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CANONICAL AGLER DECOMPOSITIONS AND TRANSFER FUNCTION REALIZATIONS

KELLY BICKEL AND GREG KNESE

ABSTRACT. A seminal result of Agler proves that the natural de Branges-Rovnyak kernel function associated to a bounded analytic function on the bidisk can be decomposed into two shift-invariant pieces. Agler's decomposition is non-constructive—a problem remedied by work of Ball-Sadosky-Vinnikov, which uses scattering systems to produce Agler decompositions through concrete Hilbert space geometry. This method, while constructive, so far has not revealed the rich structure shown to be present for special classes of functions—inner and rational inner functions. In this paper, we show that most of the important structure present in these special cases extends to general bounded analytic functions. We give characterizations of all Agler decompositions, we prove the existence of coisometric transfer function realizations with natural state spaces, and we characterize when Schur functions on the bidisk possess analytic extensions past the boundary in terms of associated Hilbert spaces.

1. INTRODUCTION

Let E and E_* be separable Hilbert spaces and recall that the *Schur class* $\mathcal{S}_d(E, E_*)$ is the set of holomorphic functions $\Phi : \mathbb{D}^d \rightarrow \mathcal{L}(E, E_*)$ such that each $\Phi(z) : E \rightarrow E_*$ is a linear contraction. In one variable, the structure of these functions is well-understood and they play key roles in many areas of both pure and applied mathematics. For example, they are objects of interest in H^∞ control theory, act as scattering functions of single-evolution Lax-Phillips scattering systems, and serve

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as the transfer functions of one-dimensional dissipative, linear, discrete-time input/state/output (i/s/o) systems [14, 22, 23]. Moreover, every $\Phi \in \mathcal{S}_1(E, E_*)$ can actually be realized as both a scattering function of a Lax-Phillips scattering system and a transfer function of a dissipative, linear, discrete-time i/s/o system. For simplicity, we omit the discussion of the connection to the interesting topic of von Neumann inequalities; see [4, 14, 24].

The situation in several variables is more complicated; although Schur functions are still the scattering functions of d -evolution scattering systems and transfer functions of d -dimensional dissipative, linear, discrete-time i/s/o systems, the converse is not always true; there are functions in $\mathcal{S}_d(E, E_*)$ that cannot be realized as transfer functions of dissipative i/s/o systems. To make this precise, let $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_d$ be a separable Hilbert space, and for each $z \in \mathbb{D}^d$, define the multiplication operator $\mathcal{E}_z := z_1 P_{\mathcal{M}_1} + \cdots + z_d P_{\mathcal{M}_d}$, where each $P_{\mathcal{M}_r}$ is the projection onto \mathcal{M}_r .

Definition 1.1. Let $\Phi \in \mathcal{S}_d(E, E_*)$. A *Transfer Function Realization* (T.F.R.) of Φ consists of a Hilbert space $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_d$ and a contraction $U : \mathcal{M} \oplus E \rightarrow \mathcal{M} \oplus E_*$ such that if U is written as

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ E \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{M} \\ E_* \end{bmatrix},$$

then $\Phi(z) = D + C(I_{\mathcal{M}} - \mathcal{E}_z A)^{-1} \mathcal{E}_z B$. The Hilbert space \mathcal{M} is called the *state space* and the contraction U is called the *colligation*. One can associate a d -dimensional dissipative, linear, discrete-time i/s/o system with the pair (\mathcal{M}, U) . The transfer function realization is called isometric, coisometric, or unitary whenever U is isometric, coisometric, or unitary.

In [1, 2], J. Agler showed that every function in $\mathcal{S}_2(E, E_*)$ has a T.F.R. and used the realizations to generalize the Pick interpolation theorem to two variables. Since Agler's seminal results, these formulas have been used frequently to both generalize one-variable results and address strictly multivariate questions on the polydisc as in [3, 5, 6, 8, 15, 25, 27, 30]. There is also a simple relationship between transfer function realizations and positive kernels:

Theorem 1.2. (Agler [2]). *Let $\Phi \in \mathcal{S}_d(E, E_*)$. Then, Φ has a transfer function realization if and only if there are positive holomorphic kernels $K_1, \dots, K_d : \mathbb{D}^d \times \mathbb{D}^d \rightarrow \mathcal{L}(E_*)$ such that for all $z, w \in \mathbb{D}^d$*

$$I_{E_*} - \Phi(z)\Phi(w)^* = (1 - z_1 \bar{w}_1)K_1(z, w) + \cdots + (1 - z_d \bar{w}_d)K_d(z, w).$$

This decomposition using positive kernels is called an *Agler decomposition* of Φ . In two variables, it is convenient to reverse the ordering, and throughout this paper, positive kernels (K_1, K_2) are called *Agler kernels of $\Phi \in \mathcal{S}_2(E, E_*)$* if for all $z, w \in \mathbb{D}^2$

$$(1.1) \quad I_{E_*} - \Phi(z)\Phi(w)^* = (1 - z_1\bar{w}_1)K_2(z, w) + (1 - z_2\bar{w}_2)K_1(z, w).$$

Agler proved the existence of a pair of Agler kernels for each function in $\mathcal{S}_2(E, E_*)$ and then showed this gives a transfer function realization via Theorem 1.2. It is often easier to go from kernels to realizations because positive kernels immediately bring operator theory and reproducing kernel Hilbert space methods into the picture. We review some of these concepts related to positive kernels below.

Remark 1.3. Recall that $K : \Omega \times \Omega \rightarrow \mathcal{L}(E)$ is a *positive kernel on Ω* if for each $N \in \mathbb{N}$

$$\sum_{i,j=1}^N \langle K(x_i, x_j)\eta_j, \eta_i \rangle_E \geq 0$$

for all $x_1, \dots, x_N \in \Omega$ and $\eta_1, \dots, \eta_N \in E$. Similarly, \mathcal{H} is a *reproducing kernel Hilbert space on Ω* if \mathcal{H} is a Hilbert space of functions on defined Ω such that evaluation at x is a bounded linear operator for each $x \in \Omega$. Then there is a unique positive kernel $K : \Omega \times \Omega \rightarrow \mathcal{L}(E)$ with

$$\langle f, K(\cdot, y)\eta \rangle_{\mathcal{H}} = \langle f(y), \eta \rangle_E \quad \forall f \in \mathcal{H}, y \in \Omega, \text{ and } \eta \in E.$$

Conversely, given any positive kernel K on Ω , there is a reproducing kernel Hilbert space, denoted $\mathcal{H}(K)$, on Ω with K as its reproducing kernel. For details, see [16].

The kernels K_1, K_2 are written in reverse order in (1.1) because upon dividing the equation through by $(1 - z_1\bar{w}_1)(1 - z_1\bar{w}_2)$, an Agler decomposition can be given a much more natural interpretation in terms of de Branges-Rovnyak spaces.

Remark 1.4. Assume (K_1, K_2) are Agler kernels of Φ and rewrite (1.1) as follows:

$$(1.2) \quad \frac{I - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{K_1(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2(z, w)}{1 - z_2\bar{w}_2}.$$

Each term in (1.2) is a positive kernel and so, we can define the following reproducing kernel Hilbert spaces:

$$\mathcal{H}_{\Phi} := \mathcal{H} \left(\frac{I - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} \right) \quad \text{and} \quad H_j := \mathcal{H} \left(\frac{K_j(z, w)}{1 - z_j\bar{w}_j} \right),$$

for $j = 1, 2$. The Hilbert space \mathcal{H}_Φ is the two-variable de Branges-Rovnyak space associated to Φ . For $j = 1, 2$, define the function Z_j by $Z_j(z) := z_j$. Then the H_j Hilbert spaces have the following properties:

- (1) $Z_j H_j \subseteq H_j$ and multiplication by Z_j on H_j is a contraction.
- (2) The reproducing kernels of the H_j sum to the kernel of \mathcal{H}_Φ .

Basic facts about reproducing kernels imply that if Hilbert spaces H_1 and H_2 satisfy (1) and (2), then the numerators of their reproducing kernels are Agler kernels of Φ .

Agler used non-constructive methods to obtain Agler kernels, and a major stride was made in this theory when Ball-Sadosky-Vinnikov proved the existence of Agler kernels through constructive Hilbert space geometric methods. Indeed, our analysis is motivated by their work on two-evolution scattering systems and scattering subspaces associated to $\Phi \in \mathcal{S}_2(E, E_*)$. In [14], they showed that such scattering subspaces have canonical decompositions into subspaces S_1 and S_2 , each invariant under multiplication by Z_1 or Z_2 . This work was continued in [24] where a specific scattering subspace associated to Φ , denoted \mathcal{K}_Φ , was used to show that canonical decompositions of \mathcal{K}_Φ yield Agler kernels (K_1, K_2) of Φ . The analysis from [14] was also extended in [13]; here, many results from [14] are illuminated or extended via the theory of formal reproducing kernel Hilbert spaces.

While more explicit, the approaches so far do not shed much light on the actual structure of the Hilbert spaces $\mathcal{H}(K_j)$ and the functions contained therein for general Schur functions. The spaces $\mathcal{H}(K_j)$ have been shown to possess a very rich structure when Φ is an *inner* function or a *rational inner* function [17, 18, 19, 29]. This has led to applications in the study of two variable matrix monotone functions in [7] and in the study of *three* variable rational inner functions in [18]. This structure is also important in the Geronimo-Woerdeman characterizations of bivariate Fejér-Riesz factorizations as well as the related bivariate auto-regressive filter problem [21]. The theory is much simpler in these cases because Agler kernels can be constructed directly from orthogonal decompositions of \mathcal{H}_Φ .

Therefore, the major goal of this paper is to show directly that the rich Agler kernel structure present when Φ is inner is still present when Φ is not an inner function. A direct application of this will be to prove that every function in $\mathcal{S}_2(E, E_*)$ possesses a *coisometric* transfer function realization with state space $\mathcal{H}(K_1) \oplus \mathcal{H}(K_2)$ for some pair of Agler kernels (K_1, K_2) ; this construction answers a question posed by Ball and Bolotnikov in [12]. We also generalize classical work of Nagy-Foias connecting regularity of $\Phi \in \mathcal{S}_1(E, E_*)$ on the boundary to the

regularity of functions in its associated de Branges-Rovnyak space. See [33] for a discussion.

We now outline the rest of the paper. The structure of \mathcal{H}_Φ is revealed by embedding an isometric copy into the larger scattering subspace \mathcal{K}_Φ alluded to above. The reader need not know anything about scattering theory—the basic facts we need are built from scratch in Section 2. In Section 3, canonical orthogonal decompositions of \mathcal{K}_Φ are projected down to canonical decompositions of \mathcal{H}_Φ and these yield certain pairs of extremal Agler kernels of Φ denoted

$$(K_1^{max}, K_2^{min}) \quad \text{and} \quad (K_1^{min}, K_2^{max}).$$

These pairs are related by a positive kernel $G : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$

$$G(z, w) := \frac{K_1^{max}(z, w) - K_1^{min}(z, w)}{1 - z_1 \bar{w}_1} = \frac{K_2^{max}(z, w) - K_2^{min}(z, w)}{1 - z_2 \bar{w}_2}.$$

In section 4, we show that all Agler kernels of Φ can be characterized in terms of the special kernels K_1^{min}, K_2^{min}, G :

Theorem (4.2). *Let $\Phi \in \mathcal{S}_2(E, E_*)$ and let $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$. Then (K_1, K_2) are Agler kernels of Φ if and only if there are positive kernels $G_1, G_2 : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$ such that*

$$\begin{aligned} K_1(z, w) &= K_1^{min}(z, w) + (1 - z_1 \bar{w}_1)G_1(z, w) \\ K_2(z, w) &= K_2^{min}(z, w) + (1 - z_2 \bar{w}_2)G_2(z, w) \end{aligned}$$

and $G = G_1 + G_2$.

While Ball-Sadosky-Vinnikov [14] proved the existence of analogous maximal and minimal decompositions in the scattering subspace \mathcal{K}_Φ , our contribution here is to show that many of these extremality properties also hold in the space of interest \mathcal{H}_Φ . On the path to our regularity result, we obtain explicit characterizations of the spaces $\mathcal{H}(K_j^{max})$ and $\mathcal{H}(K_j^{min})$ and use those to show that all $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ are contained inside “small”, easily-studied subspaces of \mathcal{H}_Φ . Section 4.2 has the details.

In Section 5, we consider applications of this Agler kernel analysis. When Φ is square matrix valued, the containments allow us to characterize when Φ and the elements of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ extend analytically past portions of $\partial\mathbb{D}^2$, thus generalizing the regularity result of Nagy-Foias mentioned above. A key point is that \mathcal{H}_Φ is too big of a space for these characterizations, and it really is necessary to study Agler kernels to investigate the regularity of Φ .

We now state the main regularity theorem found in Section 5.1. Let $X \subseteq \mathbb{T}^2$ be an open set and define the sets

$$\begin{aligned} X_1 &:= \{x_1 \in \mathbb{T} : \text{such that } \exists x_2 \text{ with } (x_1, x_2) \in X\} \\ X_2 &:= \{x_2 \in \mathbb{T} : \text{such that } \exists x_1 \text{ with } (x_1, x_2) \in X\} \end{aligned}$$

using X and the sets $\mathbb{E} := \mathbb{C} \setminus \overline{\mathbb{D}}$ and $S := \{1/\bar{z} : \det \Phi(z) = 0\}$. Then, we obtain the following result:

Theorem (5.1). *Let $\Phi \in \mathcal{S}_2(E, E_*)$ be square matrix valued. Then the following are equivalent:*

- (i) Φ extends continuously to X and Φ is unitary valued on X .
- (ii) There is some pair (K_1, K_2) of Agler kernels of Φ such that the elements of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ extend continuously to X .
- (iii) There exists a domain Ω containing

$$\mathbb{D}^2 \cup X \cup (X_1 \times \mathbb{D}) \cup (\mathbb{D} \times X_2) \cup (\mathbb{E}^2 \setminus S)$$

such that Φ and the elements of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ extend analytically to Ω for every pair (K_1, K_2) of Agler kernels of Φ . Moreover the points in the set Ω are points of bounded evaluation of every $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$.

