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Noncommutative Borsuk-Ulam Theorems

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by
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ABSTRACT OF THE DISSERTATION

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The Borsuk-Ulam theorem in algebraic topology shows that there are significant restrictions on how any topological sphere $\mathbb{S}^k$ interacts with the antipodal $\mathbb{Z}_2$ action of reflection through the origin ($x \mapsto -x$). For example, any map $f : \mathbb{S}^k \to \mathbb{S}^k$ which is continuous and odd ($f(-x) = -f(x)$) must be homotopically nontrivial. We consider various equivalent forms of the theorem in terms of the function algebras $C(\mathbb{S}^k)$ and examine which forms generalize to certain noncommutative Banach and C^*-algebras with finite group actions.

Chapter 1 contains background material on C^*-algebras, K-theory, and group actions. Next, in Chapter 2, we examine statements related to the Borsuk-Ulam theorem that may be applied on Banach algebras with $\mathbb{Z}_2$ actions; this work indicates when roots of elements do not exist and is motivated by the results of Ali Taghavi in [49]. We see that a variant of the Borsuk-Ulam theorem on $C(\mathbb{S}^k)$ written in terms of individual odd elements of $C(\mathbb{S}^k)$ does not extend to the noncommutative setting. In Chapter 3, we show that antipodally equivariant maps between θ-deformed spheres of the same dimension are nontrivial on K-theory. This generalizes the commutative case and parallels the work of Makoto Yamashita in [55] on the q-spheres, although our methods are quite different. Finally, Chapter 4 concerns a conjecture of Ludwik Dąbrowski in [14] that seeks to generalize noncommutative Borsuk-Ulam theory to arbitrary C^*-algebras through the use of unreduced suspensions. We prove Dąbrowski's conjecture and propose a new direction for continued study.

Chapter 1

Background

In 1943, I. Gelfand and M. Naimark proved a landmark theorem that jump-started the study of abstract C^*-algebras and, in time, brought to life an entirely new field of mathematics known as noncommutative geometry.

Theorem 1.0.1 (Gelfand-Naimark)**.** If A is a commutative C^*-algebra, then there is a locally compact Hausdorff space Y such that $A \cong C_0(Y)$, and Y is unique up to homeomorphism.

This theorem establishes that commutative C^*-algebras are dual to locally compact Hausdorff spaces, and this provides a lens through which to view noncommutative C^*-algebras; they should be dual to some object that we call a noncommutative space. Many invariants on topological spaces are realizable in arbitrary C^*-algebras, such as topological K-theory, and the field of noncommutative geometry also concerns questions related to differentiable structure (see [11] and [10]). However, the main goal of this work is to describe in what ways the famed Borsuk-Ulam theorem of algebraic topology, originally found in the 1933 paper [6], extends into the world of C^*-algebras.

Theorem 1.0.2 (Borsuk-Ulam)**.** Let $\mathbb{S}^k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} : x_1^2 + \ldots + x_{k+1}^2 = 1\}$ be a topological sphere. Then any continuous map $f : \mathbb{S}^k \to \mathbb{S}^k$ which is odd (satisfies $f(-\vec{x}) = -f(\vec{x})$) is homotopically nontrivial.

This pursuit is motivated by papers of A. Taghavi, M. Yamashita, and L. Dąbrowski; I will prove some results that are analogous to (or are extensions of) their theorems as well as answer some questions posed in their papers. Some of the content from this dissertation is contained in [34] (to appear in Journal of Operator Theory) and [35] (under review).

1.1 The Gelfand-Naimark Theorem

If H is a complex Hilbert space, then $B(H) = \{T : H \to H : T \text{ is linear and bounded}\}$ is a space of linear operators with a host of algebraic operations. We may form sums $T + S$ and scalar multiples λT for operators $T, S \in B(H)$ and a scalar $\lambda \in \mathbb{C}$, and we may also compose operators, which is denoted $S \circ T$ or ST. The second notation is more concise, but it also emphasizes a mentality that composition is an abstract multiplication, and this multiplication distributes over addition. The space $B(H)$ is also equipped with a norm $||T|| = \sup\{||Th|| : h \in H, ||h|| \leq 1\}$ defined in terms of the Hilbert space norm $||h|| = \sqrt{\langle h, h\rangle}$. This norm is complete and compatible with the algebraic operations, so $B(H)$ is a Banach algebra, with additional structure from the operator adjoint. If $T \in B(H)$, then the adjoint $T^* \in B(H)$ is the unique operator satisfying the following inner product identity for all $h, k \in H$.

$$\langle Th, k\rangle = \langle h, T^*k\rangle$$

The operator adjoint is antilinear and satisfies the identities $(ST)^* = T^*S^*$, $T^{**} = T$, and $||TT^*|| = ||T||^2$, which are generalized in the definition of a C^*-algebra.

Definition 1.1.1. A C^*-algebra A is a Banach algebra with a unary operation $* : A \to A$ satisfying the following identities for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.

1. $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$
2. $(ab)^* = b^*a^*$
3. $a^{**} = a$
4. $||aa^*|| = ||a||^2$

The identity operator in $B(H)$ is a multiplicative unit, but existence of an element $1 \in A$ is not required in the definition of a C^*-algebra. Such algebras where 1 exists are called *unital* or *with unit.* Any norm-closed $*$-subalgebra of $B(H)$ is a C^*-algebra, and in general C^*-algebras are not commutative: ab is not necessarily equal to ba. Examples of commutative C^*-algebras are found in function algebras, whose operations are defined pointwise.

Definition 1.1.2. If X is a compact Hausdorff topological space, then $C(X) = \{f : X \to \mathbb{C} : f \text{ is continuous}\}$ is a commutative C^*-algebra with unit. The addition, scalar multiplication, and element multiplication operations are defined pointwise, the adjoint is defined by complex conjugation $f^*(x) = \overline{f(x)}$, and the norm is the supremum norm $||f||_{\text{sup}} = \sup\{|f(x)| : x \in X\}$. The multiplicative unit is the function which takes the constant value 1.

The function algebra $C(X)$ is commutative, and because of the assumption that X is compact and Hausdorff, the functions in $C(X)$ separate the points of X. If C is a nonempty closed subset of X, the ideal $\{f \in C(X) : f|_C = 0\}$ is norm-closed and is itself a commutative C^*-algebra. However, the multiplicative unit is no longer present, as the only constant function vanishing on C takes value 0. When C consists of a single point p, it follows that $X \setminus \{p\}$ is locally compact, but not necessarily compact, and we may attempt to reconstruct X from $X \setminus \{p\}$.

Definition 1.1.3. If Y is a locally compact Hausdorff topological space, then the *one point compactification* $Y^+ = Y \amalg \{\infty\}$ is a compact Hausdorff space obtained by adding one point to Y. The neighborhoods of ∞ are of the form $\{\infty\} \cup (Y \setminus C)$ where C is a compact subspace of Y, and the subspace topology on $Y \subset Y^+$ is exactly the original topology on Y.

A locally compact Hausdorff space Y admits its own function algebra as a maximal ideal of $C(Y^+)$. When Y is actually compact, ∞ is an isolated point, essentially adding no information.

Definition 1.1.4. If Y is locally compact and Hausdorff, then $C_0(Y)$ is the *algebra of functions on Y vanishing at infinity*, a C^*-algebra with pointwise operations defined as follows.

$$\begin{aligned} C_0(Y) &= \{f \in C(Y^+) : f(\infty) = 0\} \\ &\cong \{f : Y \to \mathbb{C} : f \text{ is continuous and for any } \varepsilon > 0, \text{ there is a compact } C \subset Y \\ &\qquad \text{such that for any } y \in Y \setminus C, |f(y)| < \varepsilon\} \end{aligned}$$

The C^*-algebra $C_0(Y)$ is unital if and only if Y is compact. In this case, $C_0(Y)$ is isomorphic to $C(Y)$.

Function algebras $C_0(Y)$ are commutative, and it turns out that all commutative C^*-algebras are function algebras. This result is the Gelfand-Naimark theorem, which relies on the structure of multiplicative linear functionals.

Definition 1.1.5. If A is a unital C^*-algebra, then a *multiplicative linear functional* on A is a map $\phi : A \to \mathbb{C}$ which is linear, satisfies $\phi(ab) = \phi(a)\phi(b)$, and is not identically zero (or equivalently satisfies $\phi(1) = 1$). Any such ϕ has operator norm 1 and also automatically satisifies $\phi(a^*) = \overline{\phi(a)}$.

The set of multiplicative linear functionals $\mathcal{M} = \mathcal{M}(A)$ is a subset of the Banach algebra dual A^*, which consists of all bounded linear functionals on A. If $\mathcal{M}$ is assigned the subspace topology of the weak-* topology of A^*, then $\mathcal{M}$ is a closed subspace of the unit ball and therefore is compact by the Banach-Alaoglu theorem. Moreover, for commutative A, $\mathcal{M}$ is in one-to-one correspondence with the maximal ideals of A via $\phi \leftrightarrow \ker(\phi)$. Proofs of these facts, as well as proofs of many of the following foundational theorems in this section, can be found in [19].

Theorem 1.1.6 (Gelfand-Naimark, unital case)**.** If A is a commutative, unital C^*-algebra, then $X = \mathcal{M}(A)$ is the unique compact Hausdorff space (up to homeomorphism) that satisfies $C(X) \cong A$.

The Gelfand-Naimark theorem establishes a contravariant isomorphism between two categories: the category **Cpt** of compact Hausdorff spaces with continuous maps, and the category $\mathbf{C^*com}_1$ of commutative unital C^*-algebras with unital $*$-homomorphisms. In other words, $X = \mathcal{M}(A)$ completely determines $C(X) = A$ and vice-versa, but the domain and codomain of morphisms are reversed. A continuous map $f : X \to Y$ generates a $*$-homomorphism $C_f : C(Y) \to C(X)$, defined by $C_f(g) = g \circ f$, and similarly, a $*$-homomorphism $\psi : A \to B$ generates a continuous map $M_\psi : \mathcal{M}(B) \to \mathcal{M}(A)$, defined by $M_\psi(\phi) = \phi \circ \psi$. All of these properties imply that $\mathbf{C^*com}_1$ and **Cpt** are opposite categories (see [17]).

If A is a C^*-algebra that is not unital, then A is a maximal ideal of a unitization $A^+ = A \oplus \mathbb{C}$, which can be given a C^*-algebra structure. If A is commutative, then so is A^+, and A^+ is isomorphic to $C(X)$ for some compact Hausdorff X. All ideals of $C(X)$ are of the form $\{f \in C(X) : f|_C = 0\}$ where $C \subseteq X$ is closed, and since $A \leq C(X)$ is also a maximal ideal, it corresponds to a space of functions which vanish at one point $p \in X$. This gives A the structure of $C_0(X \setminus \{p\})$, where we note that $X \setminus \{p\}$ is locally compact. The claim that $A \cong C_0(Y)$ for some locally compact Y is then applicable to both the unital and nonunital case, as $C_0(Y) \cong C(Y)$ when Y is compact. The following general version of the Gelfand-Naimark theorem is then the same as Theorem 1.0.1.

Theorem 1.1.7 (Gelfand-Naimark, general case)**.** If A is a commutative C^*-algebra (unital or nonunital), then there is a locally compact Hausdorff space Y such that $A \cong C_0(Y)$, and Y is unique up to homeomorphism.

The assumption of commutativity in the Gelfand-Naimark theorem is essential, as $C_0(Y)$ is always commutative. Moreover, there are noncommutative C^*-algebras whose only ideals are the full ideal and trivial ideal, such as the matrix algebras $M_n(\mathbb{C})$ for $n \geq 2$, and these C^*-algebras do not have enough multiplicative linear functionals to describe their structure. On the other hand, it is still possible to view noncommutative C^*-algebras through the lens of

Gelfand-Naimark duality. For example, "noncommutative" compact Hausdorff spaces would be objects of $\mathbf{C^*alg}_1^{op}$ (the opposite category to the category of all unital C^*-algebras) that correspond to noncommutative unital C^*-algebras. This point of view can even sometimes provide unforeseen information in the topological setting.

1.2 Trace Functions and the Spectrum

If a compact Hausdorff space X is equipped with a positive, finite Baire measure μ, then every function in $C(X)$ produces an element of the Lebesgue space $L^p(X, \mu)$ for $1 \leq p \leq \infty$. If μ has full support, then any function in $C(X)$ is completely determined by its values almost everywhere, so the vector space $C(X)$ is (isomorphic to) a subspace of $L^p(X, \mu)$, which might not be closed in the p-norm. However, since any $f \in C(X)$ is bounded, there is also an associated multiplication operator M_f on the Lebesgue space $L^p(X, \mu)$, defined as follows.

$$M_f : L^p(X, \mu) \to L^p(X, \mu) \qquad\qquad g \mapsto f \cdot g$$

Now, M_f is a bounded linear map with operator norm equal to $||g||_{\text{sup}}$, the original norm of $C(X)$. Further, when $p = 2$, the space of multiplication operators then allows us to view $C(X)$ as a norm-closed $*$-subalgebra of $B(L^2(X, \mu))$. Operator algebras $B(H)$ and their norm-closed $*$-subalgebras are exactly the objects that motivated the abstract definition of a C^*-algebra, so one can ask if forming a C^*-algebra embedding $A \hookrightarrow B(H)$ is always possible regardless of commutativity or measure structure. The GNS construction shows that this is indeed always possible.

Theorem 1.2.1 (GNS Construction)**.** If A is a C^*-algebra, then there exists a Hilbert space H and an injective $*$-homomorphism $\phi : A \to B(H)$, so that A is isomorphic to a C^*-subalgebra of $B(H)$. If A is unital, we may insist that $\phi(1) = I$.

The GNS construction is named after Gelfand, Naimark, and I. Segal, and its method of proof encapsulates the entire Gelfand-Naimark theorem, even though the statements appear to be quite different. In both theorems, the key idea is to form a space of functionals.

Definition 1.2.2. A bounded linear functional $\nu : A \to \mathbb{C}$ defined on a C^*-algebra A is called *positive* if $\nu(aa^*) \geq 0$ for all $a \in A$.

If $\nu : A \to \mathbb{C}$ is a positive, bounded linear functional, then the binary operation $\langle a, b\rangle_\nu = \nu(ab^*)$ on A satisfies most of the properties of an inner product. One key exception is nondegeneracy: $\langle a, a\rangle_\nu$ might be zero even if $a \neq 0$. However, a quotient and completion of A produce a Hilbert space $L^2(A_\nu)$ on which $\langle \cdot, \cdot\rangle_\nu$ defines the inner product. For each $a \in A$, the multiplication operator $M_a : b \in A \mapsto ab \in A$ densely defines a bounded linear operator on $L^2(A_\nu)$, giving a homomorphism from A to $B(L^2(A_\nu))$. To form an injective homomorphism, this calculation is repeated for multiple functionals ν, giving a homomorphism from A to $B\left(\bigoplus_{\nu\in\mathcal{C}} L^2(A_\nu)\right)$. If $\mathcal{C}$ consists of all bounded positive linear functionals on A, the homomorphism is certainly injective, but the codomain is much larger than necessary. On the other hand, it is sufficient to have $\mathcal{C}$ consist of extreme points in the set of positive linear functionals with norm at most 1, retaining injectivity with a codomain $B\left(\bigoplus_{\nu\in\mathcal{C}} L^2(A_\nu)\right)$ that is not too large. Moreover, in the commutative case, these extreme point functionals are exactly the multiplicative linear functionals on A, and the GNS construction envelops the Gelfand-Naimark theorem. More details on this critical construction can be found in [16].

The view of a space X in terms of measure spaces (X, μ) also gives at least two other points of view that are important for functional analysis. The first is that the multiplication operator M_f on $L^2(X, \mu)$ can be defined not just for continuous functions, but for any members of $L^\infty(X, \mu)$ as well. The associated embedding $L^\infty(X, \mu) \hookrightarrow B(L^2(X, \mu))$ identifies $L^\infty(X, \mu)$ with a $*$-subalgebra that is not only norm-closed, but weakly closed. A weakly closed $*$-subalgebra of $B(H)$ that includes the identity operator is called a *von Neumann* *algebra*, and just as the Gelfand-Naimark theorem shows commutative C^*-algebras can be written as continuous function algebras, commutative von Neumann algebras take the form of L^∞ spaces. Similarly, the study of noncommutative von Neumann algebras can be called noncommutative measure theory, which is its own highly developed area of study that uses techniques quite different than noncommutative topology (see [3]). Although we will not need to discuss von Neumann algebras themselves any further, there is still one useful idea that can be motivated in terms of measure spaces. Given (X, μ) as above, we may apply the integration operator $f \mapsto \int f \, d\mu$ for any $f \in C(X)$. The appropriate generalization to arbitrary C^*-algebras is a trace function.

Definition 1.2.3. A *trace* on a C^*-algebra A is a linear functional $\tau : A \to \mathbb{C}$ satisfying $\tau(ab) = \tau(ba)$ for $a, b \in A$. A trace τ which is also positive as a linear functional is called *faithful* if for all $a \in A \setminus \{0\}$, $\tau(aa^*) > 0$.

A μ-integration functional on $C(X)$ is certainly a continuous, positive trace, and faithfulness of a trace τ is analogous to the claim that $\operatorname{supp}(\mu) = X$, for then if $f \in C(X)$ is not identically zero, it follows that $ff^* = |f|^2$ is nonnegative and takes values $|f(x)|^2 > \varepsilon > 0$ on a positive measure set for some $\varepsilon > 0$. This implies that $\int |f|^2 \, d\mu > 0$. On the other hand, if μ does not have full support, we may form a nontrivial bump function f with support inside $X \setminus \operatorname{supp}(\mu)$ that has $\int |f|^2 \, d\mu = 0$.

Example 1.2.4. On the matrix algebra $M_n(\mathbb{C})$, $\tau([a_{ij}]_{n \times n}) = \sum_{i=1}^{n} a_{ii}$ defines a positive, continuous, and faithful trace.

The above example is the traditional trace on square matrices. The commutativity restriction $\tau(ab) = \tau(ba)$ implies that τ respects diagonalization, so $\tau(uau^{-1}) = \tau(a)$ for invertible u. For diagonalizable matrices, this implies that τ just assigns a matrix the sum of its eigenvalues repeated with multiplicity, a claim which then holds for all matrices after we consider continuity. The set of eigenvalues of a matrix is also known as the spectrum of the matrix, a concept which extends to operators on infinite dimensional spaces. However, in the general case, the spectrum may be strictly larger than the set of eigenvalues.

Definition 1.2.5. If A is a unital Banach algebra and $a \in A$, then the *spectrum* of a, denoted $\sigma(a)$, is the set of all scalars $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible under multiplication.

A function $f \in C(X)$ is invertible under multiplication if and only if f never takes the value zero, so the spectrum of f is just its range.

$$\begin{aligned}
\sigma(f) &= \{\lambda \in \mathbb{C} : f - \lambda \text{ is not invertible under multiplication}\} \\
&= \{\lambda \in \mathbb{C} : f(x) - \lambda = 0 \text{ for some } x \in X\} \\
&= \{\lambda \in \mathbb{C} : f(x) = \lambda \text{ for some } x \in X\} \\
&= \operatorname{Ran}(f)
\end{aligned}$$

In this sense, an integration trace $f \mapsto \int f \, d\mu$ sees more structure than just the spectrum, as μ may be a nonuniform measure. However, the spectrum of a C^*-algebra element still contains a large amount of information, as seen in the classical spectral theorems. A spectral theorem for bounded, normal operators on a Hilbert space follows from the Gelfand-Naimark correspondence.

Definition 1.2.6. An element a of a C^*-algebra is *normal* if $aa^* = a^*a$, or *self-adjoint* if $a = a^*$. All self-adjoint elements are certainly normal.

Definition 1.2.7. If S is a subset of a C^*-algebra A, then $C^*(S)$ denotes the *C^*-algebra generated by* S, which is the smallest C^*-subalgebra of A which contains S. If $S = \{a_1, \ldots, a_n\}$, then $C^*(S)$ is also denoted $C^*(a_1, \ldots, a_n)$. Similarly, if A is unital, the *unital C^*-algebra generated by* S is $C^*(S \cup \{1\})$, also denoted $C^*(S, 1)$.

When $a \in A$ is a normal element of a unital C^*-algebra, $C^*(a)$ and $C^*(a, 1)$ are commutative, as every element can be approximated by $*$-polynomials in a. The Gelfand-Naimark theorem then says that $C^*(a,1) \cong C(X)$ for a compact Hausdorff space X, which is a space of multiplicative linear functionals. Since $\phi(1) = 1$ for every multiplicative linear functional ϕ, invertible elements u must satisfy $\phi(u) \neq 0$ from the identity $\phi(u)\phi(u^{-1}) = 1$. This implies that every multiplicative linear functional on $C^*(a,1)$ sends a to an element of $\sigma(a)$.

$$\begin{aligned} \lambda \notin \sigma(a) &\implies a - \lambda \text{ is invertible} \\ &\implies \phi(a-\lambda) \neq 0 \\ &\implies \phi(a) - \lambda \neq 0 \\ &\implies \phi(a) \neq \lambda \end{aligned}$$

On the other hand, the value of $\phi(a)$ determines the value of ϕ on every $*$-polynomial, and by continuity, on every element of $C^*(a,1)$, so multiplicative linear functionals on $C^*(a,1)$ are determined by points in the spectrum of a. Similarly, a multiplicative linear functional can be produced for every scalar in $\sigma(a)$, and the weak-* topology on the space of functionals agrees with the topology of $\sigma(a)$. The result is a spectral theorem.

Theorem 1.2.8 (A Spectral Theorem)**.** If a is a normal element of a unital C^*-algebra, then $C^*(a,1)$ is isomorphic to $C(\sigma(a))$ via an isomorphism which sends a to the functional identity $\mathrm{Id}(z) = z$ and 1 to the multiplicative identity $\mathbb{1}(z) = 1$.

When $A = B(H)$, this is a spectral theorem for bounded normal operators, which takes a special form when the operator T is self-adjoint and compact. In this case, the spectrum $\sigma(T) \subset \mathbb{R}$ is made up of countably many eigenvalues λ_i that converge to 0, which implies each λ_i is of finite multiplicity, along with 0 if 0 is not already an eigenvalue. The isomorphism in the spectral theorem specifies that T corresponds to the function $\mathrm{Id}(x) = x$ in $C(\sigma(T))$, but this can be built from components $c_i(x) = \begin{cases} 1 & x = \lambda_i \\ 0 & x \neq \lambda_i \end{cases}$ as the sum $f = \sum_{i=1}^{\infty} \lambda_i c_i$. The operator corresponding to c_i is an orthogonal projection onto the eigenspace E_i for λ_i, giving

T the form $T = \sum_{i=1}^{\infty} \lambda_i P_{E_i}$, a condensed form of the traditional spectral theorem. See [19] and [47] for more information on spectral theory; versions of the spectral theorem which integrate against a projection-valued measure are more suited for study of von Neumann algebras.

The spectral theorem found in Theorem 1.2.8 paves the way for the continuous functional calculus on normal elements of C^*-algebras. As $C^*(a, 1)$ is isomorphic to $C(\sigma(a))$ via the specific isomorphism $a \mapsto \mathrm{Id}$ and $1 \mapsto \mathbb{1}$, any continuous function on $\sigma(a)$ may be "applied" to a based on this isomorphism. For example, if $a = a^*$ has positive spectrum $\sigma(a) \subset [0, \infty)$, then the square root function on $\sigma(a)$ corresponds to a unique element in $C^*(a, 1)$, which can be denoted $\sqrt{a}$. Note in particular that in this case, $(\sqrt{a})^2 = a$, because the square of the square root function on $\sigma(a)$ is the function composition identity, Id. Further, this functional calculus agrees with the obvious choices for certain functions: the effect of applying a polynomial $p(x) = c_0 + c_1 x + \ldots + c_n x^n$ to the element a should be $p(a) = c_0 + c_1 a + \ldots + c_n a^n$, and the complex conjugate function produces the adjoint a^*. This continuous functional calculus is more general than the Riesz functional calculus for Banach algebras (see [47], 10.21) in that it applies to more functions at the cost of additional structure required of the algebra; the Riesz functional calculus applies to any element a of a unital Banach algebra A, but $f(a) \in A$ can only be defined when $f : \sigma(a) \to \mathbb{C}$ extends to be holomorphic on a neighborhood of its domain.

The Gelfand-Naimark theorem and the spectral theorem make up part of a dictionary that reflects how topological properties appear in function algebras. For locally compact Hausdorff spaces, compactness of X is dual to unitality of $C_0(X)$, and the range of a function on X is seen as the spectrum in $C(X)$. Further, the ideals of $C(X)$ correpond to functions which vanish on a closed subset of X, meaning their appropriate domain is the complement set in X. In other words, ideals correspond to open subsets of X, and similarly, quotients by ideals correspond to closed subsets of X. Note in particular how traditional theorems in topology are reflected in this correspondence; just as a closed subset of a compact Hausdorff space is itself compact, the quotient of a unital C^*-algebra is again unital. A wonderful description of this dictionary can be found in [53]. A more complicated concern is how to encode algebraic topological data from X, which is handled in the following section.

1.3 K-Theory

If A is a C^*-algebra, then there are two abelian groups $K_0(A)$ and $K_1(A)$ that provide algebraic invariants; both are defined in terms of special elements in A and in matrix algebras $M_n(A)$. When A is a function algebra $C(X)$, these groups contain information about vector bundles over X and the cohomology of X. The standard definitions of K-theory for C^*-algebras can be found in numerous texts, such as the introductions [46] and [3], and the thorough reference [4].

Definition 1.3.1. If A is a C^*-algebra, then $p \in A$ is a *projection* if $p^* = p$ and $p^2 = p$. Similarly, if A is unital, then $u \in A$ is *unitary* if u is invertible and $u^* = u^{-1}$. Let $U_n(A)$ denote the set of unitary matrices in $M_n(A)$.

If $A = C(X)$ and X is connected, then there are no projections besides 0 and 1, as every projection is a $\{0, 1\}$-valued function on X. As such, connected components of projections in $C(X)$ do not give very much useful information. On the other hand, projections in the matrix algebra $M_n(C(X))$ abound. An element of $M_n(C(X))$ can be realized as a function from X to $M_n(\mathbb{C})$, and a projection in this algebra is a function which assigns each $x \in X$ to a projection in $M_n(\mathbb{C})$. Every projection in $M_n(\mathbb{C})$ is the orthogonal projection of $\mathbb{C}^n$ onto some subspace $E \leq \mathbb{C}^n$, so a projection in $M_n(C(X))$ is the continuous assignment of points in X to subspaces of $\mathbb{C}^n$. This is just a continuous vector bundle over X that is contained in an n-dimensional trivial bundle. Roughly speaking, $K_0(C(X))$ studies equivalence classes of these vector bundles with some additional modifications that create an abelian group.

For projections $P \in M_n(A)$ and $Q \in M_p(A)$, the direct sum $P \oplus Q \in M_{n+p}(A)$ is also a projection in a larger matrix algebra. The $\oplus$ operation is associative, but it is lacking many of the properties of an abelian group operation. The most obvious of these is the problem posed by dimension; applying a direct sum can only strictly increase the dimension of the matrix algebra, so there cannot exist an identity or an inverse operation for $\oplus$. A preliminary solution is to apply an equivalence relation, for which there are many options.

Definition 1.3.2. If A is unital and $P_1, P_2 \in M_n(A)$ are projections, then P_1 and P_2 are *unitarily equivalent* if some $U \in U_n(A)$ satisfies $UP_1U^* = P_2$. This is denoted $P_1 \sim_u P_2$.

Modding out by unitary equivalence forces $\oplus$ to become abelian, as $P \oplus Q$ and $Q \oplus P$ are unitarily equivalent, but this is not enough to form an abelian group. There must be an equivalence between projections of different dimensions, or else the dimension will only increase with direct sums and prevent the existence of an additive identity and inverse. To accomplish this, the zero matrices 0_n and the identity matrices I_n will play an important role.

Definition 1.3.3. If A is unital and $P \in M_n(A), Q \in M_p(A)$ are projections, then P and Q are *stably unitarily equivalent* if there are $q, r, s \in \mathbb{Z}^+$ with the following properties.

