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Exploring User-Provided Connectivity

Mohammad H. Afrasiabi and Roch Guerin

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Exploring User-Provided Connectivity *

M. H. Afrasiabi
Dept. Elec. & Sys. Eng., U. Pennsylvania
afram@seas.upenn.edu

R. Guérin
Dept. Comp. Sci. & Eng., Washington U. in St. Louis
guerin@wustl.edu

***Abstract*—Network services often exhibit positive and negative *externalities* that affect users' adoption decisions. One such service is "user-provided connectivity" or UPC. The service offers an alternative to traditional infrastructure-based communication services by allowing users to share their "home base" connectivity with other users, thereby increasing their access to connectivity. More users mean more connectivity alternatives, *i.e.*, a positive externality, but also greater odds of having to share one's own connectivity, *i.e.*, a negative externality. The tug of war between positive and negative externalities together with the fact that they often depend not just on how many but also *which* users adopt, make it difficult to predict the service's eventual success. Exploring this issue is the focus of this paper, which investigates not only when and why such services may be viable, but also explores how pricing can be used to effectively and practically realize them.**

I. Introduction

There is no denying that we are a networked society, and many networked goods or services exhibit strong *externalities*, *i.e.,* a change — positive or negative — in the value of one unit of good, as more people use those goods. For example, Metcalfe's law [1, p.71] captures the positive effect on a network value of having more users, while the increased congestion that arises from the added traffic contributes a negative externality. Externalities, and more generally the benefits derived from goods or services, vary across users, *i.e.,* exhibit heterogeneity. This makes predicting the impact of externalities difficult, especially when positive and negative forces interact. A basic question of interest is then to determine (ahead of time) if and how offerings of goods or services that exhibit positive and negative externalities will succeed or fail.

The original motivation for this paper was answering this question for a specific service, namely, *user provided connectivity* or UPC. The goal of UPC is to address the rising thirst for ubiquitous data connectivity fueled by the fast growing number of capable and versatile mobile devices. This growth has taxed the communication infrastructure of wireless carriers to the point where it is threatening their continued success [2]. Addressing this issue calls for either upgrading the infrastructure; a costly proposition, or exploring alternatives for "off-loading" some of the traffic. WiFi off-load solutions (*e.g.,* as embodied in the Hotspot2.0 initiative of the WiFi Alliance and the Next Generation Hotspot (NGH) of the Wireless Broadband Alliance) offer a possible option, of which FON[1] demonstrated a possible realization. FON users purchase an access router (FONERA) that they use for their own local broadband access, but with the agreement that a (small) fraction of their access bandwidth can be made available to other FON users. In exchange, they receive the same privilege[2] when roaming, *i.e.,* can connect through the access points of other FON users.

Under a UPC scheme, connectivity grows "organically" as more users join the network and improve its coverage, and the challenge is to determine if it can reach sufficient critical mass to be viable. Consider for example a FON-like service starting with no users. This makes the service unattractive to users that value ubiquitous connectivity highly, *e.g.,* users that roam frequently, because the limited coverage offers little connectivity beyond that of a user's "home base". On the other hand, sedentary users are mostly insensitive to the initial minimal coverage, and if the price is low enough can derive positive utility from the service; hence join. If enough such (sedentary) users join, coverage may increase past a point where it becomes attractive to roaming users who will start joining. This would then ensure rapid growth of the service, were it not for a negative dimension to that growth.

Specifically, as more roaming users join, they compete for connectivity and may encounter increasingly congested access points. Conversely, sedentary users end-up having to share their home access more frequently. This may be sufficient to convince them to drop the service (unlike roaming users, they do not see much added value from the better coverage). The resulting reduction in coverage would in turn affect roaming users, who could then also start leaving. Hence, after an initial period of growth, the service may experience a decline.

The extent to which such behaviors arise depends on many factors, and in particular the trade-off between service cost and users' sensitivity to the positive and negative aspects of a growing user-base. Making the service "free" would clearly maximize adoption, but unless other revenue sources are available, *e.g.,* ads, is

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[1] http://www.fon.com. See also AnyFi (www.anyfinetworks.com) or previously KeyWifi, and also more recently Comcast [3] for similarly inspired services.

[2] Alternatively, they can also be offered some form of compensation.

unlikely to allow it to be viable. Increasing the service price could affect (lower) adoption, but may improve its viability. More generally, service pricing offers a "control knob" that can be used to realize a variety of objectives, *e.g.,* maximizing overall value or welfare, or maximizing provider's profit, etc. This control knob can be complex and involve offering the service at a different price to each user, *i.e.,* discriminatory pricing [4], or very basic, *e.g.,* fixed pricing, and there is typically a trade-off between how well objectives can be met and the complexity of the control (pricing) used to meet them.

This paper develops a simple model that helps understand how these factors interact and affect the adoption of a UPC service and the welfare (sum of users' utility and provider's profit) it creates, and how that welfare can be efficiently distributed between users and the service provider. To maintain analytical tractability, the model makes a series of simplifying assumptions, many of which may arguably not hold in practice. However, the analysis affords insight that, as we demonstrate, remains valid even under more general settings. Specifically, the paper's main contributions consist of

- Formulating and solving a simple model that captures key features of a UPC type of service;
- Characterizing when and how the service's total welfare, or value, is maximized;
- Identifying practical pricing policies that realize a different trade-off between optimizing welfare and distributing it between stakeholders.
- Numerically validating the robustness of the findings, when relaxing the simplifying assumptions on which the model relies.

The rest of the paper is structured as follows. Section II presents the model we rely on to capture the properties of a UPC service. Section III explores when and how the service value (total welfare) is maximized. Section IV introduces the role of pricing in realizing different goals for the service, with subsequent sections dedicated to specific pricing policies, *i.e.,* usage-based (Section V), hybrid (Section VI), and fixed-price (Section VII). Section VIII discusses generalizations and robustness of the findings. Related works are reviewed in Section IX, before summarizing the paper's findings in Section X.

II. Model Formulation

This section introduces a model that captures key aspects of adoption of a UPC-like service by users. We first present the general form of the model in Section II-A. We then introduce a series of simplifying assumptions in Section II-B to obtain a simpler model that is analytically tractable. Verifying that the findings afforded by this simplified model remain valid in more general situations calls for a two-prong approach: (1) An explicit solution is developed that offers a qualitative understanding of and insight into what drives the success (or failure) of UPC systems; (2) The robustness of those findings is then numerically tested under configurations that emulate more general settings, *i.e.,* where the model's simplifying assumptions are relaxed and errors are present in the estimation of its parameters.

A. General form

Given the expected organic growth of a UPC service, the interplay between the coverage it realizes and its ability to attract more users is of primary interest. The service coverage κ depends on the level x of adoption in the target user population, and determines the odds that users can obtain connectivity through the service while roaming. Users are heterogeneous in their propensity to roam, as captured through a variable $\theta, 0 \leq \theta \leq 1$. A user's exact θ value is private information, but its distribution (over the user population) is known. A low θ indicates a sedentary user while a high θ corresponds to a user that frequently roams. Hence, θ determines a user's sensitivity to service coverage.

As commonly done [5], a user's service adoption decision is based on the utility she derives from the service; she decides to adopt if that utility is positive. A user's utility is denoted as $U(\Theta, \theta)$, where θ is the roaming propensity of the user herself, and Θ identifies the current set of adopters. The general form of $U(\Theta, \theta)$ is given in Eq. (1).

$$U(\Theta, \theta) = F(\theta, \kappa) + G(\theta, m) - p(\Theta, \theta)\,, \tag{1}$$

where m is the volume of roaming traffic generatedby the current set of adopters Θ.

$F(\theta, \kappa)$ reflects the overall utility of connectivity, either at home or roaming, while $G(\theta, m)$ accounts for the negative impact of roaming traffic. Finally, $p(\Theta, \theta)$ is the price charged to the user θ when the adopters' set is Θ.

Note that the price $p(\Theta, \theta)$ is a control parameter that affects service adoption, *i.e.,* it can be endogenized to achieve specific objectives. In the paper, we explore the use of pricing to maximize total welfare and/or profit. Other parameters are exogenous and can be estimated, *e.g.,* using techniques from marketing research as discussed in [6], but not controlled.

Building on Eq. (1), users adopt the service only if their utility is positive, and are myopic when evaluating the utility they expect to derive from the service, *i.e.,* they do not anticipate the impact of their own decision on other users' adoption decisions. However, adoption levels affect coverage, and as coverage changes, so does an individual user's utility and, therefore, her adoption decision.

The level of adoption x is given by

$$x = |\Theta| \triangleq \int_{\theta \in \Theta} f(\theta)\, d\theta,$$

where $f(\theta)$ is a density function and reflects the distribution of roaming characteristics over the user population.

In the next section, we specialize the different terms in the utility function of Eq. (1)

B. Assumptions and the simplified model

For analytical tractability, we make several assumptions regarding the form and range of the parameters of Eq. (1) (Section VIII explores the impact of relaxing these assumptions).

First, a user's propensity to roam, as measured by θ, is taken to be uniformly distributed in $[0, 1]$, *i.e.,*

$$f(\theta) = 1, \qquad 0 \leq \theta \leq 1.$$

This implies that given a set of adopters Θ, the adoption level, x is

$$x = \int_{\theta \in \Theta} d\theta \,. \tag{2}$$

Conversely, assuming that every user contributes one unit of traffic, the volume of roaming traffic m generated by current adopters is given by

$$m = \int_{\theta \in \Theta} \theta \, d\theta \,. \tag{3}$$

Next, we assume that the distributions of users over the service area and their roaming patterns are uniform. A uniform distribution of users implies that the adoption level x also measures the availability of connectivity to roaming users, hence $\kappa = x$. Similarly, uniform roaming patterns mean that roaming users (and traffic) are evenly distributed across users' home bases, *i.e.,* all see the same connectivity while roaming. Therefore, we can write the function $F(\theta, \kappa)$ as

$$F(\theta, \kappa) = (1 - \theta)\,\gamma + \theta\, r x \,. \tag{4}$$

The parameter $\gamma \geq 0$ measures the utility of basic home connectivity, while $r \geq 0$ reflects the utility of roaming connectivity.[3] The latter needs to be weighed by the "odds" that such connectivity is available, which are proportional to the current service coverage $\kappa = x$. Hence, rx is the (true) utility of roaming connectivity, when the level of coverage is $\kappa = x$.

The additional factors $1-\theta$ and θ in Eq. (4) capture the impact of a user's roaming characteristic in how it uses, and therefore values, home and roaming connectivity. Specifically, a user with roaming characteristic θ splits its connectivity time in the proportions θ and $1-\theta$ between roaming and home connectivity, respectively.

Further, the impact of roaming traffic is assumed proportional to its volume m, which based on the assumption of uniform roaming patterns, is equally distributed across adopters' home bases. Specifically, the (negative) utility associated with roaming traffic consuming resources in the home base of users is proportional to $-cm$, $c \geq 0$. Roaming traffic affects equally the users whose home base it uses, and the roaming users seeking connectivity through it. Hence, all users experience the same impact of the form $-\theta c m - (1-\theta) c m = -cm$, so that $G(\theta, m)$ is[4]

$$G(m) = -c\, m \,.$$

Under these assumptions, a user's utility is of the form

$$U(\Theta, \theta) = \gamma - c\, m + \theta\,(r\, x - \gamma) - p(\Theta, \theta) \,. \tag{5}$$

In the next section, we characterize the total welfare that can be created by a UPC service as a function of the service parameters (exogenous and endogenous).

III. Total Welfare

In this section, we characterize the total *welfare* (value) a UPC service can create for its adopters and provider. Adopters' welfare is through the utility they derive from the service, while the provider's welfare is from what it charges adopters for the service. Using the model introduced in the previous section, we derive analytical conditions under which the total welfare is maximized. As argued earlier, the benefit of such analytical solutions is in providing insight into when and why the service may be valuable (worth deploying). The validity of that insight is tested under more general conditions in Section VIII.

To compute the maximum welfare, we first obtain the optimal set of adopters $\Theta^*(x)$ for any given adoption level x, and then solve for the optimal x.

A. Optimal Adoption Set for Given Adoption Level

For a given adoption level x, we seek the set of adopters Θ, $|\Theta| = x$, that maximizes welfare.

Provider's welfare (or profit) W_P can be written as

$$W_P(\Theta) = \int_{\theta \in \Theta} (p(\Theta, \theta) - e)\; d\theta \,, \tag{6}$$

where $p(\Theta, \theta)$ is the price charged to a user with roaming characteristic θ given a set Θ of existing adopters, and e is the per customer cost of providing the service, *e.g.,* as incurred from billing, customer service, or equipment cost subsidies[5]. Conversely users' welfare is given by

$$W_U(\Theta) = \int_{\theta \in \Theta} U(\Theta, \theta)\, d\theta \,. \tag{7}$$

The service welfare, $V(\Theta)$, is the sum of these two quantities.

$$\begin{aligned} V(\Theta) &= W_U(\Theta) + W_P(\Theta) \\ &= \int_{\theta \in \Theta} \big(U(\Theta, \theta) + p(\Theta, \theta) - e\big)\, d\theta. \end{aligned} \tag{8}$$

For notational purposes, we denote the integrand in Eq. (8) by $v(\Theta, \theta)$,

$$v(\Theta, \theta) \triangleq U(\Theta, \theta) + p(\Theta, \theta) - e \,,$$

[3] The range of the values of roaming connectivity is taken to be $r \geq \gamma$, *i.e.,* the value of roaming connectivity is at least as high as that of home connectivity.

[4] The range of the coefficient of roaming traffic, c, is taken to be $0 \leq c < r$, *i.e.,* it is lower than the max roaming utility.

[5] Note that this cost is ultimately born by the users, as it affects the price the provider charges for the service.

which can be interpreted as the *individual value* adopter θ contributes to the service. Using Eq. (5) we can rewrite Eq. (8) as

$$V(\Theta) = \int_{\theta\in\Theta} \big(\gamma + \theta\,(rx - \gamma) - cm - e\big)\, d\theta. \quad (9)$$

Characterizing optimal welfare for a given adoption level x, calls for identifying the set $\Theta^*(x)$ of adopters of cardinality $x, |\Theta^*| = x$, which maximizes Eq. (9). This is the subject of the next lemma, which is proved in Appendix I in a more general form.

Lemma 1. *For any adoption level x, maximum welfare is always obtained with a set of adopters $\Theta^*(x)$ that exhibit contiguous roaming characteristics. Specifically, $\Theta^*(x)$ is of the form*

$$\Theta^*(x) = \begin{cases} \Theta_1^*(x) = [0,\, x) & \text{if } x < \frac{\gamma}{r-c}, \\ \Theta_2^*(x) = [1-x,\, 1] & \text{if } x \geq \frac{\gamma}{r-c}. \end{cases} \quad (10)$$

B. Optimal Adoption Level

From Lemma 1, we obtain the optimal welfare $V^*(x) \triangleq V(\Theta^*(x))$ given any adoption level x. Following the partition of Eq. (10) into two cases $x \in [0, \frac{\gamma}{r-c})$ and $x \in [\frac{\gamma}{r-c}, 1]$, we consider separately the cases of $V(\Theta_1^*(x))$ and $V(\Theta_2^*(x))$.

Using Eq. (10) in Eq. (3) gives for $x \in [0, \frac{\gamma}{r-c})$,

$$m(\Theta_1^*(x)) = \int_{\theta=0}^{x} \theta d\theta = \frac{x^2}{2},$$

and therefore by Eq. (9)

$$V(\Theta_1^*(x)) = \frac{r-c}{2}x^3 - \frac{\gamma}{2}x^2 + (\gamma - e)x.$$

Similarly, for $x \in [\frac{\gamma}{r-c}, 1]$, the roaming traffic corresponding to $\Theta_2^*(x)$ is

$$m(\Theta_2^*(x)) = \int_{\theta=1-x}^{1} \theta d\theta = \frac{1}{2}\left(2x - x^2\right),$$

and therefore by Eq. (9)

$$V(\Theta_2^*(x)) = -\frac{r-c}{2}x^3 + (\frac{\gamma}{2} + r - c)x^2 - ex.$$

Combining the above expressions, the optimal service value $V^*(x) \triangleq V(\Theta^*(x))$ for a given adoption level x is given by

$$V^*(x) = \begin{cases} \frac{r-c}{2}x^3 - \frac{\gamma}{2}x^2 + (\gamma - e)x & \text{if } x < \frac{\gamma}{r-c} \\ -\frac{r-c}{2}x^3 + (\frac{\gamma}{2} + r - c)x^2 - ex & \text{if } x \geq \frac{\gamma}{r-c}, \end{cases}$$

where $\Theta^*(x)$ and x are related by Eq. (10).

Given $V^*(x)$, we can then solve for the value x^* that maximizes $V^*(x)$. The computations are mechanical in nature and are given in Appendix B, with Fig. 1 illustrating x^* as a function of γ and e (for $r - c = 1$).

The solution can be partitioned into two different regimes based on the value of γ. When $\gamma \leq r - c$ (corresponding to $\gamma \leq 1$ in Fig. 1), optimal adoption is either $x^* = 1$ or $x^* = 0$, depending on the service cost e. If the service cost is low ($e < \frac{\gamma+r-c}{2}$), then maximum welfare occurs for $x^* = 1$, and it is

$$V^*(x = 1) = \frac{\gamma + r - c}{2} - e. \quad (11)$$

Conversely, if the service cost is high ($e \geq \frac{\gamma+r-c}{2}$), then it overshadows any benefit or utility the service produces and it is impossible to create positive welfare. In this case, the "optimal" adoption is $x^* = 0$.

In contrast, when $\gamma > r-c$ (corresponding to $\gamma > 1$ in Fig. 1), intermediate values $0 < x^* < 1$ are possible (the gradient-shaded region of Fig. 1). This is because as γ increases, sedentary users start to derive more utility and progressively become the dominant value contributors. Therefore a set of (mostly) sedentary adopters can make a large positive welfare contribution. Furthermore, because this value is negatively affected by roaming traffic, the optimal adoption level discourages frequently roaming users. Note that $r - c$ gives a tentative measure of the "net" importance of roaming (roaming utility factor less roaming traffic factor), and as such the condition $\gamma > r-c$ describes a system where home connectivity has a higher value than the overall ("net") effect of roaming connectivity. Such a system may arguably not be a prime candidate for UPC services.

In summary, the main finding that emerges from the results of this section is that when a UPC service can generate significant positive value, that value is typically maximized at full adoption (or close to full adoption[6]) Section VIII numerically tests the validity of this finding when the model's assumptions are relaxed.

While this section explored the relationship between service adoption and total welfare, and identified adoption sets that maximize total welfare, the next section focuses on how to *realize* such outcomes. As we shall see, this greatly depends on the flexibility of the pricing policy used.

IV. Role of Pricing

The analysis of Section III characterizes maximum service welfare, but does not offer a constructive method to realize it. As shown in Eq. (5), adoption and, therefore, welfare, depend on $p(\Theta, \theta)$. Hence, maximizing welfare calls for identifying a suitable *pricing policy.*

Moreover, the price $p(\Theta, \theta)$ is also the parameter that determines how welfare is divided between users and the provider. For example, if $p(\Theta, \theta) = e$, then the provider is only compensated for its expenses e (its profit is $W_P(\Theta) = 0$) and the entire welfare is realized as user's utility, $W_U(\Theta) = V(\Theta)$. Conversely, if $p(\Theta, \theta) = v(\Theta, \theta) + e$, then $U(\Theta, \theta) = 0$, *i.e.,* users derive

[6]Specifically in more general cases where coverage "saturates" with adoption, the maximum total welfare may predictably be realized at slightly below full adoption. The reason is reaching full adoption in that case would add more roaming traffic without meaningfully improving coverage. Details are given in Appendix B.

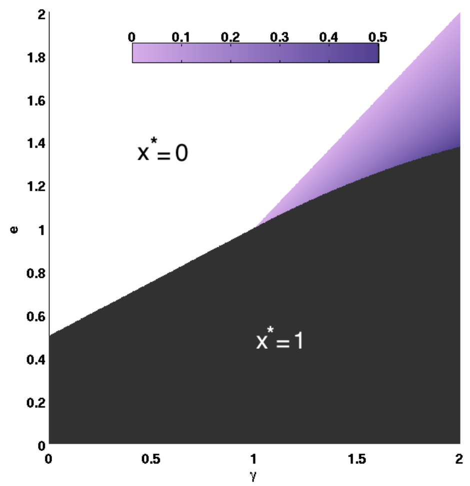


Fig. 1. Regions of optimal adoption for maximum system value. Parameters are $r = 1.6$ and $c = 0.6$ (and therefore $r - c = 1$). The gradient-shaded area corresponds to $0 < x^* < 1$, whereas the solid black and white areas correspond to $x^* = 1$ and $x^* = 0$, respectively.

zero utility (strictly speaking, prices would be set to ensure an infinitesimal but positive utility) and all of the welfare is realized as provider's profit, $W_P(\Theta) = V(\Theta)$.

Other pricing schemes are possible that distribute welfare between users and the provider. For example, a price of the form

$$\begin{aligned} p(\Theta, \theta) &= \nu(\Theta, \theta) + e - \delta \\ &= (1-\theta)\gamma + \theta r x - cm - \delta\,, \end{aligned} \tag{12}$$

which is an instance of a *discriminatory* pricing policy, leaves every user with a positive utility $U(\Theta, \theta) = \delta > 0$, hence realizing the optimal adoption level[7] $x = 1$. Therefore, the optimal welfare $V^*(1)$ of Eq. (11) is realized and by using $U([0, 1], \theta)$ in Eq. (7) it follows that the users' overall welfare is

$$W_U([0, 1]) = \delta\,.$$

This means that without affecting adoption, we can pick any $\delta > 0$ to freely vary $W_U([0, 1])$ in the range $(0, V^*(1)]$, and accordingly by Eq. (8),

$$W_P([0, 1]) = V^*(1) - W_U([0, 1]). \tag{13}$$

In short, this policy realizes two important goals

- Optimal welfare, and
- Flexible welfare distribution.

Such a discriminatory pricing policy is, however, difficult to implement in practice as it requires knowledge of individual user characteristics (θ) that may not be readily available[8], and also results in a price that varies with the adoption level x. This heterogeneity across both users and adoption levels is illustrated in Figs. 2 and 3, that plot $\nu(\Theta, \theta)$ as a function of θ and x.

In the following sections, we introduce pricing policies that offer a different trade-off between realizing maximum welfare, distributing it arbitrarily, and practicality.

V. Usage-based Pricing Policy

As mentioned above, a discriminatory pricing policy can both maximize total welfare and distribute it arbitrarily between users and the provider. It is, however, difficult to implement in practice. This section proposes a *usage-based* pricing scheme that mimics the behavior of the discriminatory policy, but makes it feasible in practice. Under a usage-based pricing scheme, users are charged based on how often they connect at home and while roaming. We present next the structure of usage-based pricing, how it is able to capture key aspects of discriminatory pricing, and also the insight that the analysis of the pricing policy affords.

A. Pricing Structure

In a UPC service, usage has two components, *home usage* denoted by z_h, and *roaming usage* denoted by z_r. A usage-based pricing policy may assign different prices to these two usage types. Assuming that p_h and p_r are unit prices for home and roaming usage, respectively, a user is charged

$$p_z(z_h, z_r) = z_h \cdot p_h + z_r \cdot p_r - a, \tag{14}$$

where a corresponds to fixed usage allowance that may be given to each user, *e.g.,* akin to the free minutes commonly included in cellular phone plans.

Eq. (14) states what a user pays for the service as a function of her usage. Next, we express this cost in terms of the user and service model of Section II. This calls for characterizing how roaming characteristics θ and the service coverage x affect a user's home and roaming usages.