In Section 5.2, we return to the setting of transfer function realizations. We use the canonical Agler kernels (K_1^{max}, K_2^{min}) to construct a T.F.R. of Φ with refined properties. Specifically we prove:

Theorem (5.4). *Let $\Phi \in \mathcal{S}_2(E, E_*)$ and consider its Agler kernels (K_1^{max}, K_2^{min}) . Then, Φ has a coisometric transfer function realization with state space $\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$.*

This construction answers a question posed by Ball and Bolotnikov in [12]. We also obtain additional information about the block operators A, B, C , and D of the associated coisometric colligation U . In Section 6, we provide an appendix outlining results concerning operator valued reproducing kernels used in the paper. We supply the commonly used symbols and table of contents below for convenience.

LIST OF SYMBOLS

(K_1, K_2)	Agler kernels of Φ , page 3
Δ, Δ_*	$(I - \Phi^*\Phi)^{1/2}, (I - \Phi\Phi^*)^{1/2}$, page 11
D_{Φ^*}	The operator $(I - \Phi P_{H^2(E)}\Phi^*)^{1/2}$, page 9
\mathcal{K}_Φ	the scattering subspace of Φ , page 13
\mathcal{H}_Φ	two-variable de Branges-Rovnyak space, page 4
$\mathcal{M}(T)$	operator range, page 8
$\mathcal{S}_d(E, E_*)$	d variable Schur Class, page 1
\mathcal{R}	The residual subspace of the scattering subspace, page 15
\mathcal{R}_j	Slight enlargements of \mathcal{R} , page 23
\mathcal{H}	de Branges-Rovnyak model for Φ , page 10
$\mathcal{W}_*, \mathcal{W}$	Incoming and outgoing subspaces, page 12
A_K	Component of isometry from H_K into K , page 17
G	Reproducing kernel for $H_{\mathcal{R}}$, page 19
H_K	Operator range of $V^* _K$, page 17
K_j^{max}, K_j^{min}	Reproducing kernels for $H_{S_j^{max} \ominus Z_j S_j^{max}}, H_{S_j^{min} \ominus Z_j S_j^{min}}$, page 19
P_+, P_-	Projection onto $H^2, L^2 \ominus H^2$, page 13
P_Φ	Projection onto \mathcal{K}_Φ , page 13
S_j^{max}, S_j^{min}	Subspaces of the scattering subspace, page 14
V	Canonical isometry from \mathcal{H}_Φ to \mathcal{K}_Φ , page 15

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2. DECOMPOSITIONS OF SCATTERING SUBSPACES

For brevity, this paper only outlines the structure of particular scattering systems defined for $\Phi \in \mathcal{S}_2(E, E_*)$. Many details of these scattering systems also appear in [14] and [13]. For a review of the general theory of one- and multi-evolution scattering systems, see [14].

2.1. Notation and Operator Ranges. Before proceeding to scattering systems, we require some notation. Let E be a Hilbert space. Then $L^2(E) := L^2(\mathbb{T}^2) \otimes E$, i.e. the space of E valued functions on \mathbb{T}^2 with square summable Fourier coefficients. Similarly, $H^2(E) := H^2(\mathbb{D}^2) \otimes E$ denotes the space of E valued holomorphic functions on \mathbb{D}^2 whose Taylor coefficients around zero are square summable. Recall that Z_1, Z_2 denote the coordinate functions $Z_j(z_1, z_2) = z_j$. We will define some standard subspaces of $L^2(E)$ according to their Fourier series support. Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_- = \{-1, -2, -3, \dots\}$. If $N \subset \mathbb{Z}^2$ and $f \in L^2(E)$, the statement $\text{supp}(\hat{f}) \subset N$ means $\hat{f}(n_1, n_2) = 0$ for $(n_1, n_2) \notin N$. Now define

$$\begin{aligned} L_{++}^2(E) &:= \{f \in L^2(E) : \text{supp}(\hat{f}) \subset \mathbb{Z}_+ \times \mathbb{Z}_+\} \\ L_{+\bullet}^2(E) &:= \{f \in L^2(E) : \text{supp}(\hat{f}) \subset \mathbb{Z}_+ \times \mathbb{Z}\} \\ L_{-\bullet}^2(E) &:= \{f \in L^2(E) : \text{supp}(\hat{f}) \subset \mathbb{Z}_- \times \mathbb{Z}\} \\ L_{+-}^2(E) &:= \{f \in L^2(E) : \text{supp}(\hat{f}) \subset \mathbb{Z}_+ \times \mathbb{Z}_-\} \\ L_{--}^2(E) &:= \{f \in L^2(E) : \text{supp}(\hat{f}) \subset \mathbb{Z}_- \times \mathbb{Z}_-\}, \end{aligned}$$

and similarly one can define $L_{\bullet+}^2(E)$, $L_{\bullet-}^2(E)$, and $L_{-+}^2(E)$. It is well-known that associating an $H^2(E)$ function f with the L^2 function whose Fourier coefficients agree with the Taylor coefficients of f maps f unitarily to its radial boundary value function in $L_{++}^2(E)$. We will denote both the function in H^2 and the associated function in L_{++}^2 by f . Similarly, given a function f , we follow convention and let f also denote the operator of multiplication by f and f^* denote multiplication by the pointwise a.e. adjoint of the function f .

We also require the following definition and simple lemma about operator ranges; for more details, see the first chapter of [33].

Definition 2.1. Let \mathcal{K} be a Hilbert space and let $T : \mathcal{K} \rightarrow \mathcal{K}$ be a bounded linear operator on \mathcal{K} . Then the *operator range* of T , denoted $\mathcal{M}(T)$, is the Hilbert space consisting of elements in the image of T endowed with the inner product

$$\langle Tx, Ty \rangle_{\mathcal{M}(T)} := \langle P_{(\ker T)^\perp} x, y \rangle_{\mathcal{K}} \quad \forall x, y \in \mathcal{K}.$$

Lemma 2.2. *Let \mathcal{K} be a Hilbert space and let $T : \mathcal{K} \rightarrow \mathcal{K}$ be a bounded linear self-adjoint operator on \mathcal{K} . Then the operator range $\mathcal{M}(T)$ is the closure of the image of T^2 in the $\mathcal{M}(T)$ norm and $\langle Tx, T^2y \rangle_{\mathcal{M}(T)} = \langle Tx, y \rangle_{\mathcal{K}}$, for all $x, y \in \mathcal{K}$.*

Proof. We show that if $\eta \in \mathcal{M}(T)$ and $\eta \perp T^2\mathcal{K}$, then $\eta \equiv 0$. Fix such an η and choose $x \in (\ker T)^\perp$ such that $Tx = \eta$. Then, for each $y \in \mathcal{K}$,

$$0 = \langle \eta, T^2y \rangle_{\mathcal{M}(T)} = \langle x, Ty \rangle_{\mathcal{K}} = \langle Tx, y \rangle_{\mathcal{K}} = \langle \eta, y \rangle_{\mathcal{K}},$$

which implies $\eta \equiv 0$. Moreover, for any $x, y \in \mathcal{K}$,

$$\langle Tx, T^2y \rangle_{\mathcal{M}(T)} = \langle P_{(\ker T)^\perp}x, Ty \rangle_{\mathcal{K}} = \langle TP_{(\ker T)^\perp}x, y \rangle_{\mathcal{K}} = \langle Tx, y \rangle_{\mathcal{K}},$$

as desired. \square

Example 2.3. Let $\Phi \in \mathcal{S}_2(E, E_*)$. The two-variable de Branges-Rovnyak space \mathcal{H}_Φ is also the operator range of the bounded linear self adjoint operator

$$D_{\Phi^*} := (I - \Phi P_{H^2(E)} \Phi^*)^{1/2} : H^2(E_*) \rightarrow H^2(E_*).$$

To see this notice first that by Lemma 2.2, $D_{\Phi^*}^2 H^2(E_*)$ is dense in $\mathcal{M}(D_{\Phi^*})$ and

$$\langle D_{\Phi^*} f, D_{\Phi^*}^2 g \rangle_{\mathcal{M}(D_{\Phi^*})} = \langle D_{\Phi^*} f, g \rangle_{H^2(E_*)}$$

for all $f, g \in H^2(E_*)$. Let k_z be the Szegő kernel on the bidisk. Then, the reproducing kernel of $H^2(E_*)$ is $k_z \otimes I_{E_*}$. Given $f \in \mathcal{M}(D_{\Phi^*})$, $z \in \mathbb{D}^2$, $v \in E_*$, we see that

$$\langle f, D_{\Phi^*}^2 k_z v \rangle_{\mathcal{M}(D_{\Phi^*})} = \langle f, k_z v \rangle_{H^2(E_*)} = \langle f(z), v \rangle_{E_*}$$

and therefore the operator range of D_{Φ^*} is a reproducing kernel Hilbert space on \mathbb{D}^2 with reproducing kernel

$$\frac{I - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)}.$$

Specifically, $\mathcal{M}(D_{\Phi^*})$ is equal to the de Branges-Rovnyak space associated to Φ , which is \mathcal{H}_Φ . This follows from the standard identity for reproducing kernels $P_{H^2} \Phi^* k_z v = \Phi(z)^* k_z v$ and the computation $D_{\Phi^*}^2 k_z v = (I - \Phi P_{H^2} \Phi^*) k_z v = k_z v - \Phi \Phi(z)^* k_z v$.

The following consequence of Douglas's lemma [20] is found on page 3 of [33].

Lemma 2.4. *Let \mathcal{K} be a Hilbert space and let $A : \mathcal{K} \rightarrow \mathcal{K}$, $B : \mathcal{K} \rightarrow \mathcal{K}$ be bounded linear operators. Then, $\mathcal{M}(A) = \mathcal{M}(B)$ isometrically if and only if $AA^* = BB^*$.*

2.2. The de Branges-Rovnyak Models. Now we proceed to scattering systems:

Definition 2.5. A *two-evolution scattering system* $\mathcal{S} = (\mathcal{H}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{F}, \mathcal{F}_*)$ consists of a Hilbert space \mathcal{H} , two unitary operators $\mathcal{U}_1, \mathcal{U}_2 : \mathcal{H} \rightarrow \mathcal{H}$, and two wandering subspaces $\mathcal{F}, \mathcal{F}_* \subseteq \mathcal{H}$ of \mathcal{U}_1 and \mathcal{U}_2 , i.e.

$$\mathcal{F} \perp \mathcal{U}_1^{n_1} \mathcal{U}_2^{n_2} \mathcal{F} \quad \text{and} \quad \mathcal{F}_* \perp \mathcal{U}_1^{n_1} \mathcal{U}_2^{n_2} \mathcal{F}_* \quad \forall (n_1, n_2) \in \mathbb{Z}^2 \setminus (0, 0).$$

Given any $\Phi \in \mathcal{S}_2(E, E_*)$, one can define the de Branges-Rovnyak model for Φ . This is a concrete transcription of the (almost) unique minimal scattering system whose scattering function coincides with Φ . See [14] for the proof and additional theory.

Definition 2.6. The *de Branges-Rovnyak model* for $\Phi \in \mathcal{S}_2(E, E_*)$ consists of the operator range, denoted \mathcal{H} , of the following bounded linear self-adjoint operator:

$$\begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} : \begin{bmatrix} L^2(E_*) \\ L^2(E) \end{bmatrix} \rightarrow \begin{bmatrix} L^2(E_*) \\ L^2(E) \end{bmatrix}.$$

Then \mathcal{H} has inner product given by

$$\left\langle \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} \begin{bmatrix} f' \\ g' \end{bmatrix} \right\rangle_{\mathcal{H}} := \left\langle P_{Q^\perp} \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f' \\ g' \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)},$$

where $Q = \ker \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2}$. Lemma 2.2 implies the image of the

operator $\begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}$ is dense in \mathcal{H} and that

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} f' \\ g' \end{bmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f' \\ g' \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)}, \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H}.$$

The de Branges-Rovnyak model also contains the following two subspaces of \mathcal{H} :

$$\mathcal{F} := \begin{bmatrix} \Phi \\ I \end{bmatrix} E = \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} 0 \\ E \end{bmatrix} \quad \text{and} \quad \mathcal{F}_* := \begin{bmatrix} I \\ \Phi^* \end{bmatrix} E_* = \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} E_* \\ 0 \end{bmatrix}$$

and the two operators $\mathcal{U}_1, \mathcal{U}_2 : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$\mathcal{U}_j := \begin{bmatrix} Z_j I_{E_*} & 0 \\ 0 & Z_j I_E \end{bmatrix} \quad \text{for } j = 1, 2.$$

Each \mathcal{U}_j is onto since

$$\mathcal{U}_j \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} = \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} \mathcal{U}_j \quad \text{and} \quad \mathcal{U}_j \left(L^2(E_*) \oplus L^2(E) \right) = L^2(E_*) \oplus L^2(E).$$

To see that \mathcal{U}_j is isometric, observe that \mathcal{U}_j preserves the \mathcal{H} norm on the image of $\begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}$ since:

$$\begin{aligned} \left\| \mathcal{U}_j \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}}^2 &= \left\langle \mathcal{U}_j \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, \mathcal{U}_j \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)} \\ &= \left\langle \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} Z_j f \\ Z_j g \end{bmatrix}, \begin{bmatrix} Z_j f \\ Z_j g \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)} \\ &= \left\| \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}}^2. \end{aligned}$$

Since said image is dense in \mathcal{H} , each \mathcal{U}_j is unitary. Observe that \mathcal{F} is *wandering* for \mathcal{U}_1 and \mathcal{U}_2 since if $\eta, \nu \in E$ and $(n_1, n_2) \neq (0, 0)$, then

$$\begin{aligned} \left\langle \begin{bmatrix} \Phi \\ I \end{bmatrix} \eta, \mathcal{U}_1^{n_1} \mathcal{U}_2^{n_2} \begin{bmatrix} \Phi \\ I \end{bmatrix} \nu \right\rangle_{\mathcal{H}} &= \left\langle \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} 0 \\ \eta \end{bmatrix}, \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} 0 \\ Z_1^{n_1} Z_2^{n_2} \nu \end{bmatrix} \right\rangle_{\mathcal{H}} \\ &= \left\langle \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} 0 \\ \eta \end{bmatrix}, \begin{bmatrix} 0 \\ Z_1^{n_1} Z_2^{n_2} \nu \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)} \\ &= \langle \eta, Z_1^{n_1} Z_2^{n_2} \nu \rangle_{L^2(E)}, \end{aligned}$$

which is zero. Analogous arguments show \mathcal{F}_* is wandering. We will usually just write $\mathcal{U}_j = Z_j$, unless we wish to emphasize the connection to scattering systems.