1. $n + q + r = p + s + r$

2. $P \oplus 0_q \oplus I_r \sim_u Q \oplus 0_s \oplus I_r$

There is no restriction on the matrix dimensions of P and Q, so in particular P and $P \oplus 0$ are stably unitarily equivalent, where 0 denotes any square matrix of zeroes. However, it is important to note that the two identity matrix summands used in the above definition are of the same dimension; the definition does not imply that P and $P \oplus I_r$ are in the same equivalence class. If we mod out by stable unitary equivalence, all 0 matrices represent the same equivalence class, and 0 is an additive identity. Unfortunately, there is not yet a way to form additive inverses, as we can see by considering the case $A = C(X)$ once again. If $P \in M_n(C(X))$ represents a vector bundle on a connected space X, then the rank of that vector bundle is a nonegative integer that is unchanged by unitary conjugation and can only increase in a direct sum $P \oplus Q$. So, there is no way that $P \oplus Q$ is stably unitarily equivalent to the zero projection unless P and Q are both zero, and we must settle for formal inverses from the Grothendieck group.

Definition 1.3.4. If A is a unital C^*-algebra, then $K_0(A)$ is an abelian group consisting of equivalence classes of formal differences $[P] - [Q]$ for projections $P \in M_n(A), Q \in M_p(A)$. Specifically, $[P_1] - [Q_1]$ and $[P_2] - [Q_2]$ represent the same class in $K_0(A)$ if and only if $P_1 \oplus Q_2$ and $P_2 \oplus Q_1$ are stably unitarily equivalent, and the group operation is the direct sum.

Remark. The same group is obtained if stable unitary equivalence is replaced with stable path connectedness. Two projections P and Q are stably path connected if there is a path of projections in some $M_t(A)$ connecting stabilizations $P \oplus 0_q \oplus I_r$ and $Q \oplus 0_s \oplus I_r$.

The above definition only applies to the unital case, and the nonunital case is obtained through a small modification. First, note that the one-dimensional algebra $\mathbb{C}$ of complex numbers has $K_0(\mathbb{C}) \cong \mathbb{Z}$; for a projection $P \in M_n(\mathbb{C})$, the associated integer in $K_0(\mathbb{C}) \cong \mathbb{Z}$ is the rank of P. Further, if ϕ is a unital $*$-homomorphism of unital C^*-algebras, then ϕ induces a homomorphism on the associated matrix algebras and also preserves projections, unitary equivalence, and stable unitary equivalence, inducing a forward map on K-theory. When A is nonunital, the unitization $A^+ = A \oplus \mathbb{C}$ admits a $*$-homomorphism $\psi : A^+ \to \mathbb{C}$ with kernel A, defined by $\psi(a, z) = z$. The induced K-theory homomorphism $\psi_* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}$ is used to define $K_0(A)$.

Definition 1.3.5. If A is a nonunital C^*-algebra, then $K_0(A)$ is the kernel of $\psi_* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}$, where $\psi : A^+ \to \mathbb{C}$ is defined by $\psi(a, z) = z$.

Every trace $\tau : A \to \mathbb{C}$ extends to a trace on the matrix algebra $M_n(A)$ by summing down the diagonal: $\tau([a_{ij}]_{n\times n}) = \sum_{i=1}^{n} \tau(a_{ii})$. This new trace respects the stabilization and unitary conjugation that define $K_0(A)$, and τ induces a homomorphism from $K_0(A)$ to $\mathbb{C}$. As such, traces can be used to distinguish elements of $K_0(A)$ from each other, as was the case when $A = \mathbb{C}$, because the rank of a projection in $M_n(\mathbb{C})$ is the value of the standard trace.

If X is a compact Hausdorff space, then the product $X \times \mathbb{R} \cong X \times (0,1)$ has an analogue in C^*-algebras that behaves extremely well with respect to numerous exact sequences of K-theory.

Definition 1.3.6. If A is a C^*-algebra, the *suspension* of A is the path space $SA = \{f \in C([0,1], A) : f(0) = f(1) = 0\}$, whose operations are defined pointwise. Iterated suspensions are denoted with a superscript, such as $S^2A = SSA$.

Because SA is an algebra of loops in A that are anchored to 0, projections in SA and $M_n(SA) \cong S(M_n(A))$ are paths of projections. The group $K_0(SA)$ appears frequently in exact sequences of K-theory, and an isomorphic group can be formulated without reference to the suspension. This is the remaining K-group $K_1(A)$.

Definition 1.3.7. If A is a unital C^*-algebra, then $K_1(A)$ is an abelian group whose elements are equivalence classes of matrices in $\bigcup_{n=1}^{\infty} U_n(A)$. Two unitary matrices $U \in M_n(A)$, $V \in M_p(A)$ represent the same element of $K_1(A)$ if there is some $q \in \mathbb{Z}^+$ such that $U \oplus I_{q-n}$ and $V \oplus I_{q-p}$ are in the same connected component of $U_q(A)$. The group operation is the direct sum.

Remark. Because polar decomposition allows any invertible matrix over A to be written as the product of a positive component and a unitary component, "unitary" may be replaced by "invertible" in the definition of $K_1(A)$.

The group operation of $K_1(A)$ is abelian because $U \oplus V$ may be connected to $V \oplus U$ within the unitary matrices by continuously changing the basis. The identity element is the identity matrix I, which is in the same equivalence class regardless of dimension. Moreover, if U and V have the same matrix dimension, then $U \oplus V$ represents the same class in $K_1(A)$ as UV. This follows becase $V \oplus I$ is in the same component of unitaries as $I \oplus V$, so since the product of two unitaries is unitary, $UV \oplus I = (U \oplus I)(V \oplus I)$ is in the same class as $(U \oplus I)(I \oplus V) = U \oplus V$. Consequently, the inverse of U in $K_1(A)$ is the multiplicative inverse U^{-1}. The stabilization process is quite powerful, as it turns the very nonabelian matrix multiplication into an abelian operation equivalent to direct sum. As usual, the nonunital case is a slight modification of the unital case.

Definition 1.3.8. If A is a nonunital C^*-algebra, then let $\widetilde{U_n}(A)$ denote the set of $n \times n$ unitary matrices over $A^+ = A \oplus \mathbb{C}$ of the restricted form $I_n + W$, where $W \in M_n(A)$. Then $K_1(A)$ consists of equivalence classes of matrices in $\bigcup_{n=1}^{\infty} \widetilde{U_n}(A)$. The matrices $U \in \widetilde{U_n}(A)$, $V \in \widetilde{U_p}(A)$ represent the same element of $K_1(A)$ if there is some $q \in \mathbb{Z}^+$ such that $U \oplus I_{q-n}$ and $V \oplus I_{q-p}$ are in the same connected component of $\widetilde{U_q}(A)$. The group operation is the direct sum.

The notation $[\cdot]_{K_j}$ or $[\cdot]_{K_j(A)}$ will denote the equivalence class of a projection or unitary in $K_j(A)$, and the relation $M \sim_{K_j} N$ or $M \sim_{K_j(A)} N$ will mean that $[M]_{K_j(A)} = [N]_{K_j(A)}$. The two K-groups $K_1(A)$ and $K_0(A)$ behave well under the suspension in that $K_0(SA) \cong K_1(A)$ and $K_1(SA) \cong K_0(A)$, and this produces the phenomenon known as Bott Periodicity.

Theorem 1.3.9 (Bott Peridocity)**.** If A is a C^*-algebra, then $K_j(SA) \cong K_{1-j}(A)$ and consequently $K_j(S^2A) \cong K_j(A)$ for $j \in \{0, 1\}$.

When $A = C(X)$, the K-groups encode some cohomological information of X, which can be obtained through the Chern Character.

Theorem 1.3.10. If X is a compact Hausdorff space, then there exist two isomorphisms $\chi_0 : K_0(C(X)) \otimes \mathbb{Q} \to \bigoplus_{n=0}^{\infty} H^{2n}(X; \mathbb{Q})$ and $\chi_1 : K_1(C(X)) \otimes \mathbb{Q} \to \bigoplus_{n=0}^{\infty} H^{2n+1}(X; \mathbb{Q})$. These are called the even and odd *Chern character*, respectively.

The cohomology above is the Čech cohomology, which applies for general spaces, but when X is a smooth manifold or CW-complex, this is equivalent to more accessible cohomology groups. The Chern character is natural (see [32]), so in particular if X and Y are compact Hausdorff spaces and $f : X \to Y$ is continuous, then the following diagram commutes for $j \in \{0, 1\}$. Here $F : C(Y) \to C(X)$ is the $*$-homomorphism associated to f, F_* is the forward map on K-theory induced by F, and f^* is the pullback on cohomology induced by f. In particular, f^* is applied individually to each cohomology group in the direct sum $\bigoplus_{n=0}^{\infty} H^{2n+j}(Y; \mathbb{Q})$.

$$\begin{tikzcd}
K_j(C(X)) \otimes \mathbb{Q} \arrow[r, "\chi_j"] & \bigoplus_{n=0}^{\infty} H^{2n+j}(X; \mathbb{Q}) \\
K_j(C(Y)) \otimes \mathbb{Q} \arrow[r, "\chi_j"] \arrow[u, "F_* \otimes \mathrm{Id}"] & \bigoplus_{n=0}^{\infty} H^{2n+j}(Y; \mathbb{Q}) \arrow[u, "f^*"']
\end{tikzcd} \tag{1.3.11}$$

Now, cohomology is ripe with exact sequences that arise when a space X is viewed as a sum of constituent parts. There do exist C^*-algebraic versions of these sequences, and one of the most important is the standard six-term exact sequence of K-theory.

Theorem 1.3.12 (Six-Term Exact Sequence)**.** If J is a closed ideal of a C^*-algebra A with inclusion map $\iota : J \to A$ and quotient map $\pi : A \to A/J$, then the following six term exact sequence of K-theory exists.

$$\begin{tikzcd}
K_0(J) \arrow[r, "\iota_*"] & K_0(A) \arrow[r, "\pi_*"] & K_0(A/J) \arrow[d] \\
K_1(A/J) \arrow[u] & K_1(A) \arrow[l, "\pi_*"'] & K_1(J) \arrow[l, "\iota_*"']
\end{tikzcd}$$

The standard six-term sequence is ubiquitous in K-theory computations, and it is the foundation of many other exact sequences.

1.4 Rieffel Deformation

A noncommutative C^*-algebra with unit is in spirit dual to a noncommutative locally compact Hausdorff space, but this point of view is a little more grounded when a noncommutative algebra is obtained from a commutative one in some continuous way. The quantum torus, which is debatably the most fundamental example in all of noncommutative geometry, arises in this fashion.

Definition 1.4.1 ([42])**.** Let $\theta \in M_n(\mathbb{R})$ be an antisymmetric matrix ($\theta^T = -\theta$). Then A_θ is the universal, unital C^*-algebra generated by unitaries $U_1, \ldots, U_n$ satisfying $U_k U_j = e^{2\pi i \theta_{jk}} U_j U_k$. The C^*-algebra A_θ is called a *quantum torus* or *noncommutative torus* of dimension n.

Numerous quantum tori are isomorphic to each other because of periodicity of the exponential function, or from changing the order of the generators. When $\theta \in M_n(\mathbb{Z})$, all of the unitary generators of A_θ commute, so $A_\theta \cong A_0$ is a commutative C^*-algebra with unit. In fact, the compact space guaranteed by the Gelfand-Naimark theorem is exactly the n-torus $\mathbb{T}^n = \{(u_1, \ldots, u_n) \in \mathbb{C}^2 : |u_1| = \ldots = |u_n| = 1\}$, and the generators of $A_0 \cong C(\mathbb{T}^n)$ are the n coordinate functions of the torus. The other A_θ are not commutative and therefore are not function algebras, but to emphasize the point of view suggested by the Gelfand-Naimark theorem, A_θ is sometimes denoted $C(\mathbb{T}^n_\theta)$. To be consistent with future notation, A_θ will be denoted $C(\mathbb{T}^n_\rho)$ in future chapters, where $\rho_{jk} = e^{2\pi i \theta_{jk}}$.

The quantum tori are indexed by a matrix θ in such a way that the noncommutativity relations vary continuously in θ. Moreover, the various A_θ can be obtained through a deformation procedure on $C(\mathbb{T}^n)$ that is now known as Rieffel deformation, developed in [44] by M. Rieffel.

Definition 1.4.2. If G is a topological group and A is a C^*-algebra, then an *action* of G on

A is a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ between G and the group of automorphisms (bijective $*$-homomorphisms) of A under composition. The action is called *strongly continuous* if for every $a \in A$, the function $g \mapsto \alpha_g(a)$ is continuous.

If $\mathbb{R}^n$ acts on the commutative torus $C(\mathbb{T}^n)$ by translation, so $\alpha_{\vec{x}}(f) \in C(\mathbb{T}^n)$ is a function whose value at $(u_1, \ldots, u_n)$ is $f(u_1 e^{2\pi i x_1}, \ldots, u_n e^{2\pi i x_n})$, then the action α is strongly continuous. Moreover, there is a dense $*$-subalgebra of smooth elements for the action, namely the smooth functions on the manifold $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, denoted $C^\infty(\mathbb{T}^n)$. In general, there is a dense $*$-subalgebra of smooth elements A^∞ of A, so $(A^\infty, \cdot, +, *, ||\cdot||)$ can be completed in only one way to form a C^*-algebra. The idea of Rieffel deformation is to modify this picture slightly by twisting the multiplication on A^∞ to form a new product $\cdot_J$ and norm $||\cdot||_J$, so the completion of $(A^\infty, \cdot_J, +, *, ||\cdot||_J)$ is a distinct C^*-algebra A_J. The result is a family of C^*-algebras indexed by J with certain continuity properties. Note that we have changed the notation very slightly from [44].

Definition 1.4.3 ([44], Chapter 2)**.** If $\alpha : \mathbb{R}^n \to A$ is a strongly continuous action of $\mathbb{R}^n$ on a C^*-algebra A, then the deformed product $\cdot_J$ is defined by the oscillatory integral

$$a \cdot_J b = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \alpha_{J\vec{u}}(a) \alpha_{\vec{v}}(b) e^{2\pi i \vec{u}\cdot\vec{v}} \, d\vec{u} \, d\vec{v}$$

for antisymmetric matrices $J \in M_n(\mathbb{R})$ and elements $a, b \in A^\infty$.

Detailed estimates in Chapter 4 of [44] show that for any fixed antisymmetric matrix $J \in \mathbb{R}^n$ and $a \in A^\infty$, a linear operator on the space of Schwarz functions on A may be used to define a norm $||\cdot||_J$ on A^∞ which is compatible with $\cdot_J$.

Definition 1.4.4 ([44], Definition 4.8, paraphrased)**.** If $\alpha : \mathbb{R}^n \to A$ is a strongly continuous action of $\mathbb{R}^n$ on a C^*-algebra A and $J \in M_n(\mathbb{R})$ is antisymmetric, then the *deformation of A by J (and α)*, denoted A_J, is the C^*-algebra formed by completing $(A^\infty, \cdot_J, +, *, ||\cdot||_J)$.

When the strongly continuous action $\alpha : \mathbb{R}^n \to \mathrm{Aut}(A)$ factors through the torus $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$ (so $\alpha_{\vec{x}+\vec{n}} = \alpha_{\vec{x}}$ whenever $\vec{n} \in \mathbb{Z}^n$), the Rieffel deformation A_J is called a *θ-deformation*, and A_J is built of *spectral subspaces*, defined for $\vec{p} \in \mathbb{Z}^n$.

$$A_{\vec{p}} = \{a \in A : \alpha_{\vec{v}}(a) = e^{2\pi i p\cdot\vec{v}}a \text{ for all } \vec{v} \in \mathbb{R}^n\}$$

Spectral subspaces are crucial to θ-deformation, as the deformed product $\cdot_J$ can be realized in terms of the original product on A ([44], Proposition 2.22).

$$a \in A_{\vec{p}}\ ,\ b \in A_{\vec{q}} \quad \Longrightarrow \quad a \cdot_J b = e^{-2\pi i\vec{p}\cdot(J\vec{q})}ab$$

The quantum torus $C(\mathbb{T}^n_\theta)$ is isomorphic to the θ-deformation of $C(\mathbb{T}^n)$ corresponding to the translation action of $\mathbb{R}^n$ and the antisymmetric matrix $J = \theta/2$ ([44], Example 10.2). The generators $U_1, \ldots, U_n$ are the coordinate functions $u_1, \ldots, u_n \in C^\infty(\mathbb{T}^n)$, which are in spectral subspaces for $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1) \in \mathbb{Z}^n$, respectively. This imposes the multiplication relation

$$u_p \cdot_J u_q = e^{-2\pi i\, e_p\cdot(Je_q)}u_pu_q = e^{2\pi i\,(Je_p)\cdot e_q}u_pu_q = e^{2\pi i\, J_{pq}}u_pu_q = e^{\pi i\,\theta_{pq}}u_pu_q$$

where concatenation denotes the usual commutative multiplication of functions on $C^\infty(\mathbb{T}^n)$. Switching the roles of p and q yields

$$u_q \cdot_J u_q = e^{\pi i\,\theta_{qp}}u_qu_p = e^{-\pi i\,\theta_{pq}}u_pu_q$$

and combining these two identites gives a relation exclusively in the product $\cdot_J$.

$$u_q \cdot_J u_p = e^{\pi i\,\theta_{pq}}u_pu_q = e^{2\pi i\,\theta_{pq}}(e^{-\pi i\,\theta_{pq}}u_pu_q) = e^{2\pi i\,\theta_{pq}}u_p \cdot_J u_q$$

The above equation shows that the coordinate functions of $C^\infty(\mathbb{T}^n)$ under $\cdot_J$ satisfy the relations required of $C(\mathbb{T}^n_\theta)$ elements, so there is a homomorphism from $C(\mathbb{T}^n_\theta)$ to $C(\mathbb{T}^n)_J$, which is actually an isomorphism. There is a similar action of $\mathbb{R}^n/\mathbb{Z}^n$ on $C(\mathbb{S}^{2n-1})$, coordinatized by $(z_1, \ldots, z_n) \in \mathbb{C}^n$, and on $C(\mathbb{S}^{2n})$, coordinatized by $(z_1, \ldots, z_n, x) \in \mathbb{C}^n \oplus \mathbb{R}$, which rotates the angular coordinates of $z_1, \ldots, z_n$. The θ-deformations are given a presentation in [33] (for the odd case, which generalizes the work of K. Matsumoto in [29]). We include a definition of matrices ρ which may be writeen $\rho_{jk} = e^{2\pi i \theta_{jk}}$ for θ antisymmetric in order to be consistent with Natsume and Olsen's conventions.

Definition 1.4.5. A matrix $\rho \in M_n(\mathbb{C})$ will be called a *parameter matrix* if it may be written $\rho_{jk} = e^{2\pi i \theta_{jk}}$ for a (nonunique) antisymmetric matrix $\theta \in M_n(\mathbb{R})$. That is, each ρ_{jk} has modulus one, $\rho_{jj} = 1$, and $\rho_{jk} = \overline{\rho_{kj}}$.

Definition 1.4.6 ([33])**.** Suppose ρ is an $n \times n$ parameter matrix. Then $C(\mathbb{S}^{2n-1}_\rho)$ is the universal, unital C^*-algebra generated by $z_1, \ldots, z_n$ with the following relations.

$$z_1 z_1^* + \ldots + z_n z_n^* = 1 \qquad z_k z_j = \rho_{jk} z_j z_k \qquad z_j z_j^* = z_j^* z_j$$

These generators automatically satisfy $z_k z_j^* = \overline{\rho_{jk}} z_j^* z_k = \rho_{kj} z_j^* z_k$. Moreover, $C(\mathbb{S}^{2n-1}_\rho)$ is isomorphic to the Rieffel deformation $C(\mathbb{S}^{2n-1})_{\theta/2}$ where $e^{2\pi i \theta_{jk}} = \rho_{jk}$.

Remark. The θ-deformed even spheres are formed as quotients $C(\mathbb{S}^{2n+1}_\omega)/\langle z_{n+1} - z_{n+1}^* \rangle$ when z_{n+1} is central. So, z_{n+1} is replaced with a self-adjoint, central generator x. These algebras were considered in [12], and projections over them were studied in [36].

An important property of Rieffel deformations can be found in [44], which states that the family of C^*-algebras A_{tJ} for $t \in [0, 1]$ forms a strict quantization. This implies that for smooth elements $f, g \in A^\infty$ (which belong to all A_{tJ} simultaneously), the multiplications $f \cdot_{tJ} g$ satisfy a differentiation identity, and the norms $||f||_{tJ}$ vary continuously in t. We will only need the following extremely weak expressions of these facts for fixed $f, g \in A^\infty$, where $\cdot = \cdot_0$ is the original multiplication and $||\cdot|| = ||\cdot||_0$ is the original norm on A.

$$\lim_{t\to 0} ||f \cdot_{tJ} g - f \cdot g||_{tJ} = 0 \qquad\qquad \lim_{t\to 0} ||f||_{tJ} = ||f|| \tag{1.4.7}$$

Moreover, the $\mathbb{R}^n$ action on A which defines a Rieffel deformation A_J also produces an action on A_J itself, allowing it to be Rieffel deformed again.

$$(A_J)_H \cong A_{J+H} \tag{1.4.8}$$

These crucial properties from [44] will play a key role in our theorems. Also, Rieffel showed in [45] that the K-groups of a deformation A_J are isomorphic to those of A.

$$K_i(A_J) \cong K_i(A) \text{ for } i \in \{0, 1\} \tag{1.4.9}$$

The K_0 group of a noncommutative 2-torus $C(\mathbb{T}^2_\rho)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and when ρ_{12} is not a root of unity, K_0 is completely determined by the values of the canonical trace on projections.

Definition 1.4.10. Any noncommutative torus $C(\mathbb{T}^n_\rho)$ admits a trace τ which is continuous, positive, and faithful, defined by $\tau\left(\sum_{\text{finite}} a_{\vec{m}} U_1^{m_1} \cdots U_n^{m_n}\right) = a_{\vec{0}}$ on the dense set of $*$-polynomials.

Remark. This trace is not usually denoted with a subscript ρ because each $\tau = \tau_\rho$ assigns the same values to members of the dense $*$-subalgebra of smooth elements. In fact, each τ is the result of passing the standard integration trace with respect to Lebesgue measure on $C(\mathbb{R}^n/\mathbb{Z}^n)$, which is unchanged by the group action, to the Rieffel deformations.

In 2011, A. Sangha studied the Rieffel deformation and (1.4.9) under the lens of KK-theory (see [7] and [4], Chapter VIII), which combines K-theory and its paired homology theory, in [48]. If $\alpha : \mathbb{R}^n \to \mathrm{Aut}(A)$ is a strongly continuous action, then $B = C([0,1], A)$ may be equipped with an $\mathbb{R}^n$ action denoted β, defined below by the images of $s \in [0,1]$.

$$\beta_{\vec{x}} f(s) = \alpha_{\sqrt{s}\vec{x}}(f(s)) \tag{1.4.11}$$

One may then form the Rieffel deformation B_J, which is equipped with a $C([0,1])$ structure $\Phi : C([0,1]) \to Z(M(B_J))$ inherited from $B = C([0,1], A)$. The fiber at $s \in [0,1]$ is by definition $B_J / \{\Phi(g) \cdot_J b : b \in B_J, g \in C([0,1]), g(s) = 0\}$, which is isomorphic to A_{sJ}. Finally, B_J is shown to be a maximal algebra of cross-sections, so B_J is denoted $\Gamma(A_{tJ})_{t\in[0,1]}$, with the quotient maps onto the fibers denoted by $\pi_h : \Gamma(A_{tJ})_{t\in[0,1]} \to A_{hJ}$. Sangha's results imply that the K-theory isomorphisms of (1.4.9) can be obtained from the section algebra $\Gamma(A_{tJ})_{t\in[0,1]}$, which imposes a kind of continuity on the K-theory in (1.4.9).

Theorem 1.4.12 ([48], Theorem 4.6, restricted to K-theory)**.** Let $h \in [0,1]$. If A is a separable C^*-algebra with a strongly continuous action of $\mathbb{R}^n$, then for any antisymmetric $J \in M_n(\mathbb{R})$, the quotient map $\pi_h : \Gamma((A_{tJ})_{t\in[0,1]}) \to A_{hJ}$ induces isomorphisms on K-theory.

One application of this theorem concerns naturality of the isomorphisms in (1.4.9). If A and B can be Rieffel deformed by actions γ and δ of $\mathbb{R}^n$, respectively, then any equivariant $*$-homomorphism $\phi : A \to B$ can itself be deformed. The map ϕ is (γ, δ)-equivariant if $\phi(\gamma_{\vec{x}}(a)) = \delta_{\vec{x}}(\phi(a))$ holds for any $\vec{x} \in \mathbb{R}^n$ and $a \in A$, and if ϕ is (γ, δ) equivariant, then ϕ induces a map $\phi_J : A_J \to B_J$ for each antisymmetric J. When the actions γ and δ are understood, ϕ is simply called equivariant, or $\mathbb{R}^n$-equivariant.

Corollary 1.4.13. Suppose A and B are separable C^*-algebras equipped with strongly continuous $\mathbb{R}^n$ actions. If J is an antisymmetric $n \times n$ matrix and $\phi : A \to B$ is $\mathbb{R}^n$-equivariant, then let $\phi_J : A_J \to B_J$ denote the corresponding homomorphism on the Rieffel deformations. Then the following K-theory diagram commutes for $j \in \{0, 1\}$.

$$\begin{array}{ccc} K_j(A) & \xrightarrow{(\pi_1)_* \circ (\pi_0)_*^{-1}} & K_j(A_J) \\ {\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle (\phi_J)_*} \\ K_j(B) & \xrightarrow{(\pi_1)_* \circ (\pi_0)_*^{-1}} & K_j(B_J) \end{array}$$

Proof. The map ϕ induces a homomorphism $\Gamma(\phi)$ between the section algebras by applying $\phi_{sJ} : A_{sJ} \to B_{sJ}$ fiberwise. This is equivalent to defining a homomorphism $\Phi : C([0,1], A) \to C([0,1], B)$ using ϕ pointwise, noting that Φ is itself $\mathbb{R}^n$-equivariant (for actions in the sense of (1.4.11)), and examining the deformed homomorphism Φ_J. We then have the following commutative diagram of homomorphisms.

$$\begin{array}{ccccc} A & \xleftarrow{\pi_0} & \Gamma((A_{tJ})_{t\in[0,1]}) & \xrightarrow{\pi_1} & A_J \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \Gamma(\phi)} & & \downarrow{\scriptstyle \phi_J} \\ B & \xleftarrow{\pi_0} & \Gamma((B_{tJ})_{t\in[0,1]}) & \xrightarrow{\pi_1} & B_J \end{array}$$

All that remains is to push this diagram to K-theory, where each $(\pi_s)_*$ is an isomorphism, and to cut out the middle. □

This small corollary appears not to be stated in Sangha's work on KK-theory, and it also appears not to have been published by others, but I am convinced a similar result was commonly known shortly after Rieffel's proof of (1.4.9).

1.5 Group Actions and the Borsuk-Ulam Theorem

The Borsuk-Ulam theorem in algebraic topology places restrictions on maps between spheres $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_{n+1}^2 = 1\}$ and Euclidean space $\mathbb{R}^n$ of the same dimension. Specifically, every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ must admit some point

$x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. The standard proof (see [22]) does not use this form of the theorem, but rather uses a reformulation in terms of maps between two spheres. First, decompose f into even and odd components.

$$\begin{aligned} f(x) &= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \\ &:= e(x) + o(x) \end{aligned} \tag{1.5.1}$$

If $f(x)$ is never equal to $f(-x)$, then the map $g(x) = \frac{o(x)}{|o(x)|}$ is defined, odd, and maps $\mathbb{S}^n$ to $\mathbb{S}^{n-1}$. The restriction of $g(x)$ to the equator $\mathbb{S}^{n-1}$ is then odd and homotopically trivial. All of the arguments above are reversible, so the theorem has four equivalent forms.

Theorem 1.5.2 (Borsuk-Ulam)**.** Each of the following conditions holds for $n \geq 2$.