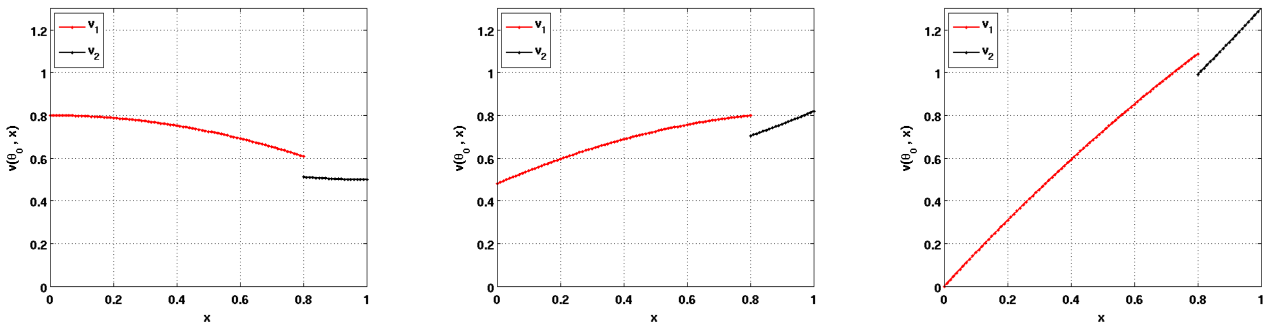
By definition, θ denotes a user's propensity to roam, *i.e.,* how often she is roaming versus at home. However, because a roaming user successfully connects only where there is coverage, her "typical" roaming usage is only $\bar{z}_r(x, \theta) = \theta x$. Conversely, her typical home usage is simply $\bar{z}_h(\theta) = 1 - \theta$ (home connectivity is always available). Replacing z_h and z_r in Eq. (14) by the typical roaming and home usages $\bar{z}_r(x, \theta)$ and $\bar{z}_h(\theta)$ of a user with roaming characteristics θ, we obtain the following expression for what she will typically be charged for using a UPC service with a coverage level of x

$$\bar{p}_z(x, \theta) = p_h(1-\theta) + p_r\,\theta x - a. \tag{15}$$

Eq. (15) has three parameters p_h, p_r and a that affect service adoption, *i.e.,* which users derive positive utility. Given our goal of emulating the discriminatory pricing policy of Eq. (12) and by comparing it to Eq. (15), we

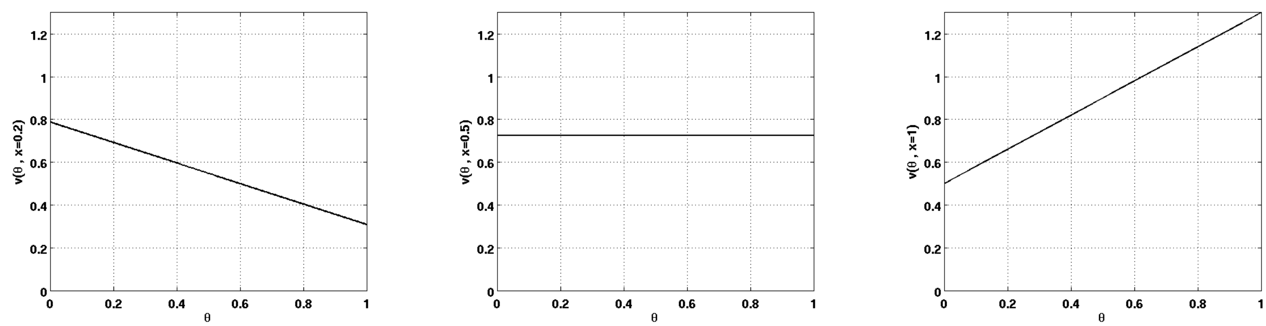
[7]When optimal adoption is not at $x = 1$, optimal welfare can still be realized by setting a high price for users who should not adopt.

[8]Even if the provider has full knowledge of individual user characteristics θ, it may not be acceptable to charge users differently.



(a) $\theta_0 = 0$ (b) $\theta_0 = 0.4$ (c) $\theta_0 = 1$

Fig. 2. System value contributed by user θ_0 as a function of x. Parameters are $\gamma = 0.8$, $e = 0$, $c = 0.6$, $b = 0$, $r = 1.6$.



(a) $x = 0.2$ (b) $x = 0.5$ (c) $x = 1$

Fig. 3. System value contribution across users, at different adoption levels. Parameters are $\gamma = 0.8$, $e = 0$, $c = 0.6$, $b = 0$, $r = 1.6$.

choose $p_h = \gamma$ and $p_r = r$, which yields the following usage-based pricing scheme

$$\overline{p}_z(x, \theta) = \gamma(1 - \theta) + r\theta x - a. \quad (16)$$

We note that the only difference between Eq. (16) and the discriminatory pricing of Eq. (12) is in the terms a versus $cm - \delta$, where the former is constant while the latter depends on the level of roaming traffic m. As we shall see next, this difference is minor, and the usage-based pricing policy of Eq. (16) is capable of realizing both maximum welfare and flexibility in how welfare is distributed across users and the provider.

B. Maximal Service Adoption

Using Eq. (16) in Eq. (5) gives the following expression for the utility derived by user θ from adopting the service

$$U(\Theta, \theta) = a - cm. \quad (17)$$

We next use Eq. (17) to identify the adoption *equilibria* under usage-based pricing. We say a set of adopters Θ comprises an equilibrium when

$$U(\Theta, \theta) > 0, \quad \text{if} \quad \theta \in \Theta, \qquad \text{and}$$
$$U(\Theta, \theta) \le 0, \quad \text{if} \quad \theta \notin \Theta.$$

Then,

Proposition 1. *Under the usage-based pricing policy of Eq. (16), full adoption, $x = 1$, is the **unique** equilibrium if $a > c/2$, and is not an equilibrium if $a \le c/2$.*

Proof: Recall that $c \ge 0$, and note that at any adoption level x (corresponding to an adopters' set Θ such that $|\Theta| = x$), the roaming traffic m satisfies $m \le 1/2$. Hence, $cm \le c/2$ and Eq. (17) yields that $U(\Theta, \theta) \ge a - c/2$. Consequently $U(\Theta, \theta) > 0$ if $a - c/2 > 0$. This is true for all values of θ and Θ, *i.e.*, all users have positive utility at all adoption levels. Therefore no other equilibrium can exist, since that would mean for some $\widehat{\Theta} \neq [0, 1]$, and for $\theta \notin \widehat{\Theta}$ the utility is negative, which is contradictory. This proves sufficiency.

On the other hand, if $a \le c/2$, then by Eq. (17) we have $U(\Theta, \theta) \le c/2 - cm$. But at full adoption $m = 1/2$ and therefore $U([0, 1], \theta) \le 0$, which means $[0, 1]$ cannot be an equilibrium. This completes the proof. ■

Proposition 1 implies that the usage-based pricing policy maximizes total welfare by realizing full adoption[9], provided the provider sets the usage allowance a higher

[9]Assuming that the parameters are such that total welfare is maximized at $x = 1$.

than the threshold $c/2$. The threshold's value $c/2$ is clearly specific to the assumptions on which the model is predicated. However, as we will see in Section VIII, such a threshold condition is present under more general conditions. In particular, as long as the usage allowance a is larger than a threshold a_0, full adoption is the unique equilibrium, while if $a \leq a_0$, full adoption is then not an equilibrium.

We explore next the policy's ability to distribute welfare between users and the provider.

C. Welfare Distribution

From Eq. (17), the utility of user θ at full adoption is

$$U([0,\, 1],\, \theta) = a - \frac{c}{2}.$$

Combining this expression with Eq. (7) gives the overall user welfare

$$W_U([0,\, 1]) = a - \frac{c}{2}\,,$$

with provider's profit given accordingly by Eq. (13).

This means that we can pick any $a > c/2$ without affecting adoption, and therefore freely vary *both* $W_U([0,\, 1])$ and $W_P([0,\, 1])$ in the full range $[0,\, V^*(1))$.

Although, as mentioned earlier, the usage-based policy does not perfectly emulate the discriminatory policy of Eq. (12), it coincides with it at full adoption through the change of variables $\delta \triangleq a - c/2$. Hence, a usage-based pricing policy offers a practical solution to realize optimality and flexibility (in distributing welfare).

Those benefits notwithstanding, implementing usage-based pricing calls for monitoring (logging) usage, which incurs a cost. In addition, some users may prefer the predictability of fixed pricing (independent of usage), even in cases where it may be less advantageous for them [7], *i.e.,* result in a lower utility. This is particularly so in the case of home-connectivity, for which fixed pricing is often the norm. For instance, Time Warner recently announced [8] that its customers would always retain the option of a flat-rate monthly pricing for broadband Internet access, with usage-based plans being optional.

For those reasons, we consider next a *hybrid* pricing policy that combines fixed and usage-based pricing, and evaluate the trade-offs it imposes.

VI. Hybrid Usage-based Pricing Policy

Consider a pricing policy that combines a fixed price for home connectivity, and a usage-based price for connectivity while roaming.

A. Pricing Structure

Using notation similar to Section V-A, let z_r denote the roaming usage of a user. The total hybrid usage-based price that a user is charged is then

$$p_y(z_r) = p_h + z_r \cdot p_r, \tag{18}$$

where the price of home usage is fixed (independent of usage) at p_h and identical for all users[10], and as before p_r is the unit usage price while roaming.

The only user-dependent term in Eq. (18) is, therefore, her roaming usage. Recalling the discussion of Section V-A, the typical roaming usage $\bar{z}_r(x,\, \theta)$ of a user with roaming profile θ when the service coverage is x is equal to θx. Hence, the typical cost to a user with profile θ for the service is given by

$$\overline{p}_y(x,\, \theta) = p_h + p_r\, \theta x, \tag{19}$$

Next, we investigate if and how p_h and p_r can be set to again emulate the discriminatory policy of Eq. (12), or more importantly achieve the same outcomes, namely, maximum welfare and flexibility in allowing distribution of welfare across users and the provider. As per the discussion of Section IV, the former calls for selecting p_h and p_r so as to ensure full adoption, *i.e.,* $x = 1$.

B. Maximal Service Adoption

Given the price structure of Eq. (19), the utility of a user can be obtained from Eq. (5) as

$$U(\Theta,\, \theta) = \gamma - cm - p_h + \theta(rx - \gamma - xp_r).$$

By applying the change of variables

$$\delta_h = \gamma - \frac{c}{2} - p_h \quad \text{and} \quad \delta_r = r - \gamma - p_r\,,$$

$U(\Theta,\, \theta)$ can be rewritten as

$$U(\Theta,\, \theta) = \frac{c}{2} - cm + \delta_h + \theta(x(\delta_r + \gamma) - \gamma). \tag{20}$$

Note that δ_h corresponds to the net residual utility for home connectivity at full adoption, and conversely δ_r is the corresponding quantity for roaming connectivity.

The next Lemma provides conditions under which full adoption is an equilibrium.

Lemma 2. *Under the hybrid pricing of Eq. (19), full adoption, $x = 1$, is an equilibrium if and only if $\delta_h > 0$ and $\delta_r > -\delta_h$.*

Proof: At full adoption we have $\Theta = [0, 1]$, $x = 1$ and $m = 1/2$. Therefore the utility of Eq. (20) becomes

$$U([0,\, 1],\, \theta) = \delta_h + \theta\delta_r.$$

For $\Theta = [0, 1]$ to be an equilibrium, all users must have positive utility. This implies

$$\delta_h + \theta\delta_r > 0\,, \quad \forall \theta \in [0, 1].$$

Since this is a linear function of θ, the inequality holds if and only if it is satisfied for both $\theta = 0$ and $\theta = 1$, *i.e.,* $\delta_h > 0$ and $\delta_h + \delta_r > 0$. ■

The conditions of Lemma 2 state that full adoption, $x = 1$, is possible only if the fixed price p_h for home connectivity is not too high, *i.e.,* $\delta_h > 0 \Rightarrow p_h < \gamma - \frac{c}{2}$, and the roaming usage-based price p_r is no higher than

[10]Note that the usage allowance a is now included in p_h.

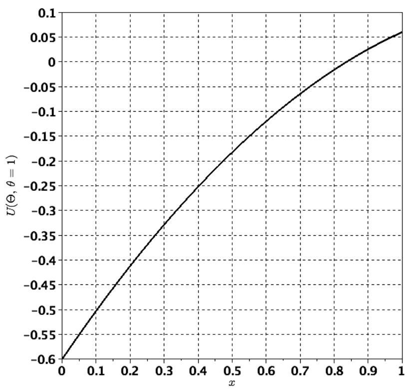


Fig. 4. Utility of a user with $\theta = 1$ as a function of coverage under hybrid pricing for $\gamma = 1$, $c = 0.7$, $\delta_h = 0.05$ and $\delta_r = 0.01$.

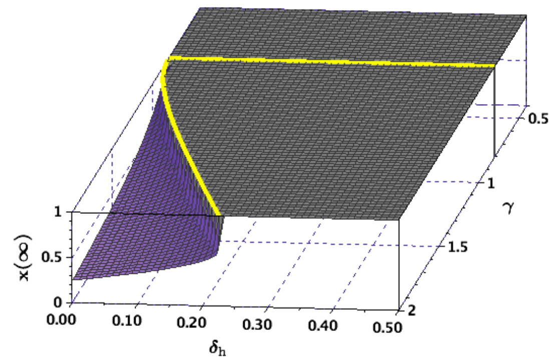


Fig. 5. Final adoption level for the hybrid pricing policy, and identification of the boundaries demarcating the regions associated with the conditions of Proposition 2. The straight line corresponds to $\gamma = c = 0.8$, and the curved line captures the condition of Eq. (22). The system's parameters are $c = 0.8$, $\delta_r = 0$, with γ and δ_h values varying.

the net roaming value at full adoption, $r - \frac{c}{2}$, minus the price p_h already charged for home connectivity, *i.e.,* $\delta_r > -\delta_h \Rightarrow p_r < r - \frac{c}{2} - p_h$.

Unlike the conditions of Proposition 1 that ensured positive utility for all users at *all* levels of coverage, Lemma 2 does not include such guarantees. In particular, and as illustrated in Fig. 4 for the $\theta = 1$ user, the utility of a user can vary from negative to positive as coverage increases, with a cross-over value of $x \approx 0.85$ in the case of Fig. 4. The $\theta = 1$ user, therefore, adopts only once coverage exceeds 0.85. Hence, her adoption depends on the adoption of enough other users ($x > 0.85$). In general, and as hinted at in Fig. 3, users with low θ values have higher utility at low coverage, and are therefore the ones joining the service when it is first offered. As they do, the service becomes more valuable for users with higher θ values, whose utility may then become positive allowing them to adopt. This progression can, however, stall before full adoption is reached, *i.e.,* adoption may stop at a level $x < 1$. This can arise even under the conditions of Lemma 2, as Lemma 2 does not guarantee the uniqueness of the $x = 1$ equilibrium.

As shown in Appendix C, when the conditions of Lemma 2 hold, $x = 1$ is the *unique* equilibrium if and only if γ satisfies

$$\gamma < c + 2\delta_h + 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)}. \tag{21}$$

This then ensures that adoption increases monotonically until reaching full adoption. The condition of Eq. (21) can be combined with Lemma 2 to obtain the equivalent of Proposition 1 for the hybrid pricing policy.

Proposition 2. *Under the hybrid pricing of Eq. (19), full adoption, x=1, is the* ***unique*** *equilibrium if and only if*

- *When* $\gamma < c$: $\quad \delta_h > 0$ *and* $\delta_r > -\delta_h$
- *When* $\gamma \geq c$: $\quad \delta_h > 0$ *and* $\delta_r > -\delta_h$ *and*

$$\delta_h > \frac{\gamma^2}{4(\gamma + \delta_r - c/2)} - c/2. \tag{22}$$

Proof: As a result of the two conditions $\delta_h > 0$ and $\delta_r > -\delta_h$ and because $c \geq 0$ it follows that $2\delta_h + 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)}$ in Eq. (21) is always positive. Therefore Eq. (21) always holds if $\gamma < c$, without further constraints on the values of δ_h and δ_r.

On the other hand, when $\gamma \geq c$, δ_h and/or δ_r need to be large enough to ensure that Eq. (21) is satisfied. Specifically, algebraic manipulation of Eq. (21) in this case yields Eq. (22). ■

Proposition 2 states that when $x = 1$ is an equilibrium under hybrid pricing, it can coexist with other equilibria when the value of home connectivity utility is high enough, *i.e.,* $\gamma \geq c$ and the condition of Eq. (22) is not satisfied. Focusing on cases when $x = 1$ maximizes total welfare, *e.g.,* e is low enough, this means that it is possible for the provider to set prices p_h and p_r (and consequently δ_h and δ_r) for which full adoption is feasible, *i.e.,* the conditions of Lemma 2 are satisfied, without ever being able to reach this target. This occurs when the provider's choice of prices allows the emergence of a *second* equilibrium $\tilde{x} < 1$, where adoption stops upon reaching it.

As Proposition 2 indicates though, it is possible to avoid such outcomes by properly selecting prices (parameters δ_h and δ_r) to comply with Eq. (22). This is illustrated in Fig. 5, which plots the system's final adoption as γ and δ_h vary for the case $c = 0.8$ (initial adoption is set to $x = 0$, and for simplicity we assume $\delta_r = 0$ and focus on the impact of varying δ_h). The figure confirms (straight boundary line at $\gamma = c = 0.8$ in the figure) that when $\gamma < c = 0.8$, any value of $\delta_h > 0$ results in full adoption. It also shows that when $\gamma \geq c = 0.8$, the system only converges to full adoption when δ_h further satisfies the condition of Eq. (22) (corresponding to δ_h values that lie to the right of the curved boundary line in the figure).

The conditions of Proposition 2 are clearly specific to the assumptions on which the model is predicated. However, we will see in Section VIII that the very same behavior arises under more general settings; specifically, a second, sub-optimal equilibrium ($\tilde{x} < 1$) can arise whenever the value of home connectivity exceeds a certain threshold, and in the process prevent the system from reaching its intended target of full adoption. In addition, overcoming this issue can again be accomplished by adjusting prices, albeit to different values than those of Proposition 2.

We note that the aspect of adjusting (lowering) prices to ensure full adoption begs the question of what would motivate the provider to do so. We explore this issue next in the broader context of the hybrid pricing policy's ability to distribute welfare between users and the provider. We first explore the pricing policy's ability to support arbitrary welfare distribution at full adoption, including maximizing the provider's profit, and then focus on the extent to which the conditions of Proposition 2 constrain this ability, and what options are available to overcome those limitations.

C. Welfare Distribution

As before, we focus on scenarios for which total welfare is maximized at full adoption, *i.e.,* combinations that, as illustrated in Fig. 1, correspond to a low enough cost e relative to the other system's parameters γ, c, and r. We explore first whether, once at full adoption (and maximum total welfare), the hybrid pricing policy allows an arbitrary distribution of welfare (as the usage-based policy did), from maximum user welfare to maximum provider profit.

Lemma 2 identifies the constraints that pricing must satisfy to ensure that full adoption is an equilibrium, *i.e.,* $\delta_h > 0$ and $\delta_r > -\delta_h$. Combining Eq. (20) and Eq. (7) gives the following expression for the users' welfare $W_U([0,1])$ at full adoption

$$W_U([0,1]) = \delta_h + \frac{\delta_r}{2}, \qquad (23)$$

with according to Eq. (13) and Eq. (11), the provider's profit given by

$$W_P([0,1]) = \frac{\gamma + r - c}{2} - e - \left(\delta_h + \frac{\delta_r}{2}\right). \qquad (24)$$

Realizing maximum user welfare calls for choosing prices such that $W_P([0,1]) = 0$, which according to Eq. (24) implies

$$\delta_h + \frac{\delta_r}{2} = \frac{\gamma + r - c}{2} - e.$$

This can be readily accomplished by choosing values of δ_h and δ_r that also satisfy Lemma 2, *e.g.,* $\delta_h = \epsilon > 0$, and $\delta_r = \gamma + r - c - 2e - 2\epsilon > -\epsilon$, where ϵ is arbitrarily small. Conversely, maximizing the provider's profit calls for setting prices that extract (nearly) all the value users realize from the system, *i.e.,* set both δ_h and δ_r equal to arbitrarily small positive values (this again satisfies the conditions of Lemma 2, namely, $\delta_h > 0$ and $\delta_r > -\delta_h$).

Intermediate distributions of welfare are also feasible simply by adjusting the values of δ_h and δ_r. Consider for example a scenario where a regulator wants all users to see the same utility value $\alpha > 0$. From Eq. (20) the utility of a user with roaming parameter θ is given by

$$U([0,1], \theta) = \delta_h + \theta\delta_r = \alpha .$$

Eliminating the dependency on θ to ensure that all users see the same utility requires $\delta_r = 0$, which then implies $\delta_h = \alpha > 0$ that again satisfies the conditions of Lemma 2. Hence, we see that *once at full adoption* (and assuming full adoption maximizes welfare), the hybrid pricing policy, like the usage-based policy, is capable of achieving any arbitrary distribution of welfare between users and the provider. However as made explicit in Proposition 2, *reaching* full adoption can, as reflected in Eq. (22), impose additional conditions on pricing, which may preclude some welfare distribution configurations. In particular, maximizing the provider's profit, which as just discussed calls for setting both δ_h and δ_r to arbitrarily small positive values, readily conflicts with the conditions of Eq. (22).

A possible approach suggested by the discussion of Section VI-B, is for the provider to offer an *introductory* pricing that satisfies the conditions of Proposition 2; thereby enabling full adoption to be reached. The motivation for the provider to do so is that once full (or nearly full[11]) adoption has been reached, it can then switch to a pricing scheme that allows it to extract a higher profit.

In the next section, we introduce a third family of pricing policies that seeks to eliminate all dependency on monitoring a user's usage; therefore simplifying implementation and possibly facilitating user acceptance.

VII. Fixed Price Policy

This section considers a pricing policy based on a fixed price that covers both home and roaming connectivity.

As mentioned earlier, the use of a fixed price is not uncommon for home connectivity, but it is arguably less so for wireless roaming access which is the other component of the service we consider. Nevertheless, a number of wireless carriers do offer fixed-price wireless services [9]. Hence it is of interest to investigate the impact such a pricing policy might have on their ability to maximize profit and on the welfare the system realizes.

[11]See Appendix C for details on how early the service provider can end the introductory pricing phase.

A. Pricing Structure

Pricing is independent of usage and based on a single parameter p,

$$p(\Theta, \theta) = p, \forall \Theta, \theta. \quad (25)$$

We investigate if and how p can be set to realize maximum welfare and flexibility in distributing it across stakeholders. As per the discussion of Section IV, the former (typically) calls for selecting p so as to ensure full adoption, *i.e.,* $x = 1$.

B. Maximum Service Adoption

Given Eq. (5) and the price structure of Eq. (25), the utility of user θ is

$$U(\Theta, \theta) = \gamma - p - c\,m + \theta\,(r\,x - \gamma)\,. \quad (26)$$

The following Lemma then gives the condition under which full adoption is an equilibrium. The proof is in Appendix D.

Lemma 3. *Under the fixed price policy of Eq. (25), full adoption is an equilibrium if and only if* $p < \gamma - c/2$.

Note that as was the case with Lemma 2, the condition of Lemma 3 does not imply uniqueness of the $x = 1$ equilibrium. In fact, as shown in Appendix D, under fixed pricing there may be as many as four equilibria, spanning combinations of stable, unstable, periodic, or chaotic equilibria. Table I summarizes possible combinations, with (•) denoting stable equilibria, (◦) unstable equilibria, (↻) equilibria associated with an "orbit" that can be either convergent, periodic, or chaotic, and (—) the absence of equilibria.

Cases	$[0, \gamma/r)$	$[\gamma/r, 1]$
1	—	—
2	•	—
2′	↻	—
3	—	•
3′	—	↻
4	•, ◦	—
5	—	•, ◦
6	•, ◦	•
7	•	•, ◦
8	•, ◦	•, ◦

TABLE I
EQUILIBRIA COMBINATIONS UNDER FIXED PRICING

Ensuring that $x = 1$ is the unique (stable) equilibrium, and therefore that the service always reaches full adoption, calls for additional constraints on p beyond those of Lemma 3. These constraints are formalized in the next Proposition, which mirrors the conditions of Proposition 1 for usage-based pricing. The proof is again in Appendix D.

Proposition 3. *Under the fixed price policy of Eq. (25), full adoption,* $x = 1$, *is the* ***unique*** *equilibrium if and only if*

$$p < \min\left(\gamma - c/2, \gamma - \frac{\gamma^2}{4r - 2c}\right)\,.$$

The conditions of Proposition 3 ensure that total welfare is maximized under a fixed price policy. Next, we see if and how these conditions limit the policy's ability to distribute welfare between users and the provider.

C. Welfare Distribution

From Eq. (26), the utility of user θ at full adoption is

$$U([0,\,1],\,\theta) = (1-\theta)\gamma + \theta r - p - c/2\,,$$

which when combined with Eq. (7), gives the following expression for user welfare

$$W_U([0,\,1]) = \frac{\gamma + r - c}{2} - p\,,$$

with Eq. (13) correspondingly giving the provider's profit as

$$W_P([0, 1]) = p - e\,.$$

As before, flexibility in distributing welfare calls for being able to vary $W_U([0,\,1])$ across the full range $(0,\,V^*(1)]$, where $V^*(1) = \frac{\gamma+r-c}{2} - e$. Clearly, this cannot be achieved without violating the conditions of Proposition 3, *e.g.,* $W_U([0, 1]) = 0$ calls for $p = \frac{\gamma+r-c}{2} \geq \gamma - c/2$ (recall that $r \geq \gamma$). Therefore the service is not capable of realizing full adoption and maximizing the provider's profit (see Appendix D for a full discussion).

Under hybrid pricing, we suggested the use of introductory prices to first realize full adoption, and then perform the desired welfare allocation. Unfortunately, this is not sufficient under fixed pricing, as certain welfare allocations are incompatible with not just Proposition 3, but also Lemma 3. In particular and as mentioned above, $W_U([0,\,1]) \approx 0$ calls for a price $p \geq \gamma - c/2$ that violates the conditions of both the Lemma and the Proposition. Hence, after an introductory price expires, it forces a drop in adoption below $x = 1$ and prevents welfare maximization.

In other words, the simplicity of the fixed price policy comes at a cost in terms of its ability to simultaneously maximize and distribute welfare. The concern is that this limitation may result in sub-optimal welfare realizations (and lower service coverage), as the provider may be tempted to set prices to maximize profit.

Fig. 6 helps assess the extent to which this may be a risk. It plots as a function of c and for a combination of parameters $\gamma = 1, r = 2$, and $e = 0.3$, the relative difference in profit between a profit maximizing choice of p and one that yields the best possible profit while also maximizing welfare, *i.e.,* maintaining $x = 1$. The figure indicates that as long as c remains relatively small (compared to γ and r), the incentive to deviate from a welfare maximizing price is small. As a matter of fact, when c is very small maximizing profit and welfare coincide even though welfare cannot be entirely realized as profit (this is an intrinsic limitation of the fixed-price policy). As the negative impact of roaming traffic, c, grows larger, it however becomes increasingly tempting

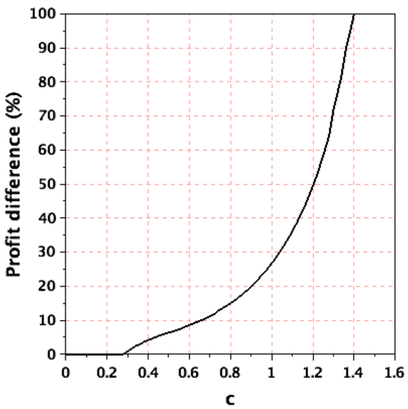


Fig. 6. Relative profit drop from profit maximization to welfare maximization (fixed-price policy $\gamma = 1, r = 2$ and $e = 0.3$).

