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Costs of Constraint Based Networks on a Sphere

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Abstract

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1. Introduction

Integrated network technologies, such as ATM, supports multimedia applications with vastly different bandwidth needs, connection request rates, and holding patterns, and makes traditional network planning techniques less relevant. A constraint-based network design model [1] is proposed, which can help create networks that can handle any combination of traffic that lies within user-specified constraints.

Previous research has shown the hardness of network design problem [2, 3, 4], and a network design tool is implemented in Java to help design networks on a Euclidian plane [1]. Because of the importance of global ATM networks, this paper studies the costs of constraint-based ATM networks on a sphere.

In the simplest case, switches are uniformly distributed on the surface of a unit sphere, and the source and sink capacity of every switch is equal to one. The impact of a specific constraint is studied, which limits the traffic between any two switches to be $\mu(u, v) = \frac{c}{n-1} \exp(-d/D)$, where c is a relaxation factor, d is the distance between switches u and v, and D is a constant. This means that there are less traffic between far apart switches.

An analytical result on the lowerbound is obtained for this case. A simulation is done on a network with fifty switches randomly distributed on a sphere. Lowerbounds with constraints are calculated by linear programming. The simulation agrees with the analytical result very well.

2. Analytical Results

When the number of switches, n, is very large, we can think of the switches as continuously spread on the surface, with a surface density of $\rho = n/4\pi$, where 4π is the total surface area of the unit sphere. Label the switch at the south pole as S, the distance between a switch of angle θ with north pole N and switch S is $d = \pi - \theta$, the number of switches within angles θ and $\theta + d\theta$ is

$$dn = \rho \cdot ds = \frac{n}{2\pi} 2\pi \sin\theta d\theta = \frac{n}{2} \sin\theta d\theta$$

A nonblocking star network can be costructed by connecting all switches to the south pole with link capacity of one in each direction. The cost is

$$
\begin{aligned}
C_{star} &= \int_0^\pi 2(\pi - \theta)\frac{n}{2}\sin\theta d\theta \\
&= n\pi
\end{aligned}
\tag{1}
$$

We define the lowerbound to be the maximum cost of traffic flow. The flow which maximizes the cost is that every switch receive traffic from the most remote switch with as much capacity as possible, then from the second remote switch, ..., until all the sink capacity of this switch is used.

Think of switch S at the south pole. Flow from one switch to S is limited to

$$\frac{c}{n}e^{-(\pi-\theta)/D}$$

Here, we replace $n-1$ with n as $n \to \infty$. Start from $\theta = 0$, send as much flow as allowed to switch S, until $\theta = \theta_0$, when the total flow to S equals to its sink capacity. So we have

$$\int_0^{\theta_0} \frac{c}{n}e^{-(\pi-\theta)/D}\frac{n}{2}\sin\theta d\theta = 1$$

Which gives

$$De^{-(\pi-\theta_0)/D}(\sin\theta_0 - D\cos\theta_0) + D^2 e^{-\pi/D} = \frac{2}{c}(1+D^2) \tag{2}$$

Given D and c, θ_0 can be calculated numerically. θ_0 will appear in Eqn. 3 for the lowerbound.

For combinations of small D and small c, θ_0 obtained from Eq. 2 may be greater than π, which just means that switch S can receive from all other switches without exhaust its sink capacity. In this case, we shall use π instead of θ_0 in Eqn. 3. We argue that the source capacity of every switch is used up exactly at θ_0. Think of the swith N at north pole. It sends $\frac{c}{n}e^{-\pi/D}$ of traffic to the switch S at south pole, $\frac{c}{n}e^{-\theta/D}$ to a switch at θ, ..., so θ_0 for source capacity is exactly the same as θ_0 for sink capacity.