1. If $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous, then there is some $x \in \mathbb{S}^n$ with $f(x) = f(-x)$.

2. If $o : \mathbb{S}^n \to \mathbb{R}^n$ is continuous and odd, then there is some $x \in \mathbb{S}^n$ with $o(x) = 0$.

3. There is no odd, continuous map $g : \mathbb{S}^n \to \mathbb{S}^{n-1}$.

4. If $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is odd and continuous, then h is homotopically nontrivial.

The standard proof of the Borsuk-Ulam theorem in [22] uses the top homology group $H_k(\mathbb{S}^k, \mathbb{Z})$ of the sphere $\mathbb{S}^k$. The group $H_k(\mathbb{S}^k, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, leading to the definition of degree for self-maps of $\mathbb{S}^k$.

Definition 1.5.3. Fix any isomorphism between $H_k(\mathbb{S}^k, \mathbb{Z})$ and $\mathbb{Z}$. Then any continuous map $f : \mathbb{S}^k \to \mathbb{S}^k$ induces a forward map $f_* : H_k(\mathbb{S}^k, \mathbb{Z}) \to H_k(\mathbb{S}^k, \mathbb{Z})$ that is equivalent to a homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$. The *degree* of f is the unique integer n such that $\phi(m) = mn$ for all $m \in \mathbb{Z}$.

The degree of a homotopically trivial map is zero. Moreover, the cohomology groups $H^k(\mathbb{S}^k; \mathbb{Z})$ are also infinite cyclic, and the degree of a map may be equivalently defined in terms of a pullback on cohomology (even with coefficients in $\mathbb{Q}$ or $\mathbb{R}$). The Borsuk-Ulam theorem is then a consequence of the following stronger theorem, which sometimes goes by the same name.

Theorem 1.5.4 (Stronger Borsuk-Ulam)**.** Any odd, continuous self-map of a sphere $\mathbb{S}^k$ must have odd degree. Therefore, the degree is nonzero and the map is homotopically nontrivial.

In the extremely interesting paper [49], Taghavi motivates the Borsuk-Ulam theorem in terms of graded algebras over finite abelian groups and presents a proof (and generalization) for the $\mathbb{S}^2$ case in this context. Perhaps the most novel part of his proof is that it deals explicitly with formulation 2 of the theorem, and not formulation 4, making particular use of the identification $\mathbb{R}^2 \cong \mathbb{C}$. The role of graded algebras is quite simple: the even/odd decomposition (1.5.1) is an example of a grading on $C(\mathbb{S}^2) = C(\mathbb{S}^2, \mathbb{C})$ by the group $\mathbb{Z}_2$.

Definition 1.5.5. If A is a Banach algebra and G is a finite group, then A is *G-graded* if it admits a decomposition $A = \bigoplus\limits_{g \in G} A_g$ into closed subspaces which satisfy $A_g \cdot A_h \subset A_{gh}$ for all $g, h \in G$. The elements of A_g are called *homogeneous*, and when $g \neq e$, *nontrivial homogeneous*.

For convenience, in this section we assume every algebra in unital. When $G = \mathbb{Z}_n$, there is a clear group action by $\mathbb{Z}_n$ on A associated to the grading, generated by the following isomorphism T, where ω is a primitive nth root of unity.

$$T : a = (a_0, \ldots, a_{n-1}) \in A \mapsto (a_0, \omega a_1, \ldots, \omega^{n-1} a_{n-1}) \tag{1.5.6}$$

In other words, A_i is prescribed as the eigenspace of T for eigenvalue ω^i, and as a result of the graded structure, such a map is not only linear, but also a continuous algebra isomorphism with $T^n = I$. An action of $\mathbb{Z}_n$ on A is then described by $\alpha_k(a) = T^k(a)$. Finally, the projections $\pi_j : A \to A_j$ take the following form, which generalizes (1.5.1).

$$a_j = \pi_j(a) = \frac{a + \omega^{-j} \cdot Ta + \omega^{-2j} \cdot T^2 a + \ldots + \omega^{-(n-1)j} \cdot T^{n-1} a}{n} \tag{1.5.7}$$

Of course, one may start with an action of $\mathbb{Z}_n$ and recover a grading by this formula.

More generally, if G is a compact, abelian, Hausdorff group which acts strongly continuously on a Banach algebra A by $\alpha : G \to \mathrm{Aut}(A)$, then for any τ in the Pontryagin dual $\widehat{G} = \{f : G \to \mathbb{S}^1 : f \text{ is a continuous homomorphism}\}$, there is a corresponding homogeneous subspace A_τ defined as follows.

$$A_\tau = \{a \in A : \text{for all } g \in G, \alpha_g(a) = \tau(g)a\} \tag{1.5.8}$$

The trivial homogeneous subspace $\{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}$ is called the *fixed point subalgebra* and is often denoted A^α. If μ denotes the unique Haar measure on G with $\mu(G) = 1$, then there is a homogeneous component projection $\pi_\tau : A \to A_\tau$ defined by an integral formula.

$$\pi_\tau(a) = \int_G \tau(g^{-1})\alpha_g(a)\, d\mu \in A_\tau \tag{1.5.9}$$

The integral above exists because its integrand is a continuous Banach-space valued function (and also bounded because G is compact), and μ is a finite Borel measure. When the group in question is $\mathbb{Z}_n$, we have that $\widehat{\mathbb{Z}_n}$ is isomorphic to $\mathbb{Z}_n$, generated by a homomorphism which sends 1 to a primitive nth root of unity, so the previous formula generalizes (1.5.7). The map $a \mapsto (\pi_\tau(a))_{\tau \in \widehat{G}}$ is injective, but we should not expect a nice formula such as $a = \int_{\widehat{G}} \pi_\tau(a)$ (integrating over a suitable Haar measure) to cleanly generalize a graded decomposition $a = \pi_0(a) + \pi_1(a) + \ldots + \pi_{n-1}(a)$ for a $\mathbb{Z}_n$ action, as such an overreaching statement would imply that every continuous function on the circle has a convergent Fourier series. In particular, $\widehat{\mathbb{S}^1} = \mathbb{Z}$ consists of the homomorphisms $z \mapsto z^n$, $n \in \mathbb{Z}$, and the natural action of $\mathbb{S}^1$ on $C(\mathbb{S}^1)$ by rotation produces the usual Fourier transform from (1.5.9) in the sense that $\pi_n(f)$ is the function mapping $z \in \mathbb{S}^1$ to $\widehat{f}(n)z^n$. As such, the reconstruction of elements of A from homogeneous components is a process enveloping all of the subtlety of Fourier series in the classical cases, and it is no surprise that dual groups provide a natural setting for a generalized Fourier transform. For more information on the role of group actions on C^*-algebras, see [37] and [38].

In [49], Taghavi proved a theorem about idempotentless Banach algebras with group actions that allowed for a new proof of the Borsuk-Ulam theorem in dimension 2.

Theorem 1.5.10. ([49], Main Theorem 1) Let A be a [unital] G-graded Banach algebra [where G is finite and abelian] with no nontrivial idempotents. Let $a \in A$ be a nontrivial homogeneous element. Then 0 belongs to the convex hull of the spectrum $\sigma(a^k)$ [for any $k \in \mathbb{Z}^+$]. Further, if A is commutative and a is invertible, then a^k and 1 do not lie in the same connected component of the space of invertible elements $G(A)$.

In Chapter 2 and [34], we prove related results on the nonexistence of roots of certain homogeneous elements, and we use different arguments to loosen conditions on the Banach algebras in Taghavi's proofs. One of these results applies to the noncommutative tori $C(\mathbb{T}^n_\rho)$, which for most ρ contain many nontrivial projections. The goal of this study was to prove the higher-dimensional Borsuk-Ulam theorem in terms of similar results on Banach or C^*-algebras, and to see if such proofs extend to the θ-deformed spheres, as any $C(\mathbb{S}^k_\rho)$ is equipped with a natural $\mathbb{Z}_2$ action that negates each generator. We find that the most natural conjecture that extends Theorem 1.5.2 statement 2 in terms of odd elements of $C(\mathbb{S}^k_\rho)$ is actually false; this negatively answers a question Taghavi posed, at least for θ-deformed spheres. However, certain versions of Theorem 1.5.2 remain true when extended to the noncommutative case, which we investigate in Chapter 3. This lines up with the results of Yamashita in [55] on the q-spheres, which are the result of a quantization process distinct from Rieffel deformation. To avoid confusion between the two distinct families of spheres, I will denote the q-spheres as $\mathfrak{C}(\mathbb{S}^k_q)$.

Definition 1.5.11 ([51])**.** For $0 < q \leq 1$, the quantum sphere $\mathfrak{C}(\mathbb{S}_q^{2n-1})$ is the univeral, unital C^*-algebra generated by elements $z_1, \ldots, z_n$ satisfying $z_j z_i = q z_i z_j$ for $i < j$, $z_j^* z_i = q z_i z_j^*$ for $i \neq j$, $z_i^* z_i = z_i z_i^* + (1-q^2)(z_{i+1} z_{i+1}^* + \ldots + z_n z_n^*)$ for $i < n$, and the identity $z_1 z_1^* + \ldots + z_n z_n^* = 1$. The quantum sphere $\mathfrak{C}(\mathbb{S}_q^{2n})$ is the quotient $\mathfrak{C}(\mathbb{S}_q^{2n+1})/\langle z_{n+1} - z_{n+1}^* \rangle$.

The q-spheres are equipped with a $\mathbb{Z}_2$ action that negates each generator z_i, and an analogue of the Borsuk-Ulam theorem holds, which Yamashita proved in [55]. Theorem 1.5.2 item 3 states that there is no continuous, odd map between $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$, or equivalently from $\mathbb{S}^n$ to $\mathbb{S}^m$ when $m < n$. As the Gelfand-Naimark theorem reverses the order of maps, this is equivalent to the fact that there is no unital $*$-homomorphism from $C(\mathbb{S}^m)$ to $C(\mathbb{S}^n)$ that is equivariant for the antipodal action.

Theorem 1.5.12 ([55], Theorem 3)**.** For any $0 < q \leq 1$ and positive integers $m < n$, there is no $\mathbb{Z}_2$-equivariant unital $*$-homomorphism from $\mathfrak{C}(\mathbb{S}_q^m)$ to $\mathfrak{C}(\mathbb{S}_q^n)$.

Yamashita's result is proved using equivariant KK-theory in a very general study of quantum homogeneous spaces. We prove the following analogous result on the θ-deformed spheres in Chapter 3 by manipulating fixed point subalgebras, and then we consider extensions to other rotation actions. The content of Chapter 3 overlaps significantly with our paper [34], and the following theorem summarizes the results that concern the antipodal action on θ-deformed spheres.

Theorem 1.5.13. Let the θ-deformed spheres be equipped with the antipodal $\mathbb{Z}_2$ action, which negates each generator z_i or x. All maps below are unital $*$-homomorphisms.

1. Any equivariant map $\phi : C(\mathbb{S}_\rho^{2n-1}) \to C(\mathbb{S}_\omega^{2n-1})$ induces a nontrivial map on $K_1 \cong \mathbb{Z}$.

2. Any equivariant map $\phi : C(\mathbb{S}_\rho^{2n}) \to C(\mathbb{S}_\omega^{2n})$ induces a nontrivial map on the component of $K_0 \cong \mathbb{Z} \oplus \mathbb{Z}$ not generated by the trivial projections.

3. Consequently, there is no equivariant map from $C(\mathbb{S}_\gamma^k)$ to $C(\mathbb{S}_\eta^{k+1})$.

Dąbrowski conjectured a possible framework in [14] to generalize the Borsuk-Ulam theorem to all C^*-algebras with suitable actions of $\mathbb{Z}_2$, citing Theorems 1.5.12 and 1.5.13 as related examples. This conjecture was also tied to other conjectures in his joint work with P. Baum and P. Hajac in [2], which primarily concerned quantum groups; [2] appeared slightly before our results. In Dąbrowski's conjecture from [14], the unreduced suspension

$$\Sigma A = \{f \in C([-1,1], A) : f(-1), f(1) \in \mathbb{C}\}$$

of a C^*-algebra A has A as a quotient algebra from evaluation at $0 \in [-1,1]$, analogous with the containment of $\mathbb{S}^k$ inside $\mathbb{S}^{k+1}$ (we have changed the domain from $[0,1]$ to $[-1,1]$ for convenience). Any $\mathbb{Z}_2$ action α extends to ΣA as

$$\mathfrak{a}(f)[t] = \alpha(f(-t))$$

and this definition is compatible with the unreduced suspension ΣX of a compact Hausdorff space X, in that $\Sigma C(X) \cong C(\Sigma X)$. In particular, $\Sigma \mathbb{S}^k \cong \mathbb{S}^{k+1}$, $\Sigma C(\mathbb{S}^k) \cong C(\mathbb{S}^{k+1})$, and the extension of the antipodal action is again antipodal. Now, $\Sigma C(\mathbb{S}^k_\rho)$ is another θ-deformed sphere, but not every θ-deformed sphere can be obtained this way.

Conjecture 1.5.14 ([14], Conjecture 3.1)**.** For a unital C^*-algebra A with a free action of $\mathbb{Z}_2$, there is no $\mathbb{Z}_2$-equivariant $*$-homomorphism $\phi : A \to \Sigma A$.

We prove this conjecture in Chapter 4 and [35] using what is essentially the weakest freeness assumption for actions on C^*-algebras, and surprisingly, the proof reduces to the original Borsuk-Ulam theorem on $\mathbb{S}^k$. Now, Dąbrowski also asked in [14] if there is a suitable definition of a "noncommutative" unreduced suspension; we define one possible version as well as a noncommutative join of a C^*-algebra and a finite cyclic group. However, these definitions differ from the noncommutative double unreduced suspension in [24] and the very general C^*-algebraic join in [2] that motivated Dąbrowski's question. A crucial component of this noncommutative unreduced suspension is the crossed product of a C^*-algebra A with $\mathbb{Z}_k$, which is defined in reference to a group action β of $\mathbb{Z}_k$. We write the action β in terms of a generating homomorphism of order dividing k, also denoted β, on A.

$$A \rtimes_\beta \mathbb{Z}_k = \{a_0 + a_1\delta + \ldots + a_{k-1}\delta^{k-1} : a_0, \ldots, a_{k-1} \in A\}$$

$$\delta^k = 1 \qquad\qquad \delta^* = \delta^{-1} \qquad\qquad a\delta = \delta\beta(a) \quad \text{for} \quad a \in A$$

The crossed product $A \rtimes_\beta \mathbb{Z}_k$ is defined for a much larger class of group actions (see [54]), but we will only need this case. There is a dual action of $\widehat{\mathbb{Z}_k} = \mathbb{Z}_k$ on $A \rtimes_\beta \mathbb{Z}_k$, denoted $\widehat{\beta}$, defined as follows when ω is a fixed primitive kth root of unity.

$$\widehat{\beta}(a_0 + a_1\delta + \ldots + a_{k-1}\delta^{k-1}) = a_0 + \omega a_1\delta + \ldots + \omega^{k-1}a_{k-1}\delta^{k-1}$$

When A is given the trivial action, this results in $A \rtimes_{\text{triv}} \mathbb{Z}_k \cong \bigoplus_{i=1}^{k} A$. In particular, when $A = \mathbb{C}$, $\mathbb{C} \rtimes_{\text{triv}} \mathbb{Z}_k$ is just the group C^*-algebra $C^*(\mathbb{Z}_k)$. We will use the crossed product in order to encode noncommutativity information with a symbol δ.

The crossed product satisfies an important identity when a $\mathbb{Z}_k$ action α on A is saturated, which means that all the homogeneous subspaces $A_m = \{a \in A : \alpha(a) = e^{2\pi i m/k}a\}$ satisfy $\overline{A_m A A_m^*} = A$. When α is saturated, the K-groups of the crossed product are isomorphic to the K-groups of the fixed point subalgebra (and this fact has a much stronger form).

$$\alpha \text{ saturated } \implies K_j(A \rtimes_\alpha \mathbb{Z}_k) \cong K_j(A^\alpha) = K_j(A_0) \text{ for } j \in \{0, 1\} \tag{1.5.15}$$

This fact is related to [43], Corollary 1.7, and reasonably explicit isomorphisms exist in the literature. First, follow [38], Definition 5.2, Definition 5.9, and Theorem 5.10 to see the strong Morita equivalence of A^α and $A \rtimes_\alpha \mathbb{Z}_k$, and then use [20], Theorem 5.3 and Theorem 5.5 to see K-theory isomorphisms for strongly Morita equivalent C^*-algebras. Saturation of an action on A is a freeness property, as an action on a space X is free if and only if the associated action on $A = C(X)$ is saturated, but for noncommutative C^*-algebras it is not the only relevant property. In general, saturation is one of the weakest freeness assumptions to require of an action on A (see [38]).

When A has an action β (saturated or not) of $\mathbb{Z}_k$, we may define a C^*-algebra

$$A *_\beta \mathbb{Z}_k := \{f \in C([0,1], A \rtimes_\beta \mathbb{Z}_k) : f(0) \in A, f(1) \in C^*(\mathbb{Z}_k)\}$$

with pointwise operations; if β is trivial and $A = C(X)$, $A *_\beta \mathbb{Z}_k$ is the continuous function algebra for the topological join of X and $\mathbb{Z}_k$. Further, $A *_\beta \mathbb{Z}_2$ is denoted $\Sigma^\beta A$, as if $k = 2$ and β is trivial, $A *_\beta \mathbb{Z}_2$ is isomorphic to the unreduced suspension ΣA. Together these ideas suggest that $A *_\beta \mathbb{Z}_k$ may be considered a noncommutative join of A and $\mathbb{Z}_k$, or a noncommutative unreduced suspension if $k = 2$. This allows us to formulate a conjecture in the same vein as the conjectures in [14] and [2] when A is equipped with another action α that is saturated and commutes with β. Note in particular that [2] includes a different definition of a general C^*-algebraic join, for the purpose of studying actions of compact quantum groups. Our version of noncommutative join only considers an action of $\mathbb{Z}_k$, but for such an action, it produces a different C^*-algebra. By considering examples based on noncommutative spheres, we see that additional restrictions on β are necessary and refine our conjecture. In this pursuit, we also prove a separate K_0-based theorem on the antipodal action for certain noncommutative unreduced suspensions of $C(\mathbb{S}^{2n-1}_\rho)$.

Theorem 1.5.16. For an $n \times n$ parameter matrix ρ, let $\mathfrak{R}^{2n}_\rho$ denote the universal, unital C^*-algebra generated by normal elements $z_1, \ldots, z_n$ and a self-adjoint element x such that $z_1 z_1^* + \ldots + z_n z_n^* + x^2 = 1$, $z_k z_j = \rho_{jk} z_j z_k$, and $x z_k = -z_k x$. Then $\mathfrak{R}^{2n}_\rho$ is isomorphic to

$\Sigma^\alpha C(\mathbb{S}_\rho^{2n-1})$ where α is the antipodal map, and $\mathfrak{R}_\rho^{2n}$ is equipped with its own antipodal map that negates each generator. If a unital $*$-homomorphism $\phi : \mathfrak{R}_\rho^{2n} \to \mathfrak{R}_\rho^{2n}$ is equivariant for the antipodal map, then ϕ induces a nontrivial map on the component of $K_0 \cong \mathbb{Z} \oplus \mathbb{Z}$ not generated by trivial projections.

The above theorem, along with some other results of Chapter 4 that are also not yet contained in any arxiv or journal articles, is proved using different techniques than those used for the θ-deformed even spheres. This appears to be necessary even though $\mathfrak{R}_\rho^{2n}$ and $C(\mathbb{S}_\rho^{2n})$ have similar presentations; the difference between a commutation relation and an anticommutation relation involving a self-adjoint element is not an issue that can be resolved continuously. In fact, by [33], Lemma 2.7, the relation $z_k x = \omega x z_k$ for $\omega \notin \mathbb{R}$ would conflict with normality and self-adjointness to imply that $z_k x = x z_k = 0$. We instead use different techniques in the proof, which is in the final section of Chapter 4 as the closing argument of this dissertation.

Chapter 2

Homogeneous Elements of Banach Algebras

2.1 Nonexistence of Roots

The Borsuk-Ulam theorem in dimension 2 states that there is no odd map from $\mathbb{S}^2$ to $\mathbb{S}^1$, or that every odd map from $\mathbb{S}^1$ to itself is homotopically nontrivial. Odd maps from $\mathbb{S}^1$ to $\mathbb{S}^1$ may be represented by the unitary elements of $C(\mathbb{S}^1)$ that are negated by the antipodal action; in other words, they are odd unitary elements of the C^*-algebra $C(\mathbb{S}^1)$. In [49], Taghavi proved an extension of the 2-dimensional Borsuk-Ulam theorem by showing similar results hold for many Banach algebras with group actions.

Theorem 2.1.1. ([49], Main Theorem 1) Let A be a [unital] G-graded Banach algebra [where G is finite and abelian] with no nontrivial idempotents. Let $a \in A$ be a nontrivial homogeneous element. Then 0 belongs to the convex hull of the spectrum $\sigma(a^k)$ [for any $k \in \mathbb{Z}^+$]. Further, if A is commutative and a is invertible, then a^k and 1 do not lie in the same connected component of the space of invertible elements $G(A)$.

Note in particular that there are no restrictions on $k \in \mathbb{Z}^+$; for example, a^k might be a trivial homogeneous element (fixed by the action). If A is equal to $C(X)$ for a compact Hausdorff space X, then X is connected if and only if A has no nontrivial idempotents. The spectrum result in Taghavi's theorem illustrates the following problem: if a is an invertible element that is nontrivial homogeneous, then in some $\mathbb{Z}_n = G/N$ grading with associated isomorphism T and primitive nth root of unity ω, $T(a) = \omega a$. Since $\sigma(a) = \sigma(Ta) = \sigma(\omega a) = \omega\sigma(a)$, if $\sigma(a)$ is missing values in any particular ray $e^{i\theta}[0, \infty)$, rotational symmetry will disconnect $\sigma(a)$ into n pieces. The holomorphic functional calculus then provides a nontrivial idempotent in A, which contradicts the assumptions. This is a proof of a more general spectral condition than Taghavi claims: the connected set $\sigma(a)$ will either include 0 or completely surround 0 in $\mathbb{C}$, so we should not expect a logarithm of a (or of a^k) using functional calculus. Taghavi's full result is a statement about (nonexistence of) logarithms that is not limited to functional calculus, and we have listed below the most general result that may be clearly distilled from the original proof; this also resolves our petty quibbles about the spectrum.

Theorem 2.1.2. ([49], Main Theorem 1, restated) Let A be a unital G-graded Banach algebra with no nontrivial idempotents, where G is a finite abelian group, and suppose $a \in A$ is a nontrivial homogeneous element. If $k \in \mathbb{Z}^+$, then there is no $b \in A$ with the following properties.

1. $g, h \in G \implies b_g b_h = b_h b_g$
2. $ab = ba$
3. $\exp(b) = a^k$

If we return to the motivating example of functional calculus, the same topological obstruction on the spectrum occurs when trying to form nth roots of invertible elements instead of logarithms, so one can ask if similar results hold for roots. Some simple counterexamples show that there must be a relationship between the size of the group $\mathbb{Z}_n$ and the order of the root.

Proposition 2.1.3. Suppose A is a unital $\mathbb{Z}_n$-graded Banach algebra with no nontrivial idempotents. If a is a nontrivial homogeneous element that is also invertible, then a cannot have an nth root b such that all the homogeneous components b_k commute.

Proof. Suppose b is such an nth root of a, so that b is also invertible and commutes with a. Consequently, if T is the isomorphism associated to the graded algebra such that $T(a) = \omega^j a$, then the fact that the homogeneous components b_k of b all commute implies that b^{-1} and Tb commute. This shows that $(b^{-1}Tb)^n = b^{-n}T(b^n)$, which is equal to $a^{-1}Ta = \omega^j$. Now, $b^{-1}Tb$ is an nth root of a constant, so by the spectral mapping theorem, its spectrum is finite. Also, the spectrum must be connected because A has no nontrivial idempotents, so $\sigma(b^{-1}Tb) = \{c\}$ and $b^{-1}Tb = c + \varepsilon$, where ε is quasinilpotent ($\sigma(\varepsilon) = \{0\}$) and $c^n = \omega^j$.

All elements that follow are in the closed subalgebra generated by elements of the form $T^k b$ or $T^k(b^{-1})$, which is commutative. The equation $b^{-1}Tb = c+\varepsilon$ implies that $Tb = b(c+\varepsilon)$, and an inductive argument shows that $T^k b = b \cdot \prod_{j=1}^{k}(c+T^{j-1}\varepsilon)$. When $k = n$ this says $b = T^n b = b \cdot \prod_{j=1}^{n}(c+T^{j-1}\varepsilon)$. Since ε is quasinilpotent, each $T^{j-1}\varepsilon$ is quasinilpotent, and the commuting product $\prod_{j=1}^{n}(c+T^{j-1}\varepsilon)$ is equal to $c^n + \delta = \omega^j + \delta$ where δ is quasinilpotent. The element δ commutes with b, so $b = b \cdot \prod_{j=1}^{n}(c+T^{j-1}\varepsilon) = b(\omega^j + \delta) = b\omega^j + \gamma$ where γ is quasinilpotent. Finally, a was a nontrivial homogeneous element, so $1 - \omega^j \neq 0$, and $(1-\omega^j)b = \gamma$ is both invertible (as b is invertible) and quasinilpotent. This is a contradiction. □

The proof technique for the previous proposition is directly inspired by Taghavi's methods. Invertibility of the element a and the relationship between the order of the group $\mathbb{Z}_n$ and the order of the root cannot be removed. These requirements can be seen in the commutative algebra $C(\mathbb{S}^1)$ with the standard $\mathbb{Z}_2$ antipodal action.

Example 2.1.4. If $\mathbb{S}^1$ is realized as the unit sphere of $\mathbb{R}^2$, then the coordinate functions x_1 and x_2 in $C(\mathbb{S}^1)$ are odd. Since $\sigma(x_i) = [-1, 1]$ and x_i is a normal element of a C^*-algebra, we may apply the continuous functional calculus for the following square root function.

$$g(t) = \begin{cases} \sqrt{t}, & t \in [0,1] \\ i\sqrt{-t}, & t \in [-1,0] \end{cases}$$

Now, $g(x_i)$ is a square root of the (non-invertible) odd element x_i.

Example 2.1.5. The invertible odd function $f(z) = z^3$ in $C(\mathbb{S}^1)$ certainly has a third root.

Proposition 2.1.3 may also be used to generalize the Borsuk-Ulam theorem in dimension 2, so there are two possible directions for generalization. First, we can modify the restrictions on the Banach algebra A to include nontrivial idempotents, allowing the results to apply to many more algebras. Alternatively, we can seek variants that allow discussion of $C(\mathbb{S}^k)$ for higher dimensions. For example, one version of the Borsuk-Ulam theorem implies that there are no odd, continuous maps from $\mathbb{S}^k$ to $\mathbb{R}^k \setminus \{0\}$, which means that if $a_1, \ldots, a_k \in C(\mathbb{S}^k)$ are self-adjoint and odd elements (i.e., real-valued odd functions), then $a_1^2 + \ldots + a_k^2$ must not be invertible under multiplication. Taghavi expressed interest in this type of condition for noncommutative spheres (C^*-algebras which are obtained from some sort of deformation procedure on $C(\mathbb{S}^k)$), but we will see that these types of conditions do not generalize to the θ-deformed spheres.

2.2 Projection Conditions

When A is a noncommutative C^*-algebra, there are often nontrivial projections in A, even if A is related in some way to a connected topological space. For example, there exist nontrivial projections in most of the quantum tori $C(\mathbb{T}^n_\rho)$, which are Rieffel deformations of $C(\mathbb{T}^n)$, and any matrix algebra $M_n(C(X))$ will have nontrivial projections from $M_n(\mathbb{C})$.

Theorem 2.2.1. Suppose A is a unital $\mathbb{Z}_2$-graded Banach algebra with the property that no idempotent P satisfies $T(P) = 1 - P$. Then if $f \in A$ is odd and invertible, there is no $g \in A$ such that $g^2 = f$ and g commutes with Tg.

