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A Q-analogue and a Symmetric Function Analogue of a Result by Carlitz, Scoville and Vaughan

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Dedicated to My Family.

Abstract of the Dissertation

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by

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We derive an equation that is analogous to a well-known symmetric function identity: $\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$. Here the elementary symmetric function e_i is the Frobenius characteristic of the representation of $\mathcal{S}_i$ on the top homology of the subset lattice B_i, whereas our identity involves the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the top homology of Segre product of B_n with itself. We then obtain a q-analogue of a polynomial identity given by Carlitz, Scoville and Vaughan through examining the Segre product of the subspace lattice $B_n(q)$ with itself. We recognize the connection between the Euler characteristic of the Segre product of $B_n(q)$ with itself and the representation on the homology of Segre product of B_n with itself by recovering our polynomial identity from specializing the identity on the representation of $\mathcal{S}_n \times \mathcal{S}_n$.

Chapter 1

Introduction

1.1 Motivation and Introduction

Poset (partially ordered set) topology is not only fundamental to many aspects of combinatorics, but it also brings together other branches of mathematics such as commutative algebra, geometry, group theory, representation theory, and topology. This thesis studies poset topology and related representation theory. In particular, we study the representation of the symmetric group on the homology of certain posets that admit edge-lexicographical labelings (EL-labeling in short). Those posets are called EL-shellable posets.

The theory of shellability was first introduced by Schläfli in the nineteenth century for computing the Euler characteristic of a convex polytope [10], and was then widely used in enumerative combinatorics in the late twentieth century (Björner [1], Stanley [15]). A shelling of a simplicial complex is a methodical way of gluing maximal faces together in a well-behaved manner. A poset whose order complex admits a shelling is called a shellable poset and an EL-labeling gives a shelling of the poset. A shelling of P gives useful information about its combinatorial, algebraic and topological properties. Björner's work [1] gives an understanding of the homotopy type of a shellable poset, which is a wedge of spheres. Later, Björner and Wachs' work [3] on edge-lexicographical labeling identifies a set of maximal chains of such a poset with those spheres.

The symmetric group $\mathcal{S}_n$ is the group of permutations on the set of n numbers $[n] := \{1, 2, ..., n\}$. We study the EL-shellable posets that have an $\mathcal{S}_n$ action. This action induces a representation of $\mathcal{S}_n$ on the reduced homology group of the poset. Then the Frobenius characteristic maps a representation of $\mathcal{S}_n$ to a homogenous degree n symmetric function. Symmetric functions provide a very convenient way to describe representations of the symmetric groups, hence are very helpful in studying representations.

The introductory Chapter 1 also provides basic definitions and simple examples of a few fundamental concepts that are required to understand this thesis.

In Chapter 2, we define the product Frobenius characteristic map, which takes a representation of $\mathcal{S}_n \times \mathcal{S}_n$ to a symmetric function in two sets of variables. This map serves as a useful tool in studying representations of $\mathcal{S}_n \times \mathcal{S}_n$. We prove that this map is a homomorphism of rings, which is also a key feature of the usual Frobenius characteristic map. Then we derive an analogue of a well-known symmetric function identity, which involves the representation of $\mathcal{S}_n$ on top homology of the Boolean algebra. Our analogue involves the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the homology of the Segre product of the Boolean algebra with itself.

In Chapter 3, we present our initial finding, Theorem 3.2.8, a q-analogue of a result given by Carlitz, Scoville, and Vaughan. Their result gives a combinatorial interpretation of the coefficients of the reciprocal J_0 Bessel function. Carlitz, Scoville and Vaughan proved that those coefficients count the number of pairs of permutations of $\mathcal{S}_n$ with no common ascent. Bessel functions are solutions to Bessel differential equations. Our q-analogue provides a combinatorial interpretation of the coefficients of the reciprocal $J_0^{(1)}$ q-Bessel function. They count the number of maximal chains whose labels are pairs of permutations of $\mathcal{S}_n$ with no common ascent in the Segre product of the subspace lattice with itself. The Segre product of of subspace lattice with itself contains pairs of subspaces with the same dimension. Those coefficients are the Euler characteristic of this Segre product poset.

Lastly, we have a short Chapter 4 demonstrating the relation of the two analogues we obtained in Chapter 2 and 3.

1.2 Preliminaries

1.2.1 Poset and Hasse diagram

A **partially ordered set** (poset) is a set P together with a binary relation $\leq$ satisfying the following axioms:

- For all $x \in P$, $x \leq x$ (reflexivity).
- If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
- If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Different from totally ordered set, two elements in a poset may be incomparable. A simple but interesting example, which we will use to illustrate various concepts, is the Boolean algebra. The Boolean algebra B_n is the collection of all subsets of $[n] := \{1, 2, ..., n\}$ ordered by containment. So $\{1, 3\} \leq \{1, 2, 3\}$, but $\{1, 3\}$ and $\{1, 2\}$ are not comparable.

Two elements $x, y \in P$ have an **upper bound** $u \in P$ if u satisfies $x \leq u$ and $y \leq u$. We call u a **least upper bound** (or **join**) of x and y, if u is an upper bound of x and y, and every upper bound v of x and y satisfies $u \leq v$. The join of x and y (if exist) is unique and is denoted $x \vee y$. The **greatest lower bound** (or **meet**) of x and y is defined dually and denoted $x \wedge y$.

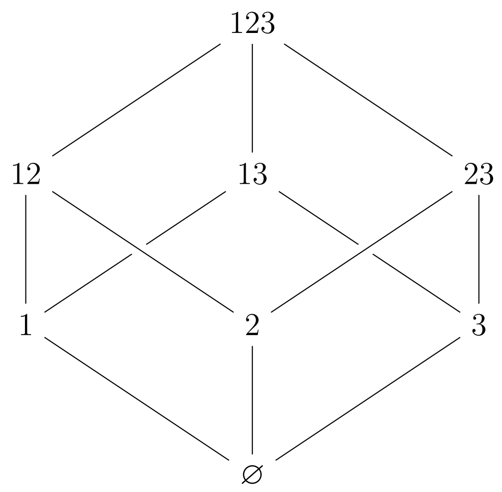

Figure 1.1. Hasse diagram of the subset lattice B_3

A poset is called a **lattice** if every pair of its elements has a join and a meet. A finite lattice clearly has a bottom (smallest) element and a top (largest) element, which are usually written as $\hat{0}$ and $\hat{1}$ respectively. The Boolean algebra B_n is a lattice with $\hat{0} = \varnothing$ and $\hat{1} = [n]$. We will also refer to B_n as the subset lattice. A poset with a top element $\hat{1}$ and a bottom element $\hat{0}$ are said to be **bounded**. We define the bounded extension of P as $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$. Note that $\hat{0}$ and $\hat{1}$ are added even if P already has a bottom or a top element.

A finite poset P can be represented by a **Hasse diagram**. Each element of P is a vertex in in the Hasse diagram of P. Let $x, y \in P$ and $x < y$. If no element $z \in P$ satisfies $x < z < y$, then we say that x is **covered by** y and write $x \lessdot y$. This cover relation is represented in the Hasse diagram by an edge connecting vertices x and y. Figure 1.1 is the Hasse diagram of a small Boolean algebra B_3. This thesis concerns only finite posets. All posets appearing here after will be assumed to be finite.

Totally ordered subsets of a poset P are called **chains**. In figure 1.1, the chain $\varnothing \lessdot 1 \lessdot 12 \lessdot 123$ is an example of a chain that is **maximal** in B_3. The **length** $l(c)$ of a finite chain c is defined to be $\#c - 1$. The **rank** of a finite poset P is $\max\{l(c) :$

c is a maximal chain of $P\}$. When every maximal chain of P is of the same length n, we call P a **graded** poset of rank n. The subset lattice B_3 is then graded of rank 3. For a graded poset of rank n, we can define a unique **rank function** $\rho : P \longrightarrow [n] \cup \{0\}$ that satisfies:

- $\rho(\hat{0}) = 0$,
- for $x, y \in P$ and $x \lessdot y$, $\rho(y) = \rho(x) + 1$.

The Boolean algebra B_n has a natural rank function that tells us the cardinality of each subset of $[n]$. A good place to learn more about poset structures is R. Stanley's book [16].

1.2.2 The order complex and (co)homology of a poset

To study the topology of a poset, we study the topology of a certain simplicial complex associated with the poset. An **abstract simplicial complex** $\triangle$ [21] on a finite vertex set V is a nonempty collection of subsets (each subset is called a **face** or a **simplex**) of V such that

- every vertex in V is an element of $\triangle$
- if $G \in \triangle$ then every subset of G is also an element of $\triangle$.

For every poset P, the **order complex** of P, denoted by $\triangle(P)$, can be realized in $\mathbb{R}^n$ using a geometric simplicial complex whose vertices are elements of P and faces are chains of P. It is important to note that every geometric simplicial complex is homeomorphic to the geometric realization of the order complex of some poset, thus studying the order complexes does not restrict us to a small subclass of topological spaces. The vertices and faces in the order complex can be seen geometrically, hence we can study its topological properties. More details on simplicial complexes can be found in [21] and [12]. Let $\bar{B}_n$ denote $B_n - \{\emptyset, [n]\}$. The geometric realization of $\triangle(\bar{B}_n)$ is in fact homeomorphic to a simple and beautiful geometric object, the $(n-2)$-sphere $\mathbb{S}^{n-2}$. See figure 1.2.