(profitable) for the provider to deviate from a welfare maximizing strategy and set a price that keeps adoption low. Arguably though, such scenarios where users are highly sensitive to the (negative) impact of roaming traffic are inherently not conducive to the large-scale deployment of a UPC like service.

The analysis of this section and its illustration in Fig. 6, are clearly dependent on the specific assumptions of the model. However, as demonstrated in Section VIII, the findings hold even under more general conditions.

In summary, although the fixed price policy exhibits clear limitations in its ability to jointly maximize welfare and profit, its simplicity still makes it an attractive candidate, at least in scenarios where users are relatively insensitive to the negative aspects of a UPC service (small c values). In addition and as discussed in Appendix D, setting the price to maximize profit can be "risky," as the optimal price is such that small errors in parameter estimation can produce a dramatic collapse in adoption and consequently profit[12]. This should make the safer welfare maximization policy more appealing to the service provider.

VIII. Generalizations and Robustness

The user adoption model reflected in the utility function of Eq. (5) is obviously highly stylized and predicated on various simplifying assumptions, namely,

- A user's propensity to roam, θ, is uniformly distributed in $[0, 1]$,
- A user's utility is a specific linear function of coverage κ and volume of roaming traffic m,
- Adoption, x, accurately measures coverage κ,
- All users see the same coverage and contribute the same amount of traffic while roaming.

Similarly, the different pricing policies discussed in the paper rely on these assumptions, as well as on an implicit knowledge (by the service provider) of the range and values of the different system parameters. This clearly raises valid questions regarding whether the paper's findings hold outside this framework.

[12] In other words, the underlying optimization is inherently fragile.

This section, and more generally Appendix F seeks to address this issue. It numerically investigates the extent to which relaxations of modeling assumptions and the introduction of estimation errors in the system's parameters affect the results. As expected, modifying the paper's assumptions produces quantitative changes in the outcomes. However, as we show next, its main qualitative findings remain valid.

More specifically, the investigation demonstrates the robustness of the paper's findings (summarized in the next section) against a broad range of perturbations. Results are presented here only for representative scenarios, with the full set of results available in Appendix F.

The rest of this section is structured as follows. Section VIII-A restates the paper's main findings for completeness. The methodology behind the robustness tests is outlined in Section VIII-B, while an illustrative example is presented in Section VIII-C.

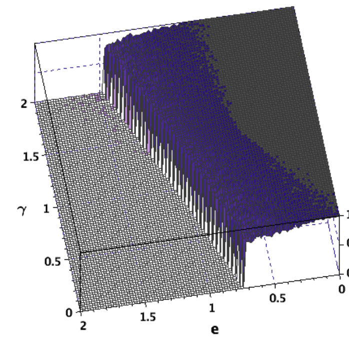
A. Main findings and insight

We briefly recall the main findings that emerged from the results of the paper's simple model.

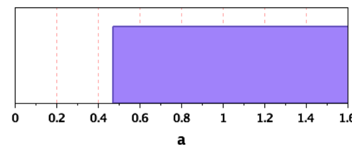
- *Maximum total welfare:* Whenever the system is capable of generating value, this value is maximized at full (or close to full) adoption;
- *Usage-based pricing:* Realizing the system's maximum value under a usage-based pricing policy calls for ensuring that users are offered a usage allowance that exceeds a minimum threshold a.
- *Hybrid usage-based pricing:* When the value of home connectivity is high, the hybrid pricing policy may not achieve maximum system value (because of the emergence of a sub-optimal equilibrium) unless prices are sufficiently discounted (high values for parameters δ_h and δ_r). Such discounts prevent the service provider from maximizing profit, unless it resorts to an introductory pricing scheme;
- *Fixed pricing:* Under a fixed price policy, profit and welfare maximization strategies typically differ unless the penalty associated with allowing roaming traffic (the parameter c) is small.

B. Robustness testing methodology

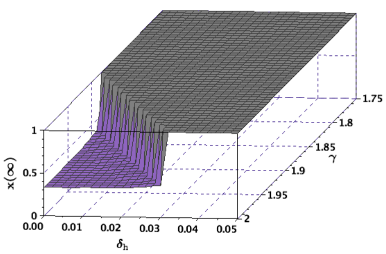
In testing for robustness, we consider perturbations to the assumptions, parameters and functional expressions of the paper's model. Because those perturbations affect the model's analytical tractability, their impact is evaluated by means of numerical simulations. The simulations also consider the effect of different types of errors in the estimation of system parameters on which the service provider relies when designing pricing strategies. We describe next the dimensions along which we perturb the



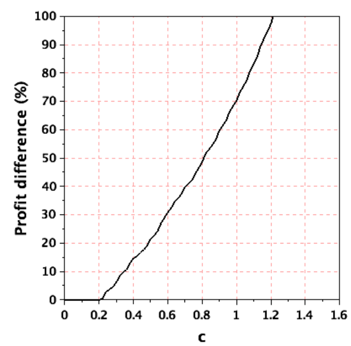
(a) Total welfare: optimal adoption for maximum system value. Compare to Fig. 1.



(b) Usage-based pricing policy: values of usage allowance a for which the system goes to full adoption.



(c) Hybrid pricing policy: final adoption level as a function of the parameters γ and δ_h. Compare to Fig. 5.



(d) Fixed price policy: relative profit drop from profit maximization to welfare maximization. Compare to Fig. 6.

Fig. 7. Impact of relaxing modeling assumptions on the paper's main findings. [1- Coverage κ is a *concave* function of adoption x that saturates as x increases; 2- Users have a non-linear utility function; 3- Users' roaming characteristics has a non-uniform distribution].

original model. Additional details can again be found in Appendix F.

- *Non-uniform roaming distributions:* We consider different probability distributions for a user's propensity to roam, θ. In particular, we consider distributions with both low and high roaming modes (fewer or more users that roam frequently).
- *Modified user utility functions:* The original model assumes a specific functional expression for users' utility that grows linearly with coverage (x) and decreases linearly with the volume of roaming traffic (m). We relax the linearity assumption, and also consider two different utility functions inspired by the models of [10].
- *Coverage saturation:* The original model assumes that coverage increases linearly with service adoption. We relax this assumption and consider a *saturation* effect for coverage, *i.e.,* coverage is now a concave function of adoption, which captures that adequate coverage may be realized with less than 100% adoption.
- *Users heterogeneity:* We consider a scenario where users belong to two "types" with different "profiles." The type of a user affects that user's utility as well as the volume of roaming traffic she generates.

C. Robustness tests

Because of space limitations, we only report on the outcome of one experiment that combines the first three perturbations of the previous section, namely, a non-uniform roaming distribution with a mode towards high roaming values, a non-linear utility function for users[13], and coverage that increases faster than adoption, *i.e.,* saturates before full adoption. We omit including different types of users in the experiment, as this additional perturbation typically masks the effect of the others. Results reporting on its effect can, however, be found in Appendix F, together with results for different utility functions and a range of other scenarios.

Fig. 7 displays the results of the evaluation. It consists of four sub-figures, with each sub-figure corresponding to one of the findings summarized in Section VIII-A, and illustrating the extent to which the corresponding finding has been affected. As we discuss next, the figures illustrate that while quantitative changes can be ob-

[13]Super-linear in a user's sensitivity to roaming traffic m, and sub-linear in her sensitivity to coverage x, *i.e.,* $U(\Theta, \theta) = \gamma - c\, m^{1.2} + \theta\,(r\, x^{0.8} - \gamma) - p(\Theta, \theta)$.

served, the overall qualitative outcomes remain similar, thereby demonstrating the robustness of the findings. A similar conclusion held across the broader range of scenarios found in Appendix F.

Consider first Fig. 7a that mirrors Fig. 1, namely, plots the adoption level that maximizes total welfare as a function of the system parameters γ and e. The figure illustrates that, as in the original model, when the system can generate positive value (the system cost e is not too high), this is achieved at or near full adoption. The wider "intermediate" area that shows welfare being maximized slightly below full adoption is intuitive in light of the assumption of coverage saturation for the system, *i.e.,* reaching full adoption adds more roaming traffic without meaningfully improving coverage.

Fig. 7b in turn displays that under the usage-based pricing policy, the system still exhibits the characteristic "threshold behavior," which had been identified in the original model. Specifically, the pricing policy needs to offer users a certain minimum usage allowance, a, to successfully realize full adoption, and therefore maximum welfare. The exact value of a is clearly different from that predicted by the original model, but the overall behavior is still present.

Fig. 7c corresponds to Fig. 5. It shows that, as before, when the value of home connectivity γ is large, the hybrid pricing policy exhibits regimes where a sub-optimal equilibrium ($\tilde{x} < 1$) can arise, thereby preventing the system from reaching full adoption. Overcoming this issue can again be accomplished by appropriately discounting the service prices. The discount values are obviously different, but the mechanism is the same.

Finally, Fig. 7d parallels Fig. 6. It displays for the fixed price policy, the gap in profit between profit maximizing and welfare maximizing strategies. As before, the gap is small when the parameter c is small, and grows large as c increases.

The above results offer evidence that the findings of the paper hold under more general settings than those of the specific and relatively simple model used to preserve analytical tractability. As mentioned earlier, further evidence of this robustness can be found in Appendix F, which also investigates the impact of various errors in the provider's estimates for the different system parameters.

IX. Related Works

The service adoption process this paper focuses on exhibits both positive and negative externalities. There is a vast literature investigating the effect of externalities, often called *network effects* [11], [12], [13], but the majority of these works focus on either positive or negative externalities separately. For example, [5] investigates the impact of positive externalities on the product adoption decisions of individuals. The effect of positive externalities on the competition between technologies is considered in [14], [15], [16], [17], [18], [19]. Conversely, the impact of negative externalities, *e.g.,* from congestion, has been extensively investigated in the context of pricing for both communication networks [20], [21], [22], [23], [24] and transportation systems [25], [26], [27], [28].

The topic of optimal pricing for systems with *both* positive and negative externalities is less studied and seems to have been first addressed in [29] that sought to optimize a combination of provider's profit and consumers' surplus. Different pricing strategies were considered, including flat pricing and pricing strategies that account for the product "amount" consumed by a user, *i.e.,* akin to the usage-based pricing model of Section V. Other works have been primarily conducted in the context of the theory of clubs first formally introduced in [30] (see [31], [32], [33] for more recent discussions). A club has a membership that shares a common good or facility, *e.g.,* a swimming pool, so that increases in membership have a positive effect (externality) by lowering the cost share of the common good, *e.g.,* lower maintenance costs of the shared swimming pool. At the same time, a larger membership also has a negative, congestion-like effect, *e.g.,* a more crowded swimming pool. In general, the coexistence of positive and negative externalities implies an optimal membership size (see also [34] for a recent interesting investigation that contrasts the outcomes of self-forming and managed memberships).

Club-like behaviors also manifest themselves in file-sharing peer-to-peer (p2p) systems. In a file-sharing p2p system, more peers increase the total resources available to store content. However, unless enough peers are willing to share their resources, more peers can also translate into a higher load on those peers willing to serve files to others, and/or a longer time for locating a desired file. This has then triggered the investigation of *incentive* mechanisms to ensure that enough peers share their resources, *e.g.,* BitTorrent "tit-for-tat" mechanism [35] or [36] that also explores a possible application to a wireless access system similar in principle to the one considered in this paper.

The model of this paper differs from these earlier works in important ways. First and foremost, it introduces a model for individual adoption decisions of a service, which allows for heterogeneity in the users' valuation of the service. In particular, certain users (roaming users) have a strong disincentive to adoption when coverage/penetration is low, while others (sedentary users) are mostly insensitive to this factor. Conversely, this heterogeneity is also present in the negative externality associated with an increase in service adoption. Its magnitude is a function of not just the number of adopters, but their identity as well, *i.e.,* roaming or sedentary users. The presence of heterogeneity in how users value the service and how they affect its value is a key aspect of a UPC–like service; one that influences its value and how to price it to realize this value.

X. Conclusion

The paper was motivated by the emergence of UPC services that feature both positive and negative externalities, and more importantly (negative) externalities that depend not just on the number of adopters, but also on which users have adopted. The goal was to develop an understanding of the conditions under which such services may succeed and the welfare they generate.

As expected given the service's strong positive externality, welfare is typically maximized when adoption is maximum. More interestingly, maximum adoption and welfare can be achieved through relatively simple pricing policies that also afford complete flexibility in deciding how welfare is distributed between users and the provider of the service. Of interest is the fact that pricing according to service usage is sufficient to capture differences in how users value the service, and successfully realize both maximum welfare and arbitrary welfare distribution.

Despite the relative simplicity of usage-based pricing, it involves monitoring overhead and may face acceptance challenges on the part of users. This motivated the investigation of alternate policies, which offer a different trade-off between implementation considerations, welfare maximization, and flexibility in welfare distribution.

The paper's main contributions are in offering new insight into the viability of UPC-like services, as well as simple (pricing) mechanisms to facilitate their successful and effective deployment.

References

[1] M. Castells, *The rise of the network society*, 2nd ed. Wiley-Blackwell, 2010, vol. I.

[2] J. Wortham, "Customers angered as iPhones overload AT&T," The New York Times, September 2, 2009.

[3] J. Linshi, "Comcast turns 50,000 homes into wi-fi hotspots," TIME, June 12, 2014, Available at http://ti.me/1lewL87.

[4] M. Armstrong, "Price discrimination," in *Handbook of Antitrust Economics*, P. Buccirossi, Ed. MIT Press, 2008.

[5] L. M. B. Cabral, "On the adoption of innovation with network externalities," *Mathematical Social Sciences*, vol. 19, 1990.

[6] P. E. Green, A. Krieger, and Y.Wind, "Thirty years of conjoint analysis: Reflections and prospects," *Interfaces*, vol. 31, no. 3, pp. S56–S73, 2001.

[7] A. Lambrecht and B. Skiera, "Paying too much and being happy about it: Existence, causes and consequences of tariff-choice biases." *Journal of Marketing Research*, vol. XLIII, pp. 212–223, 2006.

[8] J. Simmermon, "Launching an optional usage-based broadband pricing plan," http://www.twcableuntangled.com/2012/02/launching-an-optional-usage-based-pricing-plan-in-southern-texas-2/, February 27, 2012.

[9] K. Wallace, "It's a new day for unlimited data," http://newsroom.sprint.com/news-releases/its-a-new-day-for-unlimited-data.htm, August 21, 2014.

[10] J. Musacchio and J. C. Walrand, "Wifi access point pricing as a dynamic game." *IEEE/ACM Trans. Netw.*, vol. 14(2), pp. 289–301, 2006.

[11] N. Economides, "The economics of networks," *International Journal of Industrial Organization*, vol. 14, no. 2, March 1996.

[12] S. J. Liebowitz and S. E. Margolis, "Network effects and externalities," in *The New Palgrave Dictionary of Economics and the Law*, P. Newman, Ed. Stockton Press, 1998.

[13] ——, "Network effects," in *Handbook of Telecommunications Economics*, M. Cave, S. Majumdar, and I. Vogelsang, Eds. Elsevier, 2002, vol. 1.

[14] M. Katz and C. Shapiro, "Network externalities, competition and compatibility," *American Economic Review*, vol. 75, no. 3, June 1985.

[15] ——, "Technology adoption in the presence of network externalities," *Journal of Political Economy*, vol. 94, no. 3, June 1986.

[16] J. Farrell and G. Saloner, "Installed base and compatibility: Innovation, product preannouncements, and predation," *American Economic Review*, vol. 76, 1986.

[17] J. Farrell and P. Klemperer, "Coordination and lock-in: Competition with switching costs and network effects," in *Handbook of Industrial Organization, Vol. 3*, M. Armstrong and R. Porter, Eds. Elsevier, 2007.

[18] D. Joseph, N. Shetty, J. Chuang, and I. Stoica, "Modeling the adoption of new network architectures," in *CoNEXT'07 Conference*. CM, 2007.

[19] S. Sen, Y. Jin, R. Guérin, and K. Hosanagar, "Modeling the dynamics of network technology adoption and the role of converters," *IEEE/ACM Trans. Netw.*, vol. 18, no. 6, pp. 1793–1805, December 2010.

[20] R. J. Gibbens and F. P. Kelly, "Resource pricing and the evolution of congestion control," *Automatica*, vol. 35, 1999.

[21] F. P. Kelly, "Charging and rate control for elastic traffic," *European Transactions on Telecommunications*, vol. 8, 1997.

[22] J. K. MacKie-Mason and H. R. Varian, "Pricing the Internet," EconWPA, Computational Economics 9401002, January 1994. [Online]. Available: http://ideas.repec.org/p/wpa/wuwpco/9401002.html

[23] I. C. Paschalidis and J. N. Tsitsiklis, "Congestion-dependent pricing of network services," *IEEE/ACM Transactions on Networking*, vol. 8, no. 2, April 2000.

[24] R. Srikant, *The Mathematics of Internet Congestion Control*. Birkhauser, 2004.

[25] M. Beckmann, C. B. McGuire, and C. B. Winsten, *Studies in the Economics of Transportation. Cowles Commission Monograph*. Yale University Press, 1956.

[26] K. J. Button and E. T. Verhoef, Eds., *Road Pricing, Traffic Congestion and the Environment: Issues of Efficiency and Social Feasibility*. Edward Elgar Pub., 1999.

[27] F. P. Kelly, "Road pricing," *Ingenia*, vol. 29, 2006.

[28] A. Odlyzko, "The evolution of price discrimination in transportation and its implications for the Internet," *Review of Network Economics*, vol. 3, 2004.

[29] S. Chao, "Positive and negative externality effects on product pricing and capacity planning," Ph.D. dissertation, Stanford University, 1996.

[30] J. M. Buchanan, "An economic theory of clubs," *Economica*, vol. 32, no. 125, February 1965.

[31] T. Sandler and J. T. Tschirhart, "The economic theory of clubs: An evaluative survey," *Journal of Economic Literature*, vol. 18, no. 4, December 1980.

[32] G. M. Anderson, W. F. Shughart II, and E. D. Tollison, "The economic theory of clubs," in *The Encyclopedia of Public Choice*, C. K. Rowley and F. Schneider, Eds. Springer, 2003.

[33] M. Potoski and A. Prakash, Eds., *Voluntary Programs: A Club Theory Perspective*. The MIT Press, 2009.

[34] R. Johari and S. Kumar, "Congestible services and network effects," in *Proc. ACM EC 2010*, Cambridge, MA, June 2010, (Extended abstract – Full version available at http://www.stanford.edu/rjohari/uploads/posex.pdf).

[35] B. Cohen, "Incentives build robustness in BitTorrent," in *Proc. 1st Workshop Econ. Peer-to-Peer Sys.*, Berkeley, CA, June 2003.

[36] C. Courcoubetis and R. Weber, "Incentives for large peer-to-peer systems," *J. Select. Areas. Commun.*, vol. 24, no. 5, May 2006.

[37] D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems*. Cambridge University Press, 1990.

Appendix A
Discrete Dynamics

In this section we propose a discrete dynamic platform and formally describe how the equilibria of the UPC system are determined over this platform. With Eq. (5) in place, it is possible to investigate the dynamics of user adoption over time. We formulate a discrete-time model that evaluates user adoption decisions at successive epochs. For simplicity[14], at epoch $(n+1)$ all users are assumed to know the system state produced by adoption decisions at epoch n. Users with a non-negative utility then proceed to adopt. Specifically, the utility at epoch $(n+1)$, $U_{n+1}(\Theta, \theta)$, of a user with roaming value θ is given by

$$U_{n+1}(\Theta, \theta) = \gamma - c\, m_n + \theta\,(r\, x_n - \gamma) - p(\Theta, \theta)\,, \quad (27)$$

where x_n and m_n are the adoption level and volume of roaming traffic produced by adoption decisions at epoch n.

Using Eq. (27) and denoting $H(x) \equiv x_{n+1}$ as a function of $x \equiv x_n$, we can characterize the evolution of $H(x)$ and identify adoption equilibria. Equilibria can be *interior* equilibria, *i.e.*, correspond to $x \in (0,1)$, or *boundary* equilibria, *i.e.*, associated with $x = 0$ or $x = 1$. Interior equilibria satisfy the equation

$$H(x) = x\,. \quad (28)$$

Boundary equilibria need not satisfy Eq. (28) and instead verify either $x_n = 0 \geq x_{n+1}$ or $x_n = 1 \leq x_{n+1}$.

Appendix B
Derivations for the Optimal Total Welfare

Section III-B identified the optimal total welfare for a given adoption level x as

$$V^*(x) = \begin{cases} \frac{r-c}{2}x^3 - \frac{\gamma}{2}x^2 + (\gamma - e)x & \text{if } x < \frac{\gamma}{r-c} \\ -\frac{r-c}{2}x^3 + (\frac{\gamma}{2} + r - c)x^2 - ex & \text{if } x \geq \frac{\gamma}{r-c}. \end{cases}$$

Denote the above two expressions by $V_1^*(x)$ and $V_2^*(x)$ for $x < \frac{\gamma}{r-c}$ and $x \geq \frac{\gamma}{r-c}$, respectively (shown in Fig. 8a as dashed line and solid line, respectively).

Finding the maximum welfare is done in two steps. We first compute the maximum of each function $V_1^*(x)$ and $V_2^*(x)$, and then find the global maximum by comparing the two local maxima.

[14]Numerical results confirm that a more realistic, diffusion-based adoption model produces similar results.

A. Maximum of $V_1^*(x)$

For easy reference, we repeat the expression of $V_1^*(x)$ here.

$$V_1^*(x) = \frac{r-c}{2}x^3 - \frac{\gamma}{2}x^2 + (\gamma - e)x\,, \quad x < \frac{\gamma}{r-c}.$$

It is easy to find the roots of this expression as

$$\begin{cases} x = 0 \\ x = \frac{\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - 2(\gamma - e)(r-c)}}{r-c}, \end{cases} \quad (29)$$

if they exist (are real numbers). Also its derivative is

$$\frac{\partial V_1^*(x)}{\partial x} = \frac{3}{2}(r-c)x^2 - \gamma x + \gamma - e, \quad (30)$$

and the two roots of $\frac{\partial V_1^*(x)}{\partial x}$ are given by

$$\begin{cases} x_{11} = \frac{\gamma - \sqrt{\gamma^2 - 6(\gamma - e)(r-c)}}{3(r-c)} \\ x_{12} = \frac{\gamma + \sqrt{\gamma^2 - 6(\gamma - e)(r-c)}}{3(r-c)}. \end{cases} \quad (31)$$

In order to find the maximum total welfare in $x < \frac{\gamma}{r-c}$ regime, we take a step-by-step approach, with each step expressed in a lemma.

Lemma 4. *if $e \geq \gamma$, then $V_1^*(x) \leq 0$ for all values of $x \in [0, \frac{\gamma}{r-c}]$.*

Proof: First assume that $e > \gamma$. From Eq. (29) and since $r - c > 0$, the condition $e > \gamma$ guarantees that $V_1^*(x)$ has indeed three roots, $x_1 < 0$, $x_2 = 0$ and $x_3 > \frac{\gamma}{r-c}$. On the other hand, at $x = 0$ the derivative of $V_1^*(x)$ is $g - e < 0$. Therefore $V_1^*(x)$ goes from 0 to negative values for $x > 0$, and may not become non-negative again until its next root at $x_3 > \frac{\gamma}{r-c}$.

Moreover, if $e = \gamma$, then the three roots of Eq. (29) are $x_1 = 0$, $x_2 = 0$, and $x_3 = \frac{\gamma}{r-c}$. At $x = 0$ the second derivative of $V_1^*(x)$ is $-\gamma$. Therefore, as before, $V_1^*(x)$ goes from 0 to negative values for $x > 0$, and may not become non-negative again until its next root at $x = \frac{\gamma}{r-c}$. ∎

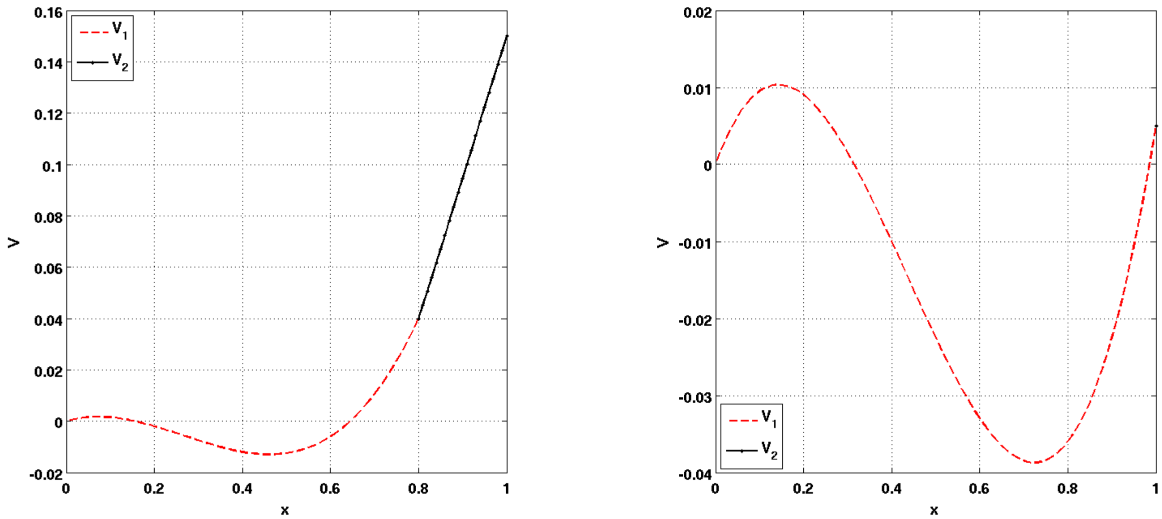
This lemma shows that total welfare is *not* positive for $x < \frac{\gamma}{r-c}$ if $e \geq \gamma$. We next look at the case where $e < \gamma$.