The following remarks detail additional facts about \mathcal{H} .

Remark 2.7. Alternate Characterization of \mathcal{H} . Define the bounded linear self-adjoint operators

$$\begin{aligned} \Delta &:= (I - \Phi^* \Phi)^{1/2} : L^2(E) \rightarrow L^2(E) \\ \Delta_* &:= (I - \Phi \Phi^*)^{1/2} : L^2(E_*) \rightarrow L^2(E_*). \end{aligned}$$

By Lemma 2.4, the factorizations

$$\begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix}^{1/2} = \begin{bmatrix} I & 0 \\ \Phi^* & \Delta \end{bmatrix} \begin{bmatrix} I & \Phi \\ 0 & \Delta \end{bmatrix} = \begin{bmatrix} \Delta_* & \Phi \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_* & 0 \\ \Phi^* & I \end{bmatrix}$$

show that

(2.1)

$$\mathcal{H} = \mathcal{M} \left(\begin{bmatrix} I & 0 \\ \Phi^* & \Delta \end{bmatrix} \right) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f \in L^2(E_*), g \in L^2(E), g - \Phi^* f \in \Delta L^2(E) \right\}$$

(2.2)

$$= \mathcal{M} \left(\begin{bmatrix} \Delta_* & \Phi \\ 0 & I \end{bmatrix} \right) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f \in L^2(E_*), g \in L^2(E), f - \Phi g \in \Delta_* L^2(E_*) \right\}.$$

where the equality is on the level of Hilbert spaces, not just as sets. These characterizations of \mathcal{H} can be used to show that the linear maps

$$\begin{bmatrix} f \\ g \end{bmatrix} \mapsto f \text{ and } \begin{bmatrix} f \\ g \end{bmatrix} \mapsto g$$

are contractive operators from \mathcal{H} onto $L^2(E_*)$ and $L^2(E)$ respectively. To see this, note that for each element in \mathcal{H} , there is an $h \in L^2(E)$ such that

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Phi^* & \Delta \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix}, \text{ where } \begin{bmatrix} f \\ h \end{bmatrix} \perp \ker \begin{bmatrix} I & 0 \\ \Phi^* & \Delta \end{bmatrix}.$$

Since \mathcal{H} and the operator range of $\begin{bmatrix} I & 0 \\ \Phi^* & \Delta \end{bmatrix}$ coincide as Hilbert spaces,

$$(2.3) \quad \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}}^2 = \|f\|_{L^2(E_*)}^2 + \|h\|_{L^2(E)}^2 \geq \|f\|_{L^2(E_*)}^2.$$

Similarly, the equality between \mathcal{H} and the operator range of $\begin{bmatrix} \Delta_* & \Phi \\ 0 & I \end{bmatrix}$ shows that for each element of \mathcal{H} ,

$$(2.4) \quad \|g\|_{L^2(E)} \leq \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}}.$$

The following remark discusses additional subspaces of \mathcal{H} that are important for the structure of the scattering system:

Remark 2.8. The Scattering Subspace \mathcal{K}_Φ . The incoming subspace \mathcal{W}_* and outgoing subspace \mathcal{W} of the de Branges-Rovnyak model are defined as follows:

$$\begin{aligned} \mathcal{W}_* &:= \bigoplus_{n \in \mathbb{Z}^2 \setminus \mathbb{Z}_+^2} \mathcal{U}_1^{n_1} \mathcal{U}_2^{n_2} \mathcal{F}_* = \begin{bmatrix} I \\ \Phi^* \end{bmatrix} L^2 \ominus H^2(E_*) \\ \mathcal{W} &:= \bigoplus_{n \in \mathbb{Z}_+^2} \mathcal{U}_1^{n_1} \mathcal{U}_2^{n_2} \mathcal{F} = \begin{bmatrix} \Phi \\ I \end{bmatrix} H^2(E). \end{aligned}$$

An easy calculation shows $\mathcal{W} \perp \mathcal{W}_*$ in \mathcal{H} . This means \mathcal{H} decomposes as

$$\mathcal{H} = \mathcal{W}_* \oplus \mathcal{K}_\Phi \oplus \mathcal{W},$$

where $\mathcal{K}_\Phi := \mathcal{H} \ominus (\mathcal{W} \oplus \mathcal{W}_*)$ is called the *scattering subspace*. A simple computation shows that

$$\begin{bmatrix} f \\ g \end{bmatrix} \perp \mathcal{W}_* \text{ iff } f \in H^2(E_*) \text{ and } \begin{bmatrix} f \\ g \end{bmatrix} \perp \mathcal{W} \text{ iff } g \in L^2 \ominus H^2(E).$$

This means that the scattering subspace

$$\begin{aligned} \mathcal{K}_\Phi &= \mathcal{H} \ominus (\mathcal{W} \oplus \mathcal{W}_*) \\ &= \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} : f \in H^2(E_*), g \in L^2 \ominus H^2(E) \right\}. \end{aligned}$$

Using the alternate characterizations of \mathcal{H} from Remark 2.7, it follows that

$$\begin{aligned} \mathcal{K}_\Phi &= \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f \in H^2(E_*), g \in L^2 \ominus H^2(E), g - \Phi^* f \in \Delta L^2(E) \right\} \\ &= \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f \in H^2(E_*), g \in L^2 \ominus H^2(E), f - \Phi g \in \Delta_* L^2(E_*) \right\}. \end{aligned}$$

The following operator gives the orthogonal projection onto \mathcal{K}_Φ :

$$P_\Phi := \begin{bmatrix} P_+ & -\Phi P_+ \\ -\Phi^* P_- & P_- \end{bmatrix},$$

where $P_+ = P_{H^2}$, and $P_- = P_{L^2 \ominus H^2}$, for either $L^2 \ominus H^2(E)$ or $L^2 \ominus H^2(E_*)$. It is easy to check that $P_\Phi^2 = P_\Phi$, $P_\Phi|_{\mathcal{K}_\Phi} \equiv I$ and $P_\Phi|_{\mathcal{W} \oplus \mathcal{W}_*} \equiv 0$.

Remark 2.9 (Inner functions). When Φ is an inner function, namely when $\Phi^* \Phi = I$, $\Phi \Phi^* = I$ a.e. on \mathbb{T}^2 , the above machinery simplifies significantly and scattering systems are not really necessary. In this case, $\Delta = 0$, $\Delta_* = 0$, so that

$$\mathcal{K}_\Phi = \left\{ \begin{bmatrix} f \\ \Phi^* f \end{bmatrix} : f \in H^2(E_*), \Phi^* f \in L^2 \ominus H^2(E) \right\}.$$

Evidently, the first component in this space is $f \in H^2(E_*)$ such that $\Phi^* f \in L^2 \ominus H^2(E)$. This is equivalent to saying $f \in H^2(E_*) \ominus \Phi H^2(E)$. This space is the usual model space associated to the inner function Φ ; it is studied in [14] and is studied in great depth in [18]. Although in this paper we recover many results from [18], there are many results related to rational inner functions in [18] that we do not mention here. In general, the paper [18] is a more accessible introduction to the present material.

2.3. Decompositions of \mathcal{K}_Φ . In [14, Theorem 5.5], Ball-Sadosky-Vinnikov prove a canonical decomposition of the scattering subspace \mathcal{K}_Φ . To state it, we need these subspaces of \mathcal{K}_Φ :

$$S_1^{max} := \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_\Phi : Z_1^k \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_\Phi \ \forall k \in \mathbb{N} \right\} \quad S_1^{min} := \text{closure } P_\Phi \begin{bmatrix} \Phi \\ I \end{bmatrix} L_{+-}^2(E)$$

$$S_2^{max} := \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_\Phi : Z_2^k \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_\Phi \ \forall k \in \mathbb{N} \right\} \quad S_2^{min} := \text{closure } P_\Phi \begin{bmatrix} \Phi \\ I \end{bmatrix} L_{-+}^2(E),$$

where each closure is taken in \mathcal{K}_Φ . The canonical decomposition of Ball-Sadosky-Vinnikov is as follows:

Theorem 2.10. *Let $\Phi \in \mathcal{S}_2(E, E_*)$. Then, each S_j^{max} and S_j^{min} is invariant under multiplication by Z_j and*

$$(2.5) \quad \mathcal{K}_\Phi = S_1^{max} \oplus S_2^{min} = S_1^{min} \oplus S_2^{max}.$$

For completeness, we include a simple proof of this canonical decomposition.

Proof. Our first observation is that S_1^{max} is equal to

$$\begin{aligned} & \left(\begin{bmatrix} I \\ \Phi^* \end{bmatrix} L^2 \ominus H^2(E_*) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L_{\bullet+}^2(E) \right)^\perp \\ & = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} : f \in H^2(E_*), g \in L_{\bullet-}^2(E) \right\} \end{aligned}$$

since $Z_1^k \begin{bmatrix} f \\ g \end{bmatrix} \perp \begin{bmatrix} \Phi \\ I \end{bmatrix} H^2(E)$ for all $k \geq 0$ if and only if $\begin{bmatrix} f \\ g \end{bmatrix} \perp \begin{bmatrix} \Phi \\ I \end{bmatrix} L_{\bullet+}^2(E)$, which is equivalent to saying $g \in L_{\bullet-}^2(E)$. Therefore, S_1^{max} is equal to

$$\begin{aligned} & \left(\begin{bmatrix} I \\ \Phi^* \end{bmatrix} L^2 \ominus H^2(E_*) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} H^2(E) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L_{-+}^2(E) \right)^\perp \\ & = \mathcal{K}_\Phi \ominus P_\Phi \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L_{-+}^2(E) \right). \end{aligned}$$

Hence,

$$\mathcal{K}_\Phi \ominus S_1^{max} = \text{closure } P_\Phi \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L_{-+}^2(E) \right) = S_2^{min},$$

which shows $\mathcal{K}_\Phi = S_1^{max} \oplus S_2^{min}$ and similarly $\mathcal{K}_\Phi = S_1^{min} \oplus S_2^{max}$. It is also clear that S_j^{max} is invariant under Z_j for $j = 1, 2$. Showing the same is true for S_j^{min} requires more work. Define the following subspace of \mathcal{H}

$$\mathcal{Q} = \left(\begin{bmatrix} I \\ \Phi^* \end{bmatrix} L_{\bullet-}^2(E_*) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L_{\bullet+}^2(E) \right)^\perp$$

and notice that \mathcal{Q} is invariant under both Z_1 and \bar{Z}_1 . Projection onto \mathcal{Q} is given by

$$P_{\mathcal{Q}} = \begin{bmatrix} P_{\bullet+} & -\Phi P_{\bullet+} \\ -\Phi^* P_{\bullet-} & P_{\bullet-} \end{bmatrix}$$

where $P_{\bullet\pm}$ is projection onto the appropriate $L^2_{\bullet\pm}$ space; the proof of this fact is similar to the proof of the formula for P_{Φ} . Now it can be directly checked that

$$P_{\Phi} \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right) = P_{\mathcal{Q}} \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right).$$

The key things to notice are that since $\Phi L^2_{+-}(E) \subset L^2_{+\bullet}(E_*)$, it follows that $P_{\bullet+} \Phi L^2_{+-}(E) = P_+ \Phi L^2_{+-}(E)$, $P_{\bullet+} L^2_{+-} = 0 = P_+ L^2_{+-}$, $P_{\bullet-} \Phi L^2_{+-}(E) = P_- \Phi L^2_{+-}(E)$, and $P_{\bullet-} L^2_{+-}(E) = P_- L^2_{+-}$. However, since \mathcal{Q} is invariant under Z_1 and \bar{Z}_1 , it follows that $P_{\mathcal{Q}}$ commutes with Z_1 . Since $\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E)$ is invariant under Z_1 , we see that

$$P_{\mathcal{Q}} \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right)$$

is invariant under Z_1 , and hence so is its closure. This shows S_1^{min} is invariant under Z_1 and the proof that S_2^{min} is invariant under Z_2 is similar. \square

Definition 2.11. The Residual Subspace \mathcal{R} . It is also useful to consider the residual subspace \mathcal{R} of \mathcal{K}_{Φ} defined initially as $\mathcal{R} := S_1^{max} \ominus S_1^{min}$. Using the decomposition in (2.5), it is basically immediate that

$$\mathcal{R} = S_2^{max} \ominus S_2^{min} = S_1^{max} \cap S_2^{max}.$$

3. CONSTRUCTING AGLER DECOMPOSITIONS

3.1. Connections between \mathcal{K}_{Φ} and \mathcal{H}_{Φ} . The decompositions of \mathcal{K}_{Φ} into S_j^{max} and S_j^{min} can be used to construct similar decompositions of \mathcal{H}_{Φ} . The following results link \mathcal{K}_{Φ} and \mathcal{H}_{Φ} .