Proof. Suppose $g^2 = f$ where g and Tg commute. Then g is invertible and

$$(T(g)g^{-1})^2 = T(g^2)(g^2)^{-1} = T(f)f^{-1} = -1$$

holds. Denote the element $T(g)g^{-1}$ by a and note that $a^2 = -1$, so $a^{-1} = -a$. However, we also have that $T(a) = -a$ by a simple calculation.

$$T(a) = T(T(g)g^{-1}) = gT(g)^{-1} = (T(g)g^{-1})^{-1} = a^{-1} = -a$$

This means a is odd, so a is an odd square root of -1. It follows that $P = \frac{1}{2} + \frac{i}{2}a$ is an idempotent with $T(P) = 1 - P$. $\square$

The condition $T(P) \neq 1-P$ is not only sufficient in the above theorem, but also necessary. If $T(P) = 1-P$, then $\pi_0(P) = \frac{P+T(P)}{2} = 1/2$, so if we examine the odd component $\pi_1(P) = b$, the idempotent equation $(1/2 + b)^2 = 1/2 + b$ implies that $b^2 = 1/4$. Consequently, $\sigma(b)$ is finite (and excludes 0) by the spectral mapping theorem. We may then form a square root c of the invertible odd element b by the holomorphic functional calculus. Since b is odd and c is in the closed, unital subalgebra generated by b and elements of the form $(b - \lambda)^{-1}$, it follows that $cT(c) = T(c)c$.

For a $\mathbb{Z}_2$ action on a C^*-algebra, if we assume $T(P) \neq 1 - P$ on the smaller class of projections (instead of all idempotents), then we obtain a similar result with a slightly weaker conclusion.

Theorem 2.2.2. Suppose A is a unital C^*-algebra with a $\mathbb{Z}_2$ action T such that no projection P satisfies $T(P) = 1 - P$. Then if $f \in A$ is an odd unitary element, there is no unitary $g \in A$ such that $g^2 = f$ and g commutes with Tg.

Proof. The proof is the same as the proof of the previous theorem, with the addition that since g is unitary, $a = T(g)g^{-1} = T(g)g^*$ satisfies $a^* = a^{-1} = -a$, and the resulting P is self-adjoint. $\square$

Remark. As in the previous theorem, the condition $T(P) \neq 1 - P$ is also necessary here. The only change to the argument is that the odd component b of a projection satisfying

$T(P) = 1 - P$ is also self-adjoint, which with the equation $b^2 = 1/4$ implies that $2b$ is unitary. Again, this element has finite spectrum, and the square root formed by the continuous functional calculus is guaranteed to be unitary.

Since the homogeneous subspaces A_0 and A_1 of a C^*-algebra with a $\mathbb{Z}_2$ action are norm-closed and closed under the adjoint operation, it follows that for any even or odd element a, aa^* and a^*a even. Further, the positive square root of either aa^* or a^*a from the continuous functional calculus is even as well (as a limit of $*$-polynomials in an even element). Similarly, the inverse of an even or odd element remains even or odd, as seen by examining the effect of the isomorphism T that generates the action. These observations show that if we start with a homogeneous invertible and scale it to form a unitary, the result is still homogeneous, giving some equivalent formulations of the projection condition.

Proposition 2.2.3. The following conditions are equivalent for a unital C^*-algebra A with a $\mathbb{Z}_2$ action defined by the isomorphism T.

1. There is a projection $P \in A$ with $T(P) = 1 - P$.
2. There is some $a \in A$ which is odd, self-adjoint, and satisfies $a^2 = 1$.
3. There is some $b \in A$ which is odd, self-adjoint, and invertible.

Proof. Condition 2 certainly implies condition 3, and the reverse implication holds by scaling b to a unitary $a = b(b^2)^{-1/2} = b|b|^{-1}$, which remains odd and self-adjoint. If P is a projection with $T(P) = 1 - P$, then its even component is $\frac{1}{2}(P + T(P)) = 1/2$, so P is of the form $1/2 + c$, where c is self-adjoint and odd. The idempotent equation $(1/2+c)^2 = 1/2+c$ implies that $c^2 = 1/4$, so $a = 2c$ satisfies $a^2 = 1$, and condition 1 implies condition 2. Similarly, if a is as in condition 2, then $P = 1/2 + a/2$ is a projection with $T(P) = 1 - P$. $\square$

When A is a graded unital Banach algebra, $M_n(A)$ is graded as well; the group action is applied entrywise, and the homogeneous subspaces consist of matrices with entries in the homogeneous subspaces of A. However, $M_n(A)$ will always have nontrivial idempotents for $n \geq 2$, so Taghavi's Main Theorem 1 in [49] and the similar result Proposition 2.1.3 in this chapter do not apply. However, the new condition $T(P) \neq 1-P$ allows for some idempotents, and the matrix dimension will play a key role. For example, if A is a C^*-algebra and there exists an $n \times n$ unitary matrix F over A which has odd entries, then $P = \begin{bmatrix} 1/2 & F/2 \\ F^*/2 & 1/2 \end{bmatrix}$ is a projection in the $2n \times 2n$ matrix algebra with $T(P) = I - P$. This dimension n encodes important information about how A interacts with its $\mathbb{Z}_2$ action.

When A is equal to $C(X)$ for some compact X with finitely many components, and T arises from an order two homeomorphism h on X, the condition that $T(P) \neq I - P$ for each projection $P \in A$ has a simple restatement. First, note that h determines a $\mathbb{Z}_2$ action on the set of components of X. If this action has no fixed point, we may define $P \in C(X)$ so that P takes value 1 on exactly one component in each orbit, and 0 on the other component. This P is a projection satisfying $T(P) = 1 - P$. Therefore, insisting that $T(P)$ is never equal to $1 - P$ means that the action on components induced by h has a fixed point. In this case, the quotient algebra of functions on this component reduces the problem to the idempotentless case, so the actual benefit of the new condition is found in noncommutative C^*-algebras. If $A = M_n(C(X))$, then a projection $P \in A$ assigns to each $x \in X$ a projection $P_x \in M_n(\mathbb{C})$, which as a linear map is the orthogonal projection onto a subspace of $\mathbb{C}^n$, forming a continuous vector bundle. If A inherits the $\mathbb{Z}_2$ action generated by $h : X \to X$, applied entrywise, and each projection $P \in A$ satisfies $T(P) \neq I - P$, then there is some $x \in X$ with $P_{h(x)} \neq I - P_x$. In other words, the vector bundle cannot assign each orbit $\{x, h(x)\}$ to a pair of orthogonal complements $\{P_x, I - P_x\}$.

A stronger version of the condition demands that if $P \in A$ is a projection with $T(P)P = 0$, then $P = 0$. With notation and restrictions as above, if $A = C(X)$, then each component of X is a fixed point of the action induced by h. Similarly, if $A = M_n(C(X))$ and $P \in A$ is a nonzero projection (vector bundle), some x must satisfy $P_{h(x)}P_x \neq 0$, meaning the subspaces assigned to x and $h(x)$ must not always be orthogonal. This requirement allows for a stronger version of Theorem 2.2.2, in which the odd unitary element that allegedly has no square root is replaced by a projection plus an odd element. This type of element occurs frequently in K-theory, as a unitary matrix F over a C^*-algebra may have odd entries, but $F \oplus I$ does not.

$$\text{Projection} + \text{Odd} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix}$$

Theorem 2.2.4. Suppose A is a unital C^*-algebra with a $\mathbb{Z}_2$ action generated by T such that every nonzero projection $P \in A$ satisfies $T(P)P \neq 0$. If f is a nonzero odd element and η is a projection such that $\eta + f$ is unitary and $\eta f = f\eta = 0$, then there is no unitary g such that $gT(g) = T(g)g$ and $g^2 = \eta + f$.

Proof. The conditions imply that $\eta f^* = f^*\eta = 0$ and $\eta + ff^* = \eta + f^*f = 1$, the last of which shows that η is even. Suppose g is unitary with $g^2 = \eta + f$ and $gT(g) = T(g)g$. Then $T(g)$ and g^{-1} also commute, allowing for the following computation.

$$(T(g)g^{-1})^2 = T(g^2)g^{-2} = T(\eta + f)(\eta + f)^* = (\eta - f)(\eta + f^*) = \eta - ff^* = 2\eta - 1$$

Since η is a projection, the spectrum of $2\eta - 1$ is contained in $\{-1, 1\}$. The spectral mapping theorem then implies that $\sigma(T(g)g^{-1}) \subset \{i, -i, -1, 1\}$. Moreover, $T(g)g^{-1}$ is unitary, so the continuous functional calculus is applicable, giving the following decomposition.

$$T(g)g^{-1} = iP - iQ - R + S \tag{2.2.5}$$

Here $P, Q, R,$ and S are mutually orthogonal projections with $P + Q + R + S = 1$. Also, $b := T(g)g^{-1}$ is a unitary element satisfying $T(b) = b^{-1} = b^*$, so the commutative C^*-algebra generated by b is also T-invariant. Consequently, P, Q, R, S, and all of their images under T pairwise commute. Next, we rephrase (2.2.5) as

$$T(g) = (iP - iQ - R + S)g \tag{2.2.6}$$

and apply T to both sides, giving the following.

$$\begin{aligned} g &= (iT(P) - iT(Q) - T(R) + T(S))T(g) \\ &= (iT(P) - iT(Q) - T(R) + T(S))(iP - iQ - R + S)g \end{aligned} \tag{2.2.7}$$

As g is invertible, expanding (2.2.7) and canceling g show that

$$1 = (-1)\beta_{-1} + i\beta_i + (-i)\beta_{-i} + \beta_1 \tag{2.2.8}$$

where the β terms are defined as follows.

$$\begin{aligned} \beta_{-1} &= T(P)P + T(Q)Q + T(R)S + T(S)R \\ \beta_i &= T(P)S + T(Q)R + T(R)Q + T(S)P \\ \beta_{-i} &= T(P)R + T(Q)S + T(R)P + T(S)Q \\ \beta_1 &= T(P)Q + T(Q)P + T(R)R + T(S)S \end{aligned} \tag{2.2.9}$$

Each of the sixteen commuting products of two projections in (2.2.9) is a projection. Moreover, any two of these sixteen projections annihilate each other: for example, $T(R)P \cdot T(Q)P = T(Q)P \cdot T(R)P = 0$ because $RQ = QR = 0$. This means that each β term is a projection, and the four projections are mutually orthogonal. Equation (2.2.8) then implies that $\beta_{-1} = 0$, so $T(P)P + T(Q)Q + T(R)S + T(S)R = 0$. As each of the summands is a projection and therefore positive, it follows that each summand is zero, so in particular $T(P)P = 0 = T(Q)Q$. By the assumptions of the theorem, both P and Q must be zero.

Finally, this gives a simpler form of (2.2.6).

$$T(g) = (-R + S)g \tag{2.2.10}$$

The projections R and S annihilate each other and commute with g, as they are in the C^*-algebra generated by $T(g)g^{-1}$. Moreover, $R + S = P + Q + R + S = 1$, so we can square both sides of (2.2.10) to reach that $f = 0$.

$$\begin{aligned} T(g^2) = (-R + S)^2 g^2 &\implies T(\eta + f) = (R + S)(\eta + f) \\ &\implies \eta - f = \eta + f \\ &\implies f = 0 \end{aligned}$$

□

2.3 Example: Noncommutative Tori

The quantum tori $C(\mathbb{T}^n_\rho)$, which are obtained as Rieffel deformations of the commutative torus $C(\mathbb{T}^n)$, are equipped with an antipodal action α which negates each generator U_i. The continuity properties of deformation will be sufficient to show that no projection P in any odd-dimensional matrix algebra over $C(\mathbb{T}^n_\rho)$ can satisfy $\alpha(P) = I - P$. Therefore, Theorem 2.2.2 applies and requires that odd unitary elements do not have certain types of square roots.

The limits in (1.4.7) and the isomorphism (1.4.8) may be rephrased as follows for any fixed smooth elements $f, g \in C^\infty(\mathbb{T}^n)$ and fixed antisymmetric matrices J and H.

$$\lim_{t\to 0} ||f \cdot_{H+tJ} g - f \cdot_H g||_{H+tJ} = 0 \qquad \lim_{t\to 0} ||f||_{H+tJ} = ||f||_H \tag{2.3.1}$$

If J is an antisymmetric matrix which is only nonzero in one pair of opposite entries, then (2.3.1) indicates that the multiplication on smooth elements of $C(\mathbb{T}^n_\rho) \cong C(\mathbb{T}^n)_J$ behaves continuously when one pair of conjugate entries in ρ is modified. To avoid switching back and forth between the two notations, denote $\cdot_J$ as $\cdot_\rho$, and let $\mathcal{C}_\omega$ denote the set of parameter matrices ρ differing from ω in one pair of conjugate entries. The following continuity restatements then apply and can also be easily extended to matrices of smooth elements.

$$||f \cdot_\omega g - f \cdot_\rho g||_\rho \to 0 \text{ as } \rho \to \omega \text{ within } \mathcal{C}_\omega \tag{2.3.2}$$

$$||f||_\rho \to ||f||_\omega \text{ as } \rho \to \omega \text{ within } \mathcal{C}_\omega \tag{2.3.3}$$

The antipodal action α is simultaneously compatible with each product $\cdot_\rho$. This is a result of the fact that the antipodal map on $C(\mathbb{T}^n)$ commutes with the $\mathbb{R}^n$ action of translation in angular coordinates, which defines the quantization. Further, the adjoint operations on various $C(\mathbb{T}^n_\rho)$ all restrict to the same operation on $C^\infty(\mathbb{T}^n)$, so any time we approximate elements of $C(\mathbb{T}^n_\rho)$ with smooth elements, we can preserve self-adjointness. Similarly, we can preserve the homogeneity classes of elements, so we prove that no projection in $M_{2m+1}(C(\mathbb{T}^n_\rho))$ satisfies $\alpha(P) = I - P$ by examining a dense subset of ρ and applying a continuity argument. When ρ consists of roots of unity, the following lemma shows that there are unitary matrices over $\mathbb{C}$ which satisfy similar noncommutativity relations as the generators of $C(\mathbb{T}^n_\rho)$.

Lemma 2.3.4. Suppose ρ is an $n \times n$ parameter matrix with each entry a root of unity. Then there are unitary matrices $A_1, \ldots, A_n$ over $\mathbb{C}$ such that $A_j A_i = \rho_{ij} A_i A_j$ for all $i, j \in \{1, \ldots, n\}$. Moreover, if m is relatively prime to the order of each ρ_{ij} as a root of unity, we may insist that the dimension of the matrices A_i is relatively prime to m.

Proof. When $n = 2$ we choose A_1 and A_2 to have dimensions $q \times q$, where q is the smallest positive integer with $\rho_{12}^q = 1$.

$$A_1 = \begin{bmatrix} 1 & & & & \\ & \rho_{12} & & & \\ & & \rho_{12}^2 & & \\ & & & \ddots & \\ & & & & \rho_{12}^{q-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & \dots & & & 0 \\ 0 & 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & & & & 0 \end{bmatrix}$$

These matrices satisfy $A_2A_1 = \rho_{12}A_1A_2$ (and the other A_jA_i relations follow from the fact that ρ is a parameter matrix). Now suppose $3 \le m \le n$, and assume we have found unitary matrices $B_1, \dots, B_{m-1}$ of some dimension $q \times q$ satisfying $B_jB_i = \rho_{ij}B_iB_j$. Let k be the least common multiple of the orders of the new noncommutativity coefficients ρ_{im}, $1 \le i \le m-1$, so each ρ_{im}^k is 1. Then define A_i, $i \in \{0, \dots, m-1\}$ and A_m in block form.

$$A_i = \begin{bmatrix} B_i & & & \\ & \rho_{im}B_i & & \\ & & \ddots & \\ & & & \rho_{im}^{k-1}B_i \end{bmatrix}, \; A_m = \begin{bmatrix} 0 & I_q & 0 & \dots & & & 0 \\ 0 & 0 & I_q & 0 & \dots & & 0 \\ 0 & 0 & 0 & I_q & 0 & \dots & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & I_q \\ I_q & 0 & \dots & & & & 0 \end{bmatrix}$$

These unitary matrices $A_1, \dots, A_m$ of dimension $qk \times qk$ satisfy the conditions $A_jA_i = \rho_{ij}A_iA_j$, completing the induction. The block dimension in each step is the least common multiple of the orders of certain ρ_{ij}, so if a prime fails to divide each of these orders, it also fails to divide $\dim(A_i)$. $\square$

Theorem 2.3.5. There is no projection P with $\alpha(P) = 1 - P$ in any $M_{2m+1}(C(\mathbb{T}_\rho^n))$.

Proof. Suppose the theorem fails, which by Proposition 2.2.3 means there is a self-adjoint odd matrix $A \in M_{2m+1}(C(\mathbb{T}^n_\rho))$ with $A \cdot_\rho A = 1$. Approximate A with a self-adjoint, odd, smooth matrix $B \in M_{2m+1}(C^\infty(\mathbb{T}^n))$ which has $||B \cdot_\rho B - 1||_\rho < 1$. When the parameter of the algebra $C(\mathbb{T}^n_\rho)$ changes, B remains odd and self-adjoint. Perturb the entries of the parameter matrix ρ using (2.3.2) and (2.3.3) multiple times to replace ρ with a parameter matrix ω, where each entry of ω is a root of unity with odd order, maintaining $||B \cdot_\omega B - 1||_\omega < 1$. So, $B \in M_{2m+1}(C^\infty(\mathbb{T}^n))$ is still odd, self-adjoint, and invertible as an element of $M_{2m+1}(C(\mathbb{T}^n_\omega))$.

We may form a homomorphism from $C(\mathbb{T}^n_\rho)$ to a matrix algebra over $C(\mathbb{T}^n)$ in a way similar to [27]. Apply Lemma 2.3.4 to find unitaries $A_1, \ldots, A_n \in M_q(\mathbb{C})$ with $A_k A_j = \omega_{jk} A_j A_k$. Moreover, because each ω_{jk} is of odd order as a root of unity, we may insist $q = 2p + 1$ is odd. The universal property of $C(\mathbb{T}^n_\omega)$ shows that there is a $*$-homomorphism

$$E : C(\mathbb{T}^n_\omega) \to M_{2p+1}(C(\mathbb{T}^n))$$

$$U_j \mapsto u_j A_j$$

where $u_j \in C(\mathbb{T}^n)$ is the jth coordinate function. Since the generators U_j of $C(\mathbb{T}^n_\omega)$ are odd, and their images $u_j A_j$ have odd functions in every entry, the map E is equivariant for the (entrywise) antipodal maps. The image $E(B) \in M_{(2p+1)(2m+1)}(C(\mathbb{T}^n))$ is then a self-adjoint, invertible matrix of odd dimension, with each entry an odd function on $\mathbb{T}^n$. The determinant of this matrix is a nowhere vanishing, real-valued, odd function on $\mathbb{T}^n$, which gives a contradiction since $\mathbb{T}^n$ is connected. $\square$

The theorem does not apply for even dimension matrix algebras, as the existence of an odd unitary $V = \bigoplus_{i=1}^{k} U_1$ in $M_k(C(\mathbb{T}^n_\rho))$ implies that $P = \frac{1}{2}\begin{bmatrix} I_q & V \\ V^* & I_q \end{bmatrix}$ is a projection in $M_{2k}(C(\mathbb{T}^n_\rho))$ with $\alpha(P) = I - P$.

2.4 Counterexample: Noncommutative Spheres

The $\mathbb{R}^n/\mathbb{Z}^n$ action which defines the Rieffel deformations $C(\mathbb{S}^{2n-1})_J \cong C(\mathbb{S}^{2n-1}_\rho)$ of $C(\mathbb{S}^{2n-1})$ only acts on the unimodular component in the polar decomposition of the coordinates $z_i = r_i u_i$, $r_i \geq 0$, $|u_i| = 1$. In [33], Natsume and Olsen prove that $C(\mathbb{S}^{2n-1}_\rho)$ may actually be written as a function algebra over the noncommutative tori.

Theorem 2.4.1. ([33], Theorem 2.5) Let $\mathbb{S}^{n-1}_+ = \{\vec{t} = (t_1, \ldots, t_n) : 0 \leq t_i \leq 1, t_1^2 + \ldots + t_n^2 = 1\}$. Then $C(\mathbb{S}^{2n-1}_\rho)$ is isomorphic to the C^*-algebra of continuous functions $f : \mathbb{S}^{n-1}_+ \to C(\mathbb{T}^n_\rho)$ which satisfy the condition that whenever $t_i = 0$, $f(\vec{t}) \in C^*(U_1, \ldots, U_{i-1}, U_{i+1}, \ldots U_n)$.

This theorem is akin to writing complex coordinates in polar form $z_i = t_i u_i$ and seeing a function on the unimodular coordinates u_i whenever the radial coordinates t_i are fixed. Moreover, when some radial coordinate is 0, the corresponding unimodular coordinate should be irrelevant. Now, the norm on the function algebra (with operations defined pointwise) is the unique C^*-norm, $||f|| = \max_{\vec{t} \in \mathbb{S}^{n-1}_+} ||f(\vec{t})||_{C(\mathbb{T}^n_\rho)}$. Also, the generators z_i take a simple form.

$$z_i(\vec{t}) = t_i U_i$$

An enormous advantage of this formulation is that since every element of $C(\mathbb{S}^{2n-1}_\rho)$ is a function on a compact space, we see the various topological joys of compact spaces (bump functions, partitions of unity, and so on) without having to pass to commutative subalgebras. Moreover, unitaries in $C(\mathbb{S}^{2n-1}_\rho)$ are paths of unitaries in $C(\mathbb{T}^n_\rho)$, a well-studied object, and any element $f \in C(\mathbb{S}^{2n-1}_\rho)$ has the property that $f(1, 0, \ldots, 0)$ belongs to the *commutative* C^*-algebra $C^*(U_1) \cong C(\mathbb{S}^1)$ (and similarly for other U_i)! The natural choice of antipodal map

$$\alpha : z_i \mapsto -z_i$$

on $C(\mathbb{S}_\rho^{2n-1})$ is also just the pointwise application of the antipodal map on $C(\mathbb{T}_\rho^n)$. Since the noncommutative tori exhibited behavior similar to $C(\mathbb{T}^n)$ with respect to the antipodal action, it seems feasible that $C(\mathbb{S}_\rho^{2n-1})$ may behave similarly to $C(\mathbb{S}^{2n-1})$.

Proposition 2.4.2. Suppose F is a matrix in $M_{2k-1}(C(\mathbb{S}_\rho^{2n-1}))$, $2n-1 \geq 3$. Then F cannot be both invertible and odd (i.e., odd in every entry).

Proof. Suppose F is invertible and odd. Then for $\vec{t} = (1, 0, \ldots, 0)$, $F(\vec{t})$ is a matrix of odd dimension over $C^*(U_1) \cong C(\mathbb{S}^1)$ such that each entry is an odd function. Its K_1 class over $C(\mathbb{S}^1)$ is determined by $\det(F(\vec{t}))$, which is an odd, nowhere vanishing function $\mathbb{S}^1 \to \mathbb{C}$. By the Borsuk-Ulam theorem, this function has odd winding number, so $F(\vec{t})$ is equivalent to U_1^a in $K_1(C^*(U_1))$, where a is odd, and certainly $F(\vec{t})$ and U_1^a are also equivalent in $K_1(C^*(U_1, U_2))$. Similarly, if $\vec{s} = (0, 1, 0, \ldots, 0)$, $F(\vec{s})$ is equivalent to U_2^b in $K_1(C^*(U_1, U_2))$, where b is odd. However, there is also a path connecting $\vec{s}$ and $\vec{t}$ within $\{\vec{r} \in \mathbb{S}_+^{n-1} : r_i = 0 \text{ for } i \geq 3\}$, so $F(\vec{s})$ and $F(\vec{t})$ are in the same component of invertibles over $C^*(U_1, U_2)$, which is isomorphic to a 2-dimensional quantum torus. This contradicts the fact that U_1^a and U_2^b are inequivalent in $K_1(C^*(U_1, U_2))$ when a or b is nonzero (see [39] for when the 2-torus $C^*(U_1, U_2)$ is given by an irrational rotation; the result on the rational torus follows from a homomorphism $C^*(U_1, U_2) \to M_p(C(\mathbb{T}^2))$ found, for example, in [27]). $\square$

Proposition 2.4.3. There are no nontrivial projections in $C(\mathbb{S}_\rho^{2n-1})$.

Proof. The θ-deformed spheres each admit a continuous, positive, and faithful trace τ, developed in [33] by integrating the usual trace on $C(\mathbb{T}_\rho^n)$ over a Borel probability measure. We may extend τ as a linear map on $M_k(C(\mathbb{S}_\rho^{2n-1}))$ in the usual way by summing over the diagonal, and since this map is invariant under unitary conjugation, this allows us to view τ as a function on $K_0(C(\mathbb{S}_\rho^{2n-1}))$. Now, $K_0(C(\mathbb{S}_\rho^{2n-1}))$ is generated by the trivial projection 1, so the only possible values of the trace on projections in $M_k(C(S_\rho^{2n-1}))$ are integers. However, faithfulness implies that any nontrivial projection in $C(\mathbb{S}_\rho^{2n-1})$ must have trace in $(0, 1)$. $\square$

Even though $C(\mathbb{S}_\rho^{2n-1})$ is often a noncommutative algebra, any element with a one-sided inverse always has a two-sided inverse. This is a property that distinguishes $M_n(\mathbb{C})$ from $B(H)$ when H is an infinite-dimensional Hilbert space, so in general a C^*-algebra A is called *finite* if whenever $x, y \in A$ have $xy = 1$, it follows that $yx = 1$. If A also has the property that $M_k(A)$ is finite for all $k \in \mathbb{Z}^+$, then A is called *stably finite* (this does not follow from finiteness of A; see [9]).

Proposition 2.4.4. Each $C(\mathbb{S}_\rho^{2n-1})$ is stably finite.

Proof. Without loss of generality, suppose $W \in M_k(C(\mathbb{S}_\rho^{2n-1}))$ is right invertible, so that $W(\vec{t})$ is right invertible in $M_k(C(\mathbb{T}_\rho^n))$ for any $\vec{t}$. Let G and G_r be defined as follows.

$$G := \{a \in C(\mathbb{T}_\rho^n) : a \text{ is invertible}\}$$

$$G_r := \{a \in C(\mathbb{T}_\rho^n) : a \text{ has a right inverse but is not invertible}\}$$

The preimage of $G_r \cup G$ under W is all of $\mathbb{S}_+^{n-1}$. Moreover, $A = W^{-1}[G]$ and $B = W^{-1}[G_r]$ are disjoint since G and G_r are disjoint. However, G and G_r are both open (see [19], Proposition 2.7), so A and B form a separation of $\mathbb{S}_+^{n-1}$ unless one of the sets is empty. Now, $W(1, 0, \ldots, 0)$ is a right invertible in the C^*-algebra $M_k(C^*(U_1)) \cong M_k(C(\mathbb{S}^1)) \cong C(\mathbb{S}^1, M_k(\mathbb{C}))$. In $M_k(\mathbb{C})$, every right invertible is invertible, so $W(1, 0, \ldots, 0)$ is invertible, A is nonempty, and $A = \mathbb{S}_+^{n-1}$. Finally, each $W(\vec{t})$ is invertible, and W is invertible. $\square$

Corollary 2.4.5. If $2n - 1 \geq 3$ and $w \in C(\mathbb{S}_\rho^{2n-1})$ is odd, then ww^* is not invertible.