Lemma 5. *If $e < \gamma$, then the maximum of $V_1^*(x)$ over values of $x \in [0, \frac{\gamma}{r-c}]$ happens at either $x = x_{11}$ (if it is real) or $x = \gamma/(r-c)$.*

Proof: If $e < \gamma$ it can be easily verified that $x_{11} > 0$ and $x_{12} < \frac{2\gamma}{3(r-c)} < \gamma/(r-c)$ (if they are real). Since $V_1^*(x)$ is an increasing function of x except for $x_{11} < x < x_{12}$, the desired result follows. ∎

consequently, we deduce that if x_{11} and x_{12} are imaginary, then the maximum of $V_1^*(x)$ over values of $x \in [0, \frac{\gamma}{r-c}]$ happens at $x = \gamma/(r-c)$. More precisely, if e satisfies

$$f_3: \; e < \gamma - \frac{\gamma^2}{6(r-c)}, \quad (32)$$



(a) $\gamma = 0.8$, $e = 0.75$, $c = 0.6$, $b = 0$, $r = 1.6$. In this case, the optimal value is achieved at $x = 1$ (Corresponding to the dark solid-colored region in Fig. 1).

(b) $\gamma = 1.3$, $e = 1.145$, $c = 0.6$, $b = 0$, $r = 1.6$. In this case, the optimal value is achieved at $x \approx 0.14$ (Corresponding to the gradient-colored region in Fig. 1).

Fig. 8. System's total value as a function of x for different sets of parameters.

then the maximum of $V_1^*(x)$ over values of $x \in [0, \frac{\gamma}{r-c}]$ happens at $x = \gamma/(r-c)$.

B. *Maximum of* $V_2^*(x)$

For easy reference, we repeat the expression of $V_2^*(x)$ here.

$$V_2^*(x) = -\frac{r-c}{2}x^3 + (\frac{\gamma}{2} + r - c)x^2 - ex\,, \quad x \geq \frac{\gamma}{r-c}.$$

It is easy to find the roots of this expression as

$$\begin{cases} x = 0 \\ x = \frac{\frac{\gamma}{2}+r-c\mp\sqrt{(\gamma/2+r-c)^2-2e(r-c)}}{r-c}, \end{cases} \tag{33}$$

if they exist (are real numbers). Also its derivative is

$$\frac{\partial V_2^*(x)}{\partial x} = -\frac{3}{2}(r-c)x^2 + (\gamma + 2r - 2c)x - e. \tag{34}$$

We now have the following lemma.

Lemma 6. *If the roots of* $\frac{\partial V_2^*(x)}{\partial x}$ *are imaginary, then* $V_2^*(x)$ *is always negative on its domain.*

Proof: If the roots are not real then the expression for derivative always has the same sign as of its first coefficient, $-\frac{3}{2}(r-c)$. Since $r-c > 0$, then the derivative is always negative, and therefore $V_2^*(x)$ is a decreasing function of x. On the other hand, since $V_2^*(x = 0) = 0$, therefore $V_2^*(x) < 0$, $\forall x \in [\gamma/(r-c), 1)$. ■

The two roots of $\frac{\partial V_2^*(x)}{\partial x}$ are given by

$$\begin{cases} x_{21} = \frac{\gamma+2r-2c-\sqrt{(\gamma+2r-2c)^2-6e(r-c)}}{3(r-c)} \\ x_{22} = \frac{\gamma+2r-2c+\sqrt{(\gamma+2r-2c)^2-6e(r-c)}}{3(r-c)}. \end{cases} \tag{35}$$

By algebraic manipulation we can show that these two roots are imaginary if and only if γ satisfies

$$-2(r-c) - \sqrt{6e(r-c)} < \gamma < -2(r-c) + \sqrt{6e(r-c)}.$$

But the first inequality is always satisfied by positivity of γ. Therefore the roots in Eq. (35) are imaginary if and only if γ satisfies

$$\gamma < -2(r-c) + \sqrt{6e(r-c)},$$

or equivalently the roots in Eq. (35) are real if and only if γ satisfies

$$f_2 : \gamma \geq -2(r-c) + \sqrt{6e(r-c)}, \tag{36}$$

which, by lemma 6 is required for positivity of $V_2^*(x)$.

Now, lets see what happens when Eq. (36) is satisfied and therefore the roots of $\frac{\partial V_2^*(x)}{\partial x}$ are real. Since $r-c > 0$, the derivative is always negative except in between its roots. Then note that as for the smaller root, $x_{21} > 0$. So $\frac{\partial V_2^*(x)}{\partial x} < 0$ at a neighborhood of $x = 0$ and therefore $V_2^*(x)$ is decreasing until a value larger than $x = 0$. After that, $V_2^*(x)$ starts increasing again until $x = x_{22}$ where it again starts to decrease and continues to decrease indefinitely. Considering that $V_2^*(x = 0) = 0$, we deduce that if $V_2^*(x)$ has a positive maximum in $x \in [\gamma/(r-c), 1)$ then it happens at $\min\{1, x_{22}\}$. On the other hand, and considering that $[\gamma/(r-c), 1)$ is only non-empty if $\gamma \leq r-c$, we can perform algebraic manipulations to show that in its valid domain, $x_{22} < 1$ if and only if $e > \gamma + (r-c)/2$.

Therefore, for all values of

$$f_1 : \gamma \geq e - (r-c)/2, \tag{37}$$

the maximum of $V_2^*(x)$ happens at $x = 1$ or $x = 0$, and is the bigger of $\frac{r-c+\gamma}{2} - e$ or 0, respectively.

As mentioned before, we finally compare the maxima of $V_1^*(x)$ and $V_2^*(x)$ for the common parameter ranges. For instance, when Eq. (32) is satisfied, it can be shown that Eq. (37) is also satisfied, and the bigger of the two maxima happens at $x = 1$. Completing the steps and using numerical comparisons when necessary, results in Fig. 1.

Appendix C
Derivations for Hybrid Usage-Based Policy

Section VI presented the hybrid usage-based pricing policy that combines a fixed price for home connectivity, and a usage-based price for connectivity while roaming.

Also, Lemma 2 provided conditions under which full adoption $x = 1$ is an equilibrium. However, that Lemma did not guarantee the uniqueness of $x = 1$ equilibrium. Indeed, the progression of adoption levels towards $x = 1$ can stall before full adoption is reached. We explore next when this arises (assuming that the conditions of Lemma 2 hold).

A. Condition for uniqueness of $x = 1$ *equilibrium*

Consider a scenario where not all users have positive utility when coverage is low, so that only a subset $\Theta \neq [0, 1]$ of users initially adopt. This initial adoption triggers other users to re-evaluate their utility $U(\Theta, \theta)$, which then determines a new set of adopters Θ^{new}, such that $\Theta^{\text{new}} = \{\theta \mid U(\Theta, \theta) > 0\}$. Basic algebraic manipulation yields that Θ^{new} comprises either all users (if $x(\delta_r + \gamma) - \gamma \geq 0$), or users that verify $\theta < \frac{c/2 - cm + \delta_h}{\gamma - x(\delta_r + \gamma)}$ (if $x(\delta_r + \gamma) - \gamma < 0$), where x and m are determined by the (old) set of adopters Θ. This implies that for any adoption level x, $0 \leq x \leq 1$, the set Θ of adopters is $[0, x]$. Using Eq. (3), this set yields a roaming traffic of the form $m = \frac{x^2}{2}$, which using Eq. (20) characterizes the utility of user θ as

$$U(\Theta, \theta) = \frac{c}{2}(1 - x^2) + \delta_h + \theta(x(\delta_r + \gamma) - \gamma).$$

Consequently, the new level of adoption $x^{\text{new}} = |\Theta^{\text{new}}|$ can be expressed as a function of the previous level x. Letting $H(x) \triangleq x^{\text{new}}$ and solving for $U(\Theta, \theta) > 0$ gives[15]

$$H(x) = \begin{cases} \frac{c/2(1-x^2)+\delta_h}{\gamma - x(\delta_r + \gamma)} & \text{if } x(\delta_r + \gamma) - \gamma < 0 \\ 1 & \text{if } x(\delta_r + \gamma) - \gamma \geq 0. \end{cases}$$

Adoption equilibria satisfy $H(x) = x$, and can, therefore, be characterized by solving this equation. It can be shown that

Lemma 7. *when the conditions of Lemma 2 hold,* $x = 1$ *is the* unique *equilibrium if and only if* γ *satisfies*

$$\gamma < c + 2\delta_h + 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)}.$$

Proof: It is easy to see that the second expression for $H(x)$ satisfies $H(x) = x$ only at $x = 1$, and therefore if there are any equilibria at $x < 1$, they must satisfy $H(x) = x$ for the first expression of $H(x)$, *i.e.,*

$$H_1(x) \triangleq \frac{c/2(1-x^2) + \delta_h}{\gamma - x(\delta_r + \gamma)} = x \quad \text{for } x(\delta_r + \gamma) - \gamma < 0.$$

We first show that if γ satisfies the condition of the Lemma, then no such equilibria may exist at $x < 1$.

Basic algebraic manipulation turns the above equation into

$$Q(x) \triangleq (\gamma + \delta_r - c/2)\, x^2 - \gamma x + c/2 + \delta_h = 0,$$

which is a quadratic equation in x and for simplicity we denote it by $Q(x) = 0$. We then compute the discriminant for this equation as

$$\begin{aligned} \Delta_x &= \gamma^2 - 4(c/2 + \delta_h)(\gamma + \delta_r - c/2) \\ &= \gamma^2 - \gamma(2c + 4\delta_h) - 4(c/2 + \delta_h)(\delta_r - c/2), \end{aligned}$$

which, in turn, is a quadratic polynomial in γ. The roots of the discriminant are

$$\begin{aligned} \gamma_1 &= c + 2\delta_h - 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)} \quad \text{and} \\ \gamma_2 &= c + 2\delta_h + 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)} \end{aligned}$$

and the discriminant is negative for γ values in the range (γ_1, γ_2).

Now consider one such γ value in the range (γ_1, γ_2), which is arbitrarily close to γ_2, *i.e.,* $\gamma = \gamma_2 - \epsilon$ for an arbitrarily small $\epsilon > 0$. Therefore, the coefficient of x^2 in $Q(x)$ becomes

$$\begin{aligned} &\gamma + \delta_r - c/2 \\ &= \left(c + 2\delta_h + 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)} - \epsilon\right) + \delta_r - c/2 \\ &= c/2 + (\delta_h + \delta_r) + \delta_h + 2\sqrt{(c/2 + \delta_h)(\delta_r + \delta_h)} - \epsilon, \end{aligned}$$

which is guaranteed to be positive if ϵ is chosen small enough, *e.g.,* $\epsilon = (\delta_h + \delta_r)/2$. (Note that $-\epsilon$ is the only negative term in this expression.) On the other hand, by the previous discussion, Δ_x is negative at $\gamma = \gamma_2 - \epsilon$. Therefore at $\gamma = \gamma_2 - \epsilon$ we have $Q(x) > 0$, $\forall x$ (Of course,

[15] For notational simplicity, we omit the constraints which ensure that like x, $H(x)$ is lower-bounded by 0 and upper-bounded by 1.

by $\forall x$ we mean values of x for which $H(x) = H_1(x)$, *i.e.,* those which satisfy $x(\delta_r + \gamma) - \gamma < 0$).

Furthermore, the only terms in $Q(x)$ that depend on γ are $\gamma(x^2 - x)$. Therefore since $x^2 - x < 0$, $\forall x$, it follows that $Q(x)$, $\forall x$ is a decreasing function of γ. Hence for any $\gamma' \leq (\gamma_2 - \epsilon)$ we also have $Q(x) > 0$, $\forall x$, which means $H_1(x) \neq x$, and it follows that $x < 1$ may not be an equilibrium.

■

Lemma 7 then ensures that adoption increases monotonically until reaching full adoption. The condition of this Lemma was previously referred to in Eq. (21).

Appendix D
Fixed Price Policy

The simplest pricing policy is one with a single fixed and flat-rate price, *i.e.,* $p(\Theta, \theta) = p$. With this pricing policy, Eq. (27) becomes

$$U_{n+1}(\Theta, \theta) = \gamma - p - c\, m_n + \theta\, (r\, x_n - \gamma)\,. \tag{38}$$

Since a user's utility is a function of the adoption set Θ, evaluating the system state calls for first characterizing Θ. The next proposition allows us to understand the composition of Θ.

Proposition 4. *For all choices of* p, γ *and* c*, the set of adopters is characterized by a range of* θ *values of the form* $[0, \hat{\theta}\,]$ *or* $[\hat{\theta}, 1], 0 \leq \hat{\theta} \leq 1$.

Proof. From Eq. (38), we have:

$$U_n(\Theta, \theta) = \beta_{n-1} + \theta\alpha_{n-1}\,,$$

where $\beta_{n-1} = \gamma - p - c\, m_{n-1}$ and $\alpha_{n-1} = r\, x_{n-1} - \gamma$. For a user to have a positive utility, and therefore adopt, its θ value must satisfy $\theta\alpha_{n-1} > -\beta_{n-1}$. This translates into different conditions depending on the sign and value of α_{n-1}.

If $\alpha_{n-1} < 0$, *i.e.,* $x_{n-1} < \gamma/r$, θ needs to satisfy $\theta < -\beta_{n-1}/\alpha_{n-1}$. Hence, the set of adopters at epoch n is either empty or corresponds to users with θ values in an interval of the form $[0, \hat{\theta}_n)$, where $\hat{\theta}_n = (-\beta_{n-1}/\alpha_{n-1})_{[0,1]}$ and we have used the notation $(x)_{[0,1]}$ to denote the projection of x on the interval $[0, 1]$.

If $\alpha_{n-1} > 0$, *i.e.,* $x_{n-1} > \gamma/r$, θ must now satisfy $\theta > -\beta_{n-1}/\alpha_{n-1}$. In this scenario, the set of adopters at epoch n is again either empty or corresponds to users with θ values in an interval of the form $(\hat{\theta}_n, 1]$ where $\hat{\theta}_n = (-\beta_{n-1}/\alpha_{n-1})_{[0,1]}$.

Finally, if $\alpha_{n-1} = 0$, *i.e.,* $x_{n-1} = \gamma/r$, then $U_n(\theta) = \beta_{n-1}, \forall\, \theta \in [0, 1]$. The set of adopters in this last case is either the empty set (if $\beta_{n-1} \leq 0$) or the entire interval $[0, 1]$ (if $\beta_{n-1} > 0$). □

As a result of proposition 4, we shall capture the adopters' set Θ_n at epoch n through an adoption vector, X_n, that includes the number of adopters, x_n, and specifies their θ values through a simple binary variable. Using Eq. (38) and denoting $H(X) \equiv X_{n+1}$ as a function of $X \equiv X_n$, we want to characterize the evolution of $H(X)$ and identify adoption equilibria. We will drop the Θ notation hence forth and simply denote the utility under a flat-rate price at epoch n by $U_n(\theta)$.

From the proof of Proposition 4, we derive expressions for x_n, for the three possible conditions on α_{n-1}.

$$x_n = \begin{cases} \hat{\theta}_n & \text{if } x_{n-1} < \gamma/r & (39a)\\ 1 - \hat{\theta}_n & \text{if } x_{n-1} > \gamma/r & (39b)\\ I_{[\beta_{n-1}]}, & \text{if } x_{n-1} = \gamma/r & (39c)\end{cases}$$

As mentioned before, proposition 4 also establishes that the adoption state at epoch n, X_n, can be represented as a two-dimensional vector $X_n = (x_n, y_n)$, where y_n is a binary variable that indicates the "type" of adoption interval of Proposition 4. Specifically,

$$y_n = \begin{cases} 0 & \text{if adopters} \in [0, \hat{\theta}_n),\ \textit{i.e.,}\ x_{n-1} < \gamma/r \\ 1 & \text{if adopters} \in [\hat{\theta}_n, 1],\ \textit{i.e.,}\ x_{n-1} \geq \gamma/r \end{cases} \tag{40}$$

where we arbitrarily took y_n to be 1 for the case where $x_{n-1} = \gamma/r$. We also note that as shown in the proof of Proposition 4, the value of y_n solely depends on x_{n-1}, *i.e.,* $y_n = 0$ when $x_{n-1} < \gamma/r$ and $y_n = 1$ when $x_{n-1} \geq \gamma/r$. In other words, the *identity* of adopters at epoch n depends on the *number* of adopters at epoch $(n-1)$.

The rest of this section is devoted to characterizing equilibria and the dynamics that lead to them. We start with a number of preliminary results on which the derivations rest.

A. Preliminary Results

Assume that when the service is introduced at $n = 0$ there are no adopters; thus $x_0 = 0$ and $m_0 = 0$. At the next epoch, $n = 1$, the utility $U_1(\theta)$ of a user with roaming value θ is

$$U_1(\theta) = \gamma - p - \theta\gamma$$

At epoch 1 adopters consist, therefore, of users with a θ value such that $\theta < (\gamma - p)/\gamma$. Hence, $x_1 = (\gamma - p)/\gamma$ when $\gamma \geq p$, and $x_1 = 0$ otherwise. In other words and as stated in Proposition 5, a positive adoption requires $\gamma > p$, *i.e.,* the price cannot exceed the utility that users derive from home base connectivity. This is likely to hold in practice, *e.g.,* the price of basic Internet connectivity is such that many have adopted the service even in the absence of a roaming option. Throughout the analysis this condition is assumed to hold. Note that under this assumption, $x = 0$ can not be an equilibrium.

Proposition 5. *Starting from an initial state of zero adoption, non-zero adoption is possible only if* $\gamma > p$.

In the next proposition, we formally establish that the vector X_n fully characterizes the adoption process, namely, that m_n can be computed once X_n is known.

Proposition 6. *The vector* $X_n = (x_n, y_n)$*, together with the parameters* γ, p, r *and* c*, are sufficient to compute a user's utility at epoch* $(n+1)$ *as expressed in Eq. (38).*

Proof. From Eq. (27), a user's utility at epoch $(n+1)$ depends on γ, p, r, c, x_n, and m_n. It therefore suffices to show that m_n can be computed based on γ, p, r, c, x_n, and y_n. We consider separately the cases $y_n = 0$ and $y_n = 1$.

If $y_n = 0$, adopters are users with $\theta \in [0, \hat{\theta}_n)$, so that $\hat{\theta}_n = x_n$ and m_n is given by:

$$m_n = \int_0^{x_n} \theta \, d\theta = \frac{1}{2} x_n^2 \,, \quad \text{if } y_n = 0 \tag{41}$$

Conversely, when $y_n = 1$ adopters are users with $\theta > \hat{\theta}_n$. Thus $\hat{\theta}_n = 1 - x_n$ and m_n is given by

$$m_n = \int_{1-x_n}^{1} \theta \, d\theta = \frac{1}{2}(-x_n^2 + 2x_n) \,, \quad \text{if } y_n = 1 \tag{42}$$

This establishes that, $U_{n+1}(\theta)$ can be computed based on X_n and the parameters γ, p, r and c. Note that this also ensures that X_{n+1} can be computed, and therefore the evolution of the adoption process can be tracked. □

B. Characterizing Adoption Evolution

We now turn to exploring the evolution of the adoption vector X_n. Our goal is to characterize adoption dynamics and identify eventual equilibria. As mentioned earlier, equilibria are either solutions of

$$H(X) = X, \tag{43}$$

or boundary points of the interval $[0, 1]$. The main difficulty in solving Eq. (43) stems from the fact that X_n is a two-dimensional vector. In particular, although Eqs. (41) and (42) show that a user's utility at epoch $(n+1)$ is solely a function of x_n, the choice of which equation to use depends on y_n or in other words on x_{n-1}, *i.e.*, $U_{n+1}(\theta)$ is a function of both x_n and x_{n-1}.

As a result, exploring adoption dynamics calls for accounting for adoption levels in the previous *two* epochs. This is reflected in the approach we describe next. Specifically, we consider separately the cases $y_n = 0$ ($x_{n-1} < \gamma/r$) and $y_n = 1$ ($x_{n-1} \geq \gamma/r$), and correspondingly introduce the notation $H_1(x) \equiv H(x, 0)$ and $H_2(x) \equiv H(x, 1)$ to investigate the evolution of adoption in these two scenarios. As we shall see, these two cases will each be divided in two further sub-cases.

1) Adoption Evolution under $H_1(x)$, i.e., $y_n = 0$*:* In this scenario, Eq. (41) is used to compute $U_{n+1}(\theta)$, which when combined with Eq. (27) gives

$$U_{n+1}(\theta) = \gamma - p - \frac{c}{2} x_n^2 + \theta\,(r\,x_n - \gamma) \,. \tag{44}$$

Eq. (44) allows us to determine the adoption threshold $\hat{\theta}_{n+1}$ at epoch $(n+1)$, *i.e.*, the θ value such that $U_{n+1}(\hat{\theta}_{n+1}) = 0$:

$$\hat{\theta}_{n+1} = \frac{\gamma - p - \frac{c}{2} x_n^2}{\gamma - r\,x_n} \,.$$

To compute the new system state X_{n+1} at epoch $(n+1)$, we distinguish between the cases $y_{n+1} = 0$ and $y_{n+1} = 1$, with Eq. (39) correspondingly identifying the expression of x_{n+1}.

When $y_{n+1} = y_n = 0$, both x_n and x_{n-1} are below γ/r. Therefore even when x_{n+1} is above γ/r, the set of adopters at epoch $(n+1)$ is still of the form $[0, \hat{\theta}_{n+1})$. Since both x_n and x_{n+1} consist of the same type of adopters, we say that adoption stays in the *"home"* region, and for convenience introduce the notation $x_{n+1} \equiv H_{1h}(x)$. Eq. (39a) then states that $x_{n+1} = \hat{\theta}_{n+1}$, so that

$$H_{1h}(x) = \frac{\gamma - p - \frac{c}{2} x^2}{\gamma - r\,x} \,. \tag{45}$$

When $y_{n+1} = 1$ and $y_n = 0$, we have $x_n \geq \gamma/r$ while x_{n-1} was below γ/r, and the set of adopters at epoch $(n+1)$ is of the form $(\hat{\theta}_{n+1}, 1]$. We denote this configuration as x_{n+1} being in the *"away"* region, and correspondingly introduce the notation $x_{n+1} \equiv H_{1a}(x)$. Eq. (39b) then states that $x_{n+1} = 1 - \hat{\theta}_{n+1}$, so that

$$H_{1a}(x) = \frac{\frac{c}{2} x^2 - r\,x + p}{\gamma - r\,x} \tag{46}$$

2) Adoption Evolution under $H_2(x)$, i.e., $y_n = 1$*:* In this scenario, Eq. (42) is used in equation Eq. (27), which gives:

$$U_{n+1}(\theta) = \gamma - p - \frac{c}{2}(-x_n^2 + 2x_n) + \theta(r\,x_n - \gamma) \,. \tag{47}$$

As before, from Eq. (47) we find the adoption threshold $\hat{\theta}_{n+1}$ for which $U_{n+1}(\hat{\theta}_{n+1}) = 0$. This gives:

$$\hat{\theta}_{n+1} = \frac{\gamma - p - \frac{c}{2}(-x_n^2 + 2x_n)}{\gamma - r\,x_n} \,.$$

Following the approach used for $H_1(x)$, we consider separately the cases where $y_{n+1} = 1$ and $y_{n+1} = 0$.

When $y_{n+1} = y_n = 1$, adopters at epoch $(n+1)$ remain characterized by a range $\theta > \hat{\theta}_{n+1}$, which as before we term the *home* region. Similarly, we let $x_{n+1} \equiv H_{2h}(x)$, which using Eq. (39b) gives

$$H_{2h}(x) = \frac{\frac{c}{2} x^2 + (r - c)x - p}{r\,x - \gamma} \tag{48}$$

When $y_{n+1} = 0$ and $y_n = 1$, x_n is now below γ/r while x_{n-1} was above γ/r, and the set of adopters at epoch $(n+1)$ is of the form $[0, \hat{\theta}_{n+1})$. We again denote this configuration as x_{n+1} being in the *away*region, with the corresponding notation $x_{n+1} \equiv H_{2a}(x)$. Eq. (39a) gives

$$H_{2a}(x) = \frac{\frac{c}{2} x^2 - c\,x + \gamma - p}{\gamma - r\,x} \tag{49}$$

In summary, the adoption state at epoch $(n+1), X_{n+1}$, has been characterized by considering the four possible combinations of adoption levels in epochs $(n-1)$ and n. In the next sections, these results are leveraged to identify possible equilibria and characterize adoption dynamics.