Suppose the link cost is linear to capacity and length, and ignore the constant factor, the total cost of this flow is

$$
\begin{aligned}
cost &= n\int_0^{\theta_0} \frac{c}{n}e^{-(\pi-\theta)/D}(\pi-\theta)\frac{n}{2}\sin\theta d\theta \\
&= \frac{cn}{2(1+D^2)^2}e^{-(\pi-\theta)/D}[D(1+D^2)(\pi-\theta)(\sin\theta - D\cos\theta) \\
&\quad + D^2(1-D^2)\sin\theta - 2D^3\cos\theta]|_0^{\theta_0} \\
&= \frac{cn}{2(1+D^2)^2}\{e^{-(\pi-\theta_0)/D}[D(1+D^2)(\pi-\theta_0)(\sin\theta_0 - D\cos\theta_0) + D^2(1-D^2)\sin\theta_0 \\
&\quad - 2D^3\cos\theta_0] + e^{-\pi/D}D^2[(1+D^2)\pi + 2D]\}
\end{aligned}
\tag{3}
$$

Here, for every switch, we only consider links terminating at it. Because every link must terminate somewhere, the links are counted exactly once.

When $D \to \infty$, Eqn. 2 is reduced into

$$\cos\theta_0 = 1 - 2/c \tag{4}$$

and Eqn. 3 is reduced into

$$cost = n[\pi + \frac{c}{2}(\theta_0\cos\theta_0 - \sin\theta_0)] \tag{5}$$

When $c \to \infty$ and $c/n < 1$, $\theta_0 = \sin\theta_0 = 2/\sqrt{c}$, Eqn. 5 can be further reduced into

$$cost = n(\pi - 2/\sqrt{c}) \tag{6}$$

So, as $c \to \infty$, *lowerbound* $\to n\pi$, the cost of a star network. This is achieved when switches in exactly opposite positions are paired up. There are $n/2$ pairs and link cost of two way traffic for each pair equals to 2π.

When the number of switches is not large enough, the simplification which treats the switches as having a surface density of ρ may not apply. When there is only flat traffic constraint, the lowerbound can be estimated by pairing up every switch with its most distant neighbor. The probability of finding the most distant neighbor between θ and $\theta + d\theta$ equals to the probability of finding one switch between θ and $\theta + d\theta$, and finding the $n - 2$ other switches between $\theta + d\theta$ and π.

$$P = \frac{n-1}{2} \sin\theta d\theta (\frac{1 + \cos\theta}{2})^{n-2}$$

The average distance to the most distant switch is

$$
\begin{aligned}
d_{avg} &= \frac{n-1}{2} \int_0^\pi (\pi - \theta)(\frac{1 + \cos\theta}{2})^{n-2} \sin\theta d\theta \\
&= \pi - \int_0^\pi \cos^{2n-2}\frac{\theta}{2} d\theta \\
&= \pi[1 - \frac{(2n-3)!!}{(2n-2)!!}]
\end{aligned}
$$

Here, $2k!! = 2k(2k-2)\cdots 2$ and $(2k-1)!! = (2k-1)(2k-3)\cdots 1$. The adjusted lowerbound is

$$
\begin{aligned}
cost &= \frac{cn}{2(1+D^2)^2}[1 - \frac{(2n-3)!!}{(2n-2)!!}]\{e^{-(\pi-\theta_0)/D}[D(1+D^2)(\pi - \theta_0)(\sin\theta_0 - D\cos\theta_0) \\
&\quad + D^2(1-D^2)\sin\theta_0 - 2D^3\cos\theta_0] + e^{-\pi/D}D^2[(1+D^2)\pi + 2D]\}
\end{aligned}
\tag{7}
$$

When there is only flat traffic constraints, the average lowerbound is

$$cost = n\pi[1 - \frac{(2n-3)!!}{(2n-2)!!}]\tag{8}$$

3. Simulation

Fifty switches are randomly distributed on the unit sphere, the limit on the traffic between two switches is calculated from $\mu(u,v) = \frac{c}{n-1}e^{-d/D}$. The lowerbound is obtained by calculating the maximum flow using linear programming. The result fits Eqn. 7 very good. In Fig. 3, the solid lines are calculated from Eqn. 3, normalized over $n\pi$, and the dotted lines are simulation, normalized over $n\pi[1 - \frac{(2\cdot50-3)!!}{(2\cdot50-2)!!}] = 0.92n\pi$.