Proof. If w is odd and ww^* is invertible, the previous proposition implies that w is invertible. This contradicts Proposition 2.4.2 for $2k - 1 = 1$. $\square$

In the commutative case, there is no odd, continuous function $F : \mathbb{S}^3 \to \mathbb{R}^3 \setminus \{0\}$. By identifying $\mathbb{R}^3 \cong \mathbb{C} \oplus \mathbb{R}$, we see that if $w, x \in C(\mathbb{S}^3)$ are odd and x is self-adjoint, $|x|^2 + |w|^2 =$

x^2+ww^* cannot be invertible. The above corollary makes a somewhat similar claim in the θ-deformed spheres when $2n-1=3$, but it is missing the self-adjoint odd element x. Further, when we try to rewrite the claim that there is no odd, continuous $F : \mathbb{S}^{2n-1} \to \mathbb{R}^{2n-1} \setminus \{0\}$ into a conjecture on elements of $C(\mathbb{S}^{2n-1}_\rho)$, there is an abundance of ambiguity. This comes from the fact that if s and t are self-adjoint, $s^2+t^2 = (s+it)(s-it) = (s+it)(s+it)^*$ only when s and t commute, which is the same as insisting $s+it$ is normal. The distinction means that the identifications $\mathbb{R}^{2n-1} \cong \mathbb{R} \oplus \bigoplus_{i=1}^{n-1} \mathbb{C}$ and $\mathbb{R}^{2n-1} \cong \bigoplus_{i=1}^{2n-1} \mathbb{R}$ lead to at least two separate questions.

Question 2.4.6. If $x \in C(\mathbb{S}^{2n-1}_\rho)$ is odd and self-adjoint, and $w_1, \ldots, w_{n-1} \in C(\mathbb{S}^{2n-1}_\rho)$ are odd, must $x^2 + w_1w_1^* + \ldots + w_{n-1}w_{n-1}^*$ fail to be invertible?

Question 2.4.7. If $f_1, \ldots, f_{2n-1} \in C(\mathbb{S}^{2n-1}_\rho)$ are odd and self-adjoint, must $f_1^2 + \ldots + f_{2n-1}^2$ fail to be invertible?

The second of these questions was posed by Taghavi in [49], as a general question about no particular family of noncommutative spheres. There are also similar questions formed by replacing some, but not all, of the expressions $w_iw_i^*$ with the square sum of two self-adjoint elements. However, regardless of formulation, the answer to each question is no.

Theorem 2.4.8. If $C(\mathbb{S}^{2n-1}_\rho)$ is noncommutative, then Questions 2.4.6 and 2.4.7, and all intermediate versions, have a negative answer.

Proof. Decompose the generators as $z_m = x_m + iy_m$ where x_m and y_m are self-adjoint, and pick two generators z_j and z_k which do not commute. Since each z_m is normal, $z_mz_m^* = x_m^2 + y_m^2$, so consider the following sum.

$$(x_j + x_k)^2 + (y_j + y_k)^2 + \sum_{m \notin \{j,k\}} z_mz_m^* \tag{2.4.9}$$

The $n-2$ elements z_m present in the sum are odd, and both x_j+x_k and y_j+y_k are self-adjoint and odd. So, (2.4.9) is of the form in Question 2.4.6, where w_1 is actually self-adjoint. After replacement of every term $z_m z_m^*$ with $x_m^2 + y_m^2$, as z_m is normal, (2.4.9) becomes the square sum of $2 + 2(n-2) = 2n-2$ odd self-adjoint elements, so it can be written in the form of Question 2.4.7 where $f_{2n-1} = 0$. For intermediate versions of the two questions, expand some (but not all) of the terms $z_m z_m^*$. To see that (2.4.9) is invertible, we first rewrite $(x_j + x_k)^2 + (y_j + y_k)^2$.

$$\begin{aligned}
(x_j + x_k)^2 + (y_j + y_k)^2 &= x_j^2 + x_k^2 + x_j x_k + x_k x_j + y_j^2 + y_k^2 + y_j y_k + y_k y_j \\
&= x_j^2 + y_j^2 + x_k^2 + y_k^2 + (x_j x_k + x_k x_j + y_j y_k + y_k y_j) \\
&= z_j z_j^* + z_k z_k^* + (x_j x_k + x_k x_j + y_j y_k + y_k y_j)
\end{aligned}$$

This gives a simpler form for the original sum (2.4.9).

$$\begin{aligned}
(x_j + x_k)^2 + (y_j + y_k)^2 + \sum_{m \notin \{j,k\}} z_m z_m^* &= (x_j x_k + x_k x_j + y_j y_k + y_k y_j) + \sum_{m=1}^{n} z_m z_m^* \\
&= (x_j x_k + x_k x_j + y_j y_k + y_k y_j) + 1
\end{aligned}$$

It suffices to show $||x_j x_k + x_k x_j + y_j y_k + y_k y_j|| < 1$. First, we rewrite the components x_j, y_j, x_k, y_k in terms of z_j and z_k via $x_m = \dfrac{z_m + z_m^*}{2}$ and $y_m = \dfrac{z_m - z_m^*}{2i}$. Then we rearrange terms and apply the adjoint noncommutativity relation $z_k z_j^* = \rho_{kj} z_j^* z_k$. Now, $x_j x_k + x_k x_j + y_j y_k + y_k y_j$ is equal to

$$\frac{z_j + z_j^*}{2} \cdot \frac{z_k + z_k^*}{2} + \frac{z_k + z_k^*}{2} \cdot \frac{z_j + z_j^*}{2} + \frac{z_j - z_j^*}{2i} \cdot \frac{z_k - z_k^*}{2i} + \frac{z_k - z_k^*}{2i} \cdot \frac{z_j - z_j^*}{2i} =$$

$$\frac{1}{4}[2 z_j z_k^* + 2 z_j^* z_k + 2 z_k^* z_j + 2 z_k z_j^*] =$$

$$\frac{1}{2}[z_j z_k^* + z_j^* z_k + \rho_{kj} z_j z_k^* + \rho_{kj} z_j^* z_k] =$$

$$\frac{1+\rho_{kj}}{2}[z_j z_k^* + z_j^* z_k]$$

and the norm of this element is determined by viewing it as a function from $\mathbb{S}_+^{n-1}$ to the noncommutative torus $C(\mathbb{T}_\rho^n)$, as in Theorem 2.4.1, where $z_i(\vec{t}) = t_i U_i$.

$$\begin{aligned}
\left|\left|\frac{1+\rho_{kj}}{2}[z_j z_k^* + z_j^* z_k]\right|\right| &= \frac{|1+\rho_{kj}|}{2} \cdot \max_{\vec{t}\in\mathbb{S}_+^{n-1}} \left\{||(t_j U_j)(t_k U_k)^* + (t_j U_j)^*(t_k U_k)||_{C(\mathbb{T}_\rho^n)}\right\} \\
&= \frac{|1+\rho_{kj}|}{2} \cdot \max_{\vec{t}\in\mathbb{S}_+^{n-1}} \{t_j t_k\} \cdot ||U_j U_k^* + U_j^* U_k||_{C(\mathbb{T}_\rho^n)} \\
&\leq \frac{|1+\rho_{kj}|}{2} \cdot \frac{1}{2} \cdot 2 \\
&< 1
\end{aligned}$$

At the last step, we have used that ρ_{kj} is unimodular, but not equal to 1, as z_j and z_k do not commute. Finally, the sum (2.4.9) is invertible. $\square$

The answers to Questions 2.4.6 and 2.4.7 (and all questions in between) for θ-deformed spheres are negative, and the proof above shows that the disconnect between the commutative and noncommutative cases is quite large. In the commutative $(2n-1)$-dimensional sphere, no square sum of $2n-1$ odd self-adjoint elements is invertible, but in a sphere where just one pair of generators fails to commute, we can form an invertible square sum using only $2n-2$ odd self-adjoint elements. Further, when $2n-1=3$, this invertible sum is of the form s^2+t^2, even though $(s+it)(s+it)^*$ cannot be invertible by Corollary 2.4.5. In other words, s and t will definitely not commute. Finally, because a counterexample was reached using fewer elements than expected, these arguments pass easily to even θ-deformed spheres $C(\mathbb{S}_\rho^{2n})$, which are quotients of $(2n+1)$-dimensional θ-deformed spheres where z_{n+1} is central.

If $C(\mathbb{S}_\rho^{2n})$ is noncommutative, then there are $2(n+1)-2=2n$ self-adjoint odd elements of $C(\mathbb{S}_\rho^{2n})$ whose square sum is invertible, in stark contrast with the commutative case.

Chapter 3

Equivariant Maps on θ-Deformed Spheres

3.1 Embeddings between Spheres

In [33], Natsume and Olsen described θ-deformed odd spheres $C(\mathbb{S}^{2n-1}_\rho)$ in two additional formats, as a universal C^*-algebra and as an algebra of functions over the quantum torus.

Definition 3.1.1 ([33], Definition 2.1)**.** If ρ is a parameter matrix, then $C(\mathbb{S}^{2n-1}_\rho)$ is the universal, unital C^*-algebra generated by $z_1, \ldots, z_n$ with the following identities.

$$z_1 z_1^* + \ldots + z_n z_n^* = 1 \qquad z_k z_j = \rho_{jk} z_j z_k \qquad z_j z_j^* = z_j^* z_j$$

These generators automatically satisfy $z_k z_j^* = \overline{\rho_{jk}} z_j^* z_k = \rho_{kj} z_j^* z_k$ ([33], Lemma 2.6), and when dealing with two separate θ-deformed spheres, we will sometimes denote the generators as $\mathfrak{z}_1, \ldots, \mathfrak{z}_n$.

Theorem 3.1.2 ([33], Theorem 2.5)**.** Let $\mathbb{S}^{n-1}_+ = \{(\vec{t} = (t_1, \ldots, t_n) : 0 \leq t_i \leq 1, t_1^2 + \ldots + t_n^2 = 1\}$. Then $C(\mathbb{S}^{2n-1}_\rho)$ is isomorphic to $\{f \in C(\mathbb{S}^{n-1}_+, C(\mathbb{T}^n_\rho)) :$ if $t_i = 0$, then $f(\vec{t}) \in C^*(U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n)\}$.

As seen in Section 2.4, the structure of $C(\mathbb{S}^{2n-1}_\rho)$ places some restrictions on the existence of odd elements and matrices, but not enough for an element-based Borsuk-Ulam theorem to hold on $C(\mathbb{S}^{2n-1}_\rho)$. In other words, the requirement that there is no odd map from $\mathbb{S}^{2n-1}$ to $\mathbb{R}^{2n-1} \setminus \{0\}$ does not have an analogue in the θ-deformed case. However, other equivalent formulations of the traditional Borsuk-Ulam theorem do hold on the θ-spheres; these results are analogous to the q-sphere results of Yamashita in [55], but use different methods.

The K-theory of $C(\mathbb{S}_\rho^{2n-1})$ is the same as that of $C(\mathbb{S}^{2n-1})$, as θ-deformation preserves K-theory, but in [33] Natsume and Olsen gave a very particular structure to $K_1(C(\mathbb{S}_\rho^{2n-1})) \cong \mathbb{Z}$ ($K_0(C(\mathbb{S}_\rho^{2n-1})) \cong \mathbb{Z}$ is generated by 1, a trivial projection). Every $U \in U_k(C(\mathbb{S}_\rho^{2n-1}))$ defines a multiplication operator on $M_k(C(\mathbb{S}_\rho^{2n-1}))$, and $C(\mathbb{S}_\rho^{2n-1})$ may be completed to a Hilbert space $L^2(\mathbb{S}_\rho^{2n-1})$. The inner product comes from a continuous, positive, and faithful trace on $C(\mathbb{S}_\rho^{2n-1})$, which integrates the standard trace τ on $C(\mathbb{T}_\rho^n)$.

$$\tau(f) = \int_{\mathbb{S}_+^{n-1}} \tau(f(\vec{t}))\, d\mu$$

$$\langle f, g \rangle = \tau(fg^*)$$

The measure μ is obtained by passing through a homeomorphism from $\mathbb{S}_+^{n-1}$ to an $(n-1)$-simplex and using normalized Lebesgue measure. Regarless, the inner product defines a completion $L^2(\mathbb{S}_\rho^{2n-1})$ of $C(\mathbb{S}_\rho^{2n-1})$, and a noncommutative Hardy space arises from the closure of analytic polynomials (no $z_1^*, \ldots, z_n^*$ terms) in the generators.

$$H^2(\mathbb{S}_\rho^{2n-1}) = \overline{\operatorname{span}_{\mathbb{C}}\{z_1^{m_1} \cdots z_n^{m_n} : m_1, \ldots, m_n \geq 0\}}$$

Natsume and Olsen showed that noncommutative Toeplitz operators

$$F \in M_k(C(\mathbb{S}_\rho^{2n-1})) \implies T_F(g) = P(F \cdot g) \text{ for } g \in \bigoplus_{i=1}^{k} H^2(\mathbb{S}_\rho^{2n-1})$$

are well-defined using the entrywise projection $P : L^2(\mathbb{S}_\rho^{2n-1}) \to H^2(\mathbb{S}_\rho^{2n-1})$, and that many (but not all) of the properties of standard Toeplitz operators carry over to this case. In particular, T_F is a Fredholm operator if and only if F is invertible, and the index of T_F completely characterizes the K_1 class of F in this case. A unitary matrix with Toeplitz index one is then necessarily a generator of $K_1(C(\mathbb{S}_\rho^{2n-1}))$, and such a matrix can be specified using a similar form to the commutative case. The standard K_1 generator of $C(\mathbb{S}^{2n-1})$ is a $2^{n-1} \times 2^{n-1}$ matrix $Z(n)$, which is defined recursively.

$$Z(1) = z_1 \qquad\qquad Z(k+1) = \begin{bmatrix} Z(k) & z_{k+1} I_{2^{k-1}} \\ -z_{k+1}^* I_{2^{k-1}} & Z(k)^* \end{bmatrix}$$

The modification Natsume and Olsen used for the θ-deformed spheres replaced $I_{2^{k-1}}$ with diagonal unitary matrices over $\mathbb{C}$.

$$Z_\rho(1) = 1 \qquad\qquad Z_\rho(k+1) = \begin{bmatrix} Z_\rho(k) & z_{k+1} D_1 \\ -z_{k+1}^* D_2 & Z_\rho(k)^* \end{bmatrix}$$

Diagonal unitaries D_1 and D_2 such that $Z_\rho(k+1)Z_\rho(k+1)^* = Z_\rho(k+1)^* Z_\rho(k+1) = \left(\sum_{i=1}^{k+1} z_i z_i^*\right) I_{2^k}$ are guaranteed to exist at each recursive step, but they are non-unique; any choice resulting in a unitary matrix $Z_\rho(n)$ produces a K_1 generator. We have chosen to use a subscript ρ here (as D_1 and D_2 depend on ρ), and in the next section we prove that this K_1 generator, which did not have uniquely specified coefficients in [33], may be chosen such that the entries of D_1 and D_2 in each stage vary continuously in ρ. This is consistent with the continuity results of Sangha in [48].

Any unital $*$-homomorphism $\phi : C(\mathbb{S}_\rho^{2n-1}) \to C(\mathbb{S}_\omega^{2n-1})$ induces a homomorphism on the K_1 groups, and any homomorphism $\phi_* : \mathbb{Z} \to \mathbb{Z}$ is associated to a unique integer n such that $\phi(m) = mn$. This number n is analogous to the degree of a function on a sphere $\mathbb{S}^{2n-1}$. One way to produce a map between θ-deformed spheres is to replace z_1 with a power z_1^k, modulo scaling issues.

Proposition 3.1.3. If $\rho = \begin{bmatrix} 1 & \rho_{12} \\ \overline{\rho_{12}} & 1 \end{bmatrix}$ and $\omega = \begin{bmatrix} 1 & \rho_{12}^k \\ \overline{\rho_{12}}^k & 1 \end{bmatrix}$ for a fixed $k \in \mathbb{Z}^+$, then $C^*(U_1^k, U_2) \leq C(\mathbb{T}_\rho^2)$ is isomorphic to $C(\mathbb{T}_\omega^2)$.

Proof. Define a map ψ from $C(\mathbb{T}^2_\omega) = C^*(V_1, V_2)$ to $C(\mathbb{T}^2_\rho) = C^*(U_1, U_2)$ as follows.

$$\psi(V_1) = U_1^k \qquad\qquad \psi(V_2) = U_2$$

The images are unitary and satisfy $U_2 U_1^k = \rho_{12}^k U_1^k U_2 = \omega_{12} U_1^k U_2$, so ψ is a well-defined $*$-homomorphism. Moreover, ψ respects the standard traces τ_ρ and τ_ω, as seen by checking on the dense set of $*$-polynomials.

$$\begin{aligned}
\tau_\rho\left(\psi\left(\sum_{\text{finite}} a_{mn} V_1^m V_2^n\right)\right) &= \tau_\rho\left(\sum_{\text{finite}} a_{mn} U_1^{km} U_2^n\right) \\
&= a_{00} \\
&= \tau_\omega\left(\sum_{\text{finite}} a_{mn} V_1^m V_2^n\right)
\end{aligned}$$

If $f \in \ker(\psi)$, then $\psi(ff^*) = 0$, and $0 = \tau_\rho(\psi(ff^*)) = \tau_\omega(ff^*)$, so by faithfulness of τ_ω, f must be 0. Now, ψ is injective, and any injective $*$-homomorphism between C^*-algebras is norm-preserving, so $\mathrm{Ran}(\psi)$ is closed. As $\mathrm{Ran}(\psi)$ includes every $*$-polynomial in U_1^k and U_2, it follows that $\mathrm{Ran}(\psi) = C^*(U_1^k, U_2)$. □

This proposition (which is undoubtedly a folklore result) allows us to view a "coarser" torus as a subalgebra of a "finer" torus, so in particular if ρ_{12} is of finite order, a copy of the commutative torus $C(\mathbb{T}^2)$ sits inside $C(\mathbb{T}^2_\rho)$. A similar embedding result applies for spheres, where again $\rho = \begin{bmatrix} 1 & \rho_{12} \\ \overline{\rho_{12}} & 1 \end{bmatrix}$ and $\omega = \begin{bmatrix} 1 & \rho_{12}^k \\ \overline{\rho_{12}}^k & 1 \end{bmatrix}$ for a fixed $k \in \mathbb{Z}^+$.

$$\begin{aligned}
C(\mathbb{S}^3_\omega) &\cong \{f \in C(\mathbb{S}^1_+, C(\mathbb{T}^2_\omega)) : f(0,1) \in C^*(V_2), f(1,0) \in C^*(V_1)\} \\
&\cong \{f \in C(\mathbb{S}^1_+, C(\mathbb{T}^2_\rho)) : f(\vec{t}) \in C^*(U_1^k, U_2), f(0,1) \in C^*(U_2), f(1,0) \in C^*(U_1^k)\} \\
&\leq C(\mathbb{S}^3_\rho)
\end{aligned}$$

Similar results follow for higher order tori and spheres, with more complicated notation. Now that it is clear embeddings of the type $C(\mathbb{S}^3_\omega) \hookrightarrow C(\mathbb{S}^3_\rho)$ exist, we now determine what effect they have on K_1. The generators of $C(\mathbb{S}^3_\omega) \leq C(\mathbb{S}^3_\rho)$ may be written easily as functions.

$$f_1(\vec{t}) = t_1 U_1^k \qquad\qquad f_2(\vec{t}) = t_2 U_2$$

In fact, f_2 is equal to z_2, and f_1 takes a very similar form to $z_1^k(t) = t_1^k U_1^k$. The $K_1(C(\mathbb{S}^3_\omega))$ generator is then $\begin{bmatrix} f_1 & f_2 \\ -\overline{\omega_{12}} f_2^* & f_1^* \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \\ -\overline{\rho_{12}}^k f_2^* & f_1^* \end{bmatrix}$, which is connected within the invertible matrices over $C(\mathbb{S}^3_\rho)$ to $\begin{bmatrix} z_1^k & z_2 \\ -\overline{\rho_{12}}^k z_2^* & z_1^{*k} \end{bmatrix}$ by scaling t_1 to t_1^k continuously.

Proposition 3.1.4. The invertible matrix $M = \begin{bmatrix} z_1^k & z_2 \\ -\overline{\rho_{12}}^k z_2^* & z_1^{*k} \end{bmatrix}$ over $C(\mathbb{S}^3_\rho)$ has $[M]_{K_1} = k$. That is, the index of the Toeplitz operator T_M is $-k$.

Proof. Because $C(\mathbb{S}^3_\rho)$ is a Rieffel deformation $C(\mathbb{S}^3)_J$ wherein z_1, z_2, z_1^*, z_2^* are smooth elements, the matrix $M = M(\rho)$ of smooth elements varies continuously in every deformed norm. This defines an invertible matrix $\widetilde{M}$ over the section algebra $\Gamma(C(\mathbb{S}^3)_{tJ})_{t\in[0,1]}$. By Theorem 1.4.12 (due to Sangha), the quotient maps $\pi_s : \Gamma(C(\mathbb{S}^3)_{tJ})_{t\in[0,1]} \to C(\mathbb{S}^3)_{sJ}$ induce automorphisms on K_1, so in particular $K_1(\Gamma(C(\mathbb{S}^3)_{tJ})_{t\in[0,1]}) \cong \mathbb{Z}$. The integer associated to $\widetilde{M}$ is the same as the integer associated to $M = \pi_1(\widetilde{M})$ and $\pi_0(\widetilde{M})$. But $\pi_0(\widetilde{M}) = \begin{bmatrix} z_1^k & z_2 \\ -z_2^* & z_1^{*k} \end{bmatrix}$ is an invertible matrix over the commutative sphere, and the K_1-class of $\pi_0(\widetilde{M})$ is the image of the $K_1(C(\mathbb{S}^3))$ generator under $(z_1, z_2) \mapsto \frac{(z_1^k, z_2)}{||(z_1^k, z_2)||}$. This map has degree k in topology ([56], Lemma 3.1), so naturality of the Chern character (1.3.11) shows that $[\psi(Z(2))]_{K_1}$ is k. $\square$

Remark. Alternatively, the kernel of the Toeplitz operator T_M is trivial, and the cokernel is spanned by $\begin{bmatrix} z_1^m \\ 0 \end{bmatrix}$ for $0 \leq m \leq k-1$. This method involves a vary tedious computation of formal power series, so it is omitted. Direct calculation of the index is also much harder (for me) to do in higher dimensions, whereas the above method generalizes to higher dimensions once we know $Z_\rho(n)$ and similar matrices in $z_1^{k_1}, \ldots, z_n^{k_n}$ may be specified continuously.

The embedding $C(\mathbb{S}^3_\omega) \hookrightarrow C(\mathbb{S}^3_\rho)$ existed whenever ω_{12} was a power of ρ_{12}, indicating that ω is a coarser rotation than ρ. It is not always possible to form an embedding in the other direction (for example, if $C(\mathbb{S}^3_\omega) = C(\mathbb{S}^3)$ is commutative but $C(\mathbb{S}^3_\rho)$ is noncommutative, because ρ_{12} is a nontrivial root of unity). However, we can instead form a map from $C(\mathbb{S}^3_\rho)$ to a matrix algebra over $C(\mathbb{S}^3_\omega)$. The key is to find matrices over $\mathbb{C}$ which satisfy noncommutativity relations from a given parameter matrix. An important example (in arbitrary dimension) is when $C(\mathbb{S}^{2n-1}_\rho)$ is formed from a parameter matrix ρ of roots of unity, and we wish to map into a matrix algebra over the commutative sphere $C(\mathbb{S}^{2n-1})$. This idea was seen in Theorem 2.3.5 and generalized a map from [27].

Lemma 3.1.5. Suppose ρ is an $n \times n$ parameter matrix with each entry a root of unity, and let $A_1, \ldots, A_n \in U_k(\mathbb{C})$ be the unitaries from Lemma 2.3.4. Then

$$E : \mathfrak{z}_i \mapsto z_i A_i$$

defines a unital $*$-homomorphism $E : C(\mathbb{S}^{2n-1}_\rho) \to M_k(C(\mathbb{S}^{2n-1}))$.

Proof. It is only necessary to check that the matrices $z_i A_i$ satisfy the relations defining $C(\mathbb{S}^{2n-1}_\rho)$. First, since A_i is unitary it is simple to check that $(z_i A_i)(z_i A_i)^*$ and $(z_i A_i)^*(z_i A_i)$ both equal $z_i z_i^* I_k$, so $z_i A_i$ is normal. The square sum is then $\sum_{i=1}^{n} (z_i A_i)(z_i A_i)^* = \sum_{i=1}^{n} z_i z_i^* I_k = I_k$. Finally, the noncommutativity conditions on A_i give similar conditions on $z_i A_i$.

$$(z_j A_j)(z_i A_i) = z_j z_i A_j A_i = z_i z_j (\rho_{ij} A_i A_j) = \rho_{ij} (z_i A_i)(z_j A_j)$$

$\square$

The map E induces a homomorphism E_* between $K_1(C(\mathbb{S}_\rho^{2n-1}))$ and $K_1(M_k(C(\mathbb{S}^{2n-1}))) \cong K_1(C(\mathbb{S}^{2n-1}))$. Since both groups are isomorphic to $\mathbb{Z}$, we seek the integer associated with $E_* : \mathbb{Z} \to \mathbb{Z}$, namely, $[E(Z_\rho(n))]_{K_1(C(\mathbb{S}^{2n-1}))}$. Here E is extended to matrix algebras over $C(\mathbb{S}_\rho^{2n-1})$ entrywise.

Definition 3.1.6. For $1 \leq m \leq n$ and $d \in \mathbb{Z}^+$, let $\mathscr{S}_m^{[d]}$ be the set of $d \times d$ matrices M over the commutative sphere $C(\mathbb{S}^{2n-1})$ such that $MM^* = M^*M = \left(\sum_{i=1}^{m} z_i z_i^*\right) I_d$. When the dimension d is understood, $\mathscr{S}_m^{[d]}$ will be denoted by $\mathscr{S}_m$. Also, for a fixed d let $\sim_m$ denote the equivalence relation from the partition of $\mathscr{S}_m = \mathscr{S}_m^{[d]}$ into path components.

Note that $\mathscr{S}_m^{[d]}$ is closed under the adjoint operation and under multiplication by $d \times d$ unitaries with scalar entries. Further, because there is a path of unitaries connecting any $U \in \mathcal{U}_d(\mathbb{C})$ to the identity, if $M \in \mathscr{S}_m^{[d]}$, then $M \sim_m UM \sim_m MU$. The following lemma describes what happens when a block matrix is in $\mathscr{S}_{m+1}^{[2d]}$.

Lemma 3.1.7. Fix $1 \leq m \leq n$ and $d \in \mathbb{Z}^+$. If $A, B \in U_d(C(\mathbb{S}^{2n-1}))$ and $U, V \in U_d(\mathbb{C})$ are unitaries of dimension d, then

$$M = \begin{bmatrix} A & z_{m+1}U \\ z_{m+1}^* V & B \end{bmatrix}$$

is in $\mathscr{S}_{m+1}^{[2d]}$ if and only if $A \in \mathscr{S}_m^{[d]}$ and $B = -VA^*U$.

Proof. Consider the following matrix multiplications and note that the lemma concerns the commutative sphere, so z_{m+1} and its adjoint commute with any matrix.

$$MM^* = \begin{bmatrix} AA^* + z_{m+1}z_{m+1}^* I_d & z_{m+1}(AV^* + UB^*) \\ z_{m+1}^*(VA^* + BU^*) & z_{m+1}z_{m+1}^* I_d + BB^* \end{bmatrix}$$

$$M^*M = \begin{bmatrix} A^*A + z_{m+1}z_{m+1}^* I_d & z_{m+1}(A^*U + V^*B) \\ z_{m+1}^*(U^*A + B^*V) & z_{m+1}z_{m+1}^* I_d + B^*B \end{bmatrix}$$

If $M \in \mathscr{S}_{m+1}^{[2d]}$, then $MM^* = M^*M = \left(\sum_{i=1}^{m+1} z_i z_i^*\right) I_{2d}$, which in block form is equal to $\begin{bmatrix} \left(\sum_{i=1}^{m+1} z_i z_i^*\right) I_d & 0 \\ 0 & \left(\sum_{i=1}^{m+1} z_i z_i^*\right) I_d \end{bmatrix}$. It follows from the top left blocks of MM^* and M^*M that $AA^* = A^*A = \left(\sum_{i=1}^{m} z_i z_i^*\right) I_d$. That is, $A \in \mathscr{S}_m^{[d]}$. The top right block of MM^* shows that $z_{m+1}(AV^* + UB^*) = 0$, and since the complement of the zero set of z_{m+1} is dense in $\mathbb{S}^{2n-1}$, this means $AV^* + UB^* = 0$. Rearranging the unitaries shows $B = -VA^*U$.