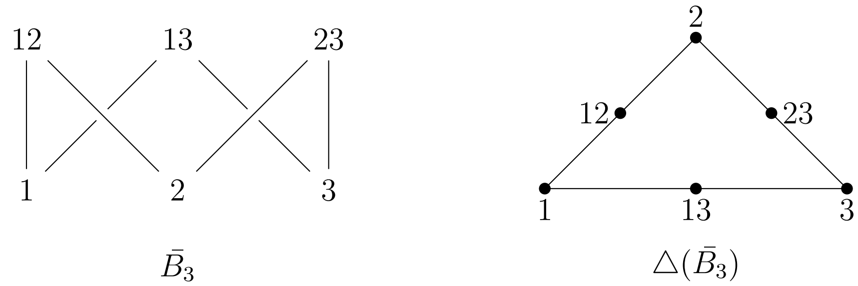

Figure 1.2. Order complex of $\bar{B}_3$, $\triangle(\bar{B}_3) \cong \mathbb{S}^1$

Given a face $F \in \triangle$, the **dimension** of F, denoted dim$F$, is $|F| - 1$. The maximal faces are called **facets**. In the case of a order complex of a poset, the dimension of a face in the complex is the length of the associated chain. When the poset is graded of rank, all facets of its order complex are of the same dimension. Such simplicial complexes are said to be **pure**.

The **homology** of a poset P is the simplicial homology of the topological space $\triangle(P)$. Usually when studying poset topology, we are interested in the *reduced* simplicial homology of a poset. In this thesis, we will only deal with reduced homology groups and will just call them homology groups for convenience. Now let us introduce those concepts in terms of chains of the poset. For a more in depth understanding of simplicial homology, please refer to *Algebraic Topology* by Allen Hatcher [7].

Let $\mathbf{k}$ be a field or the ring of integers. The jth **chain space** of a poset P is defined as follows:

$$C_j(P; \mathbf{k}) := \mathbf{k} - \text{module freely generated by } j\text{-chains (chains of length } j) \text{ of } P.$$

Elements of $C_j(P;\mathbf{k})$ are of form $\sum_i \alpha_i c_i$, where $\alpha \in \mathbf{k}$ and c_i's are j-chains of P. for a j-chain $(x_1 < ... < x_{j+1})$, define the **boundary map** $\partial_j : C_j(P;\mathbf{k}) \longrightarrow C_{j-1}(P;\mathbf{k})$ by

$$\partial_j(x_1 < ... < x_{j+1}) = \sum_{i=1}^{j+1}(-1)^i(x_1 < ... < \hat{x}_i < ... < x_{j+1}),$$

where the hat $\hat{x}_i$ means omitting the vertex x_i. Then we extend ∂_j by linearity. We can easily check that $\partial_{j-1}\partial_j = 0$. The **cycle space** $Z_j(P;\mathbf{k})$ is defined to be kernel of ∂_j and the **boundary space** $B_j(P;\mathbf{k})$ is the image of ∂_{j+1}. Define the **homology** of the poset P in the jth dimension by

$$\widetilde{H}_j(P;\mathbf{k}) := Z_j(P;\mathbf{k})/B_j(P;\mathbf{k}).$$

An **open interval** (s,t) of P is the set of all element $u \in P$ such that $s < u < t$. The cohomology of a poset is defined using a **coboundary map** $\delta_j : C_j(P;\mathbf{k}) \longrightarrow C_{j+1}(P;\mathbf{k})$ such that for all chains $x_1 < ... < x_j$,

$$\delta_j(x_1 < ... < x_j) = \sum_{i=1}^{j+1}(-1)^i \sum_{x \in (x_{i-1},x_i)} (x_1 < ...x_{i-1} < x < x_i < ... < x_j),$$

where $x_0 = \hat{0} \in \hat{P}$, $x_{j+1} = \hat{1} \in \hat{P}$, and (x_{i-1}, x_i) is an open interval of P. The **cocycle space** is defined as $Z^j(P;k) :=$ kerδ_j and the **coboundary space** is defined as $B^j(P;k) :=$ imδ_{j-1}. Similar to the homology group, the **cohomology** of the poset P in the jth dimension is

$$\widetilde{H}^j(P;k) := Z^j(P;k)/B^j(P;k).$$

Given a simplicial complex $\triangle$ and $F \in \triangle$, let $\langle F \rangle := \{G : G \subseteq F\}$. Then $\triangle$ is said to be **shellable** if its facets can be arranged in a linear order $F_1, F_2, \ldots, F_t$ such that $\left(\bigcup_{i=1}^{k-1}\langle F_i \rangle\right) \cap \langle F_k \rangle$ is a pure and (dim$F_k - 1$)-dimensional for all $k = 2, \ldots, t$. This ordering of facets is a **shelling** of $\triangle$. If $\triangle(P)$ is shellable, we say P is shellable.

Shellable posets have the homotopy type of wedges of spheres (See Björner and Wachs [3]). In this case, its homology $\widetilde{H}_j(\triangle;\mathbb{Z})$ and cohomology $\widetilde{H}^j(\triangle;\mathbb{Z})$ are the same and both are isomorphic to $\mathbb{Z}^{r_i}$, where r_i is the number of spheres of dimension i [21]. The homology $\widetilde{H}_{n-2}(\bar{B}_n)$ is therefore $\mathbb{Z}$. The rich and interesting topological properties of $\triangle(P)$ provide strong motivation for studies on poset topology.

1.2.3 Segre product posets

New posets can be formed using existing posets. Given posets P and Q, the **direct product** of P and Q is the poset $P \times Q$ on the set $(x, y) : x \in P$ and $y \in Q$ with the poset relation $(x, y) \leq (u, v)$ if only if $x \leq u$ in P and $y \leq v$ in Q. Posets studied in this thesis are a form of product poset, called the **Segre product** of posets.

Definition 1.2.1 *(A full definition can be found in* [4]*) Let P be a pure poset with a rank function ρ, then the **Segre product poset** of P with itself, denoted by $P \circ_\rho P$, is defined to be the induced subposet of the product poset $P \times P$ consisting of the pairs $(x, y) \in P \times P$ such that $\rho(x) = \rho(y)$.*

One important concept in algebra and poset topology is Cohen-Macaulayness. Cohen-Macaulay posets have very nice structures. In particular, Cohen-Macaulay posets have reduced homology groups concentrated in the top dimension as they are homotopic to a wedge of spheres [1]. The next result follows from Theorem 1 in Björner and Welker [4].

Proposition 1.2.2 *Let P be a pure poset. Let $\rho : P \longrightarrow \mathbb{N}$ be the rank function of P. If P is Cohen-Macaulay over the field k, then the Segre product poset $P \circ_\rho P$ is Cohen-Macaulay over k.*

In this thesis, P is either the subset lattice B_n or the subspace lattice $B_n(q)$. Both poset are pure with a rank function. The subspace lattice $B_n(q)$ will be introduced later. Because of the Cohen-Macaulayness of $B_n \circ_\rho B_n$ and $B_n(q) \circ_\rho B_n(q)$, those posets are well behaved, which motivated us to investigate the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the reduced homology of $B_n \circ_\rho B_n$ and related Whitney homology groups.

1.2.4 The symmetric group and group representations

The **symmetric group** $\mathcal{S}_n$ is the group of permutations on a set of n objects. It is customary to use the set $[n] := \{1, 2, ..., n\}$ of n numbers. A common way to write out a permutation is the one line notation. That is, for $\sigma \in \mathcal{S}_4$, $\sigma = 2314$ means $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, and $\sigma(4) = 4$. We will only use the one line notation in this thesis for its simplicity and correspondence with labelings of poset chains, which we will introduce later.

A matrix representation of an abstract group can give us better understanding of the group. Let Mat_d denote the set of all $d \times d$ matrices with entries in $\mathbb{C}$. Let GL_d be the complex general linear group of degree d, which is the group of all invertible (with respect to multiplication) matrices in Mat_d.

Definition 1.2.3 [13] *A **matrix representation** of a group G is a group homomorphism*

$$X : G \longrightarrow GL_d.$$

Equivalently, to each $g \in G$ is assigned $X(g) \in Mat_d$ such that

1. *$X(e) = I$ the identity matrix, and*

2. *$X(gh) = X(g)X(h)$ for all $g, h \in G$.*

*The representation has **degree**, or **dimension** d.*

For a matrix representation X of a group G, we define the **character** χ of X by

$$\chi(g) = \mathrm{tr} X(g),$$

where tr denotes the trace of a matrix and $g \in G$. The character is a key information of a representation. The following are some properties of characters (see Proposition 1.8.5 in Sagan[13]):

1. Let $d = \dim X$, then $\chi(e) = d$.

2. Elements in the same conjugacy class of G have the same character value.

Let us look at a simple example, the *defining representation* of the symmetric group $\mathcal{S}_n$. Let $\sigma \in \mathcal{S}_n$, define $X(\sigma) = (x_{i,j})_{n \times n}$ such that

$$x_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i, \\ \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked easily that the character value of σ is the number of points fixed by σ. Permutations of the same cycle type are in the same conjugacy class of $\mathcal{S}_n$ and they clearly have the same number of fixed points.

Now expand on the idea of Matrix representations. Matrices are essentially linear transformations. We can also represent elements of a group G by linear transformations of some vector space. For a vector space V, let $GL(V)$ denote the *general linear group* of V, i.e. the set of all invertible linear transformations of V to itself. The group $GL(V)$ is in fact isomorphic to GL_d for $d = dimV$.

Definition 1.2.4 *(Sagan* [13]*) Let V be a vector space and G be a group. Then V is a G-**module** if there is a group homomorphism*

$$\rho : G \longrightarrow GL(V).$$

Equivalently, V is a G-module if there is a multiplication, $g\boldsymbol{v}$, of elements of V by elements of G such that

1. $g\boldsymbol{v} \in V$,

2. $g(c\boldsymbol{v} + d\boldsymbol{w}) = c(g\boldsymbol{v}) + d(g\boldsymbol{w})$,

3. $(gh)\boldsymbol{v} = g(h\boldsymbol{v})$, *and*

4. $e\boldsymbol{v} = \boldsymbol{v}$

for all $g, h \in G$; $\boldsymbol{v}, \boldsymbol{w} \in V$; *and scalars* $c, d \in \mathbb{C}$.