Simulation also shows that with flat traffic constraint, the average lowerbound changes with the number of switches n just as predicted by Eqn. 8. This is plotted in Fig. 3

4. Closing Remarks

A formula is obtained for the average lowerbound on link costs of constraint based networks on a sphere, and fits with the simulation very well.
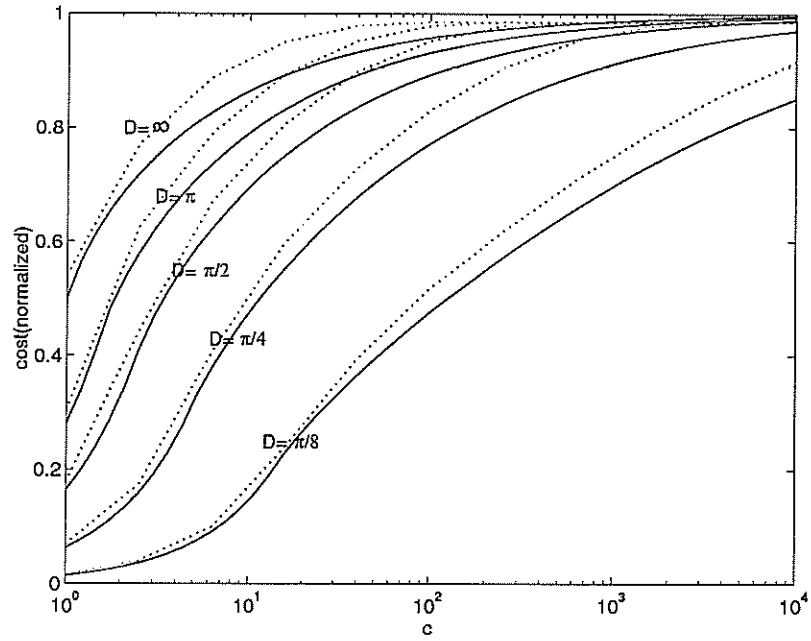
Figure 1: Lowerbound as calculated from Eqn. 3, compared with simulation on a network with 50 switches. The costs are normalized. Solid lines are calculations, dotted lines are simulations.
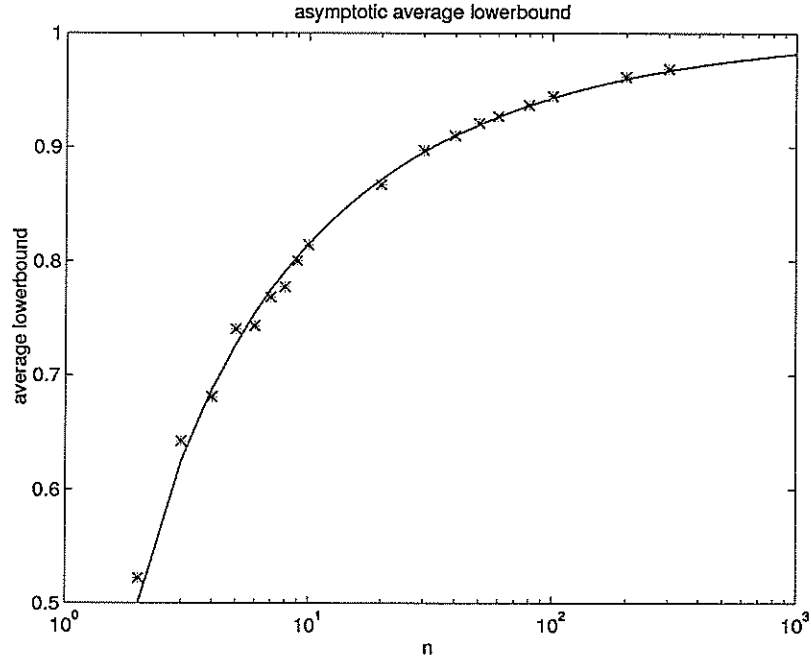

Figure 2: Lowerbound with flat traffic constraints. * is simulation, solid line is from Eqn. 8. The cost is normalized over $n\pi$.

References

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