Lemma 3.1. *There is an isometry $V : \mathcal{H}_{\Phi} \rightarrow \mathcal{K}_{\Phi}$ such that for each $f \in \mathcal{H}_{\Phi}$,*

$$Vf = \begin{bmatrix} f \\ g \end{bmatrix} \text{ for some } g \in L^2 \ominus H^2(E)$$

and

$$V^* \begin{bmatrix} f \\ g \end{bmatrix} = f \text{ for all } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_{\Phi}.$$

Proof. As was mentioned in Example 2.3, the set $D_{\Phi^*}^2 H^2(E_*)$ is dense in \mathcal{H}_{Φ} . Define the operator V on $D_{\Phi^*}^2 H^2(E_*)$ by

$$VD_{\Phi^*}^2 h = P_{\Phi} \begin{bmatrix} I \\ \Phi^* \end{bmatrix} h \quad \forall h \in H^2(E_*).$$

Notice that this equals

$$\begin{bmatrix} P_+ & -\Phi P_+ \\ -\Phi^* P_- & P_- \end{bmatrix} \begin{bmatrix} I \\ \Phi^* \end{bmatrix} h = \begin{bmatrix} D_{\Phi^*}^2 h \\ P_- \Phi^* h \end{bmatrix} = \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix}.$$

The computation

$$\begin{aligned} \left\| \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix} \right\|_{\mathcal{H}}^2 &= \left\langle \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix}, \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)} \\ &= \left\langle \begin{bmatrix} D_{\Phi^*}^2 h \\ P_- \Phi^* h \end{bmatrix}, \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)} \\ &= \langle D_{\Phi^*}^2 h, h \rangle_{L^2(E_*)} \\ &= \|D_{\Phi^*}^2 h\|_{\mathcal{H}_{\Phi}}^2 \end{aligned}$$

at once shows that V is well-defined ($D_{\Phi^*}^2 h = 0$ implies $VD_{\Phi^*}^2 h = 0$) and isometric, and therefore extends to an isometry from \mathcal{H}_{Φ} to \mathcal{K}_{Φ} . To see that the first component of Vf is always f , it suffices to notice that since the projection $\pi : \begin{bmatrix} f \\ g \end{bmatrix} \mapsto f$ is bounded from \mathcal{H} to $L^2(E_*)$ and since we have $\pi Vf = f$ for the dense set of $f \in D_{\Phi^*}^2 H^2(E_*)$, the identity $\pi Vf = f$ must hold for all $f \in \mathcal{H}_{\Phi}$ by boundedness of πV .

Now, V^* is a partial isometry from \mathcal{K}_{Φ} onto \mathcal{H}_{Φ} , and

$$\ker V^* = (\text{range } V)^{\perp} = \left\{ \begin{bmatrix} 0 \\ g \end{bmatrix} : g \in L^2 \ominus H^2(E) \cap \Delta L^2(E) \right\}.$$

The latter equality can be seen from the following computation. If

$$\begin{aligned} \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_{\Phi} \text{ is orthogonal to the range of } V, \text{ then for any } h \in H^2(E_*) \\ 0 &= \left\langle \begin{bmatrix} I & \Phi \\ \Phi^* & I \end{bmatrix} \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle_{\mathcal{K}_{\Phi}} = \left\langle \begin{bmatrix} h \\ -P_+ \Phi^* h \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle_{L^2(E_*) \oplus L^2(E)} \\ &= \langle h, f \rangle_{L^2(E_*)}, \end{aligned}$$

since $f \in H^2(E_*)$ and $g \in L^2 \ominus H^2(E)$. Upon setting $h = f$, this yields $f = 0$. On the other hand, the above computation shows that if

$\begin{bmatrix} 0 \\ g \end{bmatrix} \in \mathcal{K}_{\Phi}$, then this element is orthogonal to the range of V . So, the

action of V^* on \mathcal{K}_Φ can be directly computed as follows. Any $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_\Phi$ can be written as $Vf + \begin{bmatrix} 0 \\ h \end{bmatrix}$ for some $h \in L^2 \ominus H^2(E) \cap \Delta L^2(E)$. Then, $V^* \begin{bmatrix} f \\ g \end{bmatrix} = f$. \square

An immediate corollary of the above theorem is:

Corollary 3.2. *As sets, $\mathcal{H}_\Phi = \left\{ f \in H^2(E_*) : \text{there is a } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K}_\Phi \right\}$.*

3.2. Hilbert Spaces in \mathcal{H}_Φ . Using the partial isometry V^* and the decompositions of \mathcal{K}_Φ given in Theorem 2.10, we can construct Hilbert spaces yielding Agler decompositions. First, we make some general observations. Let K be a closed subspace of \mathcal{K}_Φ , and denote the operator range of $V^*|_K$ by H_K . Then, $f \in H_K$ if and only if there exists g such that $\begin{bmatrix} f \\ g \end{bmatrix} \in K$. Essentially by the definition of operator range, $V^*|_K$ is a unitary from $K \ominus (K \cap \ker V^*)$ onto H_K , and the inverse of this unitary will be of the form $f \mapsto \begin{bmatrix} f \\ A_K f \end{bmatrix}$ where $A_K : H_K \rightarrow L^2(E)$ is some linear operator. By (2.4), A_K is contractive, i.e. :

$$(3.1) \quad \|A_K f\|_{L^2(E)} \leq \left\| \begin{bmatrix} f \\ A_K f \end{bmatrix} \right\|_{\mathcal{H}} = \|f\|_{H_K}$$

and it is worth pointing out the following representation of the norm

$$\|f\|_{H_K} = \min \left\{ \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}} : g \text{ satisfies } \begin{bmatrix} f \\ g \end{bmatrix} \in K \right\}.$$

Let

$$k_w(z) = \frac{I}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)}$$

be the Szegő kernel on $H^2(E_*)$.

Lemma 3.3. *The reproducing kernel for H_K is given by*

$$V^* P_K V D_{\Phi^*}^2 k_w(z).$$

Moreover, if K is an orthogonal direct sum, $K = \bigoplus_{j=1}^{\infty} K_j$, then the reproducing kernel for H_K is the sum of the reproducing kernels for H_{K_j} .

Proof. Take any $f \in H_K$; this means $f = V^* \begin{bmatrix} f \\ g \end{bmatrix}$, for some $\begin{bmatrix} f \\ g \end{bmatrix} \in K \ominus [K \cap \ker V^*]$. Then, for $w \in \mathbb{D}^2$ and $v \in E_*$

$$\begin{aligned} \langle f, V^* P_K V D_{\Phi^*}^2 k_w v \rangle_{H_K} &= \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, V D_{\Phi^*}^2 k_w v \right\rangle_{\mathcal{K}_{\Phi}} \\ &= \left\langle V^* \begin{bmatrix} f \\ g \end{bmatrix}, D_{\Phi^*}^2 k_w v \right\rangle_{\mathcal{H}_{\Phi}} \\ &= \langle f, k_w v \rangle_{H^2(E_*)} = \langle f(w), v \rangle_{E_*}. \end{aligned}$$

The assertion about direct sums follows from noticing $P_K = \sum_{j=1}^{\infty} P_{K_j}$ in the strong operator topology. \square

The Hilbert spaces of primary interest are defined as follows:

Definition 3.4. Define the Hilbert spaces H_j^{max} and H_j^{min} to be the operator ranges of $V^*|_{S_j^{max}}$ and $V^*|_{S_j^{min}}$. Then

$$f \in H_j^{max} \text{ if and only if } \exists g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in S_j^{max},$$

and the H_j^{max} norm is given by

$$\|f\|_{H_j^{max}} := \left\| P_{S_j^{max} \ominus [S_j^{max} \cap \ker V^*]} \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{S_j^{max}} = \min \left\{ \left\| \begin{bmatrix} f \\ \tilde{g} \end{bmatrix} \right\|_{S_j^{max}} : \begin{bmatrix} f \\ \tilde{g} \end{bmatrix} \in S_j^{max} \right\}.$$

Lemma 3.5 (Wold decompositions).

$$\begin{aligned} S_j^{max} &= \bigoplus_{n \in \mathbb{N}} Z_j^n (S_j^{max} \ominus Z_j S_j^{max}) \oplus M_j^{max} \\ S_j^{min} &= \bigoplus_{n \in \mathbb{N}} Z_j^n (S_j^{min} \ominus Z_j S_j^{min}) \oplus M_j^{min} \end{aligned}$$

where $M_j^{max}, M_j^{min} \subset \ker V^*$.

Proof. Since multiplication by Z_j is an isometry on $S_j^{max/min}$, the classical Wold decomposition says that S_j^{max}, S_j^{min} can be decomposed as above where

$$M_j^{max} = \bigcap_{n \geq 0} Z_j^n S_j^{max} \text{ and } M_j^{min} = \bigcap_{n \geq 0} Z_j^n S_j^{min}$$

so the only thing to show is $M_j^{max} \subset \ker V^*$, since $M_j^{min} \subset M_j^{max}$. So,

if $\begin{bmatrix} f \\ g \end{bmatrix} \in \bigcap_{n \geq 0} Z_1^n S_1^{max}$, then $\bar{Z}_1^n f \in H^2(E_*)$ for all $n \geq 0$, which can only happen if $f = 0$. This shows $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker V^*$. \square

Lemma 3.6. *Let K_j^{max}, K_j^{min} be the reproducing kernels for the operator ranges of $V^*|_{S_j^{max} \ominus Z_j S_j^{max}}, V^*|_{S_j^{min} \ominus Z_j S_j^{min}}$. Then, the reproducing kernels for H_j^{max} and H_j^{min} are given by*

$$\frac{K_j^{max}(z, w)}{1 - z_j \bar{w}_j} \text{ and } \frac{K_j^{min}(z, w)}{1 - z_j \bar{w}_j}.$$

In addition, if G is the reproducing kernel for the operator range of $V^|_{\mathcal{R}}$, then*

$$(3.2) \quad \frac{K_j^{max}(z, w)}{1 - z_j \bar{w}_j} = \frac{K_j^{min}(z, w)}{1 - z_j \bar{w}_j} + G(z, w).$$

Proof. We can focus on H_1^{max} which has reproducing kernel $V^* P_{S_1^{max}} V D_{\Phi^*}^2 k_w$, by Lemma 3.3. Let P_1 denote orthogonal projection onto $S_1^{max} \ominus Z_1 S_1^{max}$. Then, the orthogonal projection onto $Z_1^n (S_1^{max} \ominus Z_1 S_1^{max})$ is given by $Z_1^n P_1 \bar{Z}_1^n$. We now claim that the reproducing kernel for the operator range of V^* restricted to $Z_1^n (S_1^{max} \ominus Z_1 S_1^{max})$ satisfies

$$V^* Z_1^n P_1 \bar{Z}_1^n V D_{\Phi^*}^2 k_w v = \bar{w}_1^n Z_1^n V^* P_1 V D_{\Phi^*}^2 k_w v.$$

Now for $\begin{bmatrix} f \\ g \end{bmatrix} \in S_1^{max}$, we have $Z_1^n V^* \begin{bmatrix} f \\ g \end{bmatrix} = V^* Z_1^n \begin{bmatrix} f \\ g \end{bmatrix}$. This means $V^* Z_1^n P_1 = Z_1^n V^* P_1$ and so, for any $f \in \mathcal{H}_{\Phi}$, $v \in E_*$,

$$\begin{aligned} \langle f, V^* Z_1^n P_1 \bar{Z}_1^n V D_{\Phi^*}^2 k_w v \rangle_{\mathcal{H}_{\Phi}} &= \langle V^* Z_1^n P_1 \bar{Z}_1^n V f, D_{\Phi^*}^2 k_w v \rangle_{\mathcal{H}_{\Phi}} \\ &= \langle Z_1^n V^* P_1 \bar{Z}_1^n V f, D_{\Phi^*}^2 k_w v \rangle_{\mathcal{H}_{\Phi}} \\ &= w_1^n \langle f, V^* Z_1^n P_1 V D_{\Phi^*}^2 k_w v \rangle_{\mathcal{H}_{\Phi}} \\ &= \langle f, \bar{w}_1^n Z_1^n V^* P_1 V D_{\Phi^*}^2 k_w v \rangle_{\mathcal{H}_{\Phi}}, \end{aligned}$$

so that $V^* Z_1^n P_1 \bar{Z}_1^n V D_{\Phi^*}^2 k_w v = \bar{w}_1^n Z_1^n V^* P_1 V D_{\Phi^*}^2 k_w v$. If we break up S_1^{max} according to its Wold decomposition, then since V^* annihilates M_1^{max} , Lemma 3.3 implies that the reproducing kernel of H_1^{max} is given by

$$\sum_{n \geq 0} \bar{w}_1^n z_1^n V^* P_1 V D_{\Phi^*}^2 k_w(z) = \frac{V^* P_1 V D_{\Phi^*}^2 k_w(z)}{1 - z_1 \bar{w}_1} = \frac{K_1^{max}(z, w)}{1 - z_1 \bar{w}_1}.$$

The formulas for H_2^{max} as well as the H_j^{min} follow similarly. The formula (3.2) follows from the orthogonal decomposition $S_j^{max} = S_j^{min} \oplus \mathcal{R}$ and Lemma 3.3. \square

3.3. Construction of Agler Kernels. As above, let K_j^{max}, K_j^{min} be the reproducing kernels for the operator ranges of $V^*|_{S_j^{max} \ominus Z_j S_j^{max}}$ and $V^*|_{S_j^{min} \ominus Z_j S_j^{min}}$ respectively.