Each group homomorphism gives a G-module, which is a representation of G. When the group G is clear in the context, we will often omit G and just use "module" for short. The *character* of a $G-module$ V is the character of a matrix representation obtained by choosing a basis for V. Though many matrix representations can correspond to one G-module V, the matrices representing the same element g will be conjugates of each other, hence their trace will be the same. The character of a G-module is then well-defined.

A key result of group representations is that we can break up large representations into smaller representations. The ones that cannot be broken up further are called **irreducible** representations. Let V be a G-module. A **submodule** of V is a subspace W that is closed under the action of G, i.e., $w \in W \Rightarrow gw \in W$ for all $g \in G$. If W is a submodule of V, we write $W \leq V$.

Theorem 1.2.5 (Maschke's Theorem, see Chapter 8 in James and Liebeck[9] and Theorem 1.5.3 in Sagan[13]) *Let G be a finite group and let V be a nonzero G-module. Then*

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)},$$

where each $W^{(i)}$ is an irreducible G-submodule of V.

Maschke's Theorem is a fundamental result in representation theory, which signifies that every nonzero G-module is a direct sum of irreducible G-submodules. There are far-reaching consequences of Maschke's Theorem. For an in depth study on representations of the symmetric group, please see the two texts from B. Sagan[13], and G. James and A. Kerber[8].

Chapter 2

A Symmetric Function Analogue

2.1 The Space of Symmetric Functions and characteristic map

Essentially, symmetric functions are power series invariant under the action of all symmetric groups. In this section, we will provide the definition and some well-known bases of the space of symmetric functions.

Consider an infinite set of variables $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$. Define the **formal power series ring** to be

$$\mathbb{C}[[\mathbf{x}]] := \{\sum_{n\geq 0} a_n x^n : a_n \in \mathbb{C} \text{ for all } n\}.$$

It is a ring with the usual addition and multiplication. We are not concerned with the convergence of those series because we will never substitute a value for x_i. The term *formal* is used to indicate that fact. The terms of a power series in $\mathbb{C}[[\mathbf{x}]]$ are monomials of forms $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} x_{i_3}^{\lambda_3} \dots x_{i_l}^{\lambda_l}$. The **degree** of such a monomial is n given $n = \sum_i \lambda_i$. For a formal power series $f(x) \in \mathbb{C}[[\mathbf{x}]]$, if every monomial in $f(x)$ has degree n, we say that $f(x)$ is **homogeneous of degree** n.

Given a permutation σ of $\mathbb{N}$, let σ act on $f(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ by permutating the indices of the variables. That is

$$\sigma f(x_1, x_2, x_3, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \dots).$$

A function that is invariant under such action of all permutations of $\mathbb{N}$ is called a **symmetric function**.

The expression $n = \sum_i \lambda_i$, in fact, gives a partition of n. A **partition** is any sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots)$$

of non-negative integers and finitely many non-zero terms satisfying:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \dots.$$

We do not distinguish two sequences that only differ by the number of zeros in the end. The positive terms λ_i of the sequence are called **parts** of a partition λ. The **length** of λ, denoted by $l(\lambda)$, is the number of parts, and the **weight** of λ, denoted by $|\lambda|$, is the sum $\sum_i \lambda_i$. If $|\lambda| = n$, we say λ is a **partition** of n or λ partitions n, and we denote this by $\lambda \vdash n$.

Sometimes we need to know the number of parts of the same size, then it is convenient to write a partition λ in the following way:

$$\lambda = (1^{m_1} 2^{m_2} \dots r^{m_r} \dots),$$

where m_i is the number of parts of λ that equal i. The number m_i is called the **multiplicity** of i in λ. For example $(2, 1, 1)$ is a partition of 4, and we write $(2, 1, 1)$ as (211) or $(1^2 2)$ for simplicity.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, the **monomial symmetric function** corresponding to λ is

$$m_\lambda = m_\lambda(\mathbf{x}) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_l}^{\lambda_l},$$

summed over all distinct monomials with exponents $\lambda_1, \dots, \lambda_l$. For example,

$$m_{(21)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \dots.$$

When $\lambda \vdash n$, m_λ is homogeneous of degree n. Clearly, the monomials m_λ are fixed under the action of any symmetric group.

Definition 2.1.1 *The* ***ring of symmetric functions*** *is*

$$\Lambda = \Lambda(\boldsymbol{x}) = \mathbb{C}m_\lambda,$$

which is the vector space spanned by all the m_λ.

It is easy to verify that Λ is in fact a subring of $\mathbb{C}[[\mathbf{x}]]$. We can decompose Λ as

$$\Lambda = \oplus_{n\geq 0}\Lambda^n,$$

where Λ^n is the space spanned by degree n monomial symmetric functions m_λ.

Proposition 2.1.2 *The set $\{m_\lambda : \lambda \vdash n\}$ forms a basis for Λ^n, the space of homogeneous degree n symmetric functions. The dimension of Λ^n is the number of partitions of n.*

Proof The m_λ are independent. ■

We will now introduce three more bases for Λ^n.

Definition 2.1.3 *For an integer $r \geq 1$, the r-th* ***elementary symmetric functions*** *e_r is the sum of all square-free monomials of degree r. That is*

$$e_r = m_{(1^r)} = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\ldots x_{i_r}.$$

The r-th ***complete homogeneous symmetric functions*** *is the sum of all monomials of degree r. That is*

$$h_r = \sum_{\lambda\vdash r} m_\lambda = \sum_{i_1\leq i_2\leq\cdots\leq i_r} x_{i_1}x_{i_2}\ldots x_{i_r}.$$

The r-th ***power sum symmetric function*** *is the sum of r-th powers of all variables. That is*

$$p_r = m_{(r)} = \sum_{i\geq 1} x_i^r.$$

We define $e_0 = h_0 = p_0 = 1$.

For example, when $r = 2$,

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + \dots,$$

$$h_2 = x_1^2 + x_2^2 + x_1x_2 + x_3^2 + x_1x_3 + x_2x_3 + x_4^2 + \dots,$$

$$p_2 = x_1^2 + x_2^2 + x_3^2 + \dots.$$

Definition 2.1.4 *For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$, define*

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots,$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots,$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots.$$

Theorem 2.1.5 (Theorem 4.3.7 in Sagan[13], and for a more detailed treatment see Macdonald[11] and Stanley [17] chapter 7.) *The following sets are bases for Λ^n the space of homogeneous degree n symmetric functions.*

1. $\{e_\lambda : \lambda \vdash n\}$.

2. $\{h_\lambda : \lambda \vdash n\}$.

3. $\{p_\lambda : \lambda \vdash n\}$.

Symmetric functions provide a convenient way of describing representations of the symmetric group. One of the properties of representations is that their character values are constant on each conjugacy class of the group. We call functions, that are constant on conjugacy classes, **class functions**. The conjugacy classes of a symmetric group $\mathcal{S}_n$ are determined by cycle types of its elements. Elements having the same cycle type are in the same class. Each cycle type corresponds to a partition of n. For instance, given

$\sigma = (12)(34)(5) \in \mathcal{S}_5$ written in cycle notation, the cycle type of σ is (221), which is a partition of 5. For an element $\sigma \in \mathcal{S}_n$ with cycle type $\mu \vdash n$, we write $\mathrm{type}(\sigma) = \mu$. We now have that each partition of n corresponds to a conjugacy class of $\mathcal{S}_n$.

Let $\mathcal{R}^n$ be the space of class functions on $\mathcal{S}_n$. Given a partition $\mu = (1^{m_1}2^{m_2}\dots)$ of n, $z_\mu := \prod_{i=1}^{i=n} i^{m_i} m_i!$.

Definition 2.1.6 *The (Frobenius)* ***characteristic map*** *is $ch^n : \mathcal{R}^n \longrightarrow \Lambda^n$ defined by*

$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\mu p_\mu,$$

where χ_μ is the value of χ on the class μ and p_μ is the power sum symmetric function.

A very important basis for Λ^n is the **Schur functions**. Schur functions s_λ are the Frobenius characteristics of irreducible $\mathcal{S}_n$-modules (see Sagan [13] and Stanley [17] chapter 7). Here we will not give a definition of the Schur functions as there are many ways to define them using different approaches, and Schur functions are not used in the main theorems of this thesis.

Now consider $\mathcal{R} = \oplus_n \mathcal{R}^n$. Let $ch = \oplus ch^n$. Let V and W be representations of $\mathcal{S}_m$ and $\mathcal{S}_n$ with characters f and g. Then $f \otimes g$ is the character of $V \otimes W$. Define the **induction product** $f \circ g$ as

$$f \circ g = f \otimes g \uparrow_{\mathcal{S}_m \times \mathcal{S}_n}^{\mathcal{S}_{m+n}} .$$

A fundamental property of the characteristic map is the following:

Proposition 2.1.7 (Stanley [17, Proposition 7.18.2]) *The Frobenius characteristic map $ch : R \longrightarrow \Lambda$ is a bijective ring homomorphism, i.e.,* ch *is one-to-one and onto, and satisfies*

$$ch(f \circ g) = ch(f)ch(g).$$

The significance of ch being a ring isomorphism is that now we can have products of characters. The induction product is the ring operation. Notice that while the tensor product $f \otimes g$ is a character of $\mathcal{S}_m \times \mathcal{S}_n$, the induction product $f \circ g$ is a character of $\mathcal{S}_{m+n}$.

The Frobenius characteristic map is often used to study a representation of the symmetric group. For instance, let $\mathcal{S}_n$ act on the top homology of the proper part of the boolean algebra, $\bar{B}_n := B_n - \{\hat{0}, \hat{1}\}$. This action induces a representation of $\mathcal{S}_n$. Let χ be the character of this representation. Then

$$ch(\chi) = e_n,$$

where e_n is the elementary symmetric function (See Example 1.3 in Sundaram [19]).

