 1} \frac{|p_t(z,w)|^2 - |\overleftarrow{p_t}(z,w)|^2}{1-t^2} \geq 0$$

but the above limit equals

$$m|p(z,w)|^2 - 2\mathrm{Re}(w\frac{\partial p}{\partial w}\overline{p(z,w)}) \geq 0$$

for $(z,w) \in \mathbb{T} \times \overline{\mathbb{D}}$. Now, since $mp = \frac{\partial \overleftarrow{p}}{\partial w} + w\frac{\partial p}{\partial w}$

$$m^2|p(z,w)|^2 - 2m\mathrm{Re}(w\frac{\partial p}{\partial w}\overline{p(z,w)}) = |\frac{\partial \overleftarrow{p}}{\partial w}|^2 - |w\frac{\partial p}{\partial w}|^2 \geq 0.$$

Therefore, any zero of $\frac{\partial \overleftarrow{p}}{\partial w}$ in $\mathbb{T} \times \overline{\mathbb{D}}$ is a zero of $w\frac{\partial p}{\partial w}$ and hence will be a zero of p, which by assumption has no zeros in $\mathbb{T} \times \mathbb{D}$. So, $\frac{\partial \overleftarrow{p}}{\partial w}$ has no zeros in $\mathbb{T} \times \mathbb{D}$.

Now, $\frac{\partial \overleftarrow{p}}{\partial w}$ can have no factors in common with $\frac{\partial p}{\partial w}$, else $w\frac{\partial p}{\partial w}$ and p have a common factor. As p is assumed to be irreducible, this is impossible. $\square$

Proof of Theorem 10.5. It is sufficient to prove the theorem for irreducible $p = \overleftarrow{p}$ since we can write the determinantal representation in terms of blocks corresponding to each irreducible factor of p. With this assumption $\frac{\partial \overleftarrow{p}}{\partial w}$ has no zeros on $\mathbb{T} \times \mathbb{D}$ and no factors in common with $w\frac{\partial p}{\partial w}$, and $mp(z,0) = \frac{\partial \overleftarrow{p}}{\partial w}(z,0)$ has n_2 zeros in $\mathbb{D}$. The proof of Theorem 10.1 says there are vector polynomials $A \in \mathbb{C}^m[z,w], B \in \mathbb{C}^{n_1}[z,w], C \in \mathbb{C}^{n_2}[z,w]$ such that

$$\frac{\partial \overleftarrow{p}}{\partial w}(z,w)\overline{\frac{\partial \overleftarrow{p}}{\partial w}(\zeta,\eta)} - w\bar{\eta}\frac{\partial p}{\partial w}(z,w)\overline{\frac{\partial p}{\partial w}(\zeta,\eta)}$$

equals

$$(1-w\bar{\eta})A(\zeta,\eta)^*A(z,w) + (1-z\bar{\zeta})(B(\zeta,\eta)^*B(z,w) - C(\zeta,\eta)^*C(z,w)).$$

By Theorem 10.1 we can choose A, B, C so that A has degree at most $(n, m-1)$ while B, C have degree at most $(n-1, m)$. Furthermore, the entries of B and C together form a linearly independent set of polynomials.

The identity $mp = \frac{\partial \overleftarrow{p}}{\partial w} + w\frac{\partial p}{\partial w}$ proves

$$m^2p\bar{p} - m(w\frac{\partial p}{\partial w}\bar{p}) - mp\eta\overline{\frac{\partial p}{\partial w}} = \frac{\partial \overleftarrow{p}}{\partial w}\overline{\frac{\partial \overleftarrow{p}}{\partial w}} - w\bar{\eta}\frac{\partial p}{\partial w}\overline{\frac{\partial p}{\partial w}}.$$

On the zero set Z_p we get the formula

$$0 = (1 - w\bar{\eta})A(\zeta, \eta)^* A(z, w) + (1 - z\bar{\zeta})(B(\zeta, \eta)^* B(z, w) - C(\zeta, \eta)^* C(z, w)).$$