Theorem 3.7. *The pairs (K_1^{max}, K_2^{min}) and (K_1^{min}, K_2^{max}) are Agler kernels of Φ , i.e. for all $z, w \in \mathbb{D}^2$,*

$$(3.3) \quad \frac{I_{E_*} - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{K_1^{max}(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2^{min}(z, w)}{1 - z_2\bar{w}_2} = \frac{K_1^{min}(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2^{max}(z, w)}{1 - z_2\bar{w}_2}.$$

Proof. The reproducing kernel of \mathcal{H}_Φ , namely

$$D_{\Phi^*}^2 k_w(z) = \frac{I_{E_*} - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)}$$

is the sum of the kernels for H_1^{max} and H_2^{min} by Lemma 3.3, and these kernels are given by

$$V^* P_{S_1^{max}} V D_{\Phi^*}^2 k_w(z) \text{ and } V^* P_{S_2^{min}} V D_{\Phi^*}^2 k_w(z).$$

By Lemma 3.6, these kernels can be computed directly in terms of the reproducing kernels of K_1^{max} and K_2^{min} to give us the formula (3.3). \square

We remark that by (3.2) we get the formula

$$\frac{I_{E_*} - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{K_1^{min}(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2^{min}(z, w)}{1 - z_2\bar{w}_2} + G(z, w)$$

where $G(z, w) = V^* P_{\mathcal{R}} V D_{\Phi^*}^2 k_w(z)$ is the reproducing kernel of $H_{\mathcal{R}}$, the operator range of $V^*|_{\mathcal{R}}$.

4. GENERAL AGLER KERNELS

4.1. Characterizations of General Agler Kernels. Assume (K_1, K_2) are Agler kernels of $\Phi \in \mathcal{S}_2(E, E_*)$ and define the Hilbert spaces

$$(4.1) \quad H_1 := \mathcal{H} \left(\frac{K_1(z, w)}{1 - z_1\bar{w}_1} \right) \text{ and } H_2 := \mathcal{H} \left(\frac{K_2(z, w)}{1 - z_2\bar{w}_2} \right).$$

Our goal is to use these auxiliary Hilbert spaces H_1 and H_2 to characterize (K_1, K_2) in terms of the extremal kernels $K_1^{max/min}$ and $K_2^{max/min}$. The first main result is the following theorem:

Theorem 4.1. *Let $\Phi \in \mathcal{S}_2(E, E_*)$ and let (K_1, K_2) be Agler kernels of Φ . Define H_1, H_2 as in (4.1). Then*

$$H_1 \subseteq H_1^{max} \text{ and } H_2 \subseteq H_2^{max}$$

and these containments are contractive, i.e. for $j = 1, 2$

$$\|f\|_{H_j^{max}} \leq \|f\|_{H_j} \quad \forall f \in H_j.$$

Proof. Let $f \in H_1$ and assume $\|f\|_{H_1} = 1$. Then for all $n \geq 0$, $Z_1^n f \in H_1 \subset \mathcal{H}_\Phi$ and $\|Z_1^n f\|_{\mathcal{H}_\Phi} \leq \|Z_1^n f\|_{H_1} \leq 1$, since multiplication by Z_1 is a contraction in H_1 . For each n we can choose $g_n \in L^2 \ominus H^2(E)$ such that $\begin{bmatrix} Z_1^n f \\ g_n \end{bmatrix} \in \mathcal{K}_\Phi \ominus \ker V^*$ and

$$\left\| \begin{bmatrix} f \\ \bar{Z}_1^n g_n \end{bmatrix} \right\|_{\mathcal{H}} = \left\| \begin{bmatrix} Z_1^n f \\ g_n \end{bmatrix} \right\|_{\mathcal{H}} = \|Z_1^n f\|_{\mathcal{H}_\Phi} \leq 1.$$

Notice $F_n := \begin{bmatrix} f \\ \bar{Z}_1^n g_n \end{bmatrix} \in \mathcal{K}_\Phi \ominus \bar{Z}_1^n \ker V^*$, since $\bar{Z}_1^n g_n \in \bar{Z}_1^n (L^2 \ominus H^2(E))$. The sequence $\{F_n\} \subset \mathcal{K}_\Phi$ is bounded in norm and therefore has a subsequence $\{F_{n_j}\}$ that converges weakly to some $F := \begin{bmatrix} f' \\ g' \end{bmatrix}$. We claim that $f = f'$ and $g' \in L^2_{\bullet-}(E)$. Since

$$\left\langle F_{n_j}, \begin{bmatrix} I \\ \Phi^* \end{bmatrix} h \right\rangle_{\mathcal{H}} = \langle f, h \rangle_{L^2(E_*)} \rightarrow \left\langle F, \begin{bmatrix} I \\ \Phi^* \end{bmatrix} h \right\rangle_{\mathcal{H}} = \langle f', h \rangle_{L^2(E_*)} \text{ as } j \rightarrow \infty$$

for all $h \in L^2(E_*)$, we see that $f = f'$. Next, for any $v \in E$ and $n \in \mathbb{Z}, m \geq 0$

$$\left\langle F_{n_j}, \begin{bmatrix} \Phi \\ I \end{bmatrix} Z_1^n Z_2^m v \right\rangle_{\mathcal{H}} = \langle \bar{Z}_1^{n_j} g_{n_j}, Z_1^n Z_2^m v \rangle_{L^2(E)} = 0$$

for j large enough that $n_j + n \geq 0$ since $g_{n_j} \perp H^2(E)$. By weak convergence, the above expression converges to

$$\left\langle F, \begin{bmatrix} \Phi \\ I \end{bmatrix} Z_1^n Z_2^m v \right\rangle_{\mathcal{H}} = \langle g', Z_1^n Z_2^m v \rangle_{L^2(E)} = \langle \hat{g}'(n, m), v \rangle_E = 0$$

so we see that $g' \perp L^2_{\bullet+}(E)$ and therefore $g' \in L^2_{\bullet-}(E)$. Hence we conclude that

$$F = \begin{bmatrix} f \\ g' \end{bmatrix} \in S_1^{max}$$

and so $f = V^* F$ must be in H_1^{max} . To show $\|f\|_{H_1^{max}} \leq 1$, observe that

$$|\langle F_{n_j}, F \rangle_{\mathcal{H}}| \rightarrow \|F\|_{\mathcal{H}}^2$$

and

$$|\langle F_{n_j}, F \rangle_{\mathcal{H}}| \leq \|F_{n_j}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}$$

so that $\|F\|_{\mathcal{H}} \leq 1$. Finally, $\|f\|_{H_1^{max}} \leq \|F\|_{\mathcal{H}} \leq 1$ as desired. Thus, H_1 is contractively contained in H_1^{max} . \square

Using the previous result, it is possible to characterize all Agler kernels in terms of the canonical kernels K_1^{min} , K_2^{min} and G as follows:

Theorem 4.2. *Let $\Phi \in \mathcal{S}_2(E, E_*)$ and let $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$. Then (K_1, K_2) are Agler kernels of Φ if and only if there are positive kernels $G_1, G_2 : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$ such that*

$$\begin{aligned} K_1(z, w) &= K_1^{\min}(z, w) + (1 - z_1\bar{w}_1)G_1(z, w) \\ K_2(z, w) &= K_2^{\min}(z, w) + (1 - z_2\bar{w}_2)G_2(z, w) \end{aligned}$$

and $G = G_1 + G_2$.

Proof. (\Rightarrow) Assume (K_1, K_2) are Agler kernels of Φ . By Theorem 4.1 and Theorem 6.3, there are positive kernels $G_1, G_2 : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$ such that

$$\begin{aligned} G_1(z, w) &= \frac{K_1^{\max}(z, w)}{1 - z_1\bar{w}_1} - \frac{K_1(z, w)}{1 - z_1\bar{w}_1} = \frac{K_1(z, w)}{1 - z_1\bar{w}_1} - \frac{K_1^{\min}(z, w)}{1 - z_1\bar{w}_1} \\ G_2(z, w) &= \frac{K_2^{\max}(z, w)}{1 - z_2\bar{w}_2} - \frac{K_2(z, w)}{1 - z_2\bar{w}_2} = \frac{K_2(z, w)}{1 - z_2\bar{w}_2} - \frac{K_2^{\min}(z, w)}{1 - z_2\bar{w}_2}. \end{aligned}$$

To show $G_1 + G_2 = G$, recall that since (K_1, K_2) are Agler kernels of Φ ,

$$\begin{aligned} \frac{K_1^{\min}(z, w)}{1 - z_1\bar{w}_1} + G_1(z, w) + \frac{K_2^{\min}(z, w)}{1 - z_2\bar{w}_2} + G_2(z, w) &= \frac{K_1(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2(z, w)}{1 - z_2\bar{w}_2} \\ &= \frac{I_{E_*} - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} \\ &= \frac{K_1^{\min}(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2^{\min}(z, w)}{1 - z_2\bar{w}_2} + G(z, w), \end{aligned}$$

which implies $G = G_1 + G_2$.

(\Leftarrow) Now assume (K_1, K_2) are positive kernels with positive kernels $G_1, G_2 : \mathbb{D}^2 \times \mathbb{D}^2 \rightarrow \mathcal{L}(E_*)$ satisfying

$$K_j(z, w) = K_j^{\min}(z, w) + (1 - z_j\bar{w}_j)G_j(z, w)$$

for $j = 1, 2$ and $G = G_1 + G_2$. Then

$$\begin{aligned} \frac{K_1(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2(z, w)}{1 - z_2\bar{w}_2} &= \frac{K_1^{\min}(z, w)}{1 - z_1\bar{w}_1} + \frac{K_2^{\min}(z, w)}{1 - z_2\bar{w}_2} + G(z, w) \\ &= \frac{I_{E_*} - \Phi(z)\Phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)}, \end{aligned}$$

which implies (K_1, K_2) are Agler kernels of Φ . \square

4.2. Containment Properties of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$. In this section, we consider the set of functions that can be contained in $\mathcal{H}(K_1)$ or $\mathcal{H}(K_2)$. This result generalizes a result about inner functions from [18]. We require two additional subspaces \mathcal{R}_1 and \mathcal{R}_2 of \mathcal{H} , defined as follows:

$$\mathcal{R}_j = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f \in H^2(E_*), g \in Z_j L^2_{--}(E), f - \Phi g \in \Delta_* L^2(E_*) \right\}$$

for $j = 1, 2$. These are slight enlargements of the residual subspace \mathcal{R} . We can now state the result:

Theorem 4.3. *Let $\Phi \in \mathcal{S}_2(E, E_*)$. Then for $j = 1, 2$*

$$\begin{aligned} \mathcal{H}(K_j^{max}) &= \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \ominus Z_j \mathcal{R} \right\} \\ \mathcal{H}(K_j^{min}) &= \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \ominus \mathcal{R} \right\}. \end{aligned}$$

If (K_1, K_2) are general Agler kernels of Φ , then for $j = 1, 2$

$$\begin{aligned} \mathcal{H}(K_j) &\subseteq \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \right\} \\ &= \left\{ f \in H^2(E_*) : f \in (\Phi Z_j L^2_{--}(E) + \Delta_* L^2(E_*)) \right\}. \end{aligned}$$

The proof of this result requires several auxiliary results about the functions in $S_j^{max} \ominus Z_j S_j^{max}$ and $S_j^{min} \ominus Z_j S_j^{min}$.

Proposition 4.4. *For $j = 1, 2$, the following equality holds:*

$$S_j^{min} \ominus Z_j S_j^{min} = \mathcal{R}_j \ominus \mathcal{R}.$$

Proof. We prove the result for S_1^{min} . We shall make use of the proof of Theorem 2.10. Recall the space \mathcal{Q} defined there:

$$\mathcal{Q} = \left(\begin{bmatrix} I \\ \Phi^* \end{bmatrix} L^2_{\bullet-}(E_*) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{\bullet+}(E) \right)^\perp.$$

We define and manipulate a related space

$$\begin{aligned} \mathcal{M} &= \left(\begin{bmatrix} I \\ \Phi^* \end{bmatrix} L^2_{\bullet-}(E_*) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2 \ominus L^2_{--}(E) \right)^\perp \\ &= \left(\begin{bmatrix} I \\ \Phi^* \end{bmatrix} L^2_{\bullet-}(E_*) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{\bullet+}(E) \right)^\perp \cap \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right)^\perp \\ &= \mathcal{Q} \ominus P_{\mathcal{Q}} \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right). \end{aligned}$$

Also, note $\mathcal{M} = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} : f \in L^2_{\bullet,+}(E_*), g \in L^2_{--}(E) \right\}$. Then,

$$\begin{aligned} \mathcal{Q} \ominus \mathcal{M} &= \text{closure}_{\mathcal{H}} P_{\mathcal{Q}} \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right) \\ &= \text{closure}_{\mathcal{H}} P_{\Phi} \left(\begin{bmatrix} \Phi \\ I \end{bmatrix} L^2_{+-}(E) \right) = S_1^{\min}, \end{aligned}$$

using the proof of Theorem 2.10. Observe that $\mathcal{M} \subseteq Z_1 \mathcal{M} \subseteq \mathcal{Q}$ and $Z_1 \mathcal{Q} = \mathcal{Q}$. Since multiplication by Z_1 is an isometry on \mathcal{H} , we can calculate

$$\begin{aligned} S_1^{\min} \ominus Z_1 S_1^{\min} &= (\mathcal{Q} \ominus \mathcal{M}) \ominus Z_1 (\mathcal{Q} \ominus \mathcal{M}) \\ &= (\mathcal{Q} \ominus \mathcal{M}) \ominus (Z_1 \mathcal{Q} \ominus Z_1 \mathcal{M}) \\ &= (\mathcal{Q} \ominus \mathcal{M}) \ominus (\mathcal{Q} \ominus Z_1 \mathcal{M}) \\ &= Z_1 \mathcal{M} \ominus \mathcal{M}. \end{aligned}$$

As $S_1^{\min} \ominus Z_1 S_1^{\min} \subseteq S_1^{\max}$, we can conclude

$$\begin{aligned} S_1^{\min} \ominus Z_1 S_1^{\min} &= (Z_1 \mathcal{M} \cap S_1^{\max}) \ominus (\mathcal{M} \cap S_1^{\max}) \\ &= \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} : f \in H^2(E_*), g \in Z_1 L^2_{--}(E) \right\} \\ &\quad \ominus \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} : f \in H^2(E_*), g \in L^2_{--}(E) \right\} \\ &= \mathcal{R}_1 \ominus \mathcal{R}, \end{aligned}$$

as desired. The proof follows similarly for S_2^{\min} . \square

We also obtain similar characterizations of $S_j^{\max} \ominus Z_j S_j^{\max}$.