2.2 The product Frobenius characteristic map

In this section we define a product Frobenius characteristic map to help understand representations of $\mathcal{S}_n \times \mathcal{S}_n$. Let us consider two sets of variables $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$.

We use $\Lambda(x) = \oplus_n \Lambda^n(x)$ and $\Lambda(y) = \oplus_n \Lambda^n(y)$ to denote the the rings of symmetric functions in variables $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$ respectively.

Definition 2.2.1 *Let χ be a class function on $\mathcal{S}_m \times \mathcal{S}_n$. The product Frobenius characteristic map $ch : \mathcal{R} \times \mathcal{R} \longrightarrow \Lambda(x) \times \Lambda(y)$ is defined as:*

$$ch(\chi) = \sum_{(\mu,\lambda)\vdash(m,n)} z_\mu^{-1} z_\lambda^{-1} \chi_{(\mu,\lambda)} p_\mu(x) p_\lambda(y), \tag{2.2.1}$$

where $\chi(\mu, \lambda)$ is the value of χ on the class (μ, λ) and p_μ, p_λ are power sum symmetric functions. The class (μ, λ) is indexed by a partition μ of m and a partition λ of n that tell us the cycle shapes of elements of $\mathcal{S}_m$ and $\mathcal{S}_n$ respectively.

The irreducible representations of $\mathcal{S}_m \times \mathcal{S}_n$ are of the form $A^{(i)} \otimes B^{(j)}$, where $A^{(i)}$ and $B^{(j)}$ are each irreducibles of $\mathcal{S}_m$ and $\mathcal{S}_n$ respectively (Sagan [13, Theorem 1.11.3]). A representation V of $\mathcal{S}_m \times \mathcal{S}_n$ can then be decomposed into a sum of irreducibles of $\mathcal{S}_m \times \mathcal{S}_n$.

Proposition 2.2.2 *Let V, a representation of $\mathcal{S}_m \times \mathcal{S}_n$, have the following decomposition: $V = \bigoplus_{i,j} c_{ij} A^{(i)} \otimes B^{(j)}$, where $A^{(i)}$'s and $B^{(j)}$'s are irreducible representations of $\mathcal{S}_m$ and $\mathcal{S}_n$ respectively and c_{ij} are non-negative integers. Then the product Frobenius characteristic of V is*

$$ch(V) = \sum_{i,j} c_{ij} ch^m(A^{(i)})(x) ch^n(B^{(j)})(y).$$

Here $ch^m(A^{(i)})(x)$ is the usual Frobenius characteristic of $A^{(i)}$ in the variable x and $ch^n(B^{(j)})(y)$ is defined similarly.

Proof Let χ denote the character of V. Let χ^{Ai} and χ^{Bj} be the characters of $A^{(i)}$ and $B^{(j)}$ respectively. Then $\chi = \sum_{i,j} c_{ij} \chi^{Ai} \otimes \chi^{Bj}$. Definition 2.2.1 gives us

$$ch(V) = \sum_{(\mu,\lambda)\vdash(m,n)} z_\mu^{-1} z_\lambda^{-1} \chi(\mu, \lambda) p_\mu(x) p_\lambda(y).$$

The character $\chi(\mu, \lambda) = \sum_{i,j} c_{ij} \big(\chi^{Ai} \otimes \chi^{Bj}(\mu, \lambda)\big)$ by [13, Corollary 1.9.4]. And $\chi^{Ai} \otimes \chi^{Bj}(\mu, \lambda) = \chi_\mu^{Ai} \chi_\lambda^{Bj}$ due to [13, Theorem 1.11.2]. Then

$$\begin{aligned} ch(V) &= \sum_{(\mu,\lambda)\vdash(m,n)} z_\mu^{-1} z_\lambda^{-1} \sum_{i,j} c_{ij} \chi_\mu^{Ai} \chi_\lambda^{Bj} p_\mu(x) p_\lambda(y) \\ &= \sum_{i,j} c_{i,j} \Big(\sum_{\mu\vdash m} z_\mu^{-1} \chi_\mu^{Ai} p_\mu(x) \Big) \Big(\sum_{\lambda\vdash n} z_\lambda^{-1} \chi_\lambda^{Bj} p_\lambda(y) \Big) \\ &= \sum_{i,j} c_{i,j} ch^m(A^{(i)})(x) ch^n(B^{(j)})(y). \end{aligned}$$

■

Because the product Frobenius characteristic map is basically an extension of the usual (Frobenius) characteristic map, we keep the notation ch for product Frobenius characteristic map even though ch was previously defined to be $\oplus_n ch^n$ in various literature (Sagan [13], Stanley [17]). The meaning of ch will be clear in the given context.

Remark 2.2.3 *Given V a representation of $\mathcal{S}_m$ with character f and W a representation of $\mathcal{S}_n$ with character g, let $V = \oplus_i a_i A^{(i)}$ and $W = \oplus_j b_j B^{(j)}$ be their decompositions into irreducibles. It can be easily verified that the product Frobenius characteristic $ch(f \otimes g) = ch(f)(x)ch(g)(y)$. It is a symmetric function in $\Lambda^m \times \Lambda^n$, while the usual Frobenius characteristic $ch(f \circ g) = ch(f)(x)ch(g)(x)$ is a symmetric function in Λ^{m+n}.*

We would like the product Frobenius characteristic map to be a homomorphism of rings as well. Given a class function ψ on $\mathcal{S}_k \times \mathcal{S}_l$ and a class function ϕ on $\mathcal{S}_m \times \mathcal{S}_n$, $\psi \otimes \phi$ is a class function on $(\mathcal{S}_k \times \mathcal{S}_l) \times (\mathcal{S}_m \times \mathcal{S}_n)$. We want to produce a class function on $\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}$.

Definition 2.2.4 *For ψ and ϕ as given above, we define the* induction product $\psi \circ \phi$ *to be $\psi \otimes \phi \uparrow_{(\mathcal{S}_k \times \mathcal{S}_l) \times (\mathcal{S}_m \times \mathcal{S}_n)}^{\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}}$.*

The following proposition will show that this induction product is a ring operation that makes $\mathcal{R} \times \mathcal{R}$ into an algebra.

Proposition 2.2.5 *Assume given ψ a class function on $\mathcal{S}_k \times \mathcal{S}_l$, and ϕ a class function on $\mathcal{S}_m \times \mathcal{S}_n$. The product Frobenius characteristic map $ch : \mathcal{R} \times \mathcal{R} \longrightarrow \Lambda(x) \times \Lambda(y)$ is an algebra isomorphism, i.e.,* ch *is one-to-one and onto, and satisfies*

$$ch(\psi \circ \phi) = ch(\psi)ch(\phi).$$

Before proving this proposition, we need to first establish a lemma:

Lemma 2.2.6 *If f is a class function on $\mathcal{S}_k \times \mathcal{S}_m$ and g is a class function on $\mathcal{S}_l \times \mathcal{S}_n$, then*

$$f \otimes g \uparrow_{(\mathcal{S}_k \times \mathcal{S}_m)\times(\mathcal{S}_l \times \mathcal{S}_n)}^{\mathcal{S}_{k+m}\times\mathcal{S}_{l+n}} = f \uparrow_{\mathcal{S}_k\times\mathcal{S}_m}^{\mathcal{S}_{k+m}} \otimes g \uparrow_{\mathcal{S}_l\times\mathcal{S}_n}^{\mathcal{S}_{l+n}} .$$

Proof Suppose $\mathcal{S}_k \times \mathcal{S}_m < \mathcal{S}_{k+m}$ has coset representatives $\{s_1, s_2, ..., s_q\}$, $q = \binom{k+m}{k}$, and $\mathcal{S}_l \times \mathcal{S}_n < \mathcal{S}_{l+n}$ has coset representatives $\{t_1, t_2, ..., t_r\}$, $r = \binom{l+n}{l}$. Then $\{(s_i, t_j)\}$, $i \in [q]$, $j \in [r]$, is a set of coset representatives for $(\mathcal{S}_k \times \mathcal{S}_m) \times (\mathcal{S}_l \times \mathcal{S}_n) < \mathcal{S}_{k+m} \times \mathcal{S}_{l+n}$. Given $H \leq G$ and a class function ϕ on H, define ϕ° to be the class function on G such that $\phi^\circ(\sigma) = \phi(\sigma)$ if $\sigma \in H$ and $\phi^\circ(\sigma) = 0$ if $\sigma \notin H$. For the class functions f and g given in the lemma, we have $(f \otimes g)^\circ = f^\circ \otimes g^\circ$. For $(\sigma_{k+m}, \sigma_{l+n}) \in \mathcal{S}_{k+m} \times \mathcal{S}_{l+n}$,

$$\begin{aligned}
f \otimes g \uparrow_{(\mathcal{S}_k \times \mathcal{S}_m)\times(\mathcal{S}_l \times \mathcal{S}_n)}^{\mathcal{S}_{k+m}\times\mathcal{S}_{l+n}} ((\sigma_{k+m}, \sigma_{l+n})) &= \sum_{i,j} (f \otimes g)^\circ\big((s_i^{-1}, t_j^{-1})(\sigma_{k+m}, \sigma_{l+n})(s_i, t_j)\big) \\
&= \sum_i f^\circ(s_i^{-1}\sigma_{k+m}s_i) \sum_j g^\circ(t_j^{-1}\sigma_{l+n}t_j) \\
&= f \uparrow_{\mathcal{S}_k\times\mathcal{S}_m}^{\mathcal{S}_{k+m}} (\sigma_{k+m}) g \uparrow_{\mathcal{S}_l\times\mathcal{S}_n}^{\mathcal{S}_{l+n}} (\sigma_{l+n}) \\
&= f \uparrow_{\mathcal{S}_k\times\mathcal{S}_m}^{\mathcal{S}_{k+m}} \otimes g \uparrow_{\mathcal{S}_l\times\mathcal{S}_n}^{\mathcal{S}_{l+n}} ((\sigma_{k+m}, \sigma_{l+n})).
\end{aligned}$$