If $A \in \mathscr{S}_m^{[d]}$ and $B = -VA^*U$, then $A^*A = AA^* = \left(\sum_{i=1}^{d} z_i z_i^*\right) I_d$ is central and $BB^* = VA^*UU^*AV^* = VA^*AV = (A^*A)(VV^*) = A^*A$. Similarly, $B^*B = AA^*$. The diagonal blocks of MM^* and M^*M are then $\left(\sum_{i=1}^{d+1} z_i z_i^*\right) I_d$. Finally, the off-diagonal blocks all vanish due to $B = -VA^*U$, and $M \in \mathscr{S}_{m+1}^{[2d]}$. $\square$

The above lemma will be useful for an inductive proof regarding the relation $\sim_m$ and the recursive definition of $Z_\rho(n)$.

Lemma 3.1.8. Suppose ρ is a parameter matrix of roots of unity and the expansion map $E : C(\mathbb{S}_\rho^{2n-1}) \to M_k(C(\mathbb{S}^{2n-1}))$ is defined as in Lemma 3.1.5 for unitaries A_i of dimension $k \times k$. Also, extend E to matrix algebras over $C(\mathbb{S}_\rho^{2n-1})$ by entrywise application. If F is an invertible matrix over $C(\mathbb{S}_\rho^{2n-1})$, then $[E(F)]_{K_1(C(\mathbb{S}^{2n-1}))} = k \cdot [F]_{K_1(C(\mathbb{S}_\rho^{2n-1}))}$.

Proof. It is only necessary to check the result on the K_1 generator, namely, to check that $[E(Z_\rho(n))]_{K_1(C(\mathbb{S}^{2n-1}))} = k$. First, for $1 \le m \le n$, the fact that the image of

$$Z_\rho(m)Z_\rho(m)^* = Z_\rho(m)^* Z_\rho(m) = \left(\sum_{i=1}^{m} \mathfrak{z}_i \mathfrak{z}_i^*\right) I_{2^{m-1}}$$

under the $*$-homomorphism E is

$$E(Z_\rho(m))E(Z_\rho(m))^* = E(Z_\rho(m))^* E(Z_\rho(m)) = \left(\sum_{i=1}^{m} z_i z_i^*\right) I_{k2^{m-1}}$$

establishes that $E(Z_\rho(m)) \in \mathscr{S}_m^{[k2^{m-1}]} := \mathscr{S}_m$. When $m = 1$,

$$E(Z_\rho(1)) = E(\mathfrak{z}_1) = z_1 A_1 \sim_1 \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_1 \end{bmatrix} = \bigoplus_{i=1}^{k} Z(1)$$

as a result of following a path connecting A_1 to the identity within $U_k(\mathbb{C})$. Now, for induction suppose $E(Z_\rho(m)) \sim_m \bigoplus_{i=1}^{k} Z(m)$. The matrix $Z_\rho(m+1)$ is of the form $\begin{bmatrix} Z_\rho(m) & \mathfrak{z}_{m+1} D_1 \\ -\mathfrak{z}_{m+1}^* D_2 & Z_\rho(m)^* \end{bmatrix}$, where D_1 and D_2 are unitary diagonal matrices with scalar entries. If we apply the expansion map to $\mathfrak{z}_{m+1} D_1$, the result is a block diagonal matrix whose diagonal blocks are unimodular scalar multiples of $z_{m+1} A_{m+1}$. A similar result applies when we expand $-\mathfrak{z}_{m+1}^* D_2$. We conclude that

$$E(Z_\rho(m+1)) = \begin{bmatrix} E(Z_\rho(m)) & z_{m+1} U \\ z_{m+1}^* V & E(Z_\rho(m)^*) \end{bmatrix}$$

for some $U, V \in U_{k2^{m-1}}(\mathbb{C})$. Since $E(Z_\rho(m+1)) \in \mathscr{S}_{m+1}$, Lemma 3.1.7 demands the following.

$$E(Z_\rho(m)) \in \mathscr{S}_m$$

$$E(Z_\rho(m)^*) = -V(E(Z_\rho(m))^*)U$$

The first of these conditions is already known, and in addition we have assumed that $E(Z_\rho(m)) \sim_m \bigoplus_{i=1}^{k} Z(m)$, so there is a path $\phi(t)$ connecting $\phi(0) = E(Z_\rho(m))$ to $\phi(1) = \bigoplus_{i=1}^{k} Z(m)$ within $\mathscr{S}_m$. In addition, let U_t and V_t denote paths in $U_{k2^{m-1}}(\mathbb{C})$ connecting $U = U_0$

to $I = U_1$ and $V = V_0$ to $-I = V_1$. Then the new path

$$\Phi(t) = \begin{bmatrix} \phi(t) & z_{m+1}U_t \\ z_{m+1}^* V_t & -V_t\phi(t)^* U_t \end{bmatrix}$$

satisfies $\Phi(t) \in \mathscr{S}_{m+1}$. By Lemma 3.1.7 again, $\Phi(t)$ demonstrates that $\Phi(0) = E(Z_\rho(m+1))$ is equivalent under $\sim_{m+1}$ to a particular block matrix.

$$\begin{aligned} E(Z_\rho(m+1)) &\sim_{m+1} \begin{bmatrix} \bigoplus_{i=1}^{k} Z(m) & z_{m+1} I_{k2^{m-1}} \\ -z_{m+1}^* I_{k2^{m-1}} & \bigoplus_{i=1}^{k} Z(m)^* \end{bmatrix} \\ &= \begin{bmatrix} \bigoplus_{i=1}^{k} Z(m) & \bigoplus_{i=1}^{k} z_{m+1} I_{2^{m-1}} \\ -\bigoplus_{i=1}^{k} z_{m+1}^* I_{2^{m-1}} & \bigoplus_{i=1}^{k} Z(m)^* \end{bmatrix} \end{aligned}$$

After a reordering of the standard orthonormal basis (which preserves the $\sim_{m+1}$ relation, as it applies a conjugation by a unitary matrix over $\mathbb{C}$), this matrix is equivalent under $\sim_{m+1}$ to $\bigoplus_{i=1}^{k} \begin{bmatrix} Z(m) & z_{m+1} I_{2^{m-1}} \\ -z_{m+1}^* I_{2^{m-1}} & Z(m)^* \end{bmatrix} = \bigoplus_{i=1}^{k} Z(m+1)$. The induction is complete, so in the end we reach $E(Z_\rho(n)) \sim_n \bigoplus_{i=1}^{k} Z(n)$. The $\sim_n$ relation is just path connectedness within unitary matrices of a fixed dimension, which in particular implies the two matrices are equivalent in K_1. Finally, $[E(Z_\rho(n))]_{K_1(C(\mathbb{S}^{2n-1}))} = k$. $\square$

For embeddings of θ-deformed spheres into (matrix algebras over) other θ-deformed spheres, the effect on K-theory is reasonable. Generally speaking, an expansion into $k \times k$ matrices shoud multiply the index by k, and embedding a sphere into a sphere defined by a finer rotation multiplies the index by a factor related to order, which we have shown for some special cases. When $C(\mathbb{S}_\rho^{2n-1})$ is given the antipodal action $\alpha : z_i \mapsto -z_i$ and this action is extended entrywise to the matrix algebras, E is an equivariant map, because it sends the odd generators to matrices with odd entries. Similarly, an embedding between spheres is equivariant if and only if the powers applied to the generators are all odd. This suggests that K_1 will behave well with respect to the antipodal map α (after all, the generator $Z_\rho(n)$ has odd entries). This is established in later sections using a continuity argument to reduce to the commutative case.

3.2 Continuity in K-Theory

In this section, we show that the K_1 generator $Z_\rho(n)$ for $K_1(C(\mathbb{S}_\rho^{2n-1}))$, which is defined by the ambiguous recurrence relation

$$Z_\rho(1) = z_1$$

$$Z_\rho(k+1) := \begin{bmatrix} Z_\rho(k) & z_{k+1}D_1 \\ -z_{k+1}^* D_2 & Z_\rho(k)^* \end{bmatrix} \tag{3.2.1}$$

where D_1 and D_2 are any diagonal matrices over $\mathbb{C}$ which make the resulting matrix satisfy $Z_\rho(k+1)Z_\rho(k+1)^* = Z_\rho(k+1)^* Z_\rho(k+1) = (z_1 z_1^* + \dots z_{k+1} z_{k+1}^*)I$, may be specified in a well-defined and continuous way. First, according to [33], regardless of the choices made for D_1 and D_2, $Z_\rho(n)$ will generate $K_1(C(\mathbb{S}_\rho^{2n-1}))$. We use a different argument than in [33], and we may even choose coefficients so that if $z_1, \dots, z_m$ generate a $(2m-1)$-sphere $C(\mathbb{S}_\omega^{2m-1})$ with the same noncommutativity relations as the first m generators $z_1, \dots, z_m$ of $C(S_\rho^{2n-1})$, $n > m$, then $Z_\rho(k) = Z_\omega(k)$ (as formal $*$-monomial matrices) for all k between 1 and m. First, we demand that any 3-sphere given by $z_2 z_1 = \rho_{12} z_1 z_2$ must have the following K_1 generator.

$$Z_\rho(2) = \begin{bmatrix} z_1 & z_2 \\ -\rho_{21} z_2^* & z_1^* \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ -\overline{\rho_{12}} z_2^* & z_1^* \end{bmatrix}$$

Together with the convention $Z_\rho(1) = z_1$, this makes a consistent, continuous choice for all 1 and 3-dimensional spheres, and that choice is compatible with extending a list of generators (z_1 alone) to form a larger sphere (in z_1 and z_2). Now, for induction we suppose we have achieved the same for all spheres of dimension up to $2(n-1)-1 = 2n-3$. If ρ is an $n \times n$ parameter matrix, then $Z_\rho(n-1)$ has been specified continuously by the inductive assumption, with upper left block $Z_\rho(n-2)$ and diagonal unitaries G_1 and G_2 over $\mathbb{C}$, as follows.

$$Z_\rho(n-1) = \begin{bmatrix} Z_\rho(n-2) & z_{n-1}G_1 \\ -z_{n-1}^* G_2 & Z_\rho(n-2)^* \end{bmatrix} \tag{3.2.2}$$

Next, form a matrix minor ω of ρ by removing row and column $n-1$, so the sphere $C(\mathbb{S}_\omega^{2n-3})$ may be given with generators relabeled as $z_1, \ldots, z_{n-2}, z_n$. Note that as formal $*$-polynomial matrices, $Z_\omega(n-2) = Z_\rho(n-2)$, as the relations on the first $n-2$ generators are the same. By the inductive assumption, diagonal unitary matrices D_1 and D_2 over $\mathbb{C}$ are specified (and vary continuously in parameter) to form $Z_\omega(n-1)$ from $Z_\omega(n-2)$.

$$Z_\omega(n-1) = \begin{bmatrix} Z_\omega(n-2) & z_n D_1 \\ -z_n^* D_2 & Z_\omega(n-2)^* \end{bmatrix} = \begin{bmatrix} Z_\rho(n-2) & z_n D_1 \\ -z_n^* D_2 & Z_\rho(n-2)^* \end{bmatrix} \tag{3.2.3}$$

The off-diagonal block entries of $Z_\omega(n-1)Z_\omega(n-1)^* = (z_1 z_1^* + \ldots + z_{n-2}z_{n-2}^* + z_n z_n^*)I$ show that

$$Z_\rho(n-2)(-z_n D_2^*) + z_n D_1 Z_\rho(n-2) = 0$$

and the equivalent equation

$$Z_\rho(n-2) z_n = D_1 z_n Z_\rho(n-2) D_2 \tag{3.2.4}$$

are constraints for D_1 and D_2, which vary continuously with ω (and therefore also with ρ).

The reverse order of multiplication, $Z_\omega(n-1)^* Z_\omega(n-1) = (z_1 z_1^* + \ldots + z_{n-2} z_{n-2}^* + z_n z_n^*)I$, then provides

$$Z_\rho(n-2)^*(z_n D_1) + (-z_n D_2^*) Z_\rho(n-2)^* = 0$$

and the equivalent equation

$$Z_\rho(n-2)^* z_n = D_2^* z_n Z_\rho(n-2)^* D_1^* \tag{3.2.5}$$

as additional constraints.

To form a matrix which satisfies similar equations in the next dimension up, let $F_1 = D_1 \oplus \rho_{n-1,n} D_2^*$ and $F_2 = D_2 \oplus \rho_{n,n-1} D_1^* = D_2 \oplus \overline{\rho_{n-1,n}} D_1^*$. The matrices G_i and D_j of (3.2.2) and (3.2.3) are diagonal with scalar entries, so they commute. We may then compute $Z_\rho(n-1) z_n$ using equations (3.2.4) and (3.2.5).

$$Z_\rho(n-1) z_n = \begin{bmatrix} Z_\rho(n-2) & z_{n-1} G_1 \\ -z_{n-1}^* G_2 & Z_\rho(n-2)^* \end{bmatrix} z_n = \begin{bmatrix} Z_\rho(n-2) z_n & z_{n-1} G_1 z_n \\ -z_{n-1}^* G_2 z_n & Z_\rho(n-2)^* z_n \end{bmatrix}$$

$$= \begin{bmatrix} D_1 z_n Z_\rho(n-2) D_2 & \rho_{n,n-1} z_n z_{n-1} G_1 \\ -\rho_{n-1,n} z_n z_{n-1}^* G_2 & D_2^* z_n Z_\rho(n-2)^* D_1^* \end{bmatrix}$$

$$= (D_1 \oplus \rho_{n-1,n} D_2^*) \left(z_n \begin{bmatrix} Z_\rho(n-2) & z_{n-1} G_1 \\ -z_{n-1}^* G_2 & Z_\rho(n-2)^* \end{bmatrix} \right) (D_2 \oplus \rho_{n,n-1} D_1^*)$$

$$= F_1 (z_n Z_\rho(n-1)) F_2$$

A nearly identical computation applies to $Z_\rho(n-1)^* z_n$.

$$Z_\rho(n-1)^* z_n = \begin{bmatrix} Z_\rho(n-2)^* & -z_{n-1} G_2^* \\ z_{n-1}^* G_1^* & Z_\rho(n-2) \end{bmatrix} z_n = \begin{bmatrix} Z_\rho(n-2)^* z_n & -z_{n-1} G_2^* z_n \\ z_{n-1}^* G_1^* z_n & Z_\rho(n-2) z_n \end{bmatrix}$$

$$= \begin{bmatrix} D_2^* z_n Z_\rho(n-2)^* D_1^* & -\rho_{n,n-1} z_n z_{n-1} G_2^* \\ \rho_{n-1,n} z_n z_{n-1}^* G_1^* & D_1 z_n Z_\rho(n-2) D_2 \end{bmatrix}$$

$$= (D_2^* \oplus \rho_{n-1,n} D_1) \left(z_n \begin{bmatrix} Z_\rho(n-2)^* & -z_{n-1} G_2^* \\ z_{n-1}^* G_1^* & Z_\rho(n-2) \end{bmatrix} \right) (D_1^* \oplus \rho_{n,n-1} D_2)$$

$$= F_2^* (z_n Z_\rho(n-1)^*) F_1^*$$

Finally, these equations ensure that we can define

$$Z_\rho(n) := \begin{bmatrix} Z_\rho(n-1) & z_n F_1 \\ -z_n^* F_2 & Z_\rho(n-1)^* \end{bmatrix}$$

so that $Z_\rho(n) Z_\rho(n)^* = Z_\rho(n)^* Z_\rho(n) = \left(\sum_{i=1}^{n} z_i z_i^* \right) I = I$. The matrices F_i depend continuously on the entries of ρ, completing the induction. The result is summarized in the following proposition.

Proposition 3.2.6. If ρ is an $n \times n$ parameter matrix and $1 \leq k \leq n$, then there is a formal $*$-monomial matrix $Z_\rho(k)$ (given recursively as above) of dimension $2^{k-1} \times 2^{k-1}$ whose coefficients vary continuously in ρ. This matrix satisfies $Z_\rho(k) Z_\rho(k)^* = Z_\rho(k)^* Z_\rho(k) = (z_1 z_1^* + \ldots z_k z_k^*) I$. If ω is another parameter matrix (perhaps of different dimension) whose upper left $k \times k$ submatrix agrees with that of ρ, then $Z_\omega(k)$ and $Z_\rho(k)$ are equal as formal $*$-monomial matrices. Moreover, $Z_\rho(n)$ gives a generator of the cyclic group $K_1(C(\mathbb{S}_\rho^{2n-1}))$.

From now on, any mention of $Z_\rho(k)$ will refer to this single, continuous choice of coefficients, as in the previous proposition.

Example 3.2.7. Consider a 5-sphere with parameter matrix ρ such that $z_2 z_1 = \alpha z_1 z_2$, $z_3 z_1 = \beta z_1 z_3$, and $z_3 z_2 = \gamma z_2 z_3$. We know that the 3-sphere with generators z_1 and z_2 will have the matrix

$$Z_\rho(2) = \begin{bmatrix} z_1 & z_2 \\ -\overline{\alpha} z_2^* & z_1^* \end{bmatrix}$$

as a K_1 generator, so this will be a building block for $Z_\rho(3)$. Now consider the 3-sphere whose generators follow the same relations as z_1 and z_3. Its K_1 generator

$$\begin{bmatrix} z_1 & z_3 \\ -\overline{\beta} z_3^* & z_1^* \end{bmatrix}$$

is formed from $Z_\rho(1) = z_1$ using diagonal matrices $D_1 = [1]$ in the upper right and $D_2 = [\overline{\beta}]$ in the lower left. This allows us to form $F_1 = D_1 \oplus \gamma D_2^* = \begin{bmatrix} 1 & 0 \\ 0 & \gamma\beta \end{bmatrix}$ and $F_2 = D_2 \oplus \overline{\gamma} D_1^* = \begin{bmatrix} \overline{\beta} & 0 \\ 0 & \overline{\gamma} \end{bmatrix}$, giving

$$\begin{aligned} Z_\rho(3) &= \begin{bmatrix} Z_\rho(2) & z_3 F_1 \\ -z_3^* F_2 & Z_\rho(2)^* \end{bmatrix} \\ &= \begin{bmatrix} z_1 & z_2 & z_3 & 0 \\ -\overline{\alpha} z_2^* & z_1^* & 0 & \gamma\beta z_3 \\ -\overline{\beta} z_3^* & 0 & z_1^* & -\alpha z_2 \\ 0 & -\overline{\gamma} z_3^* & z_2^* & z_1 \end{bmatrix} \end{aligned}$$

as the K_1 generator for $C(\mathbb{S}^5_\rho)$. One can verify that the noncommutativity relations above and the adjoint versions ($z_2^* z_1 = \overline{\alpha} z_1 z_2^*$, etc.) give that $Z_\rho(3)$ is, in fact, unitary.

The algebras $C(\mathbb{S}^{2n-1}_\rho)$ are obtained as Rieffel deformations $C(\mathbb{S}^{2n-1})_J$, so for any fixed $n \times n$ antisymmetric J, $(C(\mathbb{S}^{2n-1})_{tJ})_{t\in[0,1]}$ forms a continuous field of C^*-algebras ([44], Theorem 8.13). Existence of a continuous choice of K_1 generators $Z_\rho(n)$ is consistent with Theorem 1.4.12, a theorem of Sangha that concerns one-directional continuity in Rieffel deformations.

3.3 Odd Dimension

One version of the Borsuk-Ulam theorem insists that if $f : \mathbb{S}^k \to \mathbb{S}^k$ is an odd, continuous function, then f is homotopically nontrivial, and more specifically, the degree of f on top cohomology is odd. Oddness of the function f means that $f(-\vec{x}) = -f(\vec{x})$; in other words, f is equivariant for the antipodal $\mathbb{Z}_2$ action. For the odd case $k = 2n-1$, the K_1 group of $C(\mathbb{S}^{2n-1})$ is cyclic and contains information on the top cohomology of $\mathbb{S}^{2n-1}$ via the odd Chern character. The Chern character is natural (see (1.3.11)) and an isomorphism, so the Borsuk-Ulam theorem translates as follows.

Theorem 3.3.1 (Odd Dimension Borsuk-Ulam)**.** If $\Phi : C(\mathbb{S}^{2n-1}) \to C(\mathbb{S}^{2n-1})$ is a unital $*$-homomorphism that is equivariant for the antipodal $\mathbb{Z}_2$ action $\alpha : z_i \mapsto -z_i$, then $\phi_* : K_1(C(\mathbb{S}^{2n-1})) \to K_1(C(\mathbb{S}^{2n-1}))$ is nontrivial, as under the isomorphism $K_1(C(\mathbb{S}^{2n-1})) \cong \mathbb{Z}$, ϕ_* is multiplication by an odd integer.

Our goal is to extend the above theorem to the θ-deformed spheres, after which we will consider more general actions and the even-dimensional case. If $\Phi : C(\mathbb{S}^{2n-1}_\rho) \to C(\mathbb{S}^{2n-1}_\omega)$ is a unital $*$-homomorphism that is equivariant for the antipodal $*$-homomorphism $\alpha : z_i \mapsto -z_i$, then in particular it sends the K_1 generator $Z_\rho(n)$, which has each entry a multiple of z_i or z_i^*, to a matrix $\Phi(Z_\rho(n))$ which is odd in each entry. This implies that $Z_\omega(n)^* \cdot \Phi(Z_\rho(n))$ is a $2^{n-1} \times 2^{n-1}$ matrix of even elements. If we assume that every invertible matrix with even entries is represented by an even integer in K_1, then we can conclude that $\Phi(Z_\rho(n))$ corresponds to an odd integer in K_1 and is therefore nontrivial. In other words, we conclude that Φ_* is nontrivial on K_1. Note that the $\mathbb{Z}_2$ action α does not change the K_1 class of an invertible matrix, as verified by checking on the K_1 generator $Z_\rho(n)$. This matrix satisfies $\alpha(Z_\rho(n)) = -Z_\rho(n)$, which is K_1-equivalent to $Z_\rho(n)$ by scaling -1 to 1 within the nonzero constants. In other words, α is orientation preserving.

Lemma 3.3.2. If F is an invertible matrix over $C(\mathbb{S}^{2n-1}_\rho)$ and each entry of F is even, then the K_1 class of F is an even multiple of the generator.

Proof. Consider the fixed point subalgebra $C(\mathbb{S}^{2n-1})^\alpha$ of the commutative sphere and the ideal J of even functions which vanish on an equator X. Then part of the six-term exact sequence of K-theory is

$$K_1(J) \to K_1(C(\mathbb{S}^{2n-1})^\alpha) \to K_1(C(X)^\alpha)$$

where $C(X)$ is given the restricted antipodal action also denoted by α, forming the fixed point subalgebra $C(X)^\alpha$. Now, the equator X is homeomorphic to $\mathbb{S}^{2n-2}$, and $C(X)^\alpha$ is isomorphic to $C(\mathbb{RP}^{2n-2})$, which has trivial K_1 group ([1], Proposition 2.7.7). Therefore, the map $K_1(J) \to K_1(C(\mathbb{S}^{2n-1})^\alpha)$ is surjective. This implies that every unitary matrix V of even functions in $C(\mathbb{S}^{2n-1})$ may be stably connected within the unitaries to a matrix of even functions that assigns the identity on X. Such a function U may be written as a commuting product $U = F \cdot G$ where F assigns the identity on one side of X, and G assigns the identity on the other half. Moreover, $\alpha(F) = G$, so F and G are in the same class of $K_1(C(\mathbb{S}^{2n-1}))$. This implies that U (and therefore also V) is represented by an even integer in $K_1(C(\mathbb{S}^{2n-1}))$.

The antipodal action on the commutative sphere commutes with the $\mathbb{R}^n$ action of coordinatewise rotation that defines $C(\mathbb{S}^{2n-1}_\rho) \cong C(\mathbb{S}^{2n-1})_J$ for a suitable antisymmetric J. As such, the fixed point subalgebra $C(\mathbb{S}^{2n-1})^{\mathbb{Z}_2}$ is itself $\mathbb{R}^n$-equivariant, and we may form its Rieffel deformation $(C(\mathbb{S}^{2n-1})^{\mathbb{Z}_2})_J$. From the inclusion map $\iota : C(\mathbb{S}^{2n-1})^\alpha \to C(\mathbb{S}^{2n-1})$ we reach the following commutative diagram of Corollary 1.4.13.

$$\begin{tikzcd}
K_1(C(\mathbb{S}^{2n-1})^\alpha) \arrow[r, "\cong"] \arrow[d, "\iota_*"'] & K_1((C(\mathbb{S}^{2n-1})^\alpha)_J) \arrow[d, "(\iota_J)_*"] \\
K_1(C(\mathbb{S}^{2n-1}))) \arrow[r, "\cong"] & K_1(C(\mathbb{S}^{2n-1})_J)
\end{tikzcd}$$

All of the groups above are $\mathbb{Z}$; Rieffel deformations preserve K-theory, $K_1(C(\mathbb{S}^{2n-1})) \cong \mathbb{Z}$, and $K_1(C(\mathbb{S}^{2n-1})^\alpha) \cong K_1(C(\mathbb{RP}^{2n-1})) \cong \mathbb{Z}$ ([1], Proposition 2.7.7). Since ι_* has range $2\mathbb{Z}$, $(\iota_J)_*$ must also have range $2\mathbb{Z}$. On the other hand, ι_J is the inclusion map of $C(\mathbb{S}^{2n-1}_\rho)^\alpha$ into $C(\mathbb{S}^{2n-1}_\rho)$, so the $K_1(C(\mathbb{S}^{2n-1}))$ of any invertible matrix F over $C(\mathbb{S}^{2n-1}_\rho)^\alpha$ is an even integer. $\square$

This theorem is enough to determine the possible K_1 classes of odd unitaries, based solely on their dimension.

Corollary 3.3.3. Suppose $U \in U_k(C(\mathbb{S}^{2n-1}_\rho))$ is a unitary with odd entries. Then k decomposes as $k = (2m+1)2^q$ where $q \geq n-1$. Further, if $q = n-1$, then $[U]_{K_1}$ is an odd integer, and if $q \geq n$, then $[U]_{K_1}$ is an even integer.

Proof. First, suppose U has dimension $k = (2m+1)2^{n-1}$, and let $V = \bigoplus_{i=1}^{2m+1} Z_\rho(n)$, which has odd entries, has $[V]_{K_1} = 2m+1$, and is of the same dimension as U. Then UV is a matrix of even entries, so $[UV]_{K_1} = [U]_{K_1} + [V]_{K_1} = [U]_{K_1} + (2m+1)$ is an even integer. This implies that $[U]_{K_1}$ is odd.

Next, suppose U has dimension $k = (2m+1)2^q$ for $q \geq n$. Then consider $W = \bigoplus_{i=1}^{2^{q-(n-1)}} V$, where V is defined as above. Then W has the same dimension as U, has odd entries, and has $[W]_{K_1} = 2^{q-(n-1)}(2m+1)$, an even integer. Since UW has even entries, $[UW]_{K_1}$ is again even and equal to $[U]_{K_1} + [W]_{K_1}$, so $[U]_{K_1}$ is even.

Finally, suppose U has dimension $k = (2m+1)2^q$ for $q \leq n-2$. Then $X = \bigoplus_{i=1}^{2^{n-1-q}} U$ has odd entries and dimension $(2m+1)2^{n-1}$, but $[X]_{K_1} = 2^{n-1-q}[U]_{K_1}$ is even. This contradicts the first paragraph of the proof. $\square$

Corollary 3.3.4. Suppose a unital $*$-homomorphism $\Phi : C(\mathbb{S}^{2n-1}_\rho) \to C(\mathbb{S}^{2n-1}_\omega)$ maps between two θ-deformed spheres of the same odd dimension. If Φ is equivariant for the antipodal maps, then Φ induces a nontrivial map on K_1. More precisely, $\Phi_* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by an odd integer.

Proof. The K_1 generator $Z_\rho(n)$ is of dimension $2^{n-1} \times 2^{n-1}$ and has odd entries, so the same applies to $\Phi(Z_\rho(n))$. Therefore, $[\Phi(Z_\rho(n))]_{K_1}$ is an odd (hence nonzero) integer. $\square$

Remark. In [49], Taghavi asks if equivariant homomorphisms on noncommutative spheres have to be homomotopically nontrivial. This corollary gives a positive answer for θ-deformed spheres of odd dimension. A more general formulation will be considered in the next chapter.