C. Characterizing Equilibria

This section leverages the results of Section D-B to identify the type of equilibria to which adoption can converge. Consistent with the discussion of the previous section, we introduce the notation $H_h(x)$ for the function defined as $H_{1h}(x)$ in the interval $[0, \gamma/r)$ and as $H_{2h}(x)$ in the interval $[\gamma/r, 1]$, and $H_a(x)$ for the function defined as $H_{2a}(x)$ in $[0, \gamma/r)$ and as $H_{1a}(x)$ in $[\gamma/r, 1]$.

Since any equilibria must satisfy $y_{n+1} = y_n$, we can rule out half of the combinations of the previous section. Specifically, when $y_n = 0$, only vectors of the form $X_{n+1} = (H_{1h}(x), 0)$ need to be considered. Conversely, when $y_n = 1$, candidate equilibria must be of the form $(H_{2h}(x), 1)$. Equilibria, therefore, correspond to either points $x \in (0, \gamma/r)$ that verify $H_{1h}(x) = x$, points $x \in [\gamma/r, 1)$ that verify $H_{2h}(x) = x$, the point $x = 0$ if $H_{1h}(0) \leq 0$, or the point $x = 1$ if $H_{2h}(1) \geq 1$.

We therefore explore the relative positions of the functions $H_{1h}(x)$ and $H_{2h}(x)$ with respect to x, and their possible intersections with x. Intersections identify equilibria or fixed points, while the position of $H_{1h}(x)$ and $H_{2h}(x)$ relative to x determines the "nature" of these fixed points, *i.e.,* stable, or unstable, or associated with orbits either periodic or chaotic[16]. The derivations are mechanical and can be found in Appendix E. We distinguish between stable fixed points ($\bullet$) with monotonic trajectories (towards the fixed point inside its attraction region), unstable fixed points ($\circ$) again with monotonic trajectories (away from the fixed point), and fixed points associated with an "orbit" ($\circlearrowright$) that can be either convergent, periodic, or chaotic for different (p, l) pairs. Table I summarizes possible combinations of equilibria in each of the intervals $[0, \gamma/r)$ and $[\gamma/r, 1]$, where — denotes the absence of fixed point in that interval.

Case 1 of Table I corresponds to a scenario where no fixed point exists. We discuss later when and why this arises, but adoption patterns essentially never stabilize. Cases $2, 2', 3$ and $3'$ are instances where a single fixed point exists in either $[0, \gamma/r)$ or in $[\gamma/r, 1]$. In Cases 2 and 3, the fixed point corresponds to a stable equilibrium, while in Cases $2'$ and $3'$ it can be associated with more complex trajectories that need not converge, *e.g.,* exhibit periodic orbits or chaotic adoption patterns. Cases 4 and 5 correspond to a scenario with both a stable and an unstable equilibrium in either $[0, \gamma/r)$ or in $[\gamma/r, 1]$, with the adoption always converging to the stable fixed point. Cases 6 and 7 exhibit different combinations of equilibria in $[0, \gamma/r)$ or in $[\gamma/r, 1]$, with one having a single stable equilibrium and the other having both a stable and an unstable equilibrium. The important feature of these two latter cases is the presence of two stable equilibria, one in $[0, \gamma/r)$ and the other in $[\gamma/r, 1]$. As a result, final adoption levels can differ based on initial adoption values, *i.e.,* they can vary based on the level of seeding when the service was first offered. A similar situation is present in Case 8, where the two ranges both have a stable and an unstable equilibrium.

[16]If either $x = 0$ or $x = 1$ are equilibria, they are stable equilibria.

In the next section, we characterize the trajectories associated with the different combinations of Table I, while Section D-E articulates implications for a UPC service offering.

D. Classifying Adoption Dynamics

Table I readily identifies several possible patterns of adoption. Specifically, adoption dynamics can be of the form:

i) Absence of convergence to an equilibrium. This arises in Cases $1, 2'$, and $3'$. In Case 1, this is *independent* of the initial adoption level, as the absence of a fixed point gives rise to chaotic adoption patterns that never converge. The situation is more subtle in Cases $2'$ and $3'$, for which a fixed point does exist. However, even when a small region of attraction exists around this fixed point, adoption trajectories typically remain outside of it, and orbit around it in either periodic or chaotic manner. Such patterns are common in dynamical systems [37]. The derivations that led to the identification of those trajectories as well as an illustrative example can again be found in Appendix E. The conditions under which they arise are discussed in Section D-E;

ii) Convergence to a single stable equilibrium in either $[0, \gamma/r)$ or $[\gamma/r, 1]$, *independent* of initial penetration. This arises in Cases $2, 3, 4$, and 5, where a single stable equilibrium exists in the entire adoption range. In those cases, adoption proceeds monotonically towards the equilibrium, either increasing or decreasing depending on the value of the initial adoption level. As it does not affect the final outcome, seeding is of no benefit in these scenarios;

iii) Convergence to one of two stable equilibria in $[0, \gamma/r)$ or $[\gamma/r, 1]$, *dependent* on initial penetration. This arises in Cases $6, 7$, and 8, where a stable equilibrium exists in both $[0, \gamma/r)$ and $[\gamma/r, 1]$. These are instances where seeding may be of value, as it can affect the final adoption level. In particular, a high enough level of seeding can allow the service to realize a much higher final adoption (in $[\gamma/r, 1]$ as opposed to $[0, \gamma/r)$). As in Case ii), trajectories are monotonic towards the final adoption level.

The trajectories of the three types of possible outcomes that have been identified can be easily constructed using a standard cobweb plot[17] based on the functions $H_h(x)$ and $H_a(x)$. For illustration purposes, we consider an example associated with Case 8 from Table I, which involves stable and unstable equilibria in both $[0, \gamma/r)$ and $[\gamma/r, 1]$. The shapes of the corresponding functions $H_h(x)$ (solid line) and $H_a(x)$ (dash-dot line) are shown in Fig. 9, together with three adoption trajectories associated with different initial adoption levels.

In the first scenario, there are no initial adopters, *i.e.*, $x_0 = 0$, and adoption increases monotonically until it reaches about 10%, the stable equilibrium in $[0, \gamma/r)$. In the second scenario, seeding has been used to create an initial adoption level $x_0 \approx 35\%$. As we can see, this is not enough to prevent adoption from declining back to 10%, the stable equilibrium in $[0, \gamma/r)$. To avoid such an outcome, seeding needs to be further increased, as done in the third scenario where initial adoption is set to around 46%. In this case, the adoption trajectory enters the interval $[\gamma/r, 1]$ and eventually converges to the higher adoption equilibrium in that interval (around 85%). The trajectory also illustrates the use of the function $H_a(x)$ when first entering $[\gamma/r, 1]$ from $[0, \gamma/r)$. We note that although a high level of adoption is ultimately realized, the associated seeding "cost" is high.

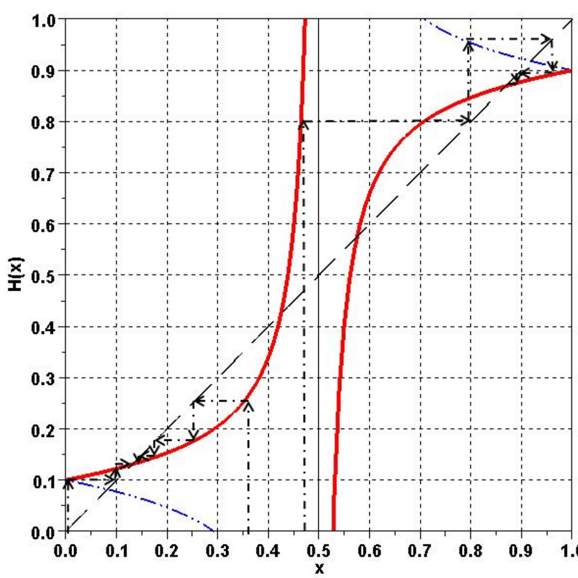


Fig. 9. $H_h(x)$ (solid) and $H_a(x)$ (dash-dot) for Case 8.

In the next section, we characterize how system parameters, in particular the price p, map to different equilibria and trajectories, and identify possible implications for UPC service offerings.

[17]See http://code.google.com/p/cobweb2008/ for an illustrative applet.

E. Interpretations

Recall that adoption trajectories and equilibria are determined by the "shape" of the functions $H_h(x)$ and $H_a(x)$ and how they intersect the line x. The shape of those functions depends in turn on the parameters γ, c, r, and p (see Eqs. (45) and (48)). As a result, it is no surprise that both adoption outcomes and trajectories are determined by values of these parameters and in particular, for any γ and r value, associated with distinct "regions" of the $(p, c)-$plane, *i.e.*, contiguous ranges of p and c values. Fig. 10 identifies the regions of the $(p, c)-$plane that map to the ten combinations of table I, and correspondingly Behaviors i), ii), and iii) used earlier to classify adoption dynamics. The boundaries of those regions are derived from constraints on the parameters, with Table II providing the corresponding functional expressions. Details on the derivations are again in Appendix E.

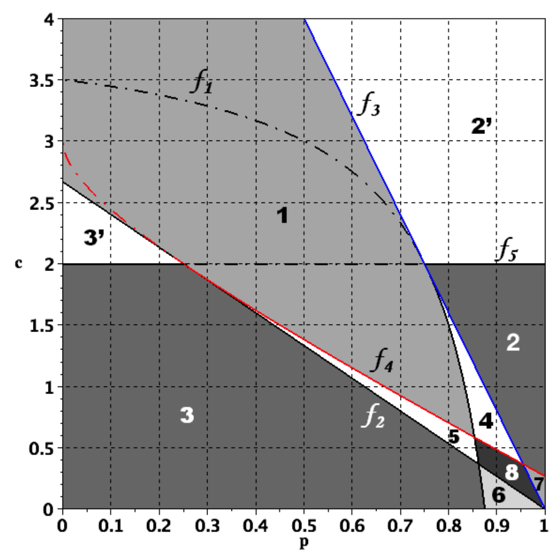


Fig. 10. Regions of the (p, c) plane corresponding to different combinations of equilibria as given by Table I. This is a sample illustration for $\gamma = 1$ and $r = 2$.

Behavior i): This maps to regions $1, 2'$ and $3'$ of Fig. 10, and is associated with configurations that do not yield convergence to an adoption equilibrium.

Region 1 consists of relatively low values of p but rather large values of c. This produces the following dynamics: When there are few or no users in the network, coverage is low and frequently-roaming users find the service unattractive despite the low p. In contrast, sedentary users are unaffected by the limited coverage, so that the low p value entices them to adopt. As they adopt, coverage improves and the service becomes attractive to roaming users. With more users adopting, coverage continues improving. The associated growth in roaming traffic, however, starts to negatively affect sedentary users that derive little benefits from the improved coverage. This leads some of them to disadopt, which reduces coverage so that eventually roaming users start leaving as well. Once roaming traffic has been sufficiently reduced, the service becomes again attractive to sedentary users, and the cycle repeats.

A similar, though more nuanced process is at work in regions $2'$ and $3'$. Region $2'$ also boasts large c values ($c \geq r$), and in the portion of that region where large values of p are allowed, it displays similar adoption patterns as region 1 to which it is adjacent. However, when p is allowed to be large, the negative effect of c never gets a chance to manifest itself. The large p prevents enough sedentary users from adopting, and the service never garners enough coverage to become attractive to frequently-roaming users. In this case, adoption converges to a low value in $[0, \gamma/r)$. As p decreases, the region of attraction of the equilibrium shrinks, and non-converging adoption patterns emerge.

The behavior in region $3'$ is similar, albeit for an equilibrium in $[\gamma/r, 1]$. Specifically, region $3'$ combines small positive p values and very large c values. The small value of p means that many users want to adopt. The very large c value, however, implies that only frequently roaming adopters derive enough benefits from the large coverage to compensate for the penalty of roaming traffic. As a result, the most sedentary users disadopt. When p is sufficiently small, this disadoption is small enough to not affect coverage to the point where frequently-roaming users start leaving as well. However, as p increases, coverage may decrease enough to trigger an exodus of frequently-roaming user, and create cyclical patterns of adoption and disadoption as in region 1.

TABLE II
REGION BOUNDARIES.

f_1	$c = 2r - \frac{\gamma^2}{2(\gamma-p)}$
f_2	$c = \frac{2r^2(\gamma-p)}{2r\gamma-\gamma^2}$
f_3	$c = \frac{2r^2(\gamma-p)}{\gamma^2}$
f_4	$c = \gamma + r - p - \sqrt{p^2 + 2p(r-\gamma)}$
f_5	$c = r$

Behavior ii): Regions 2 and 4 of Fig. 10 have a stable equilibrium in $[0, \gamma/r)$ to which adoption converges. The regions correspond to relatively high p values and relatively high values of c. The high p value is such that few sedentary users adopt and coverage never gets high enough to make the service attractive to frequently-roaming users. Hence, adoption saturates at a low level of penetration. Seeding will not help, as the rather high c value is too much of an impact even for frequently roaming users.

Conversely, in regions 3 and 5 of Fig. 10 adoption converges to a single stable equilibrium in $[\gamma/r, 1]$. The regions correspond to relatively low values of p and comparatively low c values. The low p value initially attracts sedentary users that are not deterred by the limited coverage, and once enough of them have adopted frequent roamers start joining. Because the impact of increasing roaming traffic is relatively low, few sedentary users leave and adoption stabilizes at a high level.

Behavior iii): Regions 6, 7 and 8 of Fig. 10 exhibit a stable equilibrium in both $[0, \gamma/r)$ and $[\gamma/r, 1]$. In these cases, adoption converges to either equilibrium as a function of the initial adoption level (seeding). The three regions share relatively high p values and similarly small c values.

When initial adoption (coverage) is low, frequently-roaming users are not interested in the service and the high p value limits the number of sedentary users who adopt. Hence, adoption saturates at a low level. In contrast, if seeding has produced enough initial coverage to attract frequent roamers, they will start adopting in spite of the high p value. As their number grows and coverage continues improving, some sedentary users will also adopt because of the relatively low impact that they incur from roaming traffic through their home base. As a result, overall adoption eventually stabilizes at a high level.

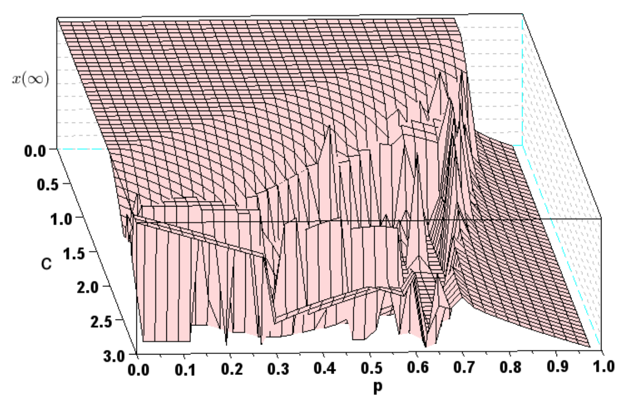


Fig. 11. Adoption Outcomes as a Function of p and c, when $\gamma = 1$ and $r = 2$

The behaviors identified in this section are illustrated in Fig. 11 that plots the "final" adoption levels for different (p, c) pairs starting from an initial adoption level of $x_0 = 0$. In scenarios where adoption does not converge, *i.e.,* Behavior i), the adoption level reported in the figure was sampled at a particular iteration. The figure clearly identifies the regions of the (p, c) plane that correspond to chaotic or at least non-converging adoption (regions $1, 2'$, and $3'$), low adoption (regions 2 and 4, as well as regions 6, 7, and 8 since $x_0 = 0$ was used), and regions of high adoption (regions 3 and 5).

F. Optimal Pricing for Provider's Profit

In a flat-price policy, all users pay the same price p. Therefore, the provider's profit (or welfare) $\Pi(p)$ that was introduced in section III-A as $W_P(\Theta)$ becomes

$$\Pi(p) = (p - e)\, x. \tag{50}$$

The UPC provider's goal is to select p so as to maximize its profit at equilibrium[18], *i.e.*, once adoption has stabilized[19]. In other words, the provider seeks to identify p^* such that

$$\Pi(p^*) = \max_p \{\Pi(p)\}\,.$$

Note that in Eq. (50) the service adoption level x is itself a function of p and the exogenous parameters of Eq. (38). $\Pi(p)$ can, therefore, be expressed as a function with p as its only variable. More precisely, because adoption equilibria have different functional expressions depending on whether adoption is low or high, we also have two distinct expressions for $\Pi(p)$. The first is associated with an equilibrium in the low-adoption region, while the second corresponds to an equilibrium in the high-adoption region.

For the sake of analytical tractability, we keep Π as a function of x (rather than p). This yields two expressions, $\Pi_L^{(1)}(x)$ and $\Pi_H^{(1)}(x)$, for the provider's profit corresponding to equilibria in $[0, 1/2)$ (low adoption), and $[1/2, 1]$ (high adoption). Derivations are mechanical in nature, and the resulting expressions are given in Eqs. (51) and (52) for completeness.

$$\Pi_L^{(1)}(x) = \frac{4-c}{2}\, x^3 - x^2 + (\gamma - e)\, x \tag{51}$$

$$\Pi_H^{(1)}(x) = \frac{c-4}{2}\, x^3 + (3-c)\, x^2 + (\gamma - e - 1)\, x \tag{52}$$

Both equations are cubic polynomials in x. Differentiating them yields expressions for the x values that maximize them, *i.e.*, $\hat{x}_L$ and $\hat{x}_H$.

The next step calls for determining which of $\Pi_L^{(1)}(\hat{x}_L)$ and $\Pi_H^{(1)}(\hat{x}_H)$ is higher, and consequently decide how to best price the service. The answer can change based on the combination of exogenous parameters, *e.g.*, the service's intrinsic value, γ, the impact of roaming traffic, c, and the value of the service cost e. For instance, it can be shown that the γ value at which high-adoption becomes more profitable than low-adoption increases with c. This is intuitive since a larger c means that sedentary users are more sensitive to roaming traffic. Hence, the service needs to be intrinsically more valuable to allow enough of them to join *and* stay as roaming traffic grows with adoption.

[18]This forces a price selection that ensures the existence of an equilibrium.

[19]Note that this implicitly assumes a recurring pricing model, as is common with most service offerings.

Appendix E
Derivations of Equilibria under the Fixed Price Policy

The intersections of $H_h(x)$ and x correspond to interior equilibria in adoption levels, *i.e.*, equilibria in $(0, 1)$, and the relative positions of $H_h(x)$ and x at $x = 0$ and $x = 1$ determine whether or not either are boundary equilibria. We consider equilibria in the intervals $[0, \gamma/r)$ and $[\gamma/r, 1]$ separately. During the analysis we may use $k \triangleq \gamma - p$ for notational conciseness.

A. Equilibria in $[0, \gamma/r)$

From Eq. (45), $H_{1h}(0) = k/\gamma > 0$, given the earlier assumption that $k > 0$. Therefore, the condition $H_{1h}(0) \leq 0$ is never met and $x = 0$ is not an equilibrium. Next, we consider interior points, *i.e.*, points in $(0, \gamma/r)$.

From Eq. (45), $H_{1h}(x) = x$ yields

$$\left(-\frac{c}{2} + r\right) x^2 - \gamma\, x + k = 0\,. \tag{53}$$

We assume $-c/2 + r > 0$ or $2r > c$, which is a reasonable and hardly restrictive assumption in our model; This means that at full adoption, the most frequently roaming user (with $\theta = 1$) will derive more utility from the ability to roam than be impacted by the external roaming traffic. Under the assumption that $2r > c$, Eq. (53) has (at most) two roots given by

$$x = \frac{\gamma \pm \sqrt{\Delta_1}}{2r - c}$$

where $\Delta_1 = \gamma^2 + 2kc - 4kr$. The inequality $H_{1h}(x) < x$ holds (only) between the two roots. We distinguish three cases:

i) $\Delta_1 < 0$ or $c < 2r - \frac{\gamma^2}{2k}$.

In this case, Eq. (53) does not have any roots and $H_{1h}(x) > x$ holds $\forall x \in [0, \gamma/r)$. In other words, there are no equilibria in $[0, \gamma/r)$. A sample illustration can be seen in Fig. 10 where we have chosen $\gamma = 1$ and $r = 2$. This criterion corresponds to the points in the (p, c) plane where $c < f_1$. The functional expressions of the different curves are given in Table II.

ii) $\Delta_1 = 0$ or $c = 2r - \frac{\gamma^2}{2k}$

In this case Eq. (53) holds at $x = \frac{\gamma}{2r-c} = 2k/\gamma$, so $H_{1h}(x)$ and x touch once in $[0, \gamma/r)$ if $k < \frac{\gamma^2}{2r}$. In this case, there is only one equilibrium $x_1 \in [0, \gamma/r)$ which is easily seen to be stable from the left and unstable from the right. This is because $H_h(x) > x$ when $x < x_1$ (adoption levels increase towards x_1 in each iteration), and $H_h(x) > x$ when $x > x_1$ as well (adoption levels continue increasing once x_1 is exceeded).

iii) $\Delta_1 > 0$ or $c > 2r - \frac{\gamma^2}{2k}$
In this case Eq. (53) has two real roots, so that $H_{1h}(x)$ and x intersect twice. These two intersections may or may not indeed be in $[0, \gamma/r)$. Next, we determine conditions for either of these intersections to lie in $[0, \gamma/r)$ and characterize the equilibria they give rise to.

1) Intersection x_{1s}: Intersection x_{1s} is the smaller of the two roots of Eq. (53) and is given by:

$$x_{1s} = \frac{\gamma - \sqrt{\gamma^2 + 2kc - 4kr}}{2r - c}. \tag{54}$$

For x_{1s} to be an equilibrium, it must be in the interval $[0, \gamma/r)$. The earlier assumptions $k > 0$ and $2r > c$ ensure that $x_{1s} > 0$. For $x_{1s} < \gamma/r$ we need:

$$\frac{-\gamma(r-c)}{r} < \sqrt{\gamma^2 + 2kc - 4kr}$$

This is trivially true if $r - c > 0$. If on the other hand $r - c < 0$, we need

$$\gamma^2 c^2 - (2r\gamma^2 + 2r^2k)c + 4kr^3 < 0. \tag{55}$$

The left side is a quadratic equation in c and the inequality holds between the two (possible) roots, which are given by:

$$\begin{aligned} c &= \frac{(r\gamma^2 + r^2k) \mp \sqrt{(r\gamma^2 - r^2k)^2}}{\gamma^2} \\ &= \frac{(r\gamma^2 + r^2k) \mp |r\gamma^2 - r^2k|}{\gamma^2}. \end{aligned}$$

Based on the sign of $r\gamma^2 - r^2k$ the interval between the two roots of Eq. (55) can be specified. We have

$$c \in \begin{cases} \left(\frac{2r^2k}{\gamma^2},\, 2r\right) & \text{if } \gamma^2 \geq rk \\ \left(2r,\, \frac{2r^2k}{\gamma^2}\right) & \text{if } \gamma^2 < rk. \end{cases}$$

But because of our previous assumption that $2r > c$, the second case above cannot happen.

This shows that when $\Delta_1 > 0$, the intersection x_{1s} will be an equilibrium in $[0, \gamma/r)$ if either $c < r$ or $c \in \left(\frac{2r^2k}{\gamma^2},\, 2r\right)$ and $k \in [0, \gamma^2/r]$. These criteria correspond to (p, c) being in Regions 2′, 2, 4, 6, 7 and 8 of Fig. 10, again with the functional expressions of the different curves given in Table II.

When (p, c) is in any of the Regions 2, 4, 6, 7 or 8, then x_{1s} can be shown to be a stable equilibrium. This is because $x_{1s} > H_h(x) > x$ (adoption increases towards x_{1s} in the next iteration), and $x_{1s} < H_h(x) < x$ (adoption decreases towards x_{1s} in the next iteration). On the other hand if (p, c) is in the Region 2′, then x_{1s} is an "orbital" equilibrium. An orbital equilibrium may have a non-empty region of attraction[20], but exhibit cyclical adoption patterns (periodic or chaotic) outside of that neighborhood. Orbital behaviors arise when $H_h(x) > x_{1s} > x$ (adoption increases beyond x_{1s} in the next iteration), and $H_h(x) < x_{1s} < x$ (adoption drops below x_{1s} in the next iteration). This gives rise to cyclical trajectories, which may or may not converge to x_{1s} depending on the slope of $H_{1h}(x)$ at $x = x_{1s}$ and the initial distance between x and x_{1s}. Note also that if $H_{1h}(x) > \gamma/r$ for some $x < x_{1s}$, the next adoption level will be determined using $H_{2a}(x)$ instead of $H_{1h}(x)$, since we have left the interval $[0, \gamma/r)$.

2) Intersection x_{1u}: Intersection x_{1u} is the larger of the two roots of Eq. (53) and is given by

$$x_{1u} = \frac{\gamma + \sqrt{\gamma^2 + 2kc - 4kr}}{2r - c}.$$

Again, for x_{1u} to be an equilibrium, it must be in $[0, \gamma/r)$. Since $2r - c > 0$, we have $x_{1u} > x_{1s} > 0$, and therefore we only need to verify when the condition $x_{1u} < \gamma/r$ holds. For this we need:

$$\sqrt{\gamma^2 + 2kc - 4kr} < \frac{\gamma(r-c)}{r}.$$

This never holds if $r - c < 0$. When $r - c > 0$, the condition becomes

$$\gamma^2 c^2 - (2r\gamma^2 + 2r^2k)c + 4kr^3 > 0.$$

which is the symmetric of Eq. (55), and thus it holds for values of c outside the roots of the corresponding quadratic equation.

We also note that equilibrium x_{1u} is unstable. This is because $H_h(x) < x$ when $x < x_{1u}$ (adoption levels keep decreasing once they have dropped below x_{1u}), and $H_h(x) > x$ when $x > x_{1u}$ (adoption levels keep increasing once they have exceeded x_{1u}).