The second and fourth equalities come from [13, Theorem 1.11.2]. ■

Proof of Proposition 2.2.5

Since the Frobenius characteristic map is bijective, so is the product Frobenius characteristic map. It is sufficient to show that the map is a homomorphism. Suppose $\psi = \sum_{i,j} a_{ij}\psi_k^{(i)} \otimes \psi_l^{(j)}$ with $\psi_k^{(i)}$'s and $\psi_l^{(j)}$'s are irreducible characters of representations of $\mathcal{S}_k$ and $\mathcal{S}_l$ respectively. Similarly, $\phi = \sum_{u,v} b_{uv}\phi_m^{(u)} \otimes \phi_n^{(v)}$. For any $\sigma_k \in \mathcal{S}_k$, $\sigma_l \in \mathcal{S}_l$, $\omega_m \in \mathcal{S}_m$, and $\omega_n \in \mathcal{S}_n$, as above, we have

$$\begin{aligned}
\psi \otimes \phi\big((\sigma_k, \sigma_l), (\omega_m, \omega_n)\big) &= \big(\sum_{i,j} a_{ij}\psi_k^{(i)}(\sigma_k)\psi_l^{(j)}(\sigma_l)\big)\big(\sum_{u,v} b_{uv}\phi_m^{(u)}(\omega_m)\phi_n^{(v)}(\omega_n)\big) \\
&= \sum_{i,j,u,v} a_{ij}b_{uv}\psi_k^{(i)}(\sigma_k)\phi_m^{(u)}(\omega_m)\psi_l^{(j)}(\sigma_l)\phi_n^{(v)}(\omega_n) \\
&= \sum_{i,j,u,v} a_{ij}b_{uv}(\psi_k^{(i)} \otimes \phi_m^{(u)}) \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)})(\sigma_k, \omega_m, \sigma_l, \omega_n).
\end{aligned}$$

Thus, $\psi \otimes \phi = \sum_{i,j,u,v} a_{ij}b_{uv}(\psi_k^{(i)} \otimes \phi_m^{(u)}) \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)})$. So

$$\begin{aligned}
\psi \circ \phi &= \psi \otimes \phi \uparrow_{(\mathcal{S}_k \times \mathcal{S}_l)\times(\mathcal{S}_m \times \mathcal{S}_n)}^{\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}} \\
&= \sum_{i,j,u,v} a_{ij}b_{uv}(\psi_k^{(i)} \otimes \phi_m^{(u)}) \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)}) \uparrow_{\mathcal{S}_k \times \mathcal{S}_m \times \mathcal{S}_l \times \mathcal{S}_n}^{\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}} \\
&= \sum_{i,j,u,v} a_{ij}b_{uv}(\psi_k^{(i)} \otimes \phi_m^{(u)}) \uparrow_{\mathcal{S}_k \times \mathcal{S}_m}^{\mathcal{S}_{k+m}} \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)}) \uparrow_{\mathcal{S}_l \times \mathcal{S}_n}^{\mathcal{S}_{l+n}} \\
&= \sum_{i,j,u,v} a_{ij}b_{uv}(\psi_k^{(i)} \circ \phi_m^{(u)}) \otimes (\psi_l^{(j)} \circ \phi_n^{(v)})
\end{aligned}$$

by Lemma 2.2.6. Now take the product Frobenius characteristic of both sides of the above equation. For clarity, we keep track of variables x and y. By Remark 2.2.3 and then Proposition 2.1.7 we get

$$\begin{aligned} ch(\psi \circ \phi)(x,y) &= \sum_{i,j,u,v} a_{ij} b_{uv} ch(\psi_k^{(i)} \circ \phi_m^{(u)})(x) ch(\psi_l^{(j)} \circ \phi_n^{(v)})(y) \\ &= \sum_{i,j,u,v} a_{ij} b_{uv} ch(\psi_k^{(i)})(x) ch(\phi_m^{(u)})(x) ch(\psi_l^{(j)})(y) ch(\phi_n^{(v)})(y) \\ &= \sum_{i,j} a_{ij} ch(\psi_k^{(i)})(x) ch(\psi_l^{(j)})(y) \sum_{u,v} b_{uv} ch(\phi_m^{(u)})(x) ch(\phi_n^{(v)})(y) \\ &= ch(\psi)(x,y) ch(\phi)(x,y) \end{aligned}$$

■

2.3 A symmetric function analogue

Using the product Frobenius characteristic map, we derive an equation that is analogous to a well-known symmetric function identity (see Stanley [17, equation (7.13)]): for $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.$$

The thing to note is that the elementary symmetric function e_i is the Frobenius characteristic of the representation of $\mathcal{S}_i$ on the top homology of $\bar{B}_i$ (see the example in the end of section 2.1). Our analogue will involve the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the top homology of the proper part of Segre product poset $B_n \circ_{\rho_n} B_n$.

In the proof of the following theorem, we used Whitney homology technique, which was introduced by Sundaram [19] for pure posets and then generalized by Wachs [20] for semipure posets.

Theorem 2.3.1 *For the subset lattice B_n with rank function ρ_n, let P_n be the proper part of the Segre product poset $B_n \circ_{\rho_n} B_n$. Write $\mathcal{S}_n$ for the symmetric group on $[n]$. The action of $\mathcal{S}_n \times \mathcal{S}_n$ induces a representation on the reduced top homology of P_n. Let $ch(\widetilde{H}_{n-2}(P_n))$ be the product Frobenious characteristic of this representation. Then*

$$\sum_{i=0}^{n}(-1)^i h_{n-i}(x)h_{n-i}(y)ch(\widetilde{H}_{i-2}(P_i)) = 0, \tag{2.3.1}$$

where h_k's are the complete homogeneous symmetric functions.

Proof Let Q be $P_n \cup \hat{0}$, which is Cohen-Macaulay. We consider the Whitney homology of Q as discussed in Sundaram [19]. The action of $S_n \times S_n$ on Q induces a representation of $S_n \times S_n$ on the reduced top homology of Q and its Whitney homology groups. From the work of Sundaram on Whitney homology (Sundaram [18, 19], Wachs [21]), we know that

$$\widetilde{H}_{n-2}(P_n) \cong_{S_n \times S_n} \bigoplus_{r=0}^{n-1}(-1)^{n-1+r}\mathrm{WH}_r(Q).$$

Let x be a rank r element of Q. Then the stabilizer of x is the young subgroup $(S_r \times S_{n-r}) \times (S_r \times S_{n-r})$. We can view the Whitney homology groups as representations. The poset Q is Cohen-Macaulay and has a bottom element $\hat{0}$, the Whitney homology of Q is defined to be

$$\mathrm{WH}_r(Q) = \bigoplus_{x \in Q_r/(S_n \times S_n)} \widetilde{H}_{r-2}(\hat{0}, x) \uparrow_{(S_r \times S_{n-r})\times(S_r \times S_{n-r})}^{S_n \times S_n},$$

where Q_r is the set of rank r elements in Q and $Q_r/(S_n \times S_n)$ is a set of orbit representatives (see Lecture 4.4 in Wachs' *Poset Topology* [21]). The action of $S_n \times S_n$ on Q_r is transitive. So the contribution of the rth Whitney homology to $\widetilde{H}_{n-2}(P_n)$ is the induced representation $\widetilde{H}_{r-2}(\hat{0}, x) \uparrow_{(S_r \times S_{n-r})\times(S_r \times S_{n-r})}^{S_n \times S_n}$ for any x in Q_r. The open interval $(\hat{0}, x)$ is isomorphic to the poset P_r. We then have

$$\widetilde{H}_{n-2}(P_n) \cong_{S_n \times S_n} \bigoplus_{r=0}^{n-1} (-1)^{n-1+r} \widetilde{H}_{r-2}(P_r) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n} .$$

Taking the product Frobenius characteristic of both sides of the above equation,

$$ch(\widetilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1} (-1)^{n-1+r} ch\big(\widetilde{H}_{r-2}(P_r) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n}\big). \tag{2.3.2}$$

Now we would like to relate $ch(\widetilde{H}_{r-2}(P_r))$ with the product Frobenius characteristic of the representation induced to $\mathcal{S}_n \times \mathcal{S}_n$. Let ψ_r be the character of the $(S_r \times S_r)$-module $\widetilde{H}_{r-2}(P_r)$. Write $1_{S_{n-r} \times S_{n-r}}$ for the character of the trivial representation of $S_{n-r} \times S_{n-r}$. When viewing $\widetilde{H}_{r-2}(P_r)$ as a $(S_r \times S_{n-r}) \times (S_r \times S_{n-r})$-module, its character equals $\psi_r \otimes 1_{S_{n-r} \times S_{n-r}}$ (Sagan, [13, Theorem 1.11.2]). Let $\psi_r \circ 1_{S_{n-r} \times S_{n-r}}$ denote the induction product of ψ_r and $1_{S_{n-r} \times S_{n-r}}$. Then

$$\begin{aligned} \widetilde{H}_{r-2}(P_r) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n} &= \psi_r \otimes 1_{S_{n-r} \times S_{n-r}} \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n} \\ &= \psi_r \circ 1_{S_{n-r} \times S_{n-r}}. \end{aligned}$$

It follows from Proposition 2.2.5 that the product Frobenius characteristic

$$ch(\psi_r \circ 1_{S_{n-r} \times S_{n-r}}) = ch(\psi_r) ch(1_{S_{n-r} \times S_{n-r}}).$$

Thus, equation (2.3.2) becomes

$$\begin{aligned} ch(\widetilde{H}_{n-2}(P_n)) &= \sum_{r=0}^{n-1} (-1)^{n-1+r} ch(\widetilde{H}_{r-2}(P_r)) ch(1_{S_{n-r} \times S_{n-r}}) \\ &= \sum_{r=0}^{n-1} (-1)^{n-1+r} ch(\widetilde{H}_{r-2}(P_r)) ch(1_{S_{n-r}})(x) ch(1_{S_{n-r}})(y). \end{aligned} \tag{2.3.3}$$

It is known that the Frobenius characteristic of the trivial representation of $\mathcal{S}_n$ is h_n (See Equation (7.85) in Stanley [17]). Multiplying both sides of equation (2.3.3) by $(-1)^{n-1}$, we get

$$(-1)^{n-1}ch(\widetilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1}(-1)^r ch(\widetilde{H}_{r-2}(P_r))h_{n-r}(x)h_{n-r}(y).$$

Finally, we conclude that

$$\sum_{i=0}^{n}(-1)^i h_{n-i}(x)h_{n-i}(y)ch(\widetilde{H}_{i-2}(P_i)) = 0.$$

■

Theorem 2.3.1 was motivated by our initial finding, which we will present in the next chapter. Our initial finding can be seen as a specialized case of equation (2.3.1), suggesting the truth of Theorem 2.3.1.