Proposition 4.5. *For $j = 1, 2$ the following equalities hold:*

$$S_j^{\max} \ominus Z_j S_j^{\max} = \mathcal{R}_j \ominus Z_j \mathcal{R}.$$

Proof. Recall that $S_j^{\max} = \mathcal{R} \oplus S_j^{\min}$ and $\mathcal{R}, Z_j \mathcal{R} \subseteq \mathcal{R}_j$. Now

$$\begin{aligned} S_j^{\max} &= (S_j^{\max} \ominus Z_j S_j^{\max}) \oplus Z_j S_j^{\max} \\ &= (S_j^{\max} \ominus Z_j S_j^{\max}) \oplus Z_j \mathcal{R} \oplus Z_j S_j^{\min} \end{aligned}$$

while S_j^{\max} can also be decomposed as

$$\begin{aligned} &\mathcal{R} \oplus (S_j^{\min} \ominus Z_j S_j^{\min}) \oplus Z_j S_j^{\min} \\ &= \mathcal{R} \oplus (\mathcal{R}_j \ominus \mathcal{R}) \oplus Z_j S_j^{\min} \\ &= \mathcal{R}_j \oplus Z_j S_j^{\min}. \end{aligned}$$

Together these show $S_j^{\max} \ominus Z_j S_j^{\max} = \mathcal{R}_j \ominus Z_j \mathcal{R}$. \square

Now we can prove Theorem 4.3.

Proof. The definitions of $\mathcal{H}(K_j^{max})$ and $\mathcal{H}(K_j^{min})$ combined with Propositions 4.4 and 4.5 imply that

$$\begin{aligned}\mathcal{H}(K_j^{max}) &= \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \ominus Z_j \mathcal{R} \right\} \\ \mathcal{H}(K_j^{min}) &= \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \ominus \mathcal{R} \right\},\end{aligned}$$

and then the definition of \mathcal{R}_j implies:

$$\begin{aligned}\mathcal{H}(K_j^{max/min}) &\subseteq \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \right\} \\ &= \{ f \in H^2(E_*) : f \in (\Phi Z_j L^2_{--}(E) + \Delta_* L^2(E_*)) \}.\end{aligned}$$

Now let (K_1, K_2) be any pair of Agler kernels of Φ . By Theorem 4.2, there are positive kernels G_1, G_2 such that each

$$K_j(z, w) = K_j^{min}(z, w) + (1 - z_j \bar{w}_j) G_j(z, w)$$

and $G = G_1 + G_2$. This means

$$\left(K_1^{min}(z, w) + G(z, w) \right) - K_1(z, w) = G_2(z, w) + z_1 \bar{w}_1 G_1(z, w)$$

is a positive kernel. Similar results hold for K_2 , so that Theorem 6.3 implies $\mathcal{H}(K_j)$ is contained contractively in $\mathcal{H}(K_j^{min} + G)$. But then, Theorem 6.5 implies that each $f \in \mathcal{H}(K_j)$ can be written as $f = f_1 + f_2$, for $f_1 \in \mathcal{H}(K_j^{min})$ and $f_2 \in \mathcal{H}(G)$. Our above arguments give the desired result for f_1 and the definition of $\mathcal{H}(G)$ gives the desired result for f_2 . This means

$$\begin{aligned}\mathcal{H}(K_j) &\subseteq \left\{ f : \text{there exists } g \text{ with } \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{R}_j \right\} \\ &= \{ f \in H^2(E_*) : f \in (\Phi Z_j L^2_{--}(E) + \Delta_* L^2(E_*)) \}.\end{aligned}$$

as desired. \square

5. APPLICATIONS

5.1. Analytic Extension Theorem. In this section, we restrict to the situation where E and E_* are finite dimensional with equal dimensions, so after fixing orthonormal bases of E and E_* , we can assume Φ is a square matrix of scalar valued $H^\infty(\mathbb{D}^2)$ functions. The containment results in Theorem 4.3 allow us to give conditions for when such Φ and the elements of any $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ associated to Agler kernels of Φ extend analytically past portions of $\partial\mathbb{D}^2$. We first make some

preliminary comments about defining functions in the canonical spaces outside of the bidisk.

Any Hilbert space contractively contained in $H^2(E_*)$ clearly has bounded point evaluations at points of \mathbb{D}^2 . On the other hand, for the spaces $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2$ we can construct points of bounded evaluation at certain points of \mathbb{E}^2 , where $\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}}$. Using the notation of (3.1), there is a unitary map from $H_{\mathcal{R}}$ onto $\mathcal{R} \ominus (\mathcal{R} \cap \ker V^*)$ of the form

$$f \mapsto \begin{bmatrix} f \\ A_{\mathcal{R}}f \end{bmatrix}$$

where $A_{\mathcal{R}}$ is a contractive linear map from $H_{\mathcal{R}}$ to $L^2_{--}(E)$. If $f \in H_{\mathcal{R}}$, then $\begin{bmatrix} f \\ A_{\mathcal{R}}f \end{bmatrix} \in \mathcal{H}$ and so, by (2.1),

$$f = \Phi A_{\mathcal{R}}f + (I - \Phi\Phi^*)^{1/2}h$$

for some $h \in L^2(E_*)$. Let

$$S = \{z \in \mathbb{E}^2 : \Phi(1/\bar{z}) \text{ is not invertible}\}.$$

Since $A_{\mathcal{R}}f \in L^2_{--}(E)$, we can write $A_{\mathcal{R}}f = \overline{Z_1 Z_2} g$ for $g \in H^2(E)$ and then evaluation at $z \in \mathbb{E}^2 \setminus S$ is defined by

$$(5.1) \quad f(z) := (\Phi(1/\bar{z})^*)^{-1} \frac{1}{z_1 z_2} \overline{g(1/\bar{z})}.$$

Since \mathbb{D}^2 and \mathbb{E}^2 are disjoint, for the moment this is just a formal definition. However, with additional assumptions on Φ , it is this definition of f in \mathbb{E}^2 that provides a holomorphic extension of f . This evaluation is bounded since $|g(1/\bar{z})| \leq C \|g\|_{H^2(E)} = C \|A_{\mathcal{R}}f\|_{L^2(E)}$ for some $C > 0$ and then

$$|f(z)| \leq C \frac{1}{|z_1 z_2|} \|(\Phi(1/\bar{z})^*)^{-1}\| \|A_{\mathcal{R}}f\|_{L^2(E)} \leq C \frac{1}{|z_1 z_2|} \|(\Phi(1/\bar{z})^*)^{-1}\| \|f\|_{H_{\mathcal{R}}}.$$

This shows evaluation at $z \in \mathbb{E}^2 \setminus S$ is a bounded linear functional of $H_{\mathcal{R}} = \mathcal{H}(G)$.

Analogous analysis can be applied to $\mathcal{R}_1, \mathcal{R}_2$ so that $H_{\mathcal{R}_1}, H_{\mathcal{R}_2}$ possess bounded point evaluations at points of $\mathbb{E}^2 \setminus S$. In the case of $f \in H_{\mathcal{R}_1}$, since $A_{\mathcal{R}_1}f \in Z_1 L^2_{--}$, we can write $f = Z_1 \overline{Z_1 Z_2} g = \bar{Z}_2 \bar{g}$ for some $g \in H^2(E_*)$ and then we replace (5.1) with

$$f(z) := (\Phi(1/\bar{z})^*)^{-1} \frac{1}{z_2} \overline{g(1/\bar{z})}$$

for $z \in \mathbb{E}^2 \setminus S$. For $H_{\mathcal{R}_2}$ we simply switch the roles of z_1, z_2 . Since $\mathcal{H}(K_j^{max/min})$ is contractively contained in $H_{\mathcal{R}_j}$, we can define point

evaluations at points of $\mathbb{E}^2 \setminus S$ for the canonical Agler kernel spaces as well.

We proceed to study analytic extensions of Φ past the boundary. Let $X \subseteq \mathbb{T}^2$ be an open set and define the related sets

$$\begin{aligned} X_1 &:= \{x_1 \in \mathbb{T} : \text{such that } \exists x_2 \text{ with } (x_1, x_2) \in X\} \\ X_2 &:= \{x_2 \in \mathbb{T} : \text{such that } \exists x_1 \text{ with } (x_1, x_2) \in X\}. \end{aligned}$$

Then we have the following result:

Theorem 5.1. *Let $\Phi \in \mathcal{S}_2(E, E_*)$ be square matrix valued. Then the following are equivalent:*

- (i) Φ extends continuously to X and Φ is unitary valued on X .
- (ii) There is some pair (K_1, K_2) of Agler kernels of Φ such that the elements of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ extend continuously to X .
- (iii) There exists a domain Ω containing

$$\mathbb{D}^2 \cup X \cup (X_1 \times \mathbb{D}) \cup (\mathbb{D} \times X_2) \cup (\mathbb{E}^2 \setminus S)$$

such that Φ and the elements of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ extend analytically to Ω for every pair (K_1, K_2) of Agler kernels of Φ . Moreover the points in the set Ω are points of bounded evaluation of every $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$.

Proof. We prove (i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i). A similar result for inner functions appears as Theorem 1.5 in [18]. Many of the arguments in this situation are similar. Thus, we outline the proof and provide more details on the points where the two proofs diverge.

Since most of the work occurs in (i) \Rightarrow (iii), let us consider this implication first. The proof involves 3 claims.

Claim 1: Φ extends analytically to Ω .

Since Φ extends continuously to X and is unitary valued there, there is a neighborhood $W^+ \subseteq \mathbb{D}^2$ such that Φ is invertible on W^+ and $X \subseteq \overline{W^+}$. Then

$$(5.2) \quad \Phi(z) := [\Phi(1/\bar{z})^*]^{-1}$$

defines an analytic function on $\mathbb{E}^2 \setminus S$ that is meromorphic on \mathbb{E}^2 . Define $W^- = \{1/\bar{z} : z \in W^+\}$. Then Φ is analytic on $W^+ \cup W^-$ and continuous on $W^+ \cup X \cup W^-$. By Rudin's continuous edge-of-the-wedge theorem, which appears as Theorem A in [32], there is a domain Ω_0 containing $W^+ \cup X \cup W^-$, where Φ extends analytically. This domain only depends on X, W^\pm . Also Φ is already holomorphic on \mathbb{D}^2 , meromorphic on \mathbb{E}^2 , and holomorphic on $\mathbb{E}^2 \setminus S$ using definition (5.2).

We can extend this domain further using Rudin's Theorem 4.9.1 in [31]. It roughly says that if a holomorphic function f on \mathbb{D}^2 extends analytically to a neighborhood N_x of some $x = (x_1, x_2) \in \mathbb{T}^2$, then f extends analytically to an open set containing $\{x_1\} \times \mathbb{D}$ and $\mathbb{D} \times \{x_2\}$. As the edge-of-the-wedge theorem guarantees Φ extends to a neighborhood N_x of each $x \in X$, Rudin's Theorem 4.9.1 implies Φ extends analytically to an open set Ω_1 containing $(X_1 \times \mathbb{D}) \cup (\mathbb{D} \times X_2)$. The *proof* of Theorem 4.9.1 implies that Ω_1 only depends on the $\{N_x\}_{x \in X}$. Thus, Φ extends analytically to

$$\Omega := \mathbb{D}^2 \cup \Omega_1 \cup \Omega_0 \cup (\mathbb{E}^2 \setminus S).$$

Claim 2: Elements of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ extend analytically to Ω .

Let (K_1, K_2) be Agler kernels of Φ and let $f \in \mathcal{H}(K_1)$. By the containment result in Theorem 4.3,

$$f = \Phi A_{\mathcal{R}_1} f + (I - \Phi \Phi^*)^{1/2} h,$$

for some $h \in L^2(E_*)$ and $A_{\mathcal{R}_1} f \in Z_1 L^2_{--}(E)$. Then $g := \overline{Z_2 A_{\mathcal{R}_1} f} \in H^2(E)$, and we can define f analytically on $\mathbb{E}^2 \setminus S$ as before:

$$f(z) = \Phi(z) \frac{1}{z_2} \overline{g(1/\bar{z})}.$$

Then f is analytic on $W^+ \cup W^-$ and $f = \Phi A_{\mathcal{R}_1} f$ on X . As in the proof of Theorem 1.5 in [18], we can use the distributional edge-of-the-wedge theorem, which appears as Theorem B in [32], to extend f to Ω_0 . As before, by an application of Rudin's Theorem 4.9.1 in [31], we can analytically extend f to Ω_1 , the set containing $X_1 \times \mathbb{D}$ and $\mathbb{D} \times X_2$ mentioned earlier. As f is already holomorphic in $\mathbb{D}^2 \cup (\mathbb{E}^2 \setminus S)$, we can conclude that every $f \in \mathcal{H}(K_1)$ is holomorphic in Ω .

Claim 3: Points in Ω are points of bounded evaluation in $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$.