To summarize, in Case **iii)**, *i.e.*, in the case of $c > 2r - \frac{\gamma^2}{2k}$ there can possibly be two equilibria in $[0, \gamma/r)$. When $c > r$, the root x_{1s} is the only equilibrium in $[0, \gamma/r)$ if the condition $c \in [\frac{2r^2k}{\gamma^2}, 2r]$ is also satisfied (Region 2′ in Fig. 10); Otherwise, no equilibrium is present in this interval (The portion of Region 1 in Fig. 10 for which $c > f_1$). When $c < r$, both x_{1s} and x_{1u} can be equilibria if $c < \frac{2r^2k}{\gamma^2}$ (Regions 4, 8 and 6 in Fig. 10), and otherwise x_{1s} is the only equilibrium in $[0, \gamma/r)$ (Regions 2 and 7 in Fig. 10). Again the functional expressions of the different curves are given in Table II.

B. *Equilibria in* $[\gamma/r, 1]$

For the boundary point $x = 1$ we use Eq. (48) and see that:

$$H_{2h}(1) = \frac{-\frac{c}{2} + k + r - \gamma}{r - \gamma} = \frac{-\frac{c}{2} + k}{r - \gamma} + 1.$$

Therefore the full adoption level, $x = 1$, will be an equilibrium *if and only if* $-c/2 + k \geq 0$. We will now consider the interior points, *i.e.*, the points in $[\gamma/r, 1)$.

From the equation $H_{2h}(x) = x$ we get:

$$-\left(r - \frac{c}{2}\right)x^2 + (\gamma + r - c)x - p = 0. \tag{56}$$

[20] A neighborhood of x_{1s} so that for values of x in that neighborhood, trajectories converge to x_{1s}.

Assuming $r - c/2 > 0$ as before, Eq. (56) exhibits (at most) the two roots given by

$$x = \frac{-(\gamma + r - c) \pm \sqrt{\Delta_2}}{-(2r - c)}$$

where

$$\begin{aligned}\Delta_2 &= (\gamma + r - c)^2 - 2p(2r - c) \\ &= c^2 - 2(\gamma - p + r)c + (\gamma + r)^2 - 4pr.\end{aligned}$$

The inequality $H_{2h}(x) > x$ holds[21] only between these two (possible) roots. We again distinguish three cases:
i) $\Delta_2 < 0$
This is equivalent to

$$c \in \left(-p + \gamma + r - \sqrt{Q}\,, -p + \gamma + r + \sqrt{Q}\right)$$

where

$$Q = p^2 + 2p(r - \gamma).$$

In this case, Eq. (56) does not have any roots and $H_{2h}(x) < x$ holds $\forall x \in (\gamma/r, 1]$. In other words, there are no equilibria in $(\gamma/r, 1]$.
ii) $\Delta_2 = 0$
This is equivalent to

$$c = -p + \gamma + r \mp \sqrt{p^2 + 2p(r - \gamma)}$$

and in this case Eq. (56) holds at $x = \frac{\gamma + r - c}{2r - c}$. Therefore the two curves $H_{2h}(x)$ and x touch once in $(\gamma/r, 1]$ if $c < r$. In this case, there is only one equilibrium $x_2 \in (\gamma/r, 1]$ which is easily seen to be stable from the right and unstable from the left. This is because $H_h(x) < x$ when $x > x_2$ (adoption levels decreases towards x_1 in each iteration), but $H_h(x) < x$ when $x < x_2$ as well (adoption levels keep decreasing if x goes below x_1).
iii) $\Delta_2 > 0$
This is equivalent to

$$c \notin \left[-p + \gamma + r - \sqrt{Q}\,, -p + \gamma + r + \sqrt{Q}\right]$$

where as before

$$Q = p^2 + 2p(r - \gamma).$$

In this case Eq. (56) has two real roots, and as a result it is possible for $H_{2h}(x)$ and x to intersect twice in $[0, \gamma/r]$. Next, we characterize the equilibria that these two possible intersections, x_{2u} and x_{2s}, can give rise to.

[21]Note that since $x \in [\gamma/r, 1]$, the denominator of $H_{2h}(x)$ is positive.

1) Intersection x_{2u}: Intersection x_{2u} is the smaller of the two roots of Eq. (56) and is given by:

$$x_{2u} = \frac{\gamma + r - c - \sqrt{(\gamma + r - c)^2 - 2p(2r - c)}}{2r - c}.$$

In order for x_{2u} to be an equilibrium, it must be in the interval $(\gamma/r, 1]$. It can be easily verified (under the assumptions already made for parameters c, γ and r) that $x_{2u} \leq 1$ always holds if the root exists. For x_{2u} to be greater than γ/r we need:

$$\sqrt{(\gamma + r - c)^2 - 2p(2r - c)} < (r - \gamma)\frac{r - c}{r}$$

For this equation to hold, it is necessary that $r - c > 0$. If this is the case we then need

$$(2r\gamma - \gamma^2)c^2 + (-6\gamma r^2 + 2\gamma^2 r + 2pr^2)c + 4r^3(\gamma - p) < 0 \quad (57)$$

which holds between the roots of the corresponding quadratic equation, which are given by:

$$\begin{aligned}c &= \frac{(3\gamma r^2 - \gamma^2 r - pr^2) \mp (\gamma r^2 - \gamma^2 r + pr^2)}{2r\gamma - \gamma^2}. \\ &= 2r \quad \text{and} \quad \frac{2r^2(p - \gamma)}{2r\gamma - \gamma^2}\end{aligned} \quad (58)$$

This implies that x_{2u} is an equilibrium in $(\gamma/r, 1]$ if both $\frac{2r^2(\gamma - p)}{2r\gamma - \gamma^2} < c < r$ and $c < -p + \gamma + r - \sqrt{p^2 + 2p(r - \gamma)}$, where we have taken into consideration the fact that when $c < r$ the inequality $c > -p + \gamma + r + \sqrt{p^2 + 2p(r - \gamma)}$ cannot hold. These criteria correspond to Regions 5, 8 and 7 in Fig. 10 with the functional expressions of the different curves given in Table II.

When these conditions are satisfied, x_{2u} can be shown to be an unstable equilibrium. This is because $H_h(x) < x$ when $x < x_{2u}$ (adoption levels keep decreasing once they have dropped below x_{2u}), and $H_h(x) > x$ when $x > x_{2u}$ (adoption levels keep increasing once they have exceeded x_{2u}).

2) Intersection x_{2s}: Intersection x_{2s} is the larger of the two roots of Eq. (56) and is given by

$$x_{2s} = \frac{\gamma + r - c + \sqrt{(\gamma + r - c)^2 - 2p(2r - c)}}{2r - c}. \quad (59)$$

Again, for x_{2s} to be an equilibrium, it must be greater than γ/r. Note that $x_{2s} < 1$ is not necessary, since a x_{2s} value that is larger than 1 will be projected down to the boundary point $x = 1$. For $x_{2s} > \gamma/r$ we need:

$$-(r - \gamma)\frac{r - c}{r} < \sqrt{(\gamma + r - c)^2 - 2p(2r - c)}.$$

This always holds if $r - c > 0$. When $c > r$, the condition becomes

$$(2r\gamma - \gamma^2)c^2 + (-6\gamma r^2 + 2\gamma^2 r + 2pr^2)c + 4r^3(\gamma - p) > 0$$

which is the symmetric of the inequality in Eq. (57), and thus it holds for values of c outside the roots of the corresponding quadratic equation. This condition reduces to $c < \frac{2r^2(\gamma-p)}{2r\gamma-\gamma^2}$ (Region 3′ in Fig. 10).

Thus x_{2s} results in an equilibrium if (p, c) is in any of the Regions 3, 3′, 5, 6, 7 and 8 of Fig. 10.

When (p, c) is in any of the Regions 3, 5, 6, 7 and 8, then x_{2s} can be shown to be a stable equilibrium. This is because $x_{2s} > H_h(x) > x$ (adoption increases towards x_{2s} in the next iteration), and $x_{2s} < H_h(x) < x$ (adoption decreases towards x_{2s} in the next iteration). On the other hand if (p, c) is in the Region 3′, then x_{2s} is an "orbital" equilibrium. An orbital equilibrium may have a non-empty region of attraction[22], but exhibit cyclical adoption patterns (periodic or chaotic) outside of that neighborhood. Orbital behaviors arise when $H_h(x) > x_{2s} > x$ (adoption increases beyond x_{2s} in the next iteration), and $H_h(x) < x_{2s} < x$ (adoption drops below x_{2s} in the next iteration). This gives rise to cyclical trajectories, which may or may not converge to x_{2s} depending on the slope of $H_{2h}(x)$ at $x = x_{2s}$ and the initial distance between x and x_{2s}.

To summarize, as for the equilibria in $[\gamma/r, 1]$, when $c < \frac{2r^2(\gamma-p)}{2r\gamma-\gamma^2}$, the root x_{2s} is the only equilibrium in $(\gamma/r, 1]$ (Regions 3′, 3 and 6 in Fig. 10). When $c > \frac{2r^2(\gamma-p)}{2r\gamma-\gamma^2}$, both x_{2s} and x_{2u} equilibria will exist if the condition $c < \min(r, -p+\gamma+r-\sqrt{p^2+2p(r-\gamma)})$ is also satisfied (Regions 5, 7 and 8 in Fig. 10). Otherwise, no equilibrium is present in $(\gamma/r, 1]$ (Regions 1, 2′, 2 and 4 in Fig. 10)

Appendix F
Model perturbations for robustness testing

Our original models make specific assumptions with regards to the magnitude and range of various parameters, functional expressions of the user utilities, and the extent to which information is considered to be known to the service provider. In order to gauge how much these assumptions affect the models' results and more importantly findings, as well as determine how robust the findings are to variations in those assumptions, we consider a series of perturbations to the original models that relax/modify one or more of those specific assumptions.

In this section, we describe perturbations that directly affect the parameters and functional expressions of the models. All scenarios are investigated by means of numerical simulations, and the results are presented in Appendix G (See Appendix I for one example of analytical generalization). Appendix G also evaluates the impact of another type of perturbations, namely, that of errors in estimates of the model's parameters on the part of the service provider. Overall, the results demonstrate that the paper's main findings are relatively robust to a wide range of perturbations.

[22]A neighborhood of x_{2s} so that for values of x in that neighborhood, trajectories converge to x_{2s}.

A. User propensity to roam θ

Our original models assume that users' propensity to roam, θ, follows a uniform distribution, *i.e.,* it is uniformly distributed in [0, 1]:

$$f(\theta) = 1, \qquad 0 \le \theta \le 1.$$

We introduce a perturbation to that assumption by considering different probability distributions for the roaming variable θ. There are obviously many possible distributions to choose from; we consider two representative examples, one with a higher density of sedentary users, and the other with a higher density of roaming users. These two choices cover the effect of both overestimating and underestimating roaming patterns. We present next the details of these two distributions.

The distributions are truncated and modified versions of an exponential distribution, and their density functions are plotted in Fig. 12. The low-mode distribution with a mode at $x = 0$ has a density function $f_{\text{Low-Mode}}(x; \lambda) = \frac{\lambda}{1-e^{-\lambda}} e^{-\lambda x}$, $0 \le x \le 1$.). Conversely the high-mode distribution with a mode at $x = 1$ has a density function $f_{\text{High-Mode}}(x; \lambda) = \frac{\lambda}{e^{\lambda}-1} e^{\lambda x}$, $0 \le x \le 1$.). The parameter λ is taken to be 1.5.

As mentioned earlier, Appendix G presents the results on how these perturbations affect the paper's findings.

B. Modified user utility functions

The original model assumes a specific functional expression for users' utility that grows linearly with coverage κ (as measured[23] by x) and decreases linearly with the volume of roaming traffic m.

We first relax the linear dependency assumption, and then consider two different utility functions inspired by the *Web Browsing Model* and the *File Transfer Model* of [10]. As before, Appendix G presents the results of this investigation.

The original utility function is stated in Eq. (5), which we restate below for convenience.

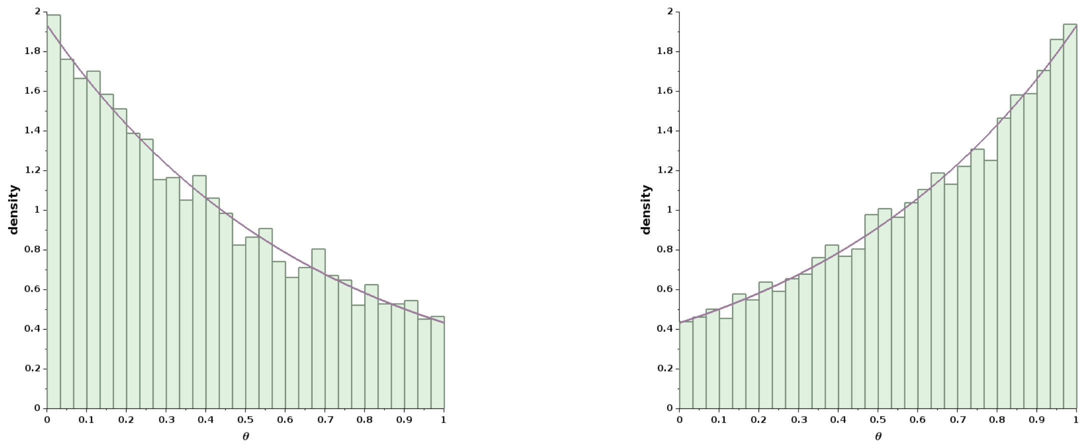
$$U(\Theta, \theta) = \gamma - c\,m + \theta\,(r\,x - \gamma) - p(\Theta, \theta)$$

1) Non-linear utility function: In order to relax the linear dependency assumption, we consider the following "perturbed" utility function:

$$U(\Theta, \theta) = \gamma - c\,m^{1.2} + \theta\,(r\,x^{0.8} - \gamma) - p(\Theta, \theta)\,.$$

The non-linear terms $m^{1.2}$ and $x^{0.8}$ are arguably only one of many possible types of non-linearities, but they

[23]As mentioned before, in Section F-D we do numerically consider scenarios where coverage κ is not equal to x and instead saturates as x grows).



(a) Low-mode: Truncated exponential distribution with parameter $\lambda = 1.5$. (high concentration of sedentary users.)

(b) High-mode: Inverted truncated exponential distribution with parameter $\lambda = 1.5$. (high concentration of roaming users.)

Fig. 12. Density functions and sample realizations for for non-uniform θ distributions.

offer a reasonable evaluation of the effect of non-linearities.

Next we introduce two different utility functions inspired by the models of [10].

2) Upper-bounded roaming: The *Web Browsing Model* from [10] considers a utility that increases with the connection duration, as long as the connection duration is not longer than an upper-bound τ (which is the duration that a user intends to browse the web).

In the context of [10] the connection duration is the main contributor to a user's utility, while in our model the roaming frequency θ determines the rate at which a user accesses the higher-valued roaming connectivity. Therefore the connection duration of [10] readily maps to roaming frequency in our model.

Hence, in order to emulate the Web Browsing Model from [10], we modify our original utility function to upper-bound the roaming frequency of the users. In a manner similar to Eq. (1) of [10] which includes a term $\min(T, \tau)$, we replace the roaming factor θ with $\min(\theta, \tau)$.

The new utility function is then given by

$$U(\Theta, \theta) = \gamma - c\,m + \min(\theta, \tau) \cdot (r\,x - \gamma) - p[\Theta, \min(\theta, \tau)], \tag{60}$$

where $0 < \tau < 1$. In the numerical tests of Section G we take $\tau = 0.8$.

3) Minimum useful coverage: The *File Transfer Model* from [10] considers a utility function with a threshold behavior, *i.e.*, it yields zero value when the connection duration is too short to download a file. Therefore the connection duration has to be longer than a certain threshold to yield a positive utility.

As mentioned before, in our context, users' utility is directly related to the ability to connect while roaming. Therefore, to emulate the File Transfer Model from [10], we modify our utility function to implement a threshold behavior based on roaming connectivity. Namely, a user experiences zero roaming utility, unless the odds of roaming connectivity are above a certain threshold, or equivalently, the system's coverage κ is above a threshold κ_{th}.

The new utility function is then

$$U(\Theta, \theta) = \gamma - c\,m + \theta\,(r\,\widehat{\kappa} - \gamma) - p(\Theta, \theta) \tag{61}$$

where $\widehat{\kappa}$ is the *perceived level* of coverage and is given by

$$\widehat{\kappa} = \begin{cases} 0 & \text{if } x < \kappa_{th}, \\ x & \text{if } x \geq \kappa_{th}. \end{cases}$$

The threshold κ_{th} satisfies $0 < \kappa_{th} < 1$. In the numerical tests of Section G we use $\kappa_{th} = 0.2$.

C. Heterogeneous population

In the original models, users are assumed to all have the same utility function, and share a common profile in how much traffic they generate, including while roaming. We relax those assumptions by considering a scenario where users belong to two types with different "profiles." The type of a user, T_1 or T_2, affects that user's utility and the volume of roaming traffic she generates as a function of her roaming parameter θ.

Users are randomly assigned a given type, so that the user population is divided into two groups of identical size. The utility functions of users of type T_1 and type T_2 are then given by:

$$U(\Theta, \theta) = \begin{cases} \gamma - c\,m + \theta\,(r\,x - \gamma) - p(\Theta, \theta) & \text{for } T_1 \text{ users} \\ 1.1\,\gamma - c\,m + \theta\,(0.9\,r\,x - 1.1\,\gamma) - p(\Theta, \theta) & \text{for } T_2 \text{ users.} \end{cases}$$

In other words, users of type T_2 exhibit a difference of 10% with type T_1 users in how much more (less) they value home (roaming) connectivity (they have a larger γ and smaller r). Moreover, a user's type also affects the volume of traffic she generates while roaming, as follows

$$\text{Contribution to roaming traffic} = \begin{cases} \theta & \text{for } T_1 \text{ users} \\ \theta^{0.7} & \text{for } T_2 \text{ users,} \end{cases}$$

In other words, given two users of types T_1 and T_2 with the same roaming parameter θ, the user of type T_2 generates more roaming traffic while roaming (since $\theta^{0.7} > \theta$ for $\theta \in [0, 1]$). As mentioned earlier, this can account for differences induced by the type of equipment each type of users uses (*e.g.,* tablet vs. smartphone). The overall roaming traffic m is then given by

$$m = \int_{T_1} \theta f(\theta)\, d\theta + \int_{T_2} \theta^{0.7} f(\theta)\, d\theta\,.$$

Results are again presented in Appendix G.

D. Coverage saturation

The original models assume that coverage κ increases linearly with the level x of service adoption. In particular, we assume that $\kappa = x$. In this section, we relax this assumption and consider a *saturation* effect for coverage. This means that while coverage initially expands in proportion to the adoption level x, its growth slows down ("levels off") as x grows large. In order to capture this effect, we assume a relation between coverage and adoption of the form $\kappa = \sin(\frac{\pi}{2}x)$ (see Fig. 13). Results illustrating how this difference in the evolution of coverage affects the paper's conclusions are again in Appendix G.

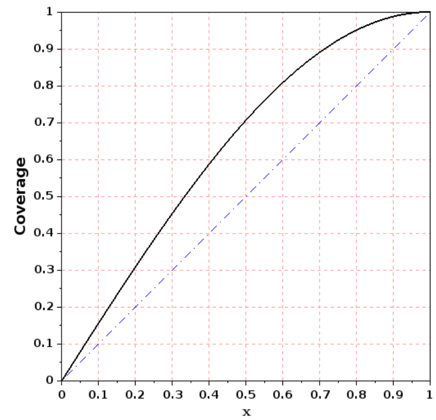


Fig. 13. Coverage saturates as adoption x grows large.

Appendix G
Numerical simulations

Appendix F introduced a series of perturbations to our original models. In this Appendix, we report on the results of numerical simulations used to investigate the impact of those perturbations. The results demonstrate that the paper's findings are robust with regards to those perturbations and errors in the modelling assumptions.

Recall that the paper's main findings belong to two broad categories. The first is concerned with the system's ability to create value, *i.e.,* the total system *welfare*. They establish that when the system is capable of creating positive value, the maximum of that value is often realized at full adoption. The second category of findings is concerned with *realizing* that potential: how to use pricing schemes to realize the optimal adoption level and the corresponding total welfare, as well as distribute the total welfare between the users and the provider.

In testing the robustness of that second group of findings, *i.e.,* those regarding pricing schemes, it is important to specify how much knowledge the provider has about potential discrepancies between the model it is using to determine (optimal) prices, and the actual model and its parameters. This is because that knowledge will affect the provider's ability to set prices that realize its goals. Therefore, throughout this section, when presenting results related to pricing policies, we also specify the extent to which the provider is aware of the perturbations.

For purposes of clarity, we consider each one of the perturbations of Appendix F in isolation, *i.e.,* we perturb one aspect of the model while keeping others intact, and report on its impact on the paper's findings. We discuss first how different perturbations affect the paper's main conclusions regarding total system welfare.

A. Optimal total welfare

The paper's main finding when it comes to total system welfare was that total welfare (value) is usually maximized when the adoption level is either $x = 1$ or $x = 0$. In other words, whenever the system is capable of generating positive value, this positive value is realized at full adoption $x = 1$.

The result was obtained under the simplifying assumptions of the system's model, but in this section we demonstrate that even under more general conditions, *i.e.,* when various aspects of the original model are perturbed,[24] this finding remains valid.

G-A.(a) Original model

A plot of the optimal adoption level x for maximizing system value was given in the paper for the original model, and is repeated for convenience in Fig. 14a. The figure indeed shows that for most values of parameters γ and e, the optimal adoption level is either $x = 1$ or $x = 0$. An optimal adoption level of $x = 0$ means that the system cannot create positive value.

G-A.(b, c) Modified roaming distribution

We changed the distribution of the roaming parameter θ as per the description of Section F-A. Under this perturbation, Figs. 14b and 14c demonstrate that the maximum total welfare is again mostly achieved at either $x = 1$ or $x = 0$.

Other remarks

Figs. 14b and 14c identify the region in the $\gamma - e$ plane where a positive total welfare is possible. The regions in the two figures are slightly different: For large values of home connectivity utility γ, the system with more sedentary users (Fig. 14b) can tolerate a larger deployment cost e while still yielding a positive value. For instance when $\gamma = 2$, the system with more sedentary users (Fig. 14b) allows $e \lessapprox 1.6$, whereas the system with a large roaming population (Fig. 14c) allows only $e \lessapprox 1.4$. This is intuitive as a higher population of sedentary users means more people will enjoy the high home connectivity utility.

On the other hand, for small values of γ the roles are reversed. The system with more roaming users (Fig. 14c) can tolerate a larger deployment cost e while still yielding a positive value. For instance when $\gamma = 0$, the system with more sedentary users (Fig. 14b) allows $e \lessapprox 0.4$; however, the system with a large roaming population (Fig. 14c) is understandably less affected by the small home connectivity utility γ, and allows $e \lessapprox 0.6$.

G-A.(d) Non-linear utility functions

We now consider the effect of non-linearities in users' utility functions using the utility function introduced in Section F-B1. The resulting optimal adoption level for maximizing total welfare is given in Fig. 14d. It shows that the maximum total welfare continues to be achieved mostly at either $x = 1$ or $x = 0$.

G-A.(e) Utility function with upper-bounded roaming

We used the new utility function given in Section F-B2 with an upper-bound value of $\tau = 0.8$. Under this new utility function, Fig. 14e displays the optimal adoption level x. Although the figure exhibits small differences with Fig. 14a, it shows that the maximum total welfare continues to be achieved mostly at either $x = 1$ or $x = 0$.

[24] Note that because total welfare is only concerned with the system's overall value and not how to realize it, the extent to which the service provider is aware of any discrepancies between the model and the actual system has no impact.

G-A.(c) Utility function with minimum useful coverage

We use the new utility function of Section F-B3 with a threshold value of $\kappa_{th} = 0.2$. Under this new utility function, Fig. 14f demonstrates that the maximum total welfare is again mostly achieved at either $x = 1$ or $x = 0$, in a manner very similar to Fig. 14a.

Fig. 14f is almost identical to Fig. 14a, because the values of optimal adoption x in Fig. 14a mostly correspond to a coverage level that is already above the coverage threshold κ_{th}, and therefore are not affected by imposing the criterion of minimum useful coverage in Fig. 14f. Therefore the regions of Fig. 14a where the optimal adoption is at $x = 0$ or $x > \kappa_{th}$ are exactly replicated in Fig. 14f. This constitutes most of the points in the figure.

G-A.(g) Heterogeneous population

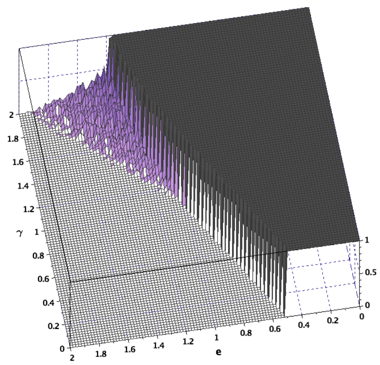
In this section, we consider the effect of a heterogeneous user population, as per the two-type user population of Section F-C. Recall that a user's type affects both her utility function and the roaming traffic she generates. Fig. 14g reports the adoption levels associated with maximum welfare for such a configuration. It again shows that the maximum total welfare is usually achieved at either $x = 1$ or $x = 0$.