Chapter 3

q-analogue of a result by Carlitz, Scoville, and Vaughan

3.1 Introduction — Carlitz, Scoville, and Vaughan's result

Consider the power series $f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{n!n!}$ and define the numbers ω_0, ω_1, ω_2,...by $\frac{1}{f(z)} = \sum_{n=0}^{\infty} \omega_n \frac{z^n}{n!n!}$. It follows quickly from the definition that

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 \omega_k = 0. \tag{3.1.1}$$

Given $\sigma \in \mathcal{S}_n$ a permutation of $[n] := \{1, 2, \ldots, n\}$. We call $i \in [n-1]$ an **ascent** of σ if $\sigma(i) < \sigma(i+1)$. Carlitz, Scoville and Vaughan [5] proved that the number ω_k in equation (3.1.1) is the number of pairs of permutations of $\mathcal{S}_k$ with no common ascent. For example, $\omega_2 = 3$. The three pairs of permutations of [2] with no common ascent written in one-line notation are $(12, 21)$, $(21, 12)$, $(21, 21)$. The Bessel function $J_0(z)$ is essentially $f(z^2)$ (See Section 1.2 in Gasper and Rahman [6]). Carlitz, Scoville and Vaughan's result provided a combinatorial interpretation of the coefficients ω_k in the reciprocal Bessel function, which in turn, gives a method to compute those coefficients.

Recall that $[n]_q := q^{n-1} + q^{n-2} + ... + 1$ is the q-analogue of the natural number n and $[n]_q! := \prod_{i=1}^{n} [i]_q$. Then the q-analogue of $\binom{n}{k}$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$. For a permutation $\sigma \in \mathcal{S}_n$, the **inversion statistic** is defined by

$$inv(\sigma) := |\{(i,j) : 1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)\}|.$$

In this chapter we will prove the following q-analogue of Carlitz, Scoville and Vaughan's result. Let $\mathcal{D}_n$ denote the set $\{(\sigma, \omega) \in \mathcal{S}_n \times \mathcal{S}_n \mid \sigma \text{ and } \omega \text{ have no common ascent}\}$. Define $W_i(q) = \sum_{(\sigma,\omega)\in\mathcal{D}_i} q^{inv(\sigma)+inv(\omega)}$, then

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q) = 0. \tag{3.1.2}$$

Put $F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{[n]_q![n]_q!}$. The function $F\Big((\frac{z}{2(1-q)})^2\Big)$ is the q-Bessel function $J_0^{(1)}(z;q)$. The q-Bessel functions were first introduced by F. H. Jackson in 1905 and can be found in later literature (see Gasper and Rahman [6]). By Equation (3.1.2), the polynomials $W_n(q)$ satisfy $\frac{1}{F(z)} = \sum_{n=0}^{\infty} W_n(q) \frac{z^n}{[n]_q![n]_q!}$. We have thus found a combinatorial meaning for coefficients of the reciprocal q-Bessel function $1/J_0^{(1)}(z;q)$, and a formula for computing those coefficients. The coefficient $W_n(q)$ is in fact the reduced Euler characteristic of the Segre product poset $B_n(q) \circ_\rho B_n(q)$.

3.2 A q-analogue of Carlitz, Scoville, and Vaughan's result

Recall the definition of $B_n(q)$. Let q be a prime power and $\mathbb{F}_q$ be the finite field of q elements. Consider the n-dimensional vector space $\mathbb{F}_q^n$ and its subspaces, then $B_n(q)$ is the lattice of those subspaces ordered by inclusion. The poset $B_n(q)$ is a geometric lattice whose every subspace is a span of its atoms (Stanley [16, Example 3.10.2]). It is graded with a rank function $\rho(W) :=$ the dimension of the subspace W.

An **edge labeling** of a bounded poset P is a map $\lambda : \mathcal{E}(P) \longrightarrow \Sigma$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of P and Σ is some poset. A maximal chain $c = (\hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_t \lessdot \hat{1})$ is **increasing** if $\lambda(\hat{0}, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_t, \hat{1})$ in Σ, and **decreasing** if $\lambda(\hat{0}, x_1) \geq \lambda(x_1, x_2) \geq \cdots \geq \lambda(x_t, \hat{1})$ in Σ. A chain c is associated with a word

$$\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2) \cdots \lambda(x_t, \hat{1}).$$

If $\lambda(c_1)$ lexicographically precedes $\lambda(c_2)$, we say that c_1 lexicographically precedes c_2 and we denote this by $c_1 <_L c_2$. Let us review the definition of an EL-labeling of a poset.

Definition 3.2.1 *(Björner and Wachs* [2, Definition 2.1]*) An edge labeling is called an* ***EL-labeling*** *(edge lexicographical labeling) if for every interval* $[x, y]$ *in* P*,*

(1) there is a unique increasing maximal chain c in $[x, y]$*, and*

(2) $c <_L c'$ *for all other maximal chains* c' *in* $[x, y]$*.*

It is well known that $B_n(q)$ admits an EL-labeling (See [21, Exercise 3.4.7]). The lexicographic ordering of all maximal chains in an EL-labeling gives a shelling order of all facets of the order complex. A poset that admits an EL-labeling is shellable. Here we describe a specific EL-labeling of $B_n(q)$, which we will use to prove our results, in the following steps:

1. For a 1-dimensional subspace X of $\mathbb{F}_q^n$ (an atom of $B_n(q)$), let x, a row vector, be a basis element of X. Let A denote the set of all atoms of the subspace lattice $B_n(q)$. We define a map f: $A \longrightarrow [n]$, $f(X) =$ the index of the right most non-zero coordinate of x. For example, in $B_3(3)$, if $X =$ span of $\{< 1, 0, 1 >\}$, $Y =$ span of $\{< 2, 1, 0 >\}$, $f(X) = 3$ and $f(Y) = 2$.

2. For X any subspace of $\mathbb{F}_q^n$, let $A(X)$ denote the set of atoms whose span is X. Let Y be an element of $B_n(q)$ that covers X, then $A(Y) \supset A(X)$. Denote the set

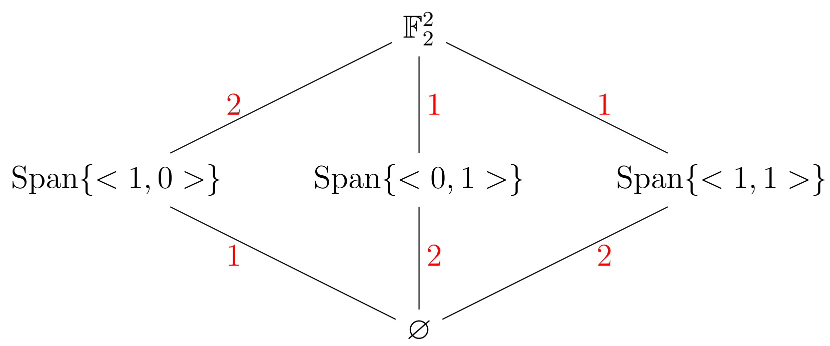

Figure 3.1. An EL-labeling of $B_2(2)$

$f(A(Y))\backslash f(A(X))$ by $\mathcal{L}$. Let ρ be the rank function of $B_n(q)$, which is defined by the dimensions of the subspaces. Because $\rho(Y) - \rho(X) = 1$, the set $\mathcal{L}$, which is a subset of $[n]$, must have exactly one element. This number is the new dimension added to Y. And that will be the label of the edge (X, Y).

Example 3.2.2 *Figure 3.1 is an edge labeling of the subspace lattice $B_2(2)$ following the rules described above. We will show in the following proposition that this labeling is an EL-labeling.*

Proposition 3.2.3 *The edge labeling described above is an EL-labeling on the subspace lattice $B_n(q)$.*

Proof Edges in the same chain cannot take duplicate labels since $\mathbb{F}_q^n$ is n-dimensional and any maximal chain must take all labels in $\{1, 2, \ldots, n\}$. Let $[X, Y]$ be a closed interval in $B_n(q)$. All maximal chains of $[X, Y]$ will take labels from the set $\mathcal{L} := f(A(Y))\backslash f(A(X))$. Let $0 < a_1 < a_2 < \cdots < a_l \leq n$ be all the elements of $\mathcal{L}$. For each i, $1 \leq i \leq l$, there is a 1-dimensional subspace V_i of $\mathbb{F}_q^n$ with $f(V_i) = a_i$ and $V_i \vee X$, the join of V_i and X, is in $[X, Y]$. The chain $c = (X \lessdot X \vee V_1 \lessdot \cdots \lessdot X \vee V_1 \vee V_2 \vee \cdots \vee V_l = Y)$ is an increasing maximal chain of $[X, Y]$. Any other 1-dimensional subspace V_i' satisfying

$f(V_i') = a_i$ and $X \vee V_1 \vee \cdots V_{i-1} \vee V_i' \in [X, Y]$ must equal $X \vee V_1 \vee \cdots \vee V_i$. Since there is only one way to arrange the a_i's increasingly, c satisfies definition 3.2.1 condition (1).