This noncommutative Borsuk-Ulam theorem shows that maps between θ-deformed odd spheres behave in much the same way as maps between topological spheres with respect to the standard $\mathbb{Z}_2$ action. The following corollary also places restrictions on maps between spheres and matrix algebras over spheres. Recall that in Lemma 3.1.5, an expansion map E between $C(\mathbb{S}^{2n-1}_\rho)$ and $M_q(C(\mathbb{S}^{2n-1}))$ was defined, which placed noncommutativity information from the generators of $C(\mathbb{S}^{2n-1}_\rho)$ (where ρ consists of roots of unity) onto $q \times q$ matrices with complex entries. The map E can be defined in a similar fashion even when the matrix algebra $M_q(C(\mathbb{S}^{2n-1}))$ is replaced with a θ-deformation $M_q(C(\mathbb{S}^{2n-1}_\omega))$, and ρ is no longer required to have entries of finite order; the map makes sense so long as the matrix expansion $\mathfrak{z}_i \mapsto z_i V_i$ is compatible with the noncommutativity relations. Regardless of the dimension q, E is equivariant for the antipodal maps, as it sends odd generators to matrices with odd entries. On the other hand, a reverse order map cannot always happen.

Theorem 3.3.5. Suppose $q \in 2\mathbb{Z}$. Then if $\psi : M_q(C(\mathbb{S}^{2n-1}_\rho)) \to C(\mathbb{S}^{2n-1}_\omega)$ is a unital $*$-homomorphism, ψ is not equivariant for the antipodal maps.

Proof. Suppose ψ is equivariant. Define a map $\phi : C(\mathbb{S}^{2n-1}_\rho) \to M_q(C(\mathbb{S}^{2n-1}_\rho))$ by $\phi(g) = \begin{bmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{bmatrix}$. Then ϕ is a unital $*$-homomorphism that is equivariant for the antipodal maps, and moreover $\phi(Z_\rho(n))$ is unitarily equivalent to q copies of $Z_\rho(n)$. It follows that

$\psi \circ \phi : C(\mathbb{S}^{2n-1}_\rho) \to C(\mathbb{S}^{2n-1}_\omega)$ is also equivariant, and $\psi(\phi(Z_\rho(n))) \sim_{K_1} \psi\left(\bigoplus_{i=1}^{q} Z_\rho(n)\right) \sim_{K_1} \bigoplus_{i=1}^{q} \psi(Z_\rho(n))$ represents an even integer in K_1, contradicting Corollary 3.3.4. $\square$

Now, just as the antipodal map α on the noncommutative sphere $C(\mathbb{S}^{2n-1}_\rho)$ supplies a $\mathbb{Z}_2$ action which generalizes the topological antipodal map $(z_1, \ldots, z_n) \mapsto (-z_1, \ldots, -z_n)$ on $\mathbb{S}^{2n-1}$, there is nothing stopping us from defining a similar map for higher order rotations on each coordinate z_i. Let $\beta_1, \ldots, \beta_n$ be primitive kth roots of unity, $k \geq 2$. Then there is a unital $*$-homomorphism

$$R : C(\mathbb{S}^{2n-1}_\rho) \to C(\mathbb{S}^{2n-1}_\rho) \tag{3.3.6}$$

$$z_i \mapsto \beta_i z_i$$

which generalizes coordinatewise rotation (with the same finite order on each coordinate) on the sphere $\mathbb{S}^{2n-1}$. Again, R exists due to the fact that the elements $\beta_i z_i$ satisfy relations defining $C(\mathbb{S}^{2n-1}_\rho)$ as a universal C^*-algebra, and this action is again just a remnant of the $\mathbb{R}^n$ rotation action which deforms $C(\mathbb{S}^{2n-1})$. The K_1 generator $Z_\rho(n)$ is usually not homogeneous for R; the entries are all homogeneous, but the homogeneity class changes by entry. This differs from the $\mathbb{Z}_2$ case, in which we could simply observe that each entry was odd. However, a quick inductive argument on the recursive definition $Z_\rho(1) = z_1$, $Z_\rho(k+1) = \begin{bmatrix} Z_\rho(k) & z_{k+1}D_1 \\ -z_{k+1}^* D_2 & Z_\rho(k)^* \end{bmatrix}$ shows that independent of ρ, there are diagonal unitary matrices A and B over $\mathbb{C}$ with $A^k = B^k = I$ for which

$$R(Z_\rho(n)) = AZ_\rho(n)B \tag{3.3.7}$$

holds. These matrices encode the homogeneity classes of the different entries of $Z_\rho(n)$. Further, since A and B have scalar entries, $R(Z_\rho(n))$ and $Z_\rho(n)$ are equivalent in $K_1(C(\mathbb{S}^{2n-1}_\rho))$. It follows that R preserves the K_1 class of any invertible matrix. As usual, we have extended

R to matrix algebras by entrywise application.

Because $Z_\rho(n)$ does not have consistent homogeneity among its entries, it is not immediately clear how to start with an arbitrary unital $*$-homomorphism $\Phi : C(\mathbb{S}^{2n-1}_\rho) \to C(\mathbb{S}^{2n-1}_\omega)$ which respects a rotation and form an element that is fixed by the same rotation. This was the main trick of the previous results: if Φ respects the antipodal map, then since $Z_\rho(n)$ and $Z_\omega(n)$ are odd, $Z_\omega(n)^* \cdot \Phi(Z_\rho(n))$ is even. Equation (3.3.7) includes a matrix multiplication on both sides, so $R(Z_\omega(n)^* \cdot \Phi(Z_\rho(n))) = B^* Z_\omega(n)^* \Phi(Z_\rho(n)) B$, and there is still a scalar matrix conjugation present. Because of this complication, we should examine the fixed point subalgebras of the various actions

$$R_U : M \mapsto U^* R(M) U$$

for unitaries $U \in U_d(\mathbb{C})$ whose orders divide k (the order of R). When U is fixed, but we wish to increase the dimension of M, we allow $M \in M_{qd}(C(\mathbb{S}^{2n-1}_\rho))$ and let R_U act on each $d \times d$ block, or equivalently apply a conjugation by a diagonal block matrix of q copies of U.

$$R_U : M \mapsto \left(\bigoplus_{i=1}^{q} U^* \right) R(M) \left(\bigoplus_{i=1}^{q} U \right)$$

It is not hard to show that for large q, there is a $qd \times qd$ invertible matrix, fixed by R_U, whose class in $K_1(C(\mathbb{S}^{2n-1})) \cong \mathbb{Z}$ is represented by k. First, take a K_1 generator $G = Z(n) \oplus I$ of size $qd \times qd$, scale G by a unitary matrix over $\mathbb{C}$ to assign the identity matrix at a pole, and then form a continuous path of invertibles that starts with G and ends with a matrix H that assigns the identity outside of a small neighborhood of the opposite pole. If the neighborhood is small enough that it does not intersect any of its images under the $\mathbb{Z}_k$ rotation R, then the product $G \cdot R_U(G) \cdots {R_U}^{k-1}(G)$ will commute and produce an R_U-invariant matrix with K_1 class equal to $k \in \mathbb{Z}$. This matches with our intuition from the antipodal map, where even invertibles were assigned even integers.

Lemma 3.3.8. If $U \in U_d(\mathbb{C})$ is a scalar unitary whose order divides k (the order of the rotation R), and $M \in GL_d(C(\mathbb{S}^{2n-1}))$ is an invertible matrix over the commutative sphere with $R_U(M) = M$, then the $K_1(C(\mathbb{S}^{2n-1}))$ class of M is in $k\mathbb{Z}$. Further, the fixed point subalgebra $M_d(C(\mathbb{S}^{2n-1}))^{R_U}$ has K_1 group isomorphic to $\mathbb{Z}$.

Proof. Let $X_n = \{(z_1, \ldots, z_n) \in \mathbb{S}^{2n-1} : z_n = 0 \text{ or } \left(\frac{z_n}{|z_n|}\right)^k = 1\}$, so X_1 is a finite set and X_n, $n \geq 2$, is a union of k closed balls $\overline{\mathbb{B}^{2n-2}}$ which intersect only on their boundaries. In any case, X_n is an invariant set for any kth order coordinate rotation. Let J_n denote the ideal of the fixed point subalgebra $M_d(C(\mathbb{S}^{2n-1}))^{R_U}$ that consists of matrix functions vanishing on X_n. Now, J_n is isomorphic to $C_0(\mathbb{B}^{2n-1})$, since $\mathbb{S}^{2n-1} \setminus X_n$ is k disjoint copies of $\mathbb{B}^{2n-1}$ which are orbits of a single ball under R. This produces an exact sequence from part of the six-term sequence for J_n.

$$K_1(C_0(\mathbb{B}^{2n-1})) \xrightarrow{\psi} K_1(M_d(C(\mathbb{S}^{2n-1}))^{R_U}) \to K_1(M_d(C(X_n))^{R_U}) \tag{3.3.9}$$

We induct on the claim that the final group $K_1(M_d(C(X_n))^{R_U})$ in the sequence is trivial. For the base case $n = 1$, this is trivial for all choices of R and U (of appropriate order) because X_1 has k points, and invariant functions on X_1 are determined by values at only one point. Now, whenever we know the final group of (3.3.9) is trivial, this implies the first map ψ is surjective, and $K_1(M_d(C(\mathbb{S}^{2n-1}))^{R_U})$ is the surjective image of a cyclic group, making it cyclic as well. Any image of ψ may always be written in the form of a commuting product $G \cdot R_U(G) \cdots {R_U}^{k-1}(G)$, where G assigns the identity matrix on all but one component of $\mathbb{S}^{2n-1} \setminus X_n$. All elements of the product are $K_1(C(\mathbb{S}^{2n-1}))$-equivalent, meaning the product's class in $K_1(C(\mathbb{S}^{2n-1}))$ must lie in $k\mathbb{Z}$. If q is chosen such that $qd \geq 2^{n-1}$, there is always an example of a $qd \times qd$ matrix M which is R_U-invariant, invertible, and represented by $k \neq 0$ in $K_1(C(\mathbb{S}^{2n-1})) \cong \mathbb{Z}$, so this implies that the induced map $K_1(M_d(C(\mathbb{S}^{2n-1}))^{R_U}) \to K_1(C(\mathbb{S}^{2n-1}))$ from inclusion is an injective map between *infinite* cyclic groups, with image exactly $k\mathbb{Z}$.

To complete the induction, assume for a fixed n that the final group of (3.3.9) is trivial for all coordinate rotations and unitaries U of the appropriate order. Let S be a rotation on $C(\mathbb{S}^{2n+1})$ of order $k \geq 2$, with R denoting the rotation on $C(\mathbb{S}^{2n-1})$ obtained from restricting S via the inclusion $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 0)$. Note that since X_{n+1} is the union of k copies of $\overline{\mathbb{B}^{2n}}$ which overlap only on their boundaries, it follows that $M_d(C(X_{n+1}))^{S_U}$ is isomorphic to $\{F \in M_d(C(\overline{\mathbb{B}^{2n}})) : F|_{\mathbb{S}^{2n-1}} \text{ is invariant under } R_U\}$. Again, examine part of a six term sequence.

$$K_1(C_0(\mathbb{B}^{2n})) \to K_1(M_d(C(X_{n+1}))^{S_U}) \xrightarrow{\phi} K_1(M_d(C(\mathbb{S}^{2n-1}))^{R_U})$$

The inductive assumption shows that the final group is infinite cyclic and has an injective image into $K_1(C(\mathbb{S}^{2n-1}))$ via the inclusion map. This immediately implies that ϕ is trivial, as every image of ϕ comes from the boundary data of a function on $\overline{\mathbb{B}^{2n}}$. Since $K_1(C_0(\mathbb{B}^{2n}))$ is also trivial, it follows that $K_1(M_d(C(X_{n+1}))^{S_U})$ is trivial, and the induction is complete. $\square$

The above lemma is what one would expect given the $\mathbb{Z}_2$ case, where the additional conjugation by U is treated as merely a technical annoyance. If U is the identity matrix, the computations include terms for the odd K-theory of lens spaces. Moreover, the role of functions on a closed ball with boundary symmetry is somewhat reminiscent of the discussion in section 6.2 of [28] (which proves a generalization from [18] of the Borsuk-Ulam theorem to other free actions by groups on $\mathbb{S}^k$), although the context and conclusions are different.

Theorem 3.3.10. If $U \in U_d(\mathbb{C})$ is a scalar unitary whose order divides k (the order of the rotation R), and $M \in GL_d(C(\mathbb{S}^{2n-1}_\rho))$ is an invertible matrix with $R_U(M) = M$, then the $K_1(C(\mathbb{S}^{2n-1}_\rho))$ class of M is in $k\mathbb{Z}$.

Proof. The action R_U commutes with the $\mathbb{R}^n$ action of entrywise rotation on $M_d(C(\mathbb{S}^{2n-1}))$, so we may deform the fixed point subalgebra $M_d(C(\mathbb{S}^{2n-1}))^{R_U}$ using the restricted action.

With the previous lemma establishing the commutative case, the proof is identical to that of Lemma 3.3.2. □

The unitary conjugation present in this section's results serves to solve the following dilemma. For rotations of order $k > 2$, the K_1 generator of the noncommutative sphere is not homogeneous, so when we consider a homomorphism Φ between two spheres, it is difficult to construct a matrix fixed by R.

Corollary 3.3.11. Suppose a unital $*$-homomorphism $\Phi : C(\mathbb{S}_\rho^{2n-1}) \to C(\mathbb{S}_\omega^{2n-1})$ satisfies $R \circ \Phi = \Phi \circ R$. Here R (on either sphere) denotes a rotation map defined in (3.3.6) for the same list of primitive kth roots of unity $\beta_1, \ldots, \beta_n$, where $k \geq 2$. Then Φ_* is nontrivial on K_1. Specifically, it is given by multiplication by an integer in $k\mathbb{Z} + 1$.

Proof. Equation (3.3.7) gives that $R(Z_\rho(n)) = AZ_\rho(n)B$ and $R(Z_\omega(n)) = AZ_\omega(n)B$, where A and B are diagonal unitaries with scalar entries that do not change with the sphere parameter, and further $A^k = B^k = I$. Since Φ is a unital $*$-homomorphism and respects the rotation maps, this implies that

$$\begin{aligned} R(Z_\omega(n)^* \cdot \Phi(Z_\rho(n))) &= R(Z_\omega(n))^* \cdot \Phi(R(Z_\rho(n))) \\ &= (AZ_\omega(n)B)^* \cdot \Phi(AZ_\rho(n)B) \\ &= B^* Z_\omega(n)^* A^* \cdot A\Phi(Z_\rho(n))B \\ &= B^*(Z_\omega(n)^* \cdot \Phi(Z_\rho(n)))B \end{aligned}$$

and $Z_\omega(n)^* \cdot \Phi(Z_\rho(n))$ is fixed by the operation $R_{B^*} : M \mapsto BR(M)B^*$. Since B^* is a unitary over $\mathbb{C}$ with $B^{*k} = I$, by the previous theorem the K_1 class of $Z_\omega(n)^* \cdot \Phi(Z_\rho(n))$ is in $k\mathbb{Z}$. Finally, the K_1 class of $\Phi(Z_\rho(n))$ is congruent to 1 mod k. □

The above corollary is a noncommutative version of the following fact: if $\beta_1, \ldots, \beta_n$ are primitive kth roots of unity ($k \geq 2$), and $f : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ is continuous and respects the rotation $(z_1, \ldots, z_n) \mapsto (\beta_1 z_1, \ldots, \beta_n z_n)$, then f is homotopically nontrivial. Just as in the $\mathbb{Z}_2$ case, there is a consequence regarding spheres of different dimensions. Specifically, there exists no $g : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-3}$ which is continuous and equivariant for the coordinatewise rotations sending $(z_1, \ldots, z_n)$ to $(\beta_1 z_1, \ldots, \beta_n z_n)$ and $(z_1, \ldots, z_{n-1})$ to $(\beta_1 z_1, \ldots, \beta_{n-1} z_{n-1})$. This result is only necessary to state when k is an odd prime (which gives the result when k is not prime, but has an odd prime divisor), as we have already stated the usual Borsuk-Ulam theorem for $k = 2$. In [50], Z. Tang showcases a proof of this topological result using the reduced K-theory of lens spaces; in contrast, our proofs in the noncommutative setting work entirely with K_1. The nonexistence of equivariant $g : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-3}$ can also be shown using homology; see [26] for this type of approach (and a generalization). The analogous result on θ-deformed spheres is as follows.

Corollary 3.3.12. Suppose $k \geq 2$ and $\beta_1, \ldots, \beta_n$ are primitive kth roots of unity. If $\Psi : C(\mathbb{S}^{2n-3}_\omega) \to C(\mathbb{S}^{2n-1}_\rho)$ is a unital $*$-homomorphism, then $R \circ \Psi \neq \Psi \circ R'$, where R denotes the rotation map for $\beta_1, \ldots, \beta_n$ and R' denotes the rotation map for the first $n-1$ of these scalars $\beta_1, \ldots, \beta_{n-1}$.

Proof. Suppose Ψ satisfies $R \circ \Psi = \Psi \circ R'$. Choose an $n \times n$ parameter matrix Ω which contains ω in the upper left, and let $\pi : C(\mathbb{S}^{2n-1}_\Omega) \to C(\mathbb{S}^{2n-3}_\omega)$ be the map defined by $z_i \mapsto z_i$ for $i \leq n-1$ and $z_n \mapsto 0$. The map π is K_1-trivial because $\pi(Z_\Omega(n)) = \begin{bmatrix} Z_\omega(n-1) & 0 \\ 0 & Z_\omega(n-1)^* \end{bmatrix}$, which is equivalent in $K_1(C(\mathbb{S}^{2n-3}_\omega))$ to $Z_\omega(n-1) Z_\omega(n-1)^* = I$, the trivial element. Moreover, the homogeneity classes of $\pi(z_i)$ show that $R' \circ \pi = \pi \circ R$, so $\Phi = \Psi \circ \pi : C(\mathbb{S}^{2n-1}_\Omega) \to C(\mathbb{S}^{2n-1}_\rho)$ is K_1-trivial and has $\Phi \circ R = R \circ \Phi$. This contradicts Corollary 3.3.11. $\square$

If $k = 2$, this is not the full strength of the Borsuk-Ulam theorem, which is instead found in Corollary 3.4.8. Similarly, if k is even, equivariant maps for order k rotations are also equivariant for the antipodal map, so the antipodal results are often preferable. However, if k is odd, the map R on $C(\mathbb{S}^{2n-1}_\rho)$ relies heavily on the complex coordinates z_i, so the results cannot be stated using the even dimensional spheres. In this sense both Corollary 3.3.11 and Corollary 3.3.12 may be viewed as full-strength noncommutative $\mathbb{Z}_k$ Borsuk-Ulam theorems when k is odd.

3.4 Even Dimension

The even-dimensional θ-deformed spheres $C(\mathbb{S}^{2n}_\rho)$ can be defined as a quotient of certain odd-dimensional spheres, but this also gives them their own presentation. If ρ is an $n \times n$ parameter matrix, then $C(\mathbb{S}^{2n}_\rho)$ is generated by $z_1, \ldots, z_n, x$ satisfying the following relations.

$$\begin{gathered} z_j z_j^* = z_j^* z_j \qquad\qquad x = x^* \\ z_k z_j = \rho_{jk} z_j z_k \qquad\qquad x z_j = z_j x \\ x^2 + z_1 z_1^* + z_2 z_2^* + \ldots + z_n z_n^* = 1 \end{gathered}$$

The presence of the central, self-adjoint generator x is reflected in the fact that the $\mathbb{R}^n/\mathbb{Z}^n$ action on $C(\mathbb{S}^{2n})$ defining its θ-deformations ignores the final coordinate on $\mathbb{S}^{2n}$. The K-theory of $C(\mathbb{S}^{2n}_\theta)$ also matches that of the commutative sphere, as θ-deformation preserves K-theory. This can be seen from the generators and relations in [36]; see also [13] and [12] for further discussion of these algebras.

$$K_0(C(\mathbb{S}^{2n}_\rho)) \cong \mathbb{Z}^2 \qquad\qquad K_1(C(\mathbb{S}^{2n}_\rho)) \cong \{0\}$$

As noted by Dąbrowski in [14], the generator x may be used to write $C(\mathbb{S}^{2n}_\rho)$ as a space of paths into $C(\mathbb{S}^{2n-1}_\rho)$, forming an algebra known as the unreduced suspension. Chapter 4 investigates this topic further and proves a conjecture from [14]. Because we will need to use a similar idea for odd dimension, below are two lemmas defining isomorphisms with function algebras; these types of examples have undoubtedly been known since early results on Rieffel deformations.

Lemma 3.4.1. Let $C(\mathbb{T}^{n+1}_\rho)$ be a quantum torus such that U_{n+1} is central. If ω is the top left $n \times n$ minor of ρ, then $C(\mathbb{T}^{n+1}_\rho)$ is isomorphic to the C^*-algebra $C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$, which has pointwise operations.

Proof. Let $S_1, \ldots, S_n \in C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ be defined by $S_i(e^{i\theta}) = U_i$, and let $S_{n+1} \in C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ be defined by $S_{n+1}(e^{i\theta}) = e^{i\theta}$. Then these $n+1$ elements are unitary and satisfy the relations in the presentation of $C(\mathbb{T}^{n+1}_\rho)$, so there is a $*$-homomorphism $\psi : C(\mathbb{T}^{n+1}_\rho) \to C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ defined by $\psi(U_i) = S_i$, which will turn out to be an isomorphism.

Define a trace on $C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ by $\mathrm{tr}(f) = \frac{1}{2\pi} \int\limits_0^{2\pi} \tau_n(f(e^{i\theta}))\, d\theta$, where τ_n is the usual trace of $C(\mathbb{T}^n_\omega)$. Now, if τ_{n+1} denotes the usual trace of $C(\mathbb{T}^{n+1}_\rho)$, then $\mathrm{tr} \circ \psi = \tau_{n+1}$, as seen by checking on $*$-polynomials of the generators. Since τ_{n+1} is a continuous, positive, faithful trace, ψ must be injective, as otherwise there would be a nonzero $a \in C(\mathbb{T}^{n+1}_\rho)$ such that $\psi(a) = 0 = \psi(aa^*)$ and $\tau_{n+1}(aa^*) = \mathrm{tr}(\psi(aa^*)) = \mathrm{tr}(0) = 0$.

To show ψ is surjective, it suffices to show that ψ has dense range, as every injective $*$-homomorphism between C^*-algebras is norm-preserving and therefore has closed range. If $f \in C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ is of the form $e^{i\theta} \mapsto \sum\limits_{\text{finite}} f_m(e^{i\theta}) U_1^{m_1} \ldots U_n^{m_n}$ where $f_m(e^{i\theta})$ is scalar-valued and has an absolutely convergent Fourier series for each $m \in \mathbb{Z}^n$, then $f \in \mathrm{Ran}(\psi)$. Any $f \in C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ may be approximated by continuous functions from $\mathbb{S}^1$ to $C(\mathbb{T}^n_\omega)$ which are piecewise linear in the angular variable θ and whose values on a prescribed finite set are $*$-polynomials. These may be decomposed into the form $\sum\limits_{\text{finite}} f_m(e^{i\theta}) U_1^{m_1} \ldots U_n^{m_n}$ where the scalar functions f_m are continous and piecewise linear (in θ), meaning they satisfy a Lipschitz condition and have absolutely convergent, therefore uniformly convergent, Fourier series by [25], Theorem 6.3. Finally, this implies that the closed set $\mathrm{Ran}(\psi)$ is dense (as it contains these finite sums), and therefore ψ is surjective. $\square$

Because the θ-deformed sphere $C(\mathbb{S}^{2n-1}_\omega)$ is isomorphic to the set of continuous functions f from $\mathbb{S}^{n-1}_+ = \{(t_1,\ldots,t_n) \in \mathbb{R}^n : t_i \geq 0 \text{ and } t_1^2 + \ldots t_n^2 = 1\}$ to $C(\mathbb{T}^n_\omega)$ satisfying the boundary condition $f(\vec{t}) \in C^*(U_1,\ldots U_{i-1}, U_{i+1},\ldots,U_n)$ whenver $t_i = 0$, a similar lemma holds for spheres.

Lemma 3.4.2. If $C(\mathbb{S}^{2n+1}_\rho)$ is such that z_{n+1} is central, and ω is the top-left $n \times n$ minor of ρ, then $C(\mathbb{S}^{2n+1}_\rho)$ is isomorphic to the following C^*-algebra with pointwise operations.

$$\mathcal{D}_\rho = \{f \in C(\overline{\mathbb{D}}, C(\mathbb{S}^{2n-1}_\omega)) : \text{for all } w \in \partial\mathbb{D}, f(w) \in \mathbb{C}\}$$

Proof. Let $C_i \in \mathcal{D}_\rho$ be defined by $C_i(w) = \sqrt{1-|w|^2}z_i$ for $i \in \{1,\ldots,n\}$ and by $C_{n+1}(w) = w$. The functions C_i satisfy the relations demanded by the generators of $C(\mathbb{S}^{2n+1}_\rho)$, so there is a $*$-homomorphism $\phi : C(\mathbb{S}^{2n+1}_\rho) \to \mathcal{D}_\rho$ defined by $\phi(z_i) = C_i$, which will have an inverse and therefore be an isomorphism.

If $f \in \mathcal{D}_\rho$, then for each $w \in \overline{\mathbb{D}}$, $f(w) \in C(\mathbb{S}^{2n-1}_\omega)$ may be realized as a continuous function from $\mathbb{S}^{n-1}_+$ to $C(\mathbb{T}^n_\omega)$ satisfying boundary conditions. Define $F : \mathbb{S}^n_+ \to C(\mathbb{S}^1, C(\mathbb{T}^n_\omega)) \cong C(\mathbb{T}^{n+1}_\rho)$, where F represents an element of $C(\mathbb{S}^{2n+1}_\rho)$ and a candidate for $\phi^{-1}(f)$, as follows.

$$F(t_1,\ldots,t_{n+1}) : e^{i\theta} \in \mathbb{S}^1 \mapsto f(t_{n+1}e^{i\theta})\left[\frac{t_1}{\sqrt{1-t_{n+1}^2}},\ldots,\frac{t_n}{\sqrt{1-t_{n+1}^2}}\right]$$

The first point to be checked is that this function is well-defined. When $t_{n+1} = 1$, $f(e^{i\theta})$ is a scalar in $C(\mathbb{S}^{2n-1}_\omega)$ and therefore is a constant function on $\mathbb{S}^{n-1}_+$. For each individual choice of $(t_1,\ldots,t_{n+1})$, including the special case $(0,\ldots,0,1)$, $F(t_1,\ldots,t_{n+1})$ is a continuous function of $e^{i\theta} \in \mathbb{S}^1$ with image in $C(\mathbb{T}^n_\omega)$ because f is by definition (uniformly) continuous. Further, $F(t_1,\ldots,t_{n+1})$ varies uniformly continuously in $(t_1,\ldots,t_{n+1})$ away from the removable singularity, and the following inequality demonstrates continuity near $(0,\ldots,0,1)$. This inequality, valid for $(t_1,\ldots,t_{n+1}) \neq (0,\ldots,0,1)$, relies on the fact that if $w \in \partial\mathbb{D}$, then $f(w)$

is a scalar, or a constant function, and can therefore be evaluated at any input point.

$$\left|\left|f(t_{n+1}e^{i\theta})\left[\frac{t_1}{\sqrt{1-t_{n+1}^2}},\ldots,\frac{t_n}{\sqrt{1-t_{n+1}^2}}\right]-f(e^{i\theta})\right|\right| =$$

$$\left|\left|f(t_{n+1}e^{i\theta})\left[\frac{t_1}{\sqrt{1-t_{n+1}^2}},\ldots,\frac{t_n}{\sqrt{1-t_{n+1}^2}}\right]-f(e^{i\theta})\left[\frac{t_1}{\sqrt{1-t_{n+1}^2}},\ldots,\frac{t_n}{\sqrt{1-t_{n+1}^2}}\right]\right|\right| \leq$$

$$||f(t_{n+1}e^{i\theta})-f(e^{i\theta})||$$

Finally, uniform continuity of f and this inequality show that F also varies continuously at $(0,\ldots,0,1)$. Further, F satisfies the desired boundary conditions: when $t_{n+1}=0$, $F(t_1,\ldots,t_{n+1})$ is independent of θ, and when $t_i = 0$ for some $i \leq n$, the range values of $F(t_1,\ldots,t_{n+1})$ exclude U_i terms by the boundary conditions on $C(\mathbb{S}_\omega^{2n-1})$. This means that under the isomorphism $C(\mathbb{S}^1, C(\mathbb{T}_\omega^n)) \cong C(\mathbb{T}_\rho^{n+1})$, F represents a unique element of $C(\mathbb{S}_\rho^{2n+1})$.