The proof for inner functions given in [18] essentially goes through to give bounded point evaluations in Ω . Recall from the previous section that points of \mathbb{D}^2 and $\mathbb{E}^2 \setminus S$ are points of bounded evaluation for $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$. The next step is to show that the set of points of bounded evaluation is relatively closed in Ω . This follows using the uniform boundedness principle as in [18]. To show evaluation at points of Ω_0 are bounded, we merely note as we did in [18] that the proof of the edge-of-the-wedge theorem in [32] produces the extended values via an

integral over a compact subset K of $W^+ \cup X \cup W^-$. Since evaluation at any point of K is bounded in $\mathcal{H}(K_j)$ and since elements of $\mathcal{H}(K_j)$ are analytic in a neighborhood of K ,

$$\sup\{\|f(z)\|_{E_*} : z \in K\} < \infty$$

for each $f \in \mathcal{H}(K_j)$ and therefore by the uniform boundedness principle there exists M such that

$$\|f(z)\|_{E_*} \leq M\|f\|_{\mathcal{H}(K_j)} \quad \forall f \in \mathcal{H}(K_j)$$

and $z \in K$. So, since values of f in Ω_0 are given by an integral of f over K , it follows that evaluation at points in Ω_0 are bounded in $\mathcal{H}(K_j)$. Now consider the points in Ω_1 . As Rudin's Theorem 4.9.1 in [31] also constructs the extension of f using values of f at points in compact sets $K \subset \Omega_0$, the uniform boundedness principle implies that the points in Ω_1 are also points of bounded evaluation.

(iii) \Rightarrow (ii) is immediate.

Now consider (ii) \Rightarrow (i).

First, we will show that there is a point $w \in \mathbb{D}^2$ where $\Phi(w)$ is invertible. To do this, take any sequence $\{z^n\} \subset \mathbb{D}^2$ converging to a point $x \in X \subset \mathbb{T}^2$. Since elements of $\mathcal{H}(K_j)$ extend continuously to X , for each fixed $f \in \mathcal{H}(K_j)$ the set

$$\{\|f(z^n)\|_{E_*} : n = 1, 2, \dots\}$$

is bounded. Therefore by the uniform boundedness principle for each $j = 1, 2$ the set

$$\{\|f(z^n)\|_{E_*} : f \in \mathcal{H}(K_j), \|f\|_{\mathcal{H}(K_j)} \leq 1, n = 1, 2, \dots\}$$

is bounded by say $M > 0$, and this is enough to show evaluation at $x \in X$ is bounded in $\mathcal{H}(K_j)$ and

$$\|K_j(z^n, z^n)\|_{E_* \rightarrow E_*} \leq M^2 \text{ for each } n \text{ and } \|K_j(x, x)\|_{E_* \rightarrow E_*} \leq M^2$$

for $j = 1, 2$. It follows immediately that

$$\limsup_{n \rightarrow \infty} (1 - |z_1^n|^2) K_2(z^n, z^n) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} (1 - |z_2^n|^2) K_1(z^n, z^n) = 0.$$

This shows that

$$(5.3) \quad \lim_{n \rightarrow \infty} I - \Phi(z^n) \Phi(z^n)^* = \lim_{n \rightarrow \infty} (1 - |z_1^n|^2) K_2(z^n, z^n) + (1 - |z_2^n|^2) K_1(z^n, z^n) = 0$$

and therefore for some $N \in \mathbb{N}$, $I - \Phi(z^N)\Phi(z^N)^* \leq \frac{1}{2}I$, which implies $\Phi(z^N)$ is invertible. Set $w = z^N$. Since Φ satisfies

$$I - \Phi(z)\Phi(w)^* = (1 - z_1\bar{w}_1)K_{2,w}(z) + (1 - z_2\bar{w}_2)K_{1,w}(z)$$

we can extend Φ continuously to X via the formula

$$\Phi(z) = (I - (1 - z_1\bar{w}_1)K_{2,w}(z) - (1 - z_2\bar{w}_2)K_{1,w}(z))(\Phi(w)^*)^{-1}$$

since the right hand side is assumed to be continuous. Finally, (5.3) and the continuity of Φ at x implies that $\Phi(x)$ is unitary. \square

5.2. Canonical Realizations. Unlike the previous section, we no longer assume E, E_* are finite dimensional. Let $\Phi \in \mathcal{S}_1(E, E_*)$ and define its de Branges-Rovnyak space \mathcal{H}_Φ to be the Hilbert space with reproducing kernel

$$K_\Phi(z, w) := \frac{I - \Phi(z)\Phi(w)^*}{1 - z\bar{w}}.$$

Then, Φ has an (almost) unique coisometric transfer function realization with state space equal to \mathcal{H}_Φ and colligation defined by

$$U := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}_\Phi \\ E \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_\Phi \\ E_* \end{bmatrix}$$

with block operators given by

$$\begin{aligned} A : f(z) &\mapsto \frac{f(z) - f(0)}{z} & B : e &\mapsto \frac{\Phi(z) - \Phi(0)}{z}e \\ C : f(z) &\mapsto f(0) & D : e &\mapsto \Phi(0)e. \end{aligned}$$

Then, $\Phi(z) = D + Cz(I - Az)^{-1}B$, and this representation is unique up to a minimality condition and unitary equivalence [12].

In two variables, transfer function realizations are more complicated and rarely unique. Traditionally, T.F.R.'s associated to $\Phi \in \mathcal{S}_2(E, E_*)$ are constructed using Agler kernels (K_1, K_2) of Φ . In [12], Ball-Bolotnikov studied T.F.R.'s defined using pairs of Agler kernels and obtained partial characterizations of the associated block operators A, B, C , and D . Refined results about unitary T.F.R.'s for a subclass of $\mathcal{S}_d(\mathbb{D}^d)$ appear in [13]; these are constructed in the related, but different setting of minimal augmented Agler decompositions.

Nevertheless, open questions about the structure of Agler kernels often go hand in hand with open questions about the structure of T.F.R.'s. In this section, we use our previous analysis to clear up one such question. Specifically, we use the concrete Agler kernels (K_1^{max}, K_2^{min}) to construct a coisometric T.F.R. with an explicit state space \mathcal{M} and colligation U . The construction answers a question posed by Ball and Bolotnikov in [12].

Remark 5.2. Constructing Transfer Function Realizations. There is a canonical way to obtain transfer function realizations from Agler kernels. To illustrate this method, let (K_1, K_2) be Agler kernels of Φ . Then, they satisfy

$$(5.4) \quad I_{E_*} - \Phi(z)\Phi(w)^* = (1 - z_1\bar{w}_1)K_2(z, w) + (1 - z_2\bar{w}_2)K_1(z, w).$$

Define the kernel functions $K_{j,w}\nu(z) := K_j(z, w)\nu$ and define the operator V by

$$V : \begin{bmatrix} \bar{w}_1 K_{2,w}\nu \\ \bar{w}_2 K_{1,w}\nu \\ \nu \end{bmatrix} \mapsto \begin{bmatrix} K_{2,w}\nu \\ K_{1,w}\nu \\ \Phi(w)^*\nu \end{bmatrix} \quad \forall w \in \mathbb{D}^2, \nu \in E_*.$$

Then (5.4) guarantees that V can be extended to an isometry mapping the space

$$\mathcal{D}_V := \bigvee_{w \in \mathbb{D}^2, \nu \in E_*} \begin{bmatrix} \bar{w}_1 K_{2,w}\nu \\ \bar{w}_2 K_{1,w}\nu \\ \nu \end{bmatrix} \subseteq \mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E_*$$

onto the space

$$\mathcal{R}_V := \bigvee_{w \in \mathbb{D}^2, \nu \in E_*} \begin{bmatrix} K_{2,w}\nu \\ K_{1,w}\nu \\ \Phi(w)^*\nu \end{bmatrix} \subseteq \mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E.$$

Transfer function realizations with state space $\mathcal{H}(K_2) \oplus \mathcal{H}(K_1)$ are obtained by extending V to a contraction from

$$\mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E \rightarrow \mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E_*$$

and setting $U = V^*$. In Ball-Bolotnikov [12], such a U is called a *canonical functional model (c.f.m.) colligation* of Φ associated to (K_1, K_2) . Similarly, coisometric transfer function realizations are obtained by extending V to an isometry mapping

$$\mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus \mathcal{H} \oplus E \rightarrow \mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus \mathcal{H} \oplus E_*,$$

where \mathcal{H} is an arbitrary infinite dimensional Hilbert space only added in when required, and U is defined to be V^* . Observe that if \mathcal{H} is needed, then Φ has many coisometric transfer function realizations that can be constructed starting from (K_1, K_2) .

Moreover, notice that if the codimension of \mathcal{D}_V does not exceed the codimension of \mathcal{R}_V , then V can be extended to an isometry mapping

$$\mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E \rightarrow \mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E_*.$$

Specifically, if this codimension inequality holds, then the infinite-dimensional space \mathcal{H} is not needed to obtain a coisometric transfer function realization of Φ . This motivates the following question:

Question 5.3. Let $\Phi \in \mathcal{S}(E, E_*)$ and (K_1, K_2) be Agler kernels of Φ . One can ask: Is there a coisometric transfer function realization of Φ with state space $\mathcal{H}(K_2) \oplus \mathcal{H}(K_1)$ for every pair of Agler kernels (K_1, K_2) . This is the same as asking: does the V constructed in Remark 5.2 extend to an isometry mapping

$$\mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E \rightarrow \mathcal{H}(K_2) \oplus \mathcal{H}(K_1) \oplus E_*$$

Currently, this is an open question. Ball-Bolotnikov asked the following related question (in the d -variable setting) in Section 3.2 of [12]:

Let $\Phi \in \mathcal{S}_2(E, E_*)$. Is there *any* pair of Agler kernels (K_1, K_2) of Φ such that Φ has a *coisometric* c.f.m. colligation associated to (K_1, K_2) ?

This is equivalent to asking if the construction in Remark 5.2 gives a coisometric transfer function realization of Φ with state space $\mathcal{H}(K_2) \oplus \mathcal{H}(K_1)$, i.e. if the infinite dimensional space \mathcal{H} is not required. As mentioned above, this will occur if the codimension of \mathcal{D}_V is less than that of \mathcal{R}_V .

The following theorem answers Question 5.3 in the affirmative. Specifically, it shows that if one does the construction in Remark 5.2 with (K_1^{max}, K_2^{min}) , then the codimension of \mathcal{D}_V is zero. As mentioned earlier, this implies Φ has a coisometric c.f.m. colligation associated to (K_1^{max}, K_2^{min}) .

Theorem 5.4. *Let $\Phi \in \mathcal{S}_2(E, E_*)$ and consider its Agler kernels (K_1^{max}, K_2^{min}) . The construction in Remark 5.2 gives a unique, coisometric transfer function realization of Φ with state space $\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$.*

Proof. Consider the construction in Remark 5.2 using Agler kernels (K_1^{max}, K_2^{min}) . The operator V is initially defined by

$$V : \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \\ \nu \end{bmatrix} \mapsto \begin{bmatrix} K_{2,w}^{min} \nu \\ K_{1,w}^{max} \nu \\ \Phi(w)^* \nu \end{bmatrix} \quad \forall w \in \mathbb{D}^2, \nu \in E_*$$

and extended to an isometry on the space

$$\mathcal{D}_V := \bigvee_{w \in \mathbb{D}^2, \nu \in E_*} \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \\ \nu \end{bmatrix} \subseteq \mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max}) \oplus E_*.$$

Then, transfer function realizations with state space $\mathcal{H}(K_2) \oplus \mathcal{H}(K_1)$ are obtained by extending V to a contraction on $\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max}) \oplus E_*$. We will show $\mathcal{D}_V = \mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max}) \oplus E_*$. Then, the result will

follow because V will already be an isometry on $\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max}) \oplus E_*$ and so we can immediately set $U = V^*$. Define

$$\mathcal{D} := \bigvee_{w \in \mathbb{D}^2, \nu \in E_*} \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \end{bmatrix} \subseteq \mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max}).$$

Examining the case $w = 0$ shows that \mathcal{D}_V coincides with $\mathcal{D} \oplus E_*$, so it suffices to show $\mathcal{D} = \mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$. Assume

$$\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} \in [\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})] \ominus \mathcal{D}.$$

Then for each $w \in \mathbb{D}^2$ and $\nu \in E_*$,

$$\begin{aligned} 0 &= \left\langle \begin{bmatrix} f_2 \\ f_1 \end{bmatrix}, \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \end{bmatrix} \right\rangle_{\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})} \\ &= w_1 \langle f_2, K_{2,w}^{min} \nu \rangle_{\mathcal{H}(K_2^{min})} + w_2 \langle f_1, K_{1,w}^{max} \nu \rangle_{\mathcal{H}(K_1^{max})} \\ &= \langle w_1 f_2(w) + w_2 f_1(w), \nu \rangle_{E_*}, \end{aligned}$$

which implies $Z_1 f_2 + Z_2 f_1 = 0$. Thus, there is some $F \in H^2(E_*)$ such that $f_1 = Z_1 F$. Now, since $f_1 \in \mathcal{H}(K_1^{max})$, there is a $g_1 \in Z_1 L^2_{--}(E)$ such that

$$(5.5) \quad \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \in \mathcal{R}_1 \ominus Z_1 \mathcal{R}.$$

This also gives $g_1 - \Phi^* f_1 \in \Delta L^2(E)$ and a $G \in L^2_{--}(E)$ with $g = Z_1 G$. Since $\Delta L^2(E)$ is invariant under Z_1^* , it is clear that $G - \Phi^* F \in \Delta L^2(E)$ as well. Then

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = Z_1 \begin{bmatrix} F \\ G \end{bmatrix} \text{ and } \begin{bmatrix} F \\ G \end{bmatrix} \in \mathcal{R}.$$

Given this, (5.5) forces $f_1 \equiv 0$. As $Z_1 f_2 + Z_2 f_1 = 0$, we must have $f_2 \equiv 0$. This immediately implies $\mathcal{D} = \mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$. \square

The arguments in the proof of Theorem 5.4 can also be used to obtain additional information about $\mathcal{H}(K_1^{max})$. Specifically, we have:

Proposition 5.5. *The set $\{(0, w_2) : w_2 \in \mathbb{D} \setminus \{0\}\}$ is a set of uniqueness for $\mathcal{H}(K_1^{max})$.*