Suppose there is another maximal chain $c' = (X = W_0 \lessdot W_1 \lessdot \cdots \lessdot W_l = Y)$. Let $f(A(W_i)) \backslash f(A(W_{i-1})) = b_i$ for all $i \in [l]$. Let k, $1 \leq k \leq l$, be the smallest integer such that $b_k \neq a_k$. We know that b_k must be in $\mathcal{L}$ and $b_k \neq a_1, a_2, \ldots, a_k$. Also $a_1, a_2, ..., a_k$ are the smallest k elements of $\mathcal{L}$ arranged increasingly. It follows immediately that $b_k > a_k$. Therefore condition (2) in the definition of EL-labeling is also satisfied.

■

Under this EL-labeling, each maximal chain of the subspace lattice $B_n(q)$ can then be identified with a permutation σ of S_n. See section 3.1 for the definition of inversion statistic $inv(\sigma)$.

Lemma 3.2.4 *The number of maximal chains of $B_n(q)$ assigned label $\sigma \in S_n$ is $q^{inv(\sigma)}$.*

Proof Let $\sigma \in S_n$, for each 1-dimensional subspace of $\mathbb{F}_q^n$, we can take the vector whose right most non-zero coordinate is 1 as its basis element. For each $i \in [n-1]$, let $inv(\sigma(i))$ denote the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. The number of ways to choose an atom W_1 such that the edge $(0, W_1)$ takes label $\sigma(1)$ is $q^{inv(\sigma(1))}$. Let $k \in [n]$, assume the chain $0 \lessdot W_1 \lessdot ... \lessdot W_{k-1}$, has label $\sigma(1)\sigma(2)...\sigma(k-1)$. For each $i \in [k-1]$, pick an atom $V_i \in A(W_{k-1})$ with $f(V_i) = \sigma(i)$ and v_i the basis element of V_i. The vectors v_1, v_2, ..., v_{k-1} are linearly independent hence form a basis of W_{k-1}. In order for the edge (W_{k-1}, W_k) to take label $\sigma(k)$, W_k needs to be the join of W_{k-1} and an atom whose basis element, call it v_k, has 1 on the $\sigma(k)$th coordinate and all 0's after the $\sigma(k)$th coordinate. Then v_1, v_2, ..., v_k will form a basis for W_k. So we need to find the number of ways to choose a v_k that each results in a distinct W_k.

The vector $e_{\sigma(k)} =< 0, ..., 0, 1, 0, ...0 >$ who has 1 on the $\sigma(k)$th coordinate and 0 everywhere else certainly is a choice for v_k. For each j, such that $1 \le k < j \le n$ and $\sigma(k) > \sigma(j)$, the edge label $\sigma(j)$ comes after $\sigma(k)$ in this chain label σ. So W_{k-1} has no vectors whose right most non-zero coordinate is the $\sigma(j)$th. And the $\sigma(j)$th coordinate appears before the $\sigma(k)$th in a vector. So varying the $\sigma(j)$th coordinate of $e_{\sigma(k)}$ will produce new vectors that are not in the span of $\{v_1, ..., v_{k-1}, e_{\sigma(k)}\}$. There are $inv(\sigma(k))$ choices for j, and for each j, there are q choices for the value of the jth coordinate. Each choice will produce a distinct v_k that is linearly independent of $v_1, v_2, \ldots, v_{k-1}$, thus a distinct W_k. Therefore for any given chain $0 \lessdot W_1 \lessdot ... \lessdot W_{k-1}$ assigned label $\sigma(1)\sigma(2)...\sigma(k-1)$, there are $q^{inv(\sigma(k))}$ choices for W_k such that the edge (W_{k-1}, W_k) takes label $\sigma(k)$. Hence the number of maximal chains assigned label σ is $\prod_{i=1}^{i=n} q^{inv(\sigma(i))} = q^{\sum_{i=1}^{i=n} inv(\sigma(i))} = q^{inv(\sigma)}$. ■

The following theorem from Björner and Wachs is essential to connecting the permutations of $\mathcal{S}_n$ with the Segre product poset $B_n(q) \circ_\rho B_n(q)$:

Theorem 3.2.5 *(Björner and Wachs* [3, Theorem 4.1]*, see also Wachs* [21, Theorem 3.2.4]*). Suppose P is a poset for which $\hat{P}$ admits an EL-labeling. Then P has the homotopy type of a wedge of spheres, where the number of i-spheres is the number of decreasing maximal $(i+2)$-chains of $\hat{P}$.*

Now consider the Segre product of $B_n(q)$ with itself. Denote the proper part of this Segre product by $P_n(q)$. Using the EL-labeling of $B_n(q)$ described right after definition 3.2.1, the Segre product poset $B_n(q) \circ_\rho B_n(q)$ admits an edge-labeling in which the labels are ordered pairs from the poset $[n] \times [n]$. A label $(i, j) \in [n] \times [n] \le (k, l)$ if and only if $i \le k$ and $j \le l$. It is easy to verify that this labeling of $B_n(q) \circ_\rho B_n(q)$ is an EL-labeling. The decreasing chains are labeled with pairs of permutations with no common ascent.

Given a pair of permutations (σ, ω), the number of decreasing maximal chains assigned label (σ, ω) is $q^{inv(\sigma)} \cdot q^{inv(\omega)}$ from Lemma 3.2.4. Recall that $\mathcal{D}_n$ denotes the set of pairs of permutations $(\sigma, \omega) \in \mathcal{S}_n \times \mathcal{S}_n$ with no common ascent. We immediately arrive at the following proposition:

Proposition 3.2.6 *Let $W_n(q)$ be the total number of decreasing maximal chains of $B_n(q) \circ_\rho B_n(q)$. Then*

$$W_n(q) = \sum_{(\sigma,\omega)\in\mathcal{D}_n} q^{(inv(\sigma)+inv(\omega))}.$$

Remark 3.2.7 *The Segre product poset $B_n(q) \circ B_n(q)$ is the q-analogue of the Segre product poset $B_n \circ B_n$, agreeing with the formal definition of a q-analogue in R. Simion's paper* [14]. *She showed that the q-analogue of an EL-shellable poset is also EL-shellable. This particular EL-labeling of $B_n(q) \circ B_n(q)$ provided intuition and a combinatorial interpretation for $W_n(q)$.*

Theorem 3.2.8 *Let $P_n(q)$ be the proper part of the Segre product poset $B_n(q) \circ_\rho B_n(q)$. Let $\begin{bmatrix} n \\ i \end{bmatrix}_q$ be the q-analogue of $\binom{n}{i}$ and $W_n(q)$ be the total number of decreasing maximal chains of $B_n(q) \circ_\rho B_n(q)$. Then*

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q) = 0. \tag{3.2.1}$$

Proof The poset $P_n(q)$ is pure. By Theorem 3.2.5, $P_n(q)$ has the homotopy type of a wedge of $(n-2)$-spheres, and the number of decreasing maximal $(n-2)$-chains is the number of spheres. Then following from Proposition 3.2.6, $W_n(q)$ is the number of $n-2$-dimensional faces of $P_n(q)$ and the reduced Euler characteristic of $\triangle(P_n(q))$ is

$$\widetilde{\chi}(\Delta(P_n(q))) = (-1)^n W_n(q). \tag{3.2.2}$$

We know that the Möbius number of a poset is the same as its reduced Euler Characteristic by Philip Hall's theorem (Stanley [16, Proposition 3.8.6]), which gives us

$$\mu_{\widehat{P_n(q)}}(\hat{0}, \hat{1}) = (-1)^n W_n(q) = \widetilde{\chi}(\Delta(P_n(q))). \tag{3.2.3}$$

On the other hand, by the definition of the möbius function,

$$\mu(\hat{0}, \hat{1}) = -\sum_{\hat{0} \leq x < \hat{1}} \mu(\hat{0}, x).$$

Each x in $P_n(q)$ is a subspace of $\mathbb{F}_q^n \times \mathbb{F}_q^n$, which is the product of two k-dimensional subspaces X_1, X_2 of $\mathbb{F}_q^n$ for some k with $0 \leq k < n$. But the intervals $[\hat{0}, X_1]$ and $[\hat{0}, X_2]$ are isomorphic to the poset $B_k(q)$, hence $\mu(\hat{0}, x)$ is just $\mu_{\widehat{P_k(q)}}(\hat{0}, \hat{1})$, where $P_k(q) = B_k(q) \circ_\rho B_k(q) \setminus \{\hat{0}, \hat{1}\}$. The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (Stanley [16, Proposition 1.7.2]), the q-analogue of $\binom{n}{k}$. So the number of distinct $x = (X_1, X_2)$ where X_1 and X_2 are k-dimensional subspaces is $\begin{bmatrix} n \\ k \end{bmatrix}_q^2$. Therefore we have

$$\mu_{\widehat{P_n(q)}}(\hat{0}, \hat{1}) = -\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 \mu_{\widehat{P_i(q)}}(\hat{0}, \hat{1}) = -\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q).$$

Consequently,

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q) = 0.$$

By Proposition 3.2.6, $W_i(q) = \sum_{(\sigma,\omega) \in \mathcal{D}_i} q^{(inv(\sigma)+inv(\omega))}$ is the number of decreasing maximal chains of $P_i(q)$, where $\mathcal{D}_i$ denotes the set of pairs of permutations $(\sigma, \omega) \in \mathcal{S}_i \times \mathcal{S}_i$ with no common ascent. ■

Corollary 3.2.9 *The Euler characteristic of the Segre product of the subspace lattice $B_n(q) \circ_\rho B_n(q)$ is $(-1)^n W_n(q)$.*

Proof See equation (3.2.2) in the proof of theorem 3.2.8. ■

3.3 An alternative proof of Carlitz, Scoville, and Vaughan's result

In [5], Carlitz, Scoville and Vaughan gave the coefficients ω_k of the reciprocal of the Bessel function $J_0(z)$ a combinatorial explanation. They showed that ω_k is the number of pairs of k-permutations with no common ascent. When letting $q = 1$ in our q-analogue (3.2.1), the subspaces of $\mathbb{F}_q^n$ become subsets of $\{1, 2, ..., n\}$. The value $W_n(1) = \sum_{(\sigma,\omega)\in\mathcal{D}_n} 1^{inv(\sigma)+inv(\omega)}$ simply counts the number of pairs of permutations of $[n]$ with no common ascent, i.e. ω_n. We then obtain the above result from Carlitz, Scoville and Vaughan.