G-A.(h) Coverage saturation with adoption

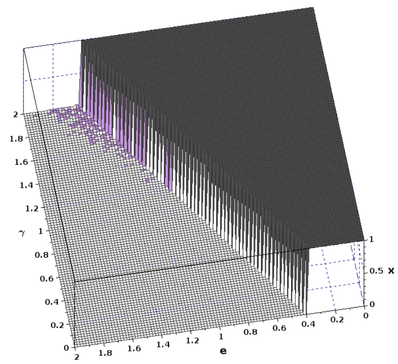
The last perturbation we consider involves a scenario where coverage saturates as the system approaches full adoption $x = 1$ (as described in section F-D). The results are shown in Fig. 14h.

Fig. 14h highlights some minor differences with the paper's original findings of Fig. 14a. Specifically, while maximum total welfare is still often achieved at either $x = 1$ or $x = 0$, an intermediate region has emerged for which the optimal adoption level, while still high and close to 1, is nevertheless slightly lower. The difference is small and quite intuitive, as we explain next.

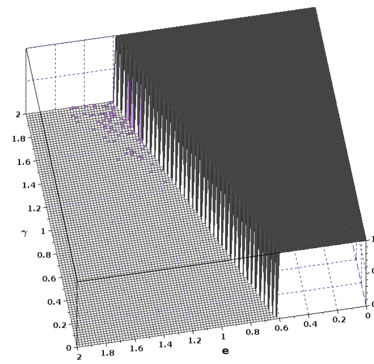
Recall the two effects of increasing adoption. On one hand, an increase in adoption improves total welfare, both because it improves coverage, which favorably affects the utility of all users, and because the new users themselves contribute to the total welfare. On the other hand, more users means more roaming traffic, which adversely affects all users' utility and, therefore, welfare. The combined contributions of these opposing effects determines whether higher adoption increases or decreases total welfare. When coverage saturates earlier, new users still contribute to the system welfare, but their impact on improving coverage is now diminished while the negative contribution of their roaming traffic is unchanged. Hence, it is to be expected that under a model where coverage saturates before full adoption, maximum welfare may be realized slightly below full adoption as seen in the "blue" region of Fig. 14h. We highlight this dependency in the paper, when discussing



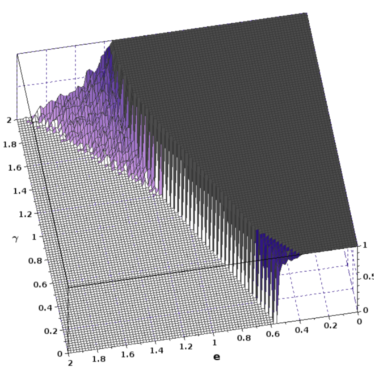
(a) Original model



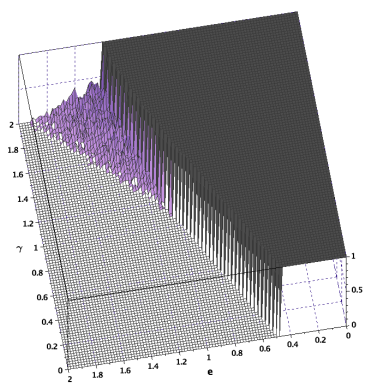
(b) Low-mode θ distribution with parameter $\lambda = 1.5$.



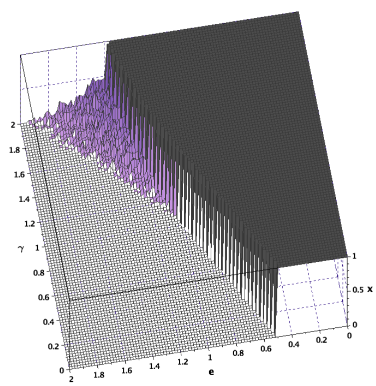
(c) High-mode θ distribution with parameter $\lambda = 1.5$.



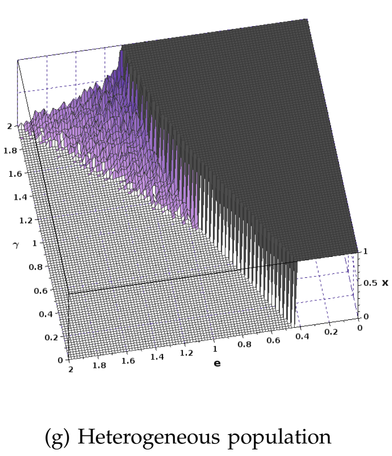
(d) Non-linear utility function



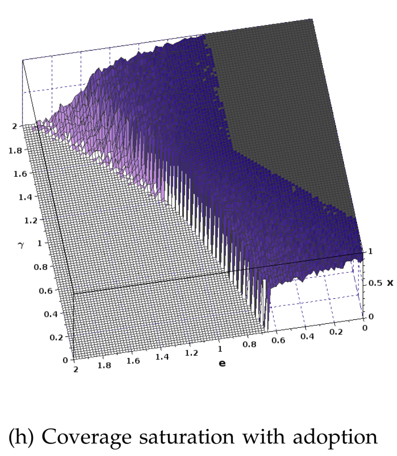
(e) Upper-bounded roaming ($\tau = 0.8$)



(f) Minimum useful coverage ($\kappa_{th} = 0.2$)



(g) Heterogeneous population



(h) Coverage saturation with adoption

Fig. 14. Values of optimal adoption x for maximum total welfare under different perturbations. Parameters are $r = 1.6$ and $c = 0.6$ (and therefore $r - c = 1$).

the result that total welfare is typically achieved at full adoption.

B. Usage-based pricing

Under the original model, we concluded that for the usage-based pricing policy, full adoption $x = 1$ is the unique equilibrium of the system if and only if the usage allowance a is larger than a threshold value.

In this section we demonstrate that even under more general conditions, *i.e.,* when various aspects of the original model are perturbed[25], this finding remains valid.

Throughout the simulations of this section, we fix the parameters $c = 0.8$, $\gamma = 1$, $r = 1.6$ and find the final adoption level that the system converges to, as the value a of usage allowance varies. By observing the final adoption level we can determine whether $x = 1$ is the unique equilibrium of the system. The details of the simulations are as follows: At each value of a, we start the system from zero adoption. After each iteration in the simulation, users evaluate their utility and those with a positive utility adopt. The simulation stops once consecutive iterations yield the same set of adopters. At this point the final adoption level is recorded.

G-B.(a) Original model

Under the paper's original model, full adoption $x = 1$ is the unique equilibrium if and only if the value of usage allowance satisfies $a > c/2$ (Proposition 2). This is illustrated in Fig. 15a which shows the values of a for which full adoption $x = 1$ is the unique equilibrium (recall that $c/2 = 0.4$). The figure shows that there exists a threshold value a_0 such that for $a > a_0$, full adoption $x = 1$ is the unique equilibrium of the system, and for $a < a_0$, full adoption is not an equilibrium.

G-B.(b, c) Modified roaming distribution

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the modified θ distribution and assumes the θ distribution is still uniform.

The roaming distribution is modified as per the description of Section F-A. We see from Figs. 15b and 15c that under the two new roaming distributions of Section F-A (low and high mode), the outcome is similar to that of the original model, *i.e.,* there exists a threshold value such that for values of a above it $x = 1$ is the unique equilibrium, and for values of a below it, $x = 1$ is not an equilibrium.

[25] Unlike section G-A that only dealt with maximizing the system value, this section and all subsequent ones are concerned with pricing the service. Prices are set by the provider, and as a result the information available to the provider about the system's characteristics is important. In the remainder, we therefore mention not only perturbations to the original model, but also the provider's knowledge of those perturbations.

G-B.(d) Non-linear utility function

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the non-linearity of the utility function and assumes the original function is valid.

We now consider the effect of non-linearities in the utility function, as discussed in Section F-B1. The outcome is shown, again as a function of the usage allowance a, in Fig. 15d, which exhibits a similar pattern as Fig. 15a, *i.e.,* there exists a threshold value such that for values of a above it $x = 1$ is the unique equilibrium of the system and for values of a below it, $x = 1$ is not an equilibrium.

G-B.(e) Utility function with upper-bounded roaming

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the modified utility function and assumes the original function is valid.

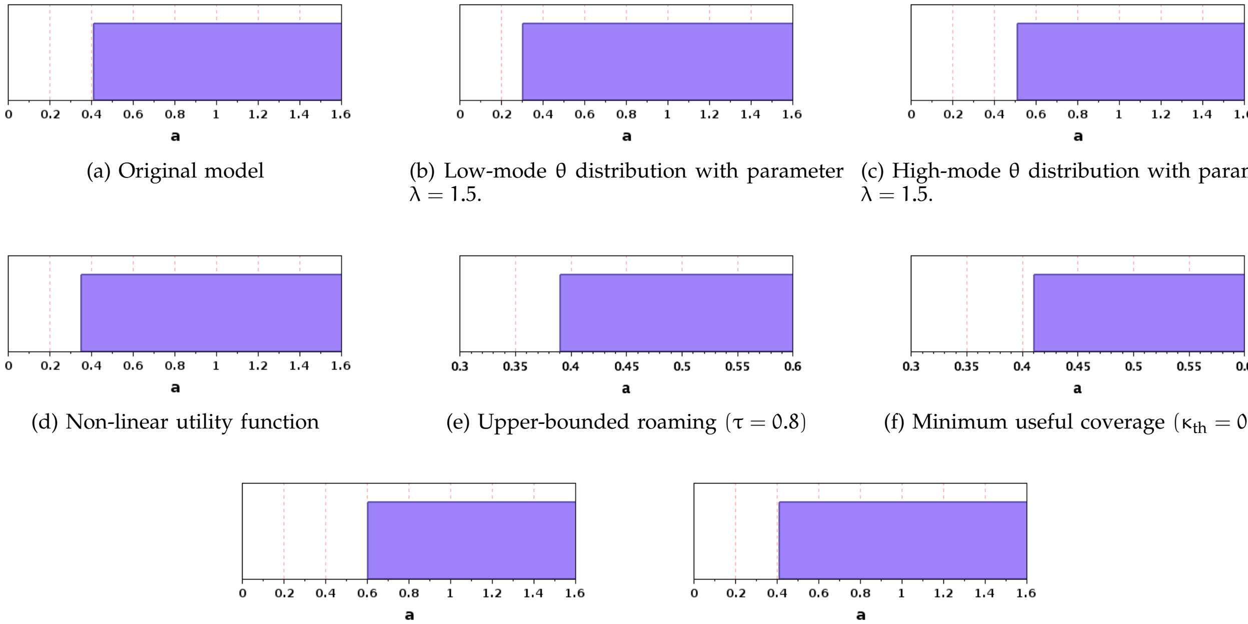
We use the new utility function of Section F-B2 with an upper-bound value of $\tau = 0.8$. We see from Fig. 15e that under this new utility function, the outcome is similar to that of the original model, *i.e.,* there exists a threshold value, albeit a different one, such that for values of a above it $x = 1$ is the unique equilibrium, and for values of a below it, $x = 1$ is not an equilibrium.

G-B.(f) Utility function with minimum useful coverage

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the modified utility function and assumes the original function is valid.

As before, we use the new utility function of Section F-B3 with a coverage threshold of $\kappa_{th} = 0.2$. We see from Fig. 15f that under this new utility function, the outcome is very similar to that of the original model, *i.e.,* there exists a threshold in the values of usage allowance a, such that for values of a above it $x = 1$ is the unique equilibrium, and for values of a below it, $x = 1$ is not an equilibrium.

In fact, the allowance threshold value in Fig. 15f is identical to that of the original model in Fig. 15a. This is because, as shown in Appendix H, the outcome of the usage-based pricing is very robust to this change in the utility function. Nevertheless, differences in the outcome would naturally arise under more drastic changes, *i.e.,* by



(a) Original model

(b) Low-mode θ distribution with parameter $\lambda = 1.5$.

(c) High-mode θ distribution with parameter $\lambda = 1.5$.

(d) Non-linear utility function

(e) Upper-bounded roaming ($\tau = 0.8$)

(f) Minimum useful coverage ($\kappa_{th} = 0.2$)

(g) Heterogeneous population

(h) Coverage saturation with adoption

Fig. 15. Usage-based pricing policy: values of usage allowance a for which full adoption $x = 1$ is the (unique) equilibrium of the system, under different perturbations. Parameters are $c = 0.8$, $\gamma = 1$, $r = 1.6$.

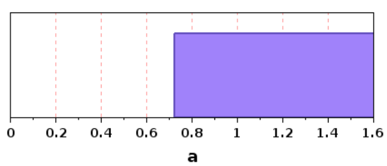


Fig. 16. Results for more drastic changes in the minimum useful coverage ($\kappa_{th} = 0.6$). Values of usage allowance a in the usage-based pricing policy for which full adoption $x = 1$ is the (unique) equilibrium of the system. Parameters are $c = 0.8$, $\gamma = 1$, $r = 1.6$.

considering significantly larger values for the coverage threshold.

For instance, as the value for the coverage threshold κ_{th} is changed to $\kappa_{th} = 0.6$ (roaming users do not consider the system valuable until coverage exceeds 60%), differences appear in the adoption outcomes. This is shown in Fig. 16. Nevertheless, the figure also shows that even under this more drastic change, the overall behavior remains consistent with that of the original model.

G-B.(g) Heterogeneous population

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about users of type 2 and assumes that everyone is a type 1 user.

This scenario assumes that the users' population is heterogeneous and split into two sub-populations of different type, as described in Section F-C. Fig. 15g reports the results, which are again consistent with those of the original model, *i.e.*, there exists a threshold value such that for values of a above it $x = 1$ is the unique equilibrium of the system and for values of a below it, $x = 1$ is not an equilibrium.

G-B.(h) Coverage saturation with adoption

Provider's knowledge of the perturbations:
The provider is not assumed to have any knowledge of the coverage saturation (of course, in practice the provider may be able to estimate coverage, but the simulations do not assume such knowledge).

As with the case of optimal welfare, the last perturbation we consider involves a scenario where coverage saturates as the system approaches full adoption $x = 1$ (as described in section F-D). The results are shown in Fig. 15h, and again yield a similar outcome as in the original model, *i.e.*, there exists a threshold value such that for values of a above it $x = 1$ is the unique equilibrium of the system and for values of a below it, $x = 1$ is not an equilibrium.

We also note that unlike what happened with optimal welfare where optimal adoption could end-up slightly lower than full adoption, the threshold value is unchanged when compared to that of the original model. This is because the usage based pricing (by its nature) does not require knowledge of the actual service coverage by the provider, and is, therefore, insensitive to errors in the coverage level.

C. Hybrid pricing tests with partial provider's knowledge

Under the original model, we concluded that for the hybrid pricing policy, there are values of home connectivity utility γ for which the system has an equilibrium at $x < 1$, which would prevent the system from reaching full adoption, hence resulting in a sub-optimal total welfare. The hybrid pricing policy, however, offers a way to eliminate the lower equilibria and allow the system to reach full adoption. This is possible by adjusting the value of the discount parameters δ_h or δ_r (for simplicity, we focus on adjusting δ_h).

In this section, we demonstrate that even under more general conditions, *i.e.,* when various aspects of the original model are perturbed, the system also exhibits regimes where a sub-optimal equilibrium $(x < 1)$ can arise, thereby preventing the system from reaching full adoption. In addition, overcoming this issue can again be accomplished by adjusting the value of δ_h, albeit typically with a different discount value.

Throughout the simulations of this section, we fix the parameters $c = 0.8$, $\delta_r = 0$ and find the final adoption level, denoted by $x(\infty)$, as we vary γ and δ_h values. The details of the simulations are as follows: At each point (γ, δ_h), we start the system from zero adoption. After each iteration in the simulation, users evaluate their utility and those with a positive utility adopt. The simulation stops once consecutive iterations yield the same set of adopters. At this point the final adoption level is recorded.

Moreover, throughout the simulations, the price parameters of the hybrid policy are computed as:

$$p_h = \gamma - c\,\alpha - \delta_h, \quad \text{and}$$
$$p_r = r - \gamma - \delta_r,$$

where α is the estimate for overall intensity of roaming traffic m at full adoption (for the original model we had $\alpha = 1/2$, which gives $p_h = \gamma - c/2 - \delta_h$). The simulations of this section assume that the provider can accurately estimate the value of α. (We will further eliminate this assumption in section G-D where we assume that the provider has no knowledge of α.)

G-C.(a) Original model

Fig. 17a shows the final adoption level for the hybrid pricing policy under the original model. The figure illustrates the presence of a region of (γ, δ_h) values where the system does not go to full adoption, and shows that by increasing the discount factor δ_h we can avoid that region, hence realizing full adoption.

This can be seen from the three sample points indicated by pins in Fig. 17a. Pin *a* indicates a point where the system reaches full adoption. Pin *b*, on the other hand, is at a point where the system converges to a lower equilibrium and full adoption is not possible. However, by increasing the value of δ_h, we move to Pin *c* where, once again, the system converges to full adoption.

G-C.(b, c) Modified roaming distribution

Provider's knowledge of the perturbations:
The simulations assume that the provider can accurately estimate α (the intensity of roaming traffic m at full adoption). We relax this in section G-D. Other than that, the provider does *not* have any knowledge about the modified θ distribution.

The roaming distribution is modified as per the description of Section F-A. We see from Figs. 17b and 17c that adoption outcomes are similar to those of the original model, *i.e.,* the system exhibits regimes where the final adoption is at a sub-optimal level $x < 1$, and that full adoption can be realized by adjusting the value of δ_h.

As expected, the level of discount δ_h required to realize full adoption is different in Fig. 17b and Fig. 17c, as the exact amount depends on the exact specifications of the system. However, the overall behavior is similar.

G-C.(d) Non-linear utility function

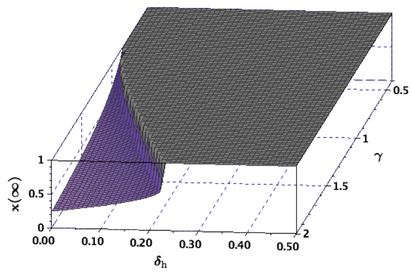
Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the non-linearity of the utility function and assumes the original function is valid.

We now consider the effect of non-linearities in the utility function as introduced in section F-B1. The final adoption level is given in Fig. 17d, which again yields a similar outcome, *i.e.,* the system exhibits regimes where the final adoption is at a sub-optimal level $x < 1$, but full adoption can be realized by adjusting the value of the discount δ_h.

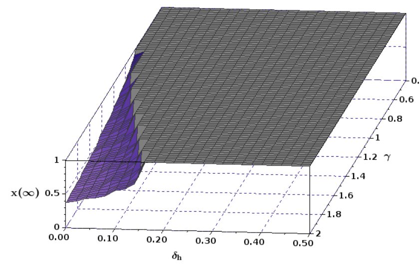
G-C.(e) Utility function with upper-bounded roaming

Provider's knowledge of the perturbations:
The simulations assume that the provider can accurately estimate α (the intensity of roaming traffic m at full adoption). We relax this in section G-D. Other than that, the provider does *not* have any knowledge about the new utility function.

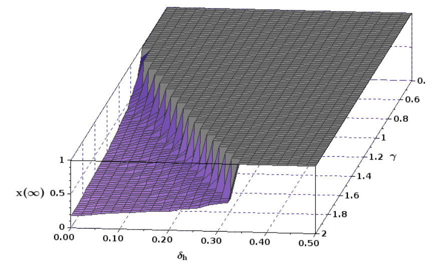
We use the new utility function of Section F-B2 with an upper-bound value of $\tau = 0.8$. We see from Fig. 17e



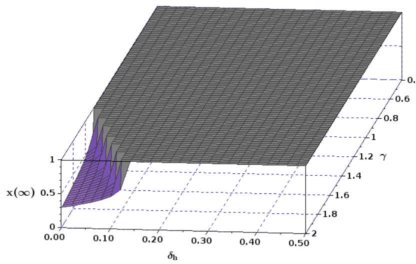
(a) Original model.



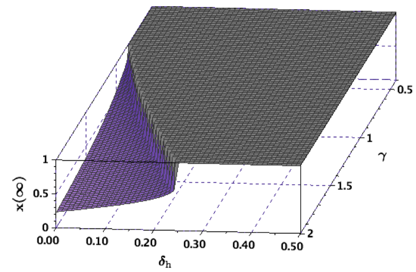
(b) Low-mode θ distribution with parameter $\lambda = 1.5$.



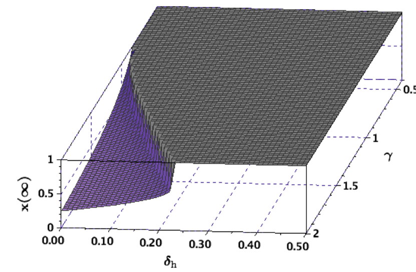
(c) High-mode θ distribution with parameter $\lambda = 1.5$.



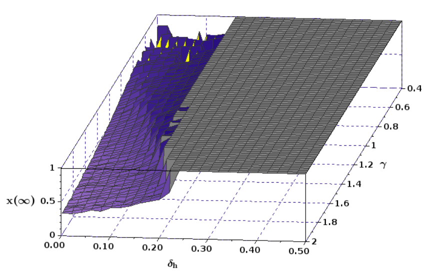
(d) Non-linear utility function



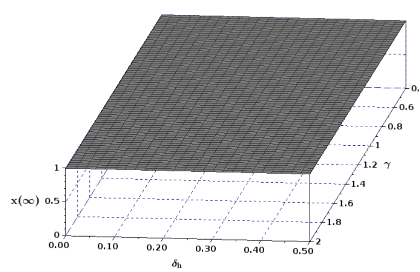
(e) Upper-bounded roaming ($\tau = 0.8$)



(f) Minimum useful coverage ($\kappa_{th} = 0.2$)



(g) Heterogeneous population



(h) Coverage saturation with adoption

Fig. 17. Final adoption level for the hybrid pricing policy under different perturbations. Parameters are $c = 0.8$, $\delta_r = 0$, with γ and δ_h values varying.

that adoption outcomes under this new utility function are very similar to those of the original model, *i.e.,* the system exhibits regimes where the final adoption is at a sub-optimal level $x < 1$, and that full adoption can be realized by adjusting the value of δ_h.

Note that, as mentioned above, the exact values of discount δ_h required to realize full adoption in Fig. 17e, are very close to that of the original model (Fig. 17a). However, greater differences would obviously arise under more drastic changes, *i.e.,* by considering a significantly smaller upper-bound value τ.

For instance, as the value for the upper-bound τ of Section F-B2 is changed to $\tau = 0.15$ (no user roams more than 15% of the time), greater differences arise. This is shown in Fig. 18. Nevertheless, the figure also shows that even under this more drastic change, the overall behavior remains consistent with that of the original model.

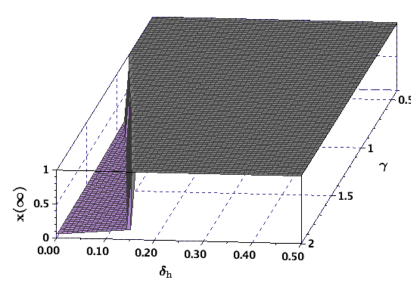


Fig. 18. Result for more drastic changes in the utility function with upper-bounded roaming ($\tau = 0.15$). Compare to Fig. 17e.

G-C.(f) Utility function with minimum useful coverage

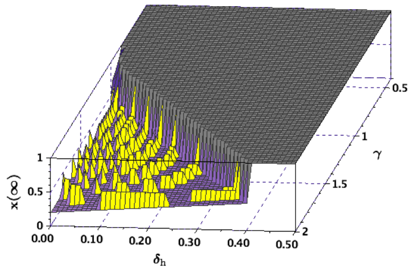


Fig. 19. Result for more drastic changes in the utility function with minimum useful coverage $\kappa_{th} = 0.4$. Compare to Fig. 17f.

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the new utility function.

We use the new utility function of Section F-B3 with a threshold value of $\kappa_{th} = 0.2$. We see from Fig. 17f that adoption outcomes under this new utility function are very similar to those of the original model, *i.e.,* the system exhibits regimes where the final adoption is at a sub-optimal level $x < 1$, and that full adoption can be realized by adjusting the value of δ_h.

Note that, as mentioned above, the exact values of discount δ_h required to realize full adoption in Fig. 17f, are very close to that of the original model (Fig. 17a). However, as seen earlier, greater differences would obviously arise under more drastic changes, *i.e.,* by considering a significantly larger threshold value κ_{th}.

For instance, as the value for the threshold κ_{th} of Section F-B3 is changed to $\kappa_{th} = 0.4$ (roaming users do not consider the system valuable until coverage exceeds 40%), greater differences arise. This is shown in Fig. 19. Nevertheless, the figure also shows that even under this more drastic change, the overall behavior remains consistent[26] with that of the original model.

G-C.(g) Heterogeneous population

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about the users of type T_2 and assumes that everyone is a user of type T_1. But the simulations assume that the provider can accurately estimate α (the intensity of roaming traffic m at full adoption). We relax this in section G-D.

In this section, we consider the effect of a heterogeneous user population, as per the two-type user population of Section F-C. The results are shown in Fig. 17g. We see that again the system exhibits regimes where the final adoption is at a sub-optimal level $x < 1$, but that we can still realize full adoption by adjusting the value of the discount δ_h.

There are, however, unavoidable differences between Fig. 17g and Fig. 17a. Notably, we now need a positive discount ($\delta_h \gtrapprox 0.18$) to reach full adoption at *all* γ values. This is because the provider is totally unaware of the existence of the type T2 users, which introduces relatively big errors in the pricing policy. As a result and because we need to compensate for those large errors, reaching full adoption now requires a bigger discount factor δ_h than before. In general, the larger the errors in the assumptions used to set prices, the bigger the discount "margin" required to compensate for them. Nevertheless, the *structure* of the system remains unchanged.