A lurking isometry argument now produces the formulas we want. First, we rearrange

$$\begin{aligned} &w\bar{\eta}A(\zeta, \eta)^* A(z, w) + z\bar{\zeta}B(\zeta, \eta)^* B(z, w) + C(\zeta, \eta)^* C(z, w) \\ &= A(\zeta, \eta)^* A(z, w) + B(\zeta, \eta)^* B(z, w) + z\bar{\zeta}C(\zeta, \eta)^* C(z, w) \end{aligned}$$

for $(z, w), (\zeta, \eta) \in Z_p$. Then, the map

$$\begin{pmatrix} wA(z,w) \\ zB(z,w) \\ C(z,w) \end{pmatrix} \mapsto \begin{pmatrix} A(z,w) \\ B(z,w) \\ zC(z,w) \end{pmatrix}$$

extends to a well-defined unitary on the span of the elements on the left (as (z, w) varies over Z_p) to the span of the elements on the right. Since the ambient spaces have the same dimension we can extend to an $(m + n) \times (m + n)$ unitary U such that

$$U \begin{pmatrix} wA(z,w) \\ zB(z,w) \\ C(z,w) \end{pmatrix} = \begin{pmatrix} A(z,w) \\ B(z,w) \\ zC(z,w) \end{pmatrix}$$

on Z_p. Then,

$$(U\Delta(z, w) - \Gamma(z, w)) \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

and since A, B, C vanish at only finitely many points in Z_p, we get

$$\det (U\Delta(z, w) - \Gamma(z, w)) = 0. \tag{10.11}$$

The polynomial on the left has degree less than or equal to that of p and vanishes on Z_p. Since p is irreducible, it must either be a nonzero multiple of p or it must be identically zero.

Claim: The determinant in (10.11) is not identically zero.

The explanation is similar to before. If the determinant is identically zero, then by simple matrix manipulations

$$\det \left(\begin{pmatrix} wI_m & 0 & 0 \\ 0 & zI_{n_1} & 0 \\ 0 & 0 & z^{-1}I_{n_2} \end{pmatrix} - U \right) \equiv 0.$$

The coefficient of w^m is

$$\det(\begin{pmatrix} zI_{n_1} & 0 \\ 0 & z^{-1}I_{n_2} \end{pmatrix} - U_4) \equiv 0$$

where U_4 is the lower-right $n \times n$ block of U. The above determinant cannot vanish for any $z = \zeta \in \mathbb{T}$, since if it does there exists a non-zero vector $v = (v_1, v_2)$ such that

$$(\zeta v_1, \bar{\zeta} v_2) = (v_1, v_2)U_4.$$

Then, on Z_p

$$(0, \zeta v_1, \bar{\zeta} v_2) \begin{pmatrix} wA(z,w) \\ zB(z,w) \\ C(z,w) \end{pmatrix} = (0, v_1, v_2)U \begin{pmatrix} wA(z,w) \\ zB(z,w) \\ C(z,w) \end{pmatrix} = (0, v_1, v_2) \begin{pmatrix} A(z,w) \\ B(z,w) \\ zC(z,w) \end{pmatrix}$$

and so $\zeta z v_1 B(z,w) + \bar{\zeta} v_2 C(z,w) = v_1 B(z,w) + z v_2 C(z,w)$ which implies

$$(\zeta z - 1) v_1 B(z,w) + (\bar{\zeta} - z) v_2 C(z,w) = 0$$

which in turn implies

$$\zeta v_1 B(z,w) - v_2 C(z,w) = 0 \text{ on } Z_p$$

since $z = \bar{\zeta}$ for finitely many $(z,w) \in Z_p$. Since p is irreducible, p divides $\zeta v_1 B - v_2 C$. Since B and C have degree at most $(n-1, m)$, we see that $\zeta v_1 B - v_2 C = 0$, which is not possible unless v_1 and v_2 are zero vectors, which they are not.

So, the determinant in (10.11) is not identically zero. It follows that p is a multiple of the determinant in (10.11).

$\square$

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