The proof of theorem 3.2.8 can also be easily adapted to an alternative proof of Carlitz, Scoville and Vaughan's result (3.1.1) by changing $B_n(q)$ to B_n, using P_n instead of $P_n(q)$ to denote the Segre product, and recognizing that the intervals in the alternating sum for the Möbius number of $\widehat{P_n}$ are isomorphic to smaller subset lattices B_i's. Carlitz, Scoville and Vaughan's proof in [5] included general cases where occurrences of common ascent are allowed. Our proof provides a less technical approach by utilizing Björner and Wachs' work on shellability and poset homology [3].

Chapter 4

The Connection of Two Analogues

4.1 Specialization of Symmetric Functions

Let $ps : \Lambda \longrightarrow \mathbb{Q}[q]$ be the **stable principal specialization**. For a symmetric function $f(x_1, x_2, x_3, \ldots)$, $ps(f)$ is defined to be $f(1, q, q^2, ...)$. A summary of the specializations of different bases for the symmetric functions can be found in Stanley's *Enumerative Combinatorics vol.* 2 [17, proposition 7.8.3]. Consider a symmetric function f in two sets of variables $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$. We take the stable principal specialization of f in each set of variables, that is substituting $(1, q, q^2, ...)$ for both $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$. The product Frobenius characteristic of the $\mathcal{S}_n \times \mathcal{S}_n$-module $\widetilde{H}_{n-2}(P_n)$ is a symmetric function in two sets of variables. Then it is natural to ask what we can say about its specialization. It turns out that $ps(ch(\widetilde{H}_{n-2}(P_n)))$ has an interesting relation with the Euler characteristic of the Segre product poset $B_n(q) \circ_\rho B_n(q)$.

4.2 The Connection

Recall that P_n is the proper part of the Segre product of the subset lattice B_n with itself. The product Frobenius characteristic of the $\mathcal{S}_n \times \mathcal{S}_n$-module $\widetilde{H}_{n-2}(P_n)$ has an innate connection with $W_n(q)$. The following theorem provides an equation that connects the stable principal specialization of $ch(\widetilde{H}_{n-2}(P_n))$ and the Euler characteristic $W_n(q)$.

Theorem 4.2.1 *Let P_n be the proper part of Segre product $B_n \circ_\rho B_n$ and $\mathcal{S}_n$ the symmetric group. The action of $\mathcal{S}_n \times \mathcal{S}_n$ on $B_n \circ B_n$ induces a representation on the reduced top homology of P_n. Let $(-1)^n W_n(q)$ be the Euler characteristic of the Segre product $B_n(q) \circ_\rho B_n(q)$. For a symmetric function f in two sets of variables $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$, the stable principal specialization $ps(f)$ specializes both x_i and y_i to q^{i-1}. Then*

$$ps(ch(\widetilde{H}_{n-2}(P_n))) = \frac{W_n(q)}{\prod_{i=1}^{n}(1-q^i)^2}, \tag{4.2.1}$$

where $ch(V)$ is the product Frobenius characteristic of a $\mathcal{S}_n \times \mathcal{S}_n$-module V.

Proof We will use induction. The base cases $n = 2$ and $n = 3$ can be verified by hand.

$$ps(ch(\widetilde{H}_0(P_2))) = \frac{q^2 + 2q}{(1-q)^2(1-q^2)^2} = \frac{W_2(q)}{(1-q)^2(1-q^2)^2}$$

and

$$ps(ch(\widetilde{H}_1(P_3))) = \frac{q^6 + 4q^5 + 6q^4 + 6q^3 + 2q^2}{(1-q)^2(1-q^2)^2(1-q^3)^2} = \frac{W_3(q)}{(1-q)^2(1-q^2)^2(1-q^3)^2}.$$

Assume that the statement is true for $P_i, i = 1, ..., n-1$. Now let us consider the reduced top homology of P_n. Equation (2.3.1) gives us a way to express $ch(\widetilde{H}_{n-2}(P_n))$ in terms of the product Frobenius characteristic of smaller posets. That is

$$ch(\widetilde{H}_{n-2}(P_n)) = \sum_{i=0}^{n-1}(-1)^{n-1+i} h_{n-i}(x) h_{n-i}(y) ch(\widetilde{H}_{i-2}(P_i)) \tag{4.2.2}$$

Then we take the stable principal specialization of both sides of equation (4.2.2). We know from Stanley's *Enumerative Combinatorics vol. 2* that $ps(h_n) = \prod_{i=1}^{n} \frac{1}{1-q^i}$ [17]. It follows from our induction hypothesis that

$$
\begin{aligned}
ps(ch(\widetilde{H}_{n-2}(P_n))) &= \sum_{i=0}^{n-1}(-1)^{n-1+i}ps(ch(\widetilde{H}_{i-2}(P_i)))\prod_{j=1}^{n-i}\frac{1}{(1-q^j)^2} \\
&= \sum_{i=0}^{n-1}(-1)^{n-1+i}\frac{W_i(q)}{\prod_{k=1}^{i}(1-q^k)^2}\prod_{j=1}^{n-i}\frac{1}{(1-q^j)^2} \\
&= \frac{1}{\prod_{k=1}^{n}(1-q^k)^2}\cdot\sum_{i=0}^{n-1}(-1)^{n-1+i}W_i(q)\frac{\prod_{j=i+1}^{n}(1-q^j)^2}{\prod_{j=1}^{n-i}(1-q^j)^2} \\
&= \frac{1}{\prod_{k=1}^{n}(1-q^k)^2}\cdot\sum_{i=0}^{n-1}(-1)^{n-1+i}W_i(q)\begin{bmatrix} n \\ i \end{bmatrix}_q^2.
\end{aligned}
\tag{4.2.3}
$$

Finally, using the identity involving the Euler characteristic $W_n(q)$ given in Theorem 3.2.8, we obtain

$$
ps(ch(\widetilde{H}_n(P_n))) = \frac{W_n(q)}{\prod_{j=1}^{n}(1-q^j)^2}.
$$

■

When we take the stable principal specialization of Equation (2.3.1) in Theorem 2.3.1, we obtain Equation (3.2.1) in Theorem 3.2.8 and Equation (4.2.1) in Theorem 4.2.1. Therefore we can view the symmetric function analogue identity (2.3.1) as a generalization of our q-analogue identity (3.2.1) to a symmetric group representation result.

Bibliography

[1] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. AMS*, 260:159–183, 1980.

[2] A. Björner and M. L. Wachs. On lexicographically shellable posets. *Trans. AMS*, 277:323–341, 1983.

[3] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets I. *Trans. AMS*, 348:1299–1327, 1996.

[4] A. Björner and V. Welker. Segre and Rees products of posets, with ring-theoretic applications. *Journal of Pure and Applied Algebra*, 198:43–55, 2005.

[5] L. Carlitz, R. Scoville, and T. Vaughan. Enumeration of pairs of permutations. *Discrete Mathematics*, 14:215–239, 1976.

[6] G. Gasper and M. Rahman. Basic Hypergeometric Series, 2004.

[7] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 1984.

[8] G. James and A. Kerber. The Representation Theory of the Symmetric Group, 1981.

[9] G. James and M. Liebeck. *Representations and Characters of Groups.* Cambridge University Press, New York, second edition, 2001.

[10] L. Schläfli. Theorie der vielfachen Kontiutät, written 1850-1852. *Denkschriften der Schweizerischen naturforschenden Gesellschaft **38***, pages 1–237, 1901.

[11] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995.

[12] James R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 2001.

[13] B. E. Sagan. *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions.* Graduate Texts in Mathematics **203**. Springer-Verlag, New York, second edition, 2001.

[14] Rodica Simion. On q-analogues of partially ordered sets. *Journal of Combinatorial Theory, Series A*, 72(1):135 – 183, 1995.

[15] R. P. Stanley. Balanced Cohen-Macaulay complexes. *Trans. Amer. Math. Soc.* ***249***, No. 1:139–157, 1979.

[16] R. P. Stanley. *Enumerative Combinatorics, Vol. 1.* Cambridge Studies in Advanced Mathematics **49**. Cambridge University Press, Cambridge, 1997.

[17] R. P. Stanley. *Enumerative Combinatorics, Vol. 2.* Cambridge Studies in Advanced Mathematics **62**. Cambridge University Press, Cambridge, 1999.

[18] S. Sundaram. Applications of the Hopf trace formula to computing homology representations, 1994.

[19] S. Sundaram. The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice. *Advances in Math.*, 104:225–296, 1994.

[20] M. L. Wachs. Whitney homology of semipure shellable posets. *J. Algebraic Combin. 9*, No. 2:173–207, 1999.

[21] M. L. Wachs. Poset Topology: Tools and Applications. In B. Sturmfels E. Miller, V. Reiner, editor, *Geometric Combinatorics, IAS/PCMI Lecture Notes Series, vol. 13, Amer. Math. Soc., Providence, RI*, pages 497–615, 2007.