G-C.(h) Coverage saturation with adoption

Provider's knowledge of the perturbations:
The provider is not assumed to have any knowledge of the coverage saturation (of course, in practice they can measure the coverage if they want to, but our simulations do not assume that knowledge).

As before, we consider a scenario where coverage saturates as the system approaches full adoption $x = 1$ (see Section F-D). The results for this scenario are shown in Fig. 17h that displays a somewhat different structure from the other figures, namely, the system appears to always reach full adoption even with a discount of $\delta_h = 0$. This is, however, not surprising given that at any adoption level the coverage is higher than in the original model (the saturating coverage function has a concave shape). As a result of this higher coverage, more users find the service useful, and hence adopt, eventually resulting in full adoption.

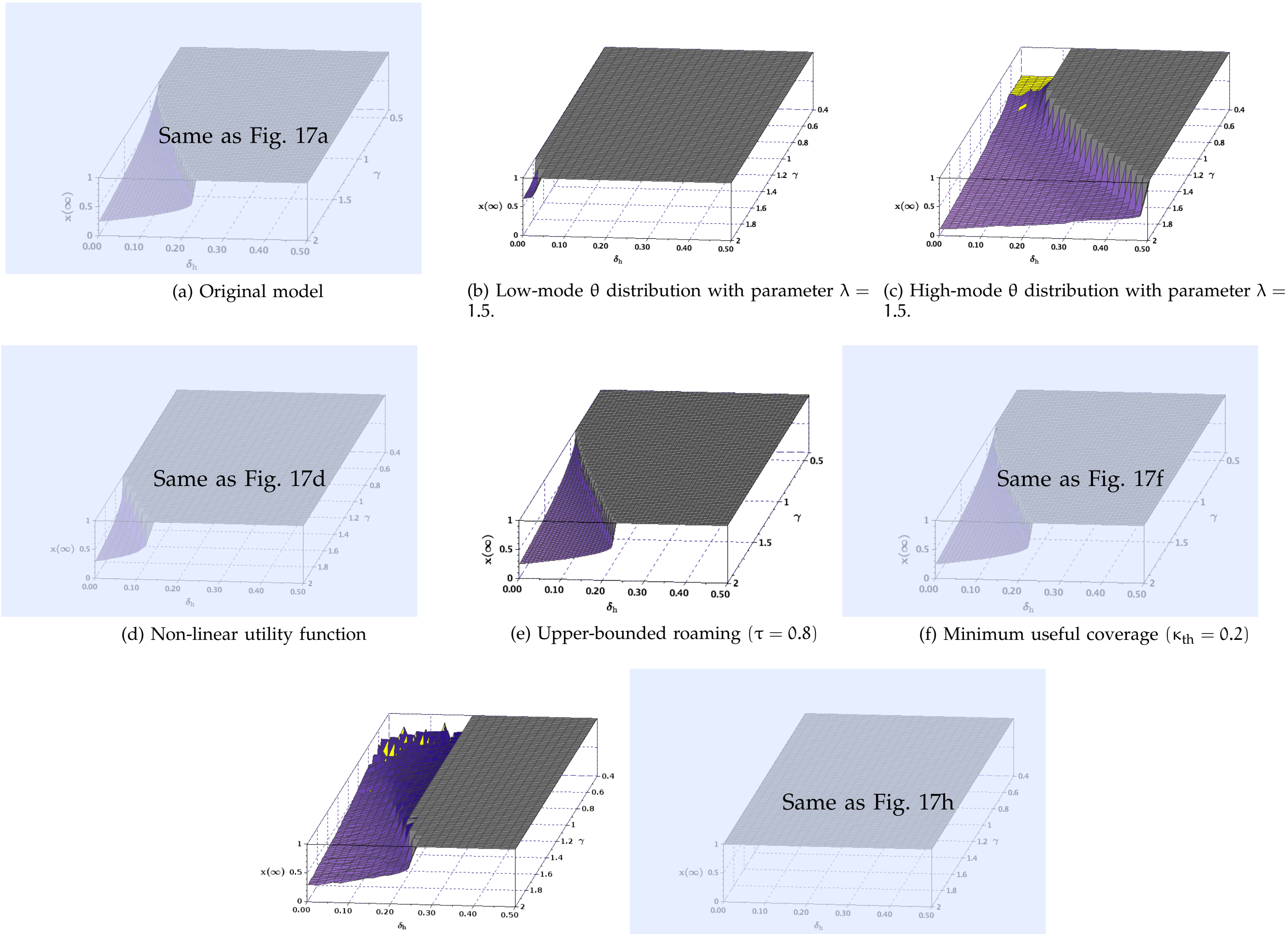
Nonetheless, the paper's analysis can help us understand this result as well. For instance, consider the case of zero discounts, *i.e.,* $\delta_h = \delta_r = 0$. The utility function for each user becomes

$$U(\Theta, \theta) = c\,(\alpha - m) + \theta\,\gamma(\kappa - 1).$$

As before, α is the estimate for the roaming traffic m at full adoption, so that $\alpha - m$ is non-negative. Similarly, because coverage κ is less than or equal to 1, it follows that $(\kappa - 1)$ is negative (or 0). Because coverage saturates earlier, the term $\theta\,\gamma(\kappa - 1)$ is greater than in the original model, hence enticing more roaming users to adopt, therefore facilitating reaching full adoption[27].

[26] The yellow stripes in Fig. 19 correspond to points where the system does not converge to an equilibrium. However, we still have the previous behavior, *i.e.,* as δ_h increases, full adoption becomes the unique equilibrium of the system.

[27] Obviously, a scenario where coverage proceeds more slowly as adoption increases, *i.e.,* a convex rather than concave coverage function, would have the opposite effect.



(a) Original model

(b) Low-mode θ distribution with parameter $\lambda = 1.5$.

(c) High-mode θ distribution with parameter $\lambda = 1.5$.

(d) Non-linear utility function

(e) Upper-bounded roaming $(\tau = 0.8)$

(f) Minimum useful coverage $(\kappa_{th} = 0.2)$

(g) Heterogeneous population

(h) Coverage saturation with adoption

Fig. 20. Final adoption level for the hybrid pricing policy (the provider does *not* know m at full adoption) under different perturbations. Parameters are $c = 0.8$, $\delta_r = 0$, with γ and δ_h values varying.

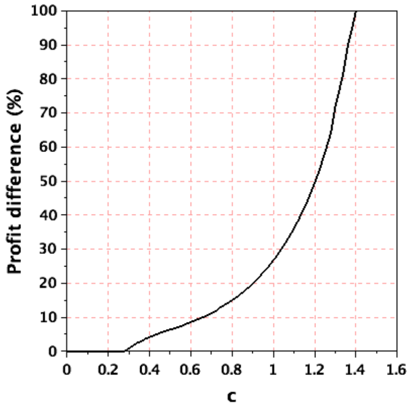
D. *Hybrid pricing tests with zero provider's knowledge*

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about any of the perturbations in this section.

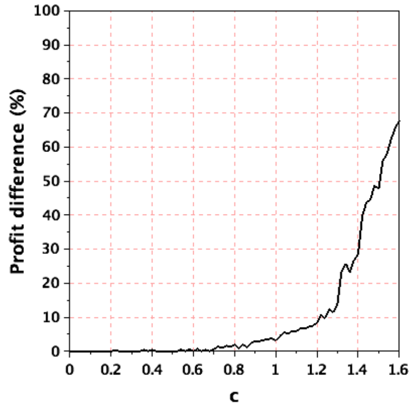
This section presents simulations similar to those of the previous section, with the difference that we assume that the provider has *no knowledge* of the system's parameters. Specifically, we *relax* the assumption that the provider can accurately estimate the actual level of roaming traffic m generated at full adoption. The results are given in Fig. 20 that parallels Fig. 17.

Note that Figures. 20d, 20f and 20h are identical to their counterparts in Fig. 17. The reason is that the perturbations associated with the scenarios of those three figures do not alter the value of m at full adoption. Hence, the provider still estimates the correct value for m. The same does not hold for the other scenarios and Figures (b), (c), (e) and (g) differ from their counterparts in Fig. 17. However, in spite of those differences, they exhibit similar overall behaviors, *i.e.*, they display regimes where a sub-optimal equilibrium $x < 1$ arises, but full adoption can still be realized by adjusting the value of the discount δ_h.

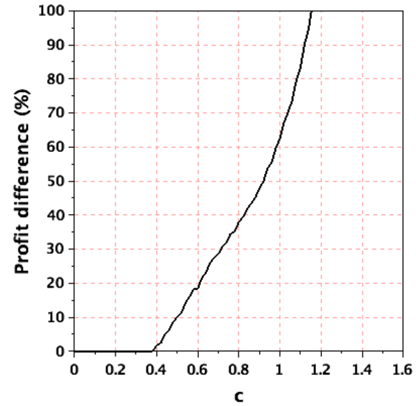
The differences between Fig. 20 and Fig. 17 are not surprising, as the perturbations now result in more severe errors in the pricing policy, due to the complete lack of insight by the provider about the system. These



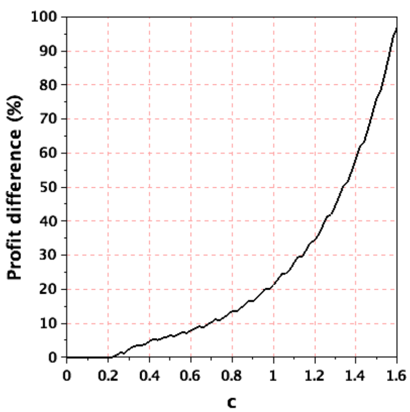
(a) Original model



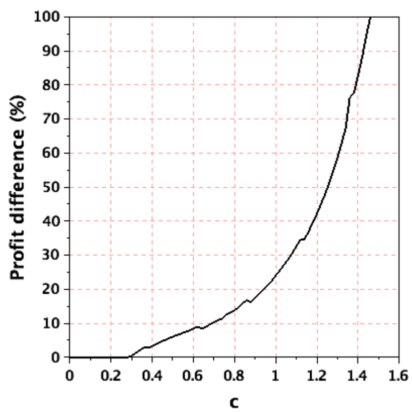
(b) Low-mode θ distribution with parameter $\lambda = 1.5$.



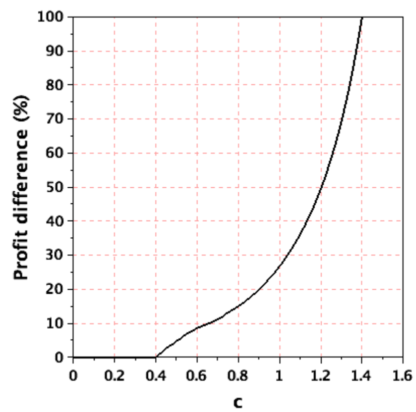
(c) High-mode θ distribution with parameter $\lambda = 1.5$.



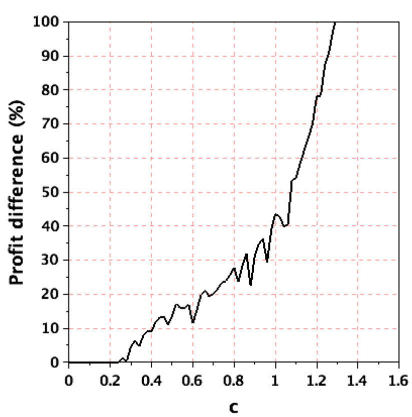
(d) Non-linear utility function



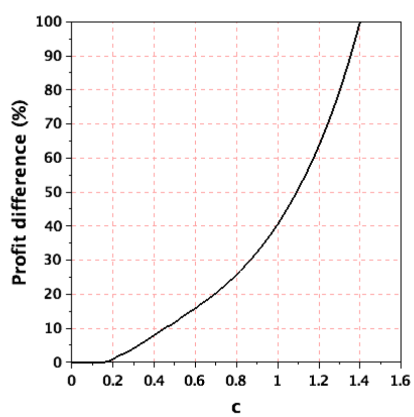
(e) Upper-bounded roaming ($\tau = 0.8$)



(f) Minimum useful coverage ($\kappa_{th} = 0.2$)



(g) Heterogeneous population



(h) Coverage saturation with adoption

Fig. 21. Relative profit drop from profit maximization to welfare maximization (fixed-price policy $\gamma = 1, r = 2$ and $e = 0.3$).

larger errors work in favor of full adoption in (b) and (e), and against it in (c) and (g). As expected, differences in errors result in different necessary discount values, even if the overall pattern and structure are preserved.

E. Fixed price policy

Provider's knowledge of the perturbations:
The provider does *not* have any knowledge about any of the perturbations in this section.

Under the original model and the fixed price policy, a profit maximizing strategy would often differ from a welfare maximizing one. In the paper, we quantified this gap by comparing the overall profit under both types of strategies. The gap was small when the parameter c was small, but grew large as c increased.

In this section, we demonstrate that even under more general conditions, *i.e.,* when various aspects of the original model are perturbed, this finding remains valid.

Throughout the simulations of this section, we fix the parameters $\gamma = 1$, $r = 2$ and $e = 0.3$ and consider a range of c values. The details of the simulations are as follows: At each point, we iterate over different values of p to find the price p^* that maximizes the provider's profit with no constraint, as well as the price $\widehat{p}$ that maximizes provider's profit with the constraint that the total welfare is also maximized. We denote the corresponding values of maximum profit by W_P^* and $\widehat{W_P}$, respectively. We then compute the relative profit drop from profit maximization to welfare maximization as

$$\text{Profit difference} = \frac{W_P^* - \widehat{W_P}}{W_P^*} \times 100\%.$$

Fig. 21 compares the resulting profit drops for both the original model and the seven different perturbations introduced in Appendix F. The figure illustrates that the overall behavior is similar across all scenarios, *i.e.,* there is no profit difference for small values of c, but the gap increases rapidly as c increases beyond some moderate threshold value.

Appendix H
Usage-based pricing and utility functions with minimum useful coverage

In this section we analyze user adoption under the usage-based pricing policy and the utility function with *minimum useful coverage* rule.

Putting the usage-based price function of Eq. (16) into the utility function of Eq. (61), the utility for user θ is found as

$$U(\Theta,\,\theta) = \begin{cases} a - cm - r\theta x & \text{if } x < \kappa_{th}, \\ a - cm & \text{if } x \geq \kappa_{th}. \end{cases} \quad (62)$$

In order to analyze the adoption dynamics in this case, we assume that at each "decision time", only the most "eager" of the users adopts (or disadopts) the service. Such a "diffusion-like" adoption mechanism prevents artifacts such as sudden oscillation in the adoption level for the current case.

We first note that by Eq. (62), at any adoption level x, the users with smaller roaming frequency θ have higher utility. Therefore, the adoption interval is always of the form $[0,\,x]$, and consequently $m = x^2/2$. Therefore the utility function of Eq. (62) becomes

$$U(\Theta,\,\theta) = \begin{cases} a - c\,\frac{x^2}{2} - r\theta x & \text{if } x < \kappa_{th}, \\ a - c\,\frac{x^2}{2} & \text{if } x \geq \kappa_{th}. \end{cases} \quad (63)$$

We want to find the conditions under which full adoption $x = 1$ is the *unique* equilibrium and the adoption levels eventually reach this equilibrium. Now assume that adoption levels are initially at $x = 0$. Because of the low adoption level, user utilities are given by the first expression in Eq. (63). As adoption levels increase, we want to consistently have the user with $\theta = x^+$ see a positive utility, hence adopt the service. The worst case happens for the user with $\theta = \kappa_{th}^-$, who at the time of her decision sees a utility of

$$\begin{aligned} U(x \approx \kappa_{th}, \theta = \kappa_{th}) &= a - (c/2)\kappa_{th} - r\kappa_{th} \\ &= a - \kappa_{th}(c/2 + r). \end{aligned}$$

Therefore we obtain the extra condition $a > \kappa_{th}(c/2{+}r)$ for $x = 1$ to be a unique equilibrium. Consequently, we get a modified form of proposition 1.

Proposition 7. *Under the usage-based pricing policy of Eq. (16), and a utility function with minimum useful coverage rule given in Eq. (61), full adoption, $x = 1$, is the **unique** equilibrium if $a > \max\{c/2,\, \kappa_{th}(c/2 + r)\}$, and is not an equilibrium if $a \leq c/2$.*

Note that if the threshold κ_{th} is such that $c/2 \geq \kappa_{th}(c/2 + r)$, then the system's adoption behavior is the same as the original model. On the other hand, If $c/2 < \kappa_{th}(c/2 + r)$, then for $c/2 < a < \kappa_{th}(c/2 + r)$, full adoption $x = 1$ is an equilibrium but not unique.

Appendix I
Contiguity of the optimal adoption set

In this section we provide analytical proof for a more general form of Lemma 1. Namely, in a setting where the users' propensity to roam, θ, has a general arbitrary distribution $f(\theta)$ in $[0, 1]$.

Under a general distribution, the adoption level x, the roaming traffic m and the total welfare $V(\Theta)$ should be computed based on their general expressions, as follows.

$$x(\Theta) = \int_{\theta \in \Theta} f(\theta)\, d\theta, \tag{64}$$

$$m(\Theta) = \int_{\theta \in \Theta} \theta\, f(\theta)\, d\theta\,, \tag{65}$$

$$V(\Theta) = \int_{\theta \in \Theta} \nu(\Theta,\, \theta)\, f(\theta)\, d\theta. \tag{66}$$

The next Lemma then gives the generalization of Lemma 1.

Generalization of Lemma 1. *Under an arbitrary roaming distribution with density $f(\theta)$ and for any adoption level x, maximum welfare is always obtained with a set of adopters $\Theta^*(x)$ that exhibit contiguous roaming characteristics. Specifically, $\Theta^*(x)$ is of the form*

$$\Theta^*(x) = \begin{cases} \Theta_1^*(x) = [0,\, x) & \text{if } x < \frac{\gamma}{r-c}, \\ \Theta_2^*(x) = [1-x,\, 1] & \text{if } x \geq \frac{\gamma}{r-c}. \end{cases} \tag{67}$$

Proof: For any given adoption level x, consider an arbitrary realization Θ^{old} of adopters such that $|\Theta^{\text{old}}| = x$. Now take any two intervals N_1 and N_2 from $[0, 1]$ such that

$$N_1 = [\theta_1,\, \theta_1 + \epsilon_1), \quad N_1 \cap \Theta^{\text{old}} = \varnothing,$$
$$N_2 = [\theta_2,\, \theta_2 + \epsilon_2), \quad N_2 \subset \Theta^{\text{old}}$$

where $\theta_2 > \theta_1$, $\epsilon_1 > 0$ and ϵ_2 is selected such that

$$x(N_1) = x(N_2) \triangleq \epsilon, \tag{68}$$

$x(\cdot)$ being the coverage generated by a particular set as defined by Eq. (64). The above conditions mean that everyone in N_1 is a non-adopter and everyone in N_2 is an adopter, and the population of these two sets is the same, taken to be ϵ. Construct a new set of adopters by having everyone in N_1 adopt and everyone in N_2 disadopt,

$$\Theta^{\text{new}} = (\Theta^{\text{old}} \cup N_1) \backslash N_2,$$

where $\backslash$ indicates the set difference operation. We investigate next the change Δ in welfare when the adopers' set changes from Θ^{old} to Θ^{new}, *i.e.*,

$$\Delta \triangleq V(\Theta^{\text{new}}) - V(\Theta^{\text{old}}). \tag{69}$$

Using Eq. (66) and splitting the bounds of the integral, we can write

$$\begin{aligned} V(\Theta^{\text{old}}) &= \int_{\Theta^{\text{old}}} \nu(\Theta^{\text{old}},\, \theta)\, f(\theta)\, d\theta \\ &= \int_{\Theta^{\text{old}} \backslash N_2} \nu(\Theta^{\text{old}},\, \theta)\, f(\theta)\, d\theta \\ &\quad + \int_{N_2} \nu(\Theta^{\text{old}},\, \theta)\, f(\theta)\, d\theta, \end{aligned} \tag{70}$$

and similarly

$$\begin{aligned} V(\Theta^{\text{new}}) &= \int_{\Theta^{\text{new}}} \nu(\Theta^{\text{new}},\, \theta)\, f(\theta)\, d\theta \\ &= \int_{\Theta^{\text{new}} \backslash N_1} \nu(\Theta^{\text{new}},\, \theta)\, f(\theta)\, d\theta \\ &\quad + \int_{N_1} \nu(\Theta^{\text{new}},\, \theta)\, f(\theta)\, d\theta, \end{aligned} \tag{71}$$

Note that

$$\Theta^{\text{old}} \backslash N_2 = \Theta^{\text{new}} \backslash N_1 = \Theta^{\text{old}} \cap \Theta^{\text{new}},$$

and therefore we can use Eq. (70) and Eq. (71) in Eq. (69) to get

$$\begin{aligned} &\Delta = \Delta_1 + \Delta_2, \quad \text{where} \\ &\Delta_1 \triangleq \int_{\Theta^{\text{new}} \cap \Theta^{\text{old}}} \Big(\nu(\Theta^{\text{new}},\, \theta) - \nu(\Theta^{\text{old}},\, \theta)\Big)\, f(\theta)\, d\theta, \\ &\Delta_2 \triangleq \int_{N_1} \nu(\Theta^{\text{new}},\, \theta)\, f(\theta)\, d\theta - \int_{N_2} \nu(\Theta^{\text{old}},\, \theta)\, f(\theta)\, d\theta. \end{aligned}$$

Moreover, from Eq. (9) we have

$$\nu(\Theta^{\text{old}},\, \theta) = \gamma + \theta\, (r x^{\text{old}} - \gamma) - c m^{\text{old}} - e,$$

where x^{old} and m^{old} are the adoption level and the volume of roaming traffic corresponding to Θ^{old}. Similarly,

$$\nu(\Theta^{\text{new}},\, \theta) = \gamma + \theta\, (r x^{\text{new}} - \gamma) - c m^{\text{new}} - e,$$

with x^{new} and m^{new} defined respective to Θ^{new}. Note that as a result of the condition in Eq. (68), we have $x^{\text{old}} = x^{\text{new}} = x$. Therefore

$$\begin{aligned} \Delta_1 &= -\int_{\Theta^{\text{new}} \cap \Theta^{\text{old}}} c\, \Big(m^{\text{new}} - m^{\text{old}}\Big)\, f(\theta)\, d\theta \\ &= -c\, \Big(m^{\text{new}} - m^{\text{old}}\Big) \int_{\Theta^{\text{new}} \cap \Theta^{\text{old}}} f(\theta)\, d\theta \\ &= -c\, (m(N_1) - m(N_2))\, (x - \epsilon) \end{aligned}$$

and

$$\begin{aligned} \Delta_2 =& (\gamma - c m^{\text{new}} - e) \int_{N_1} f(\theta)\, d\theta + (r x - \gamma) \int_{N_1} \theta\, f(\theta)\, d\theta \\ &- (\gamma - c m^{\text{old}} - e) \int_{N_2} f(\theta)\, d\theta - (r x - \gamma) \int_{N_2} \theta\, f(\theta)\, d\theta \\ =& (\gamma - c m^{\text{new}} - e)\epsilon + (r x - \gamma)\, m(N_1) \\ &- (\gamma - c m^{\text{old}} - e)\epsilon - (r x - \gamma)\, m(N_2) \\ =& -\epsilon c\, (m(N_1) - m(N_2)) + (r x - \gamma)\, (m(N_1) - m(N_2))\,, \end{aligned}$$

Where $m(\cdot)$ is as given by Eq. (65). Thus, we compute Δ as

$$\begin{aligned} \Delta =& -c x\, (m(N_1) - m(N_2)) \\ &+ (r x - \gamma)(m(N_1) - m(N_2)) \\ =& (m(N_2) - m(N_1))(c x - r x + \gamma). \end{aligned} \tag{72}$$

We also have

$$\begin{aligned} m(N_2) &= \int_{\theta_2}^{\theta_2+\epsilon_2} \theta\, f(\theta)\, d\theta \\ &> \theta_2 \int_{\theta_2}^{\theta_2+\epsilon_2} f(\theta)\, d\theta = \theta_2 \int_{\theta_1}^{\theta_1+\epsilon_1} f(\theta)\, d\theta \\ &> (\theta_1 + \epsilon_1) \int_{\theta_1}^{\theta_1+\epsilon_1} f(\theta)\, d\theta \\ &> \int_{\theta_1}^{\theta_1+\epsilon_1} \theta\, f(\theta)\, d\theta = m(N_1), \end{aligned}$$

where $\theta_2 > \theta_1 + \epsilon_1$ holds since by construction N_1 and N_2 are mutually exclusive.

Consequently $m(N_2) - m(N_1) > 0$, and Eq. (72) indicates that $\Delta > 0$ if and only if $x < \frac{\gamma}{r-c}$. But a $\Delta > 0$ (positivity independent of the specific choices of N_1 and N_2) means that welfare always increases if an interval of high-θ users leave and a same-size interval of low-θ users join. Repeating this for multiple intervals of suitable sizes will create a contiguous set of adopters in $[0, x)$ that generates more welfare than any other set. Similarly, the case of $\Delta \leq 0$ creates[28] a contiguous set of adopters in the other end of $[0, 1]$ interval, *i.e.*, $[1-x, 1]$. ■

The generalization of Lemma 1 characterizes the structure of optimal adoption set for any given x and establishes that is a contiguous set of adopters.

[28] When $\Delta = 0$, this optimal contiguous Θ is not the only optimum.