Proof. Suppose two functions $g_1, g_2 \in \mathcal{H}(K_1^{max})$ satisfy $g_1(0, w_2) = g_2(0, w_2)$ for all $w_2 \neq 0$. By analyticity, this immediately implies $g_1(0, 0) = g_2(0, 0)$ and

$$g_1 - g_2 = Z_1 h$$

for some $h \in H^2(E_*)$. Arguments identical to those in the proof of Theorem 5.4 show that $g_1 - g_2$ must be zero, so $g_1 = g_2$. \square

Remark 5.6. The Canonical Block Operators. Let U be the operator associated to the transfer function realization given in Theorem 5.4. Much can be said about its block operators A, B, C, D . In the setting of general (K_1, K_2) , much of this analysis already appears in [11] and [12]. We will first give the formulas for A, B, C, D and then discuss the derivations. Specifically, for every $f := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$ and $\eta \in E$,

$$C : \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \mapsto f_1(0) + f_2(0) \quad \text{and} \quad D : \eta \mapsto \Phi(0)\eta.$$

For A and B , let us first simplify notation by setting

$$\begin{bmatrix} (Af)_1 \\ (Af)_2 \end{bmatrix} := A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (B\eta)_1 \\ (B\eta)_2 \end{bmatrix} := B\eta.$$

Then $(Af)_2$ and $(B\eta)_2$ are the unique functions in $\mathcal{H}(K_1^{max})$ satisfying

$$(Af)_2(0, w_2) = \frac{f_1(0, w_2) - f_1(0) + f_2(0, w_2) - f_2(0)}{w_2}$$

$$(B\eta)_2(0, w_2) = \frac{\Phi(0, w_2) - \Phi(0)}{w_2}\eta,$$

for all $w_2 \in \mathbb{D} \setminus \{0\}$, and $(Af)_1$ and $(B\eta)_1$ are the unique functions in $\mathcal{H}(K_2^{min})$ satisfying

$$(Af)_1(w) = \frac{f_1(w) - f_1(0) + f_2(w) - f_2(0) - w_2(Af)_2(w)}{w_1}$$

$$(B\eta)_1(w) = \frac{(\Phi(w) - \Phi(0))\eta - w_2(B\eta)_2(w)}{w_1},$$

for all $w \in \mathbb{D}^2$ with $w_1 \neq 0$. The results for C and D follow because, by definition

$$U^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} : \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \\ \nu \end{bmatrix} \mapsto \begin{bmatrix} K_{2,w}^{min} \nu \\ K_{1,w}^{max} \nu \\ \Phi(w)^* \nu \end{bmatrix} \quad \forall w \in \mathbb{D}^2, \nu \in E_*.$$

Setting $w = 0$ immediately implies that

$$C^* : \nu \mapsto \begin{bmatrix} K_{2,0}^{min} \nu \\ K_{1,0}^{max} \nu \end{bmatrix} \quad \text{and} \quad D^* : \nu \mapsto \Phi(0)^* \nu$$

for all $\nu \in E_*$. Then the calculations

$$\left\langle C \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \nu \right\rangle_{E_*} = \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} K_{2,0}^{min} \nu \\ K_{1,0}^{max} \nu \end{bmatrix} \right\rangle_{\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})} = \langle f_1(0) + f_2(0), \nu \rangle_{E_*}$$

and

$$\langle D\eta, \nu \rangle_{E_*} = \langle \eta, D^* \nu \rangle_E = \langle \eta, \Phi(0)^* \nu \rangle_E = \langle \Phi(0)\eta, \nu \rangle_{E_*}$$

give the formulas for C and D . Moreover, the results about C^* and D^* imply that

$$A^* : \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \end{bmatrix} \mapsto \begin{bmatrix} (K_{2,w}^{min} - K_{2,0}^{min}) \nu \\ (K_{1,w}^{max} - K_{1,0}^{max}) \nu \end{bmatrix}$$

and

$$B^* : \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \end{bmatrix} \mapsto (\Phi(w)^* - \Phi(0)^*) \nu.$$

Then

$$\begin{aligned} \langle w_1(Af)_1(w) + w_2(Af)_2(w), \nu \rangle_{E_*} &= \left\langle Af, \begin{bmatrix} \bar{w}_1 K_{2,w}^{min} \nu \\ \bar{w}_2 K_{1,w}^{max} \nu \end{bmatrix} \right\rangle_{\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})} \\ &= \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} (K_{2,w}^{min} - K_{2,0}^{min}) \nu \\ (K_{1,w}^{max} - K_{1,0}^{max}) \nu \end{bmatrix} \right\rangle_{\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})} \\ &= \langle f_1(w) - f_1(0) + f_2(w) - f_2(0), \nu \rangle_{E_*}, \end{aligned}$$

and similarly,

$$\langle w_1(B\eta)_1(w) + w_2(B\eta)_2(w), \nu \rangle_{E_*} = \langle (\Phi(w) - \Phi(0))\eta, \nu \rangle_{E_*}.$$

Therefore, we have

$$(5.6) \quad w_1(Af)_1(w) + w_2(Af)_2(w) = f_1(w) - f_1(0) + f_2(w) - f_2(0)$$

$$(5.7) \quad w_1(B\eta)_1(w) + w_2(B\eta)_2(w) = (\Phi(w) - \Phi(0))\eta.$$

Operators that solve (5.6) or (5.7) are said to solve the structured Gleason problem for $\mathcal{H}(K_2^{min}) \oplus \mathcal{H}(K_1^{max})$ or for Φ , respectively. In general, such operators are not unique. However, in this situation, A and B are uniquely determined. To see this, observe that when $w_1 = 0$ and $w_2 \neq 0$, (5.6) and (5.7) become

$$(5.8) \quad (Af)_2(0, w_2) = \frac{f_1(0, w_2) - f_1(0) + f_2(0, w_2) - f_2(0)}{w_2}$$

$$(5.9) \quad (B\eta)_2(0, w_2) = \frac{\Phi(0, w_2) - \Phi(0)}{w_2} \eta.$$

Since $(Af)_2$ and $(B\eta)_2$ are in $\mathcal{H}(K_1^{max})$, Proposition 5.5 states that the set $\{(0, w_2) : w_2 \in \mathbb{D} \setminus \{0\}\}$ is a set of uniqueness for these functions. Thus, $(Af)_2$ and $(B\eta)_2$ must be the unique functions satisfying (5.8) and (5.9) respectively. Then, the other components $(Af)_1$ and $(B\eta)_1$ are uniquely determined by (5.6) and (5.7). In one-variable, Af and $B\eta$ can be explicitly written in terms of f and η . Given that, our characterizations of A and B seem slightly unsatisfying. This motivates the question:

Question 5.7. Assume $g \in \mathcal{H}(K_1^{max})$. Is there an explicit way to construct g using only the function $g(0, w_2)$?

A clean answer would also provide nice formulas for the operators A and B . It seems possible that the refined results in [13] about unitary T.F.R.'s associated to minimal augmented Agler decompositions might suggest methods of answering this question.

6. APPENDIX: VECTOR VALUED RKHS'S

In this section, we record several facts about vector valued reproducing kernel Hilbert spaces that were used in earlier sections. The results are well-known in the scalar valued case. See, for example [10], [16], Chapter 2 in [9], and Chapter 2 in [4]. We outline how the needed vector valued results follow from the known scalar valued results. Let Ω be a set and E be a separable Hilbert space. We will frequently use the following observation:

Remark 6.1. For each function $f : \Omega \rightarrow E$ there is an associated scalar valued function $\tilde{f} : \Omega \times E \rightarrow \mathbb{C}$ defined as follows:

$$\tilde{f}(z, \eta) := \langle f(z), \eta \rangle_E.$$

If functions $f, g : \Omega \rightarrow E$ and $\tilde{f} \equiv \tilde{g}$, then $f \equiv g$.

Definition 6.2. Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space of E valued functions on Ω . For $w \in \Omega$ and $\nu \in E$, define the function $K_w\nu := K(\cdot, w)\nu$. An associated reproducing kernel Hilbert space of scalar valued functions on $\Omega \times E$ can be defined as follows: Define the set of functions

$$\mathcal{H} := \left\{ \tilde{f} : f \in \mathcal{H}(K) \right\}$$

and equip \mathcal{H} with the inner product

$$\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}(K)}.$$

It is routine to show that \mathcal{H} is a Hilbert space with this inner product and since

$$\tilde{f}(w, \nu) = \langle f(w), \nu \rangle_E = \langle f, K_w \nu \rangle_{\mathcal{H}(K)} = \left\langle \tilde{f}, \widetilde{K_w \nu} \right\rangle_{\mathcal{H}},$$

\mathcal{H} is a reproducing kernel Hilbert space with reproducing kernel

$$L((z, \eta), (w, \nu)) := \widetilde{K_w \nu}(z, \eta) = \langle K(z, w) \nu, \eta \rangle_E = \eta^* K(z, w) \nu.$$

Then $f \in \mathcal{H}(K)$ if and only if $\tilde{f} \in \mathcal{H}(L)$. It is also clear that $\|f\|_{\mathcal{H}(K)} = \|\tilde{f}\|_{\mathcal{H}(L)}$.

The following results are well-known for scalar valued reproducing kernel Hilbert spaces and follow easily for vector valued reproducing kernel Hilbert spaces.

Theorem 6.3. *Let $\mathcal{H}(K)$ and $\mathcal{H}(K_1)$ be reproducing kernel Hilbert spaces of E valued functions on Ω . Then $\mathcal{H}(K_1) \subseteq \mathcal{H}(K)$ contractively if and only if*

$$K(z, w) - K_1(z, w) \text{ is a positive kernel.}$$

Proof. As in Definition 6.2, consider the Hilbert spaces $\mathcal{H}(L)$ and $\mathcal{H}(L_1)$ of scalar valued functions on $\Omega \times E$ with reproducing kernels given by $L((z, \eta), (w, \nu)) := \eta^* K(z, w) \nu$ and $L_1((z, \eta), (w, \nu)) := \eta^* K_1(z, w) \nu$.

It is routine to show that $\mathcal{H}(K_1) \subseteq \mathcal{H}(K)$ contractively if and only if $\mathcal{H}(L_1) \subseteq \mathcal{H}(L)$ contractively. It follows from well-known scalar results, which appear on page 354 of [10], that $\mathcal{H}(L_1) \subseteq \mathcal{H}(L)$ contractively if and only if

$$L(z, w) - L_1(z, w) \text{ is a positive kernel.}$$

The result follow from the fact that $L(z, w) - L_1(z, w)$ is a positive kernel if and only if $K(z, w) - K_1(z, w)$ is a positive kernel. \square

Similarly, the following two results can be deduced from the scalar-valued case:

Theorem 6.4. *Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space of E valued functions on Ω and let $\psi : \Omega \rightarrow \mathbb{C}$. Then ψ is a multiplier of $\mathcal{H}(K)$ with multiplier norm bounded by one if and only if*

$$(1 - \psi(z)\overline{\psi(w)})K(z, w) \text{ is a positive kernel.}$$

Proof. When we say “ ψ is a multiplier of $\mathcal{H}(K)$,” we mean that $\psi \otimes I_{\mathcal{H}(K)}$ maps $\mathcal{H}(K)$ into $\mathcal{H}(K)$.

Now, using the definition of $\mathcal{H}(L)$, it is easy to show that ψ is a multiplier of $\mathcal{H}(K)$ with multiplier norm bounded by one if and only if ψ is a multiplier of $\mathcal{H}(L)$ with multiplier norm bounded by one. By the

analogous scalar valued result, which appears as Corollary 2.3.7 in [4], it follows that ψ is a multiplier of $\mathcal{H}(L)$ with multiplier norm bounded by one if and only if

$$(1 - \psi(z)\overline{\psi(w)})L((z, \eta), (w, \nu)) \text{ is a positive kernel.}$$

The result then follows by using the definition of a positive kernel to show that $(1 - \psi(z)\overline{\psi(w)})L((z, \eta), (w, \nu))$ is a positive kernel if and only if $(1 - \psi(z)\overline{\psi(w)})K(z, w)$ is a positive kernel. \square

Theorem 6.5. *Let $\mathcal{H}(K_1), \mathcal{H}(K_2)$ be reproducing kernel Hilbert spaces of E valued functions on Ω . Then $\mathcal{H}(K_1 + K_2)$ is precisely the Hilbert space composed of the set of functions*

$$\mathcal{H}(K_1) + \mathcal{H}(K_2) := \{f_1 + f_2 : f_j \in \mathcal{H}(K_j)\}.$$

equipped with the norm

$$\|f\|_{\mathcal{H}(K_1+K_2)}^2 = \min_{\substack{f=f_1+f_2 \\ f_j \in \mathcal{H}(K_j)}} \|f_1\|_{\mathcal{H}(K_1)}^2 + \|f_2\|_{\mathcal{H}(K_2)}^2.$$

Proof. As before consider the related scalar valued reproducing kernel Hilbert spaces $\mathcal{H}(L_1)$ and $\mathcal{H}(L_2)$, where

$$L_1((z, \eta), (w, \nu)) := \eta^* K_1(z, w)\nu \quad \text{and} \quad L_2((z, \eta), (w, \nu)) := \eta^* K_2(z, w)\nu.$$

The analogous scalar valued result, which appears on page 353 in [10], states $\mathcal{H}(L_1 + L_2)$ is precisely the Hilbert space composed of the set of functions

$$\mathcal{H}(L_1) + \mathcal{H}(L_2) := \{f_1 + f_2 : f_j \in \mathcal{H}(L_j)\}.$$

equipped with the norm

$$\|f\|_{\mathcal{H}(L_1+L_2)}^2 = \min_{\substack{f=f_1+f_2 \\ f_j \in \mathcal{H}(L_j)}} \|f_1\|_{\mathcal{H}(L_1)}^2 + \|f_2\|_{\mathcal{H}(L_2)}^2,$$

Using this and the connections between $\mathcal{H}(L_j)$ and $\mathcal{H}(K_j)$, it is easy to deduce the desired result. The details are left as an exercise. \square

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