The assignment $\psi : f \in \mathcal{D}_\rho \mapsto F \in C(\mathbb{S}_\rho^{2n+1})$ is a $*$-homomorphism, and ψ is injective; if two functions $f, g \in \mathcal{D}_\rho$ disagree at some point, then their corresponding functions from $\mathbb{S}_+^n$ to $C(\mathbb{S}^1, C(\mathbb{T}_\omega^n)) \cong C(\mathbb{T}_\rho^{n+1})$ disagree at some point, and $\psi(f), \psi(g) \in C(\mathbb{S}_\rho^{2n+1})$ are distinct. Since $\psi : \mathcal{D}_\rho \to C(\mathbb{S}_\rho^{2n+1})$ is injective, to prove $\psi = \phi^{-1}$ it suffices to show $\psi \circ \phi = \mathrm{Id}_{C(\mathbb{S}_\rho^{2n+1})}$, as this will establish that ψ is also surjective, and its one-sided inverse ϕ will be its two-sided inverse. Moreover, as ϕ and ψ are both $*$-homomorphisms (and automatically continuous), we need only show that $\psi(\phi(z_i)) = z_i$ for $i \in \{1,\ldots,n+1\}$. First, check for $1 \leq i \leq n$.

$$\phi(z_i) = C_i \qquad\qquad C_i : w \in \overline{\mathbb{D}} \mapsto \sqrt{1-|w|^2}z_i \in C(\mathbb{S}_\rho^{2n-1})$$

$$\begin{aligned}\psi(C_i)[t_1,\ldots,t_{n+1}] : e^{i\theta} \mapsto C_i(t_{n+1}e^{i\theta})&\left[\frac{t_1}{\sqrt{1-t_{n+1}^2}},\ldots,\frac{t_n}{\sqrt{1-t_{n+1}^2}}\right]\\ &= \sqrt{1-t_{n+1}^2}\frac{t_i}{\sqrt{1-t_{n+1}^2}}U_i\\ &= t_iU_i\end{aligned}$$

The function $(t_1, \ldots, t_{n+1}) \mapsto t_i U_i$ is exactly the form of z_i, so $\psi \circ \phi(z_i) = z_i$ for $1 \leq i \leq n$. The computation is slightly different for z_{n+1}.

$$\phi(z_{n+1}) = C_{n+1} \qquad\qquad C_{n+1} : w \in \overline{\mathbb{D}} \mapsto w \in C(\mathbb{S}^{2n-1}_\rho)$$

$$\begin{aligned}\psi(C_{n+1})[t_1, \ldots, t_{n+1}] : e^{i\theta} \mapsto C_{n+1}(t_{n+1}e^{i\theta}) &\left[\frac{t_1}{\sqrt{1-t_{n+1}^2}}, \ldots, \frac{t_n}{\sqrt{1-t_{n+1}^2}}\right] \\ = t_{n+1}e^{i\theta}&\end{aligned}$$

In the isomorphism between $C(\mathbb{S}^1, C(\mathbb{T}^n_\omega))$ and $C(\mathbb{T}^{n+1}_\rho)$, $e^{i\theta}$ corresponds to U_{n+1}, so $\psi(C_{n+1}) = \psi \circ \phi(z_{n+1})$ is the function $(t_1, \ldots, t_{n+1}) \mapsto t_{n+1}U_{n+1}$, or the generator z_{n+1}. This completes the proof. $\square$

Realizing a sphere $C(\mathbb{S}^{2n+1}_\rho)$ with z_{n+1} central as a function algebra $\{f : \overline{\mathbb{D}} \to C(\mathbb{S}^{2n-1}_\omega) : f \text{ is continuous and for all } w \in \partial\mathbb{D}, f(w) \in \mathbb{C}\}$ also gives the standard function representation of $C(\mathbb{S}^{2n}_\omega) = C(\mathbb{S}^{2n+1}_\rho)/\langle z_{n+1} - z^*_{n+1}\rangle$. To see this, note that as a function, $z_{n+1} - z^*_{n+1}$ assigns $w \in \overline{\mathbb{D}}$ to $w - \overline{w} = 2i\ \mathrm{Im}(w)$, which vanishes on $[-1, 1]$. Limits of polynomial functions in $\mathrm{Im}(w)$ with no constant term are sufficient to generate functions

$$p_n(w) = \begin{cases} 0 & \text{if } w \in \overline{\mathbb{D}} \text{ and } |\mathrm{Im}(w)| \leq \frac{1}{n} \\ n|\mathrm{Im}(w)| - 1 & \text{if } w \in \overline{\mathbb{D}} \text{ and } \frac{1}{n} \leq |\mathrm{Im}(w)| \leq \frac{2}{n} \\ 1 & \text{if } w \in \overline{\mathbb{D}} \text{ and } \frac{2}{n} \leq |\mathrm{Im}(w)| \end{cases}$$

in the ideal generated by $z_n - z^*_n$. These act as an approximate identity for functions vanishing on $[-1, 1]$: if f vanishes on $[-1, 1]$, then $||f - f \cdot p_n|| \to 0$ as $n \to \infty$ and $f \in \langle z_n - z^*_n\rangle$. This means that the ideal and its quotient are completely characterized.

$$\langle z_n - z^*_n\rangle \cong \{f \in C(\overline{\mathbb{D}}, C(\mathbb{S}^{2n-1}_\omega)) : f|_{[-1,1]} = 0, \text{ and if } w \in \partial\mathbb{D}, \text{ then } f(w) \in \mathbb{C}\}$$

$$C(\mathbb{S}^{2n}_\omega) \cong C(\mathbb{S}^{2n+1}_\rho)/\langle z_n - z_n^* \rangle \cong \{g \in C([-1,1], C(\mathbb{S}^{2n-1}_\omega)) : g(1), g(-1) \in \mathbb{C}\} \tag{3.4.3}$$

This form of $C(\mathbb{S}^{2n}_\omega)$ is called the unreduced suspension of $C(\mathbb{S}^{2n-1}_\omega)$. Unreduced suspensions and $\mathbb{Z}_2$ actions are discussed by Dąbrowski in [14], and the next chapter proves a conjecture from that paper. For now, note that the antipodal actions on $C(\mathbb{S}^{2n+1}_\rho)$ and $C(\mathbb{S}^{2n}_\omega)$ are reflected in their function algebras over $C(\mathbb{S}^{2n-1}_\omega)$ as follows. First, negate the domain variable in $\overline{\mathbb{D}}$ or $[-1,1]$, and then apply the antipodal action pointwise on $C(\mathbb{S}^{2n-1}_\omega)$.

$$\alpha_{2n+1}(f)[w] = \alpha_{2n-1}(f(-w)), w \in \overline{\mathbb{D}}$$

$$\alpha_{2n}(f)[x] = \alpha_{2n-1}(f(-x)), x \in [-1,1]$$

This fact will be useful when pushing homogeneous matrices between even and odd spheres.

If $P \in M_k(C(\mathbb{S}^{2n}_\omega))$ is a projection, then $B = 2P - I$ is a self-adjoint square root of I, or equivalently B is self-adjoint and unitary. Viewing B as a function of $x \in [-1,1]$ with (self-adjoint and unitary) values in $M_k(C(\mathbb{S}^{2n-1}_\omega))$, we may form a unitary function $\widetilde{B} : \overline{\mathbb{D}} \to M_k(C(\mathbb{S}^{2n-1}_\omega))$ by introducing an additional coordinate.

$$\widetilde{B}(x+iy) = \sqrt{1-y^2}\,B\left(\frac{x}{\sqrt{1-y^2}}\right) + iyI_k$$

The function $\widetilde{B}$ appears to be only well-defined on $\overline{\mathbb{D}} \setminus \{\pm i\}$, but the term $B\left(\frac{x}{\sqrt{1-y^2}}\right)$ is bounded, as B has norm 1, and the multiplication by $\sqrt{1-y^2}$ implies that $\widetilde{B}$ extends continuously to $\overline{\mathbb{D}}$ by the squeeze theorem. Because B is self-adjoint and unitary, $\widetilde{B}$ is unitary. Moreover, the boundary conditions on P and $B = 2P - I$ imply that if $x^2 + y^2 = 1$, then $\widetilde{B}(x+iy) \in M_k(\mathbb{C})$, so $\widetilde{B}$ represents a unitary element of $M_k(C(\mathbb{S}^{2n+1}_\rho))$. The association $P \mapsto B \mapsto \widetilde{B}$ between projections of dimension k over $C(\mathbb{S}^{2n}_\omega)$ to unitaries of dimension k over

$C(\mathbb{S}_\rho^{2n+1})$ is continuous, meaning it respects path components, and in addition it is compatible with the direct sum. Also, if P is a trivial projection $I_k \oplus 0_l$, then the function $\widetilde{B}$ takes values in $U_k(\mathbb{C})$. The domain $\overline{\mathbb{D}}$ is contractible, so $\widetilde{B}$ is connected within $U_{k+l}(C(\mathbb{S}_\rho^{2n+1}))$ to a scalar-entried matrix, which is then connected to the identity. (Note that this contraction was only possible because the range of $\widetilde{B}$ only included scalar-valued matrices, which did not conflict with boundary conditions. In general an element in $U_{k+l}(C(\mathbb{S}_\rho^{2n+1}))$ is not connected to the identity, as attempting to contract $\overline{\mathbb{D}}$ to a point conflicts with the boundary conditions.) In other words, if Ω_k denotes the function $P \mapsto \widetilde{B}$ for projections P of dimension k, then the various Ω_k are compatible and produce a single homomorphism

$$\Omega : K_0(C(\mathbb{S}_\omega^{2n})) \to K_1(C(\mathbb{S}_\rho^{2n+1}))$$

on the K-groups, and Ω has all trivial projections $I_k \oplus 0_l$ in its kernel. Moreover, each Ω_k is compatible with the $\mathbb{Z}_2$ antipodal action in the following way. If $P \in M_k(C(\mathbb{S}_\omega^{2n}))$ is a projection satisfying $\alpha_{2n}(P) = I - P$, then $B = 2P - I$ has $\alpha_{2n}(B) = -B$, or rather, B is odd. This is reflected in the function algebra as $\alpha_{2n}(B)[x] = \alpha_{2n-1}(B(-x)) = -B(x)$, which will help show that $\Omega_k(P) = \widetilde{B}$ is also odd.

$$\begin{aligned}
\alpha_{2n+1}(\widetilde{B})[x+iy] &= \alpha_{2n-1}(\widetilde{B}(-x-iy)) \\
&= \alpha_{2n-1}\left(\sqrt{1-(-y)^2}B\left(\frac{-x}{\sqrt{1-(-y)^2}}\right) + i(-y)I_k\right) \\
&= \sqrt{1-y^2}\left[\alpha_{2n-1}\left(B\left(\frac{-x}{\sqrt{1-y^2}}\right)\right)\right] - iyI_k \\
&= \sqrt{1-y^2}\left(-B\left(\frac{x}{\sqrt{1-y^2}}\right)\right) - iyI_k \\
&= -\left(\sqrt{1-y^2}B\left(\frac{x}{\sqrt{1-y^2}}\right) + iyI_k\right) \\
&= -\widetilde{B}(x+iy)
\end{aligned}$$

The above computation allows us to prove a K_0 theorem that states certain projections are nontrivial, based on how they interact with the antipodal map. Unforunately, the group operation of K_0 is only realized as the direct sum, as opposed to the group operation of K_1, which can be realized as either the direct sum or matrix multiplication. This means that if a statement in K_0 like Corollary 3.3.3 is desired, there is not a clear way to move the argument to the fixed point subalgebra by multiplication. Instead, we glean K_0 information on $C(\mathbb{S}^{2n}_\omega)$ from K_1 information of $C(\mathbb{S}^{2n+1}_\rho)$.

Theorem 3.4.4. If P is a projection matrix over $C(\mathbb{S}^{2n}_\omega)$ of dimension $(2m+1)2^n$ that satisfies $\alpha(P) = I - P$, then when P is viewed in $K_0(C(\mathbb{S}^{2n}_\rho)) \cong \mathbb{Z}^2$, P is not in the subgroup generated by trivial projections.

Proof. Suppose α is in the subgroup generated by trivial projections. Then $V := \Omega_{(2m+1)2^n}(P)$ is in $U_{(2m+1)2^n}(C(\mathbb{S}^{2n+1}_\rho))$, where ρ is an $(n+1) \times (n+1)$ parameter matrix with ω in the upper left and 1 in all other entries. Because α is in the subgroup of trivial projections in K_0, $[V]_{K_1} = \Omega([P]_{K_0})$ is the trivial element of $K_1(C(\mathbb{S}^{2n+1}_\rho))$. However, V is odd and of dimension $(2m+1)2^n$, so this contradicts Corollary 3.3.3 for spheres of dimension $2n+1 = 2(n+1) - 1$. $\square$

This theorem shows that the $\mathbb{Z}_2$ structure of even spheres is reflected in their K-theory. Just as in the odd case, we now prove a noncommutative Borsuk-Ulam theorem.

Corollary 3.4.5. Suppose $\Phi : C(\mathbb{S}^{2n}_\omega) \to C(\mathbb{S}^{2n}_\delta)$ is a unital $*$-homomorphism that is equivariant for the antipodal maps. Then the induced map on $K_0 \cong \mathbb{Z}^2$ has non-cyclic range (that is, its image is not in the subgroup generated by trivial projections).

Proof. Consider the projection $P = \begin{bmatrix} \frac{1+x}{2} I_{2^{n-1}} & Z_\omega(n) \\ Z_\omega(n)^* & \frac{1-x}{2} I_{2^{n-1}} \end{bmatrix}$. Here $Z_\omega(n)$ denotes the formal $*$-mononomial matrix in $z_1, \ldots, z_n$ that, if evaluated in the lower dimensional sphere $C(\mathbb{S}^{2n-1}_\omega)$, would produce its K_1-generator. The projection P is of dimension 2^n and satisfies

$\alpha(P) = I - P$. Because Φ is equivariant, the same applies to $\Phi(P)$, so by the previous corollary $\Phi(P)$ is not in the subgroup generated by trivial projections. $\square$

Remark. The projection P takes a similar form to projections in [15] on q-spheres of dimension 4 and in [36] for θ-deformed spheres of dimension 4.

Corollary 3.3.4 and Corollary 3.4.5 together generalize one version of the topological Borsuk-Ulam theorem in all dimensions: every odd, continuous function from $\mathbb{S}^k$ to $\mathbb{S}^k$ is homotopically nontrivial (because it is nontrivial on top cohomology). In the topological case, this is equivalent to the claim that there is no odd map from $\mathbb{S}^k$ to $\mathbb{S}^{k-1}$. The topological sphere $\mathbb{S}^{k-1}$ sits inside $\mathbb{S}^k$ as the equator, in such a way that the antipodal maps are compatible and $\mathbb{S}^{k-1}$ lies inside a contractible subset of $\mathbb{S}^k$. A similar phenomenon occurs in the θ-deformed case, so the next two definitions give algebraic versions of this topological embedding; note that the maps are automatically K_1-trivial since the even spheres have trivial K_1 groups.

Definition 3.4.6. Suppose ρ is an $n \times n$ parameter matrix with $\rho_{in} = \rho_{ni} = 1$ for all i, and let $\widetilde{\rho}$ be the minor of ρ formed by removing row and column n. Then $\pi : C(\mathbb{S}^{2n-1}_\rho) \to C(\mathbb{S}^{2n-2}_{\widetilde{\rho}})$ is the unique, unital $*$-homomorphism defined by $z_i \mapsto z_i$ for $1 \leq i \leq n-1$ and $z_n \mapsto x$.

Definition 3.4.7. Suppose ρ is an $n \times n$ parameter matrix. Then $\pi : C(\mathbb{S}^{2n}_\rho) \to C(\mathbb{S}^{2n-1}_\rho)$ is the unique, unital $*$-homomorphism defined by $z_i \mapsto z_i$ and $x \mapsto 0$.

In both cases, π exists due to the relations defining the algebras, π respects the $\mathbb{Z}_2$ structure, and π is automatically K_1-trivial. This leads to the following consequence of Corollary 3.3.4.

Corollary 3.4.8. There is no unital $*$-homomorphism $\Psi : C(\mathbb{S}^{k-1}_\rho) \to C(\mathbb{S}^k_\omega)$ which is equivariant for the antipodal maps.

Proof. Suppose the theorem fails. If $k = 2n$, then consider $\pi : C(\mathbb{S}^{2n}_{\omega}) \to C(\mathbb{S}^{2n-1}_{\omega})$. Since π is equivariant and K_1-trivial, $\Phi = \pi \circ \Psi : C(\mathbb{S}^{2n-1}_{\rho}) \to C(\mathbb{S}^{2n-1}_{\omega})$ is also equivariant and K_1-trivial. This contradicts Corollary 3.3.4.

If $k = 2n-1$, then ρ has dimensions $(n-1) \times (n-1)$. Let $\mathcal{P}$ be the $n \times n$ parameter matrix with ρ in the upper left and all other entries equal to 1, and form $\pi : C(\mathbb{S}^{2n-1}_{\mathcal{P}}) \to C(\mathbb{S}^{2n-2}_{\rho})$. Then $\Phi = \Psi \circ \pi : C(\mathbb{S}^{2n-1}_{\mathcal{P}}) \to C(\mathbb{S}^{2n-1}_{\omega})$ contradicts Corollary 3.3.4. $\square$

This corollary generalizes the Borsuk-Ulam theorem in all dimensions, as desired, and it is analogous to Yamashita's theorem on q-spheres (Theorem 1.5.12). As noted earlier, it is also possible to prove this corollary only from the K_0 theorems on even spheres. Now, the presentation of $C(\mathbb{S}^{2n}_{\omega})$ imposes that the self-adjoint generator x be central, which seems very restrictive. Certainly an arbitary noncommutation relation $xz_j = \omega_j z_j x$ would be problematic, as this would impose the restriction $xz_j = \overline{\omega_j} z_j x$ as well, since x is self-adjoint and z_j is normal ([33], Lemma 2.6). However, there is not an immediate reason why insisting x commutes with some z_j, but anticommutes with other z_j, would be flawed. The implications of this assertion are investigated in Chapter 4 as part of a study of general C^*-algebras with $\mathbb{Z}_2$ actions.

Chapter 4

Unreduced Suspensions and Anticommutation

4.1 Saturated Actions

In the previous chapter, we established noncommutative Borsuk-Ulam theorems on the θ-deformed spheres, which produced results that parallel those of Yamashita in [55] on q-spheres and partially answered a question of Taghavi in [49]. In 2015, Dąbrowski formulated a conjecture in [14] that seeks a general form of a noncommutative Borsuk-Ulam theorem, applicable for all unital C^*-algebras with suitable $\mathbb{Z}_2$ actions. If A is a unital C^*-algebra with a $\mathbb{Z}_2$ action generated by $\alpha : A \to A$, then this action extends to the unreduced suspension

$$\Sigma A = \{f \in C([-1,1], A) : f(-1), f(1) \in \mathbb{C}\}$$

in a natural way that includes an effect of the domain variable. (We use a domain of $[-1,1]$ instead of $[0,1]$ for convenience, slightly different than the convention in [14].)

$$\mathfrak{a}(f)[t] = \alpha(f(-t)) \tag{4.1.1}$$

The unreduced suspension of a unital C^*-algebra is dual to the unreduced suspension ΣX (or SX, for some authors) of a compact space X, which is the quotient of a cylinder $X \times [-1,1]$ that collapses $X \times \{1\}$ to a point and $X \times \{-1\}$ to a point. For example, $\Sigma \mathbb{S}^k = \mathbb{S}^{k+1}$, so $\Sigma C(\mathbb{S}^k) = C(\mathbb{S}^{k+1})$, and the extension of the antipodal action is again antipodal. For a θ-sphere $C(\mathbb{S}^k_\rho)$, $\Sigma C(\mathbb{S}^k_\rho)$ is another θ-sphere, but not all θ-spheres come about this way; the new generator $f(t) = t$ is central. Dąbrowski conjectured the following.

Conjecture 4.1.2 ([14], Conjecture 3.1)**.** For a unital C^*-algebra A with a free action of $\mathbb{Z}_2$, there is no $\mathbb{Z}_2$-equivariant $*$-homomorphism $\phi : A \to \Sigma A$.

While there is only one definition of freeness for a group action on a topological space X, C^*-algebras have numerous inequivalent properties that could be called freeness (see [38]); in proving this conjecture as Theorem 4.1.9, we will assume the action is *saturated*, a very weak assumption. The conjecture holds for the θ-deformed spheres because $\Sigma C(\mathbb{S}^k_\rho)$ is another θ-deformed sphere of dimension $k+1$, and Corollary 3.4.8 forbids the existence of ϕ. The commutative case $A = C(X)$ was previously known only when X is of finite covering dimension, but this restriction will not persist. Also, the suspension of a topological space X is the same as a join of X with $\mathbb{Z}_2$, and in a different paper [2], Baum, Dąbrowski, and Hajac formulated a join of C^*-algebras, which is particularly relevant when one of the algebras is a quantum group acting on the other, and the following conjecture.

Conjecture 4.1.3 ([2], Conjecture 2.3 Type 1)**.** Let A be a unital C^*-algebra with a free action of a nontrivial compact quantum group (H, Δ). Also, let $A *_\delta H$ be the equivariant noncommutative join C^*-algebra of A and H with the induced free action of (H, Δ) given by δ_Δ. Then there is no H-equivariant $*$-homomorphism from A to $A *_\delta H$.

We introduce a type of noncommutative join, for actions of $\mathbb{Z}_k$ only, that is different from that of [2] and formulate a similar conjecture (in noncommittal question form) as Question 4.3.3, with comments about special cases. Methods used for the suspension are partially applicable, proving the following topological analogue (another conjecture by Baum, Dąbrowski, and Hajac in [2]) in the special case that G has nontrivial torsion.

Conjecture 4.1.4 ([2], Conjecture 2.2)**.** Let X be a compact Hausdorff space with a continuous free action of a nontrivial compact Hausdorff group G. Then for the diagonal action of G on $X * G$, there does not exist a G-equivariant continuous map $f : X * G \to X$.

The following theorem shows the extent to which the topological version of Conjecture 4.1.2 was already known before our work.

Theorem 4.1.5. Suppose X is a compact Hausdorff space *with finite covering dimension* equipped with a free $\mathbb{Z}_2$ action α. Then there is no continuous, equivariant map from ΣX to X, where ΣX has the action $(x,t) \mapsto (\alpha(x), -t)$. Equivalently, every $\mathbb{Z}_2$-equivariant continuous map from X to itself is homotopically nontrivial.

Descriptions of the proof of the result, which applies in a similar form for $\mathbb{Z}_k$ actions, $k > 2$, may be found in [8], Theorem 4 for CW-complexes, [18] for compact Hausdorff spaces, and [55], Remark 17 for the noncommutative (KK-theoretic) point of view. The assumption of finite covering dimension of X allows discussion of top Čech cohomology, which is a homotopy invariant. Alternatively, the proof in [18] uses simplicial methods to show that finite dimensionality of X may be used to reduce from the case of free $\mathbb{Z}_k$ actions on X to standard rotation actions on spheres, for which every equivariant self-map is homotopically nontrivial. Our proof of Conjecture 4.1.2 results from an attempt to find the only "obvious" type of counterexample that might exist in the topological case: a compact, infinite-dimensional space X which includes a copy of each $\mathbb{S}^k$.

Let $\mathbb{S}^k$ map injectively into $\mathbb{S}^{k+1}$ via $x \mapsto (x, 0)$, and form the increasing union, denoted $\mathbb{S}^\infty$, by taking a direct limit. The topology of $\mathbb{S}^\infty$ is given by the final topology, meaning that a subset O of $\mathbb{S}^\infty$ is open if and only if $O \cap \mathbb{S}^k$ is open in $\mathbb{S}^k$ for all k. Now, $\mathbb{S}^\infty$ is not a (locally) compact space, but it is equal to its own unreduced suspension, and its Stone-Čech compactification $\beta\mathbb{S}^\infty$ is compact and admits a $\mathbb{Z}_2$ action generated by the antipodal map α, as follows. Note that while $\mathbb{S}^\infty$ is not locally compact, it is a Tychonoff space, so the inclusion map $\iota : \mathbb{S}^\infty \to \beta\mathbb{S}^\infty$ exists, has non-open range, and satisfies the universal property of Stone-Čech. So, since $\iota \circ \alpha : \mathbb{S}^\infty \to \beta\mathbb{S}^\infty$ is a map from $\mathbb{S}^\infty$ into a compact Hausdorff space, there exists a continuous map $\widetilde{\alpha} : \beta\mathbb{S}^\infty \to \beta\mathbb{S}^\infty$ which extends it. On the dense subset $\iota(\mathbb{S}^\infty)$ of $\beta\mathbb{S}^\infty$, $\widetilde{\alpha} \circ \widetilde{\alpha}$ is the identity map, so $\widetilde{\alpha}$ is of order 2.

$$\begin{tikzcd}
\mathbb{S}^\infty \arrow[r, "\iota"] \arrow[d, "\alpha"'] \arrow[dr] & \beta\mathbb{S}^\infty \arrow[d, dashed, "\widetilde{\alpha}"] \\
\mathbb{S}^\infty \arrow[r, "\iota"] & \beta\mathbb{S}^\infty
\end{tikzcd}$$

By the definition of the topology of $\mathbb{S}^\infty$, if a sequence of continuous maps $\psi_k : \mathbb{S}^k \to X$ is consistent with all of the inclusions $\mathbb{S}^k \hookrightarrow \mathbb{S}^{k+1}$, then the sequence produces a unique continuous map $\psi : \mathbb{S}^\infty \to X$. If X is compact and Hausdorff, this extends to $\widetilde{\psi} : \beta\mathbb{S}^\infty \to X$. If all of the ψ_k are equivariant maps for the antipodal action and a $\mathbb{Z}_2$ action on X, then ψ and $\widetilde{\psi}$ are also equivariant. Such a map $\widetilde{\psi}$ can actually be produced any time there is an equivariant map from ΣX to X by the following iteration scheme. If $f : \Sigma X \to X$ is equivariant, then we can suspend f to form an equivariant map from $\Sigma^2 X$ to ΣX, and repeated suspensions produce a sequence

$$\ldots \to \Sigma^3 X \to \Sigma^2 X \to \Sigma X \to X \tag{4.1.6}$$

of equivariant maps. Now, ΣX contains an equivariant copy of $\mathbb{Z}_2 = \mathbb{S}^0$, $\Sigma^2 X$ contains an equivariant copy of $\Sigma\mathbb{S}^0 = \mathbb{S}^1$, and so on. This forms a sequence of equivariant, continuous maps $\psi_k : \mathbb{S}^k \to X$ that are consistent with the inclusions $\mathbb{S}^k \hookrightarrow \mathbb{S}^{k+1}$, and by the above discussion, these extend to a continuous, equivariant map $\widetilde{\psi} : \beta\mathbb{S}^\infty \to X$.

From this discussion, we reach a crossroads. It was natural to consider compactifications of $\mathbb{S}^\infty$ because $\mathbb{S}^\infty$ is its own unreduced suspension (including the form of the antipodal action), and if that property were preserved by a compactification on which the antipodal action remained free, this would disprove Dąbrowski's conjecture, as Theorem 4.1.5 would truly require the assumption of finite dimensionality. However, if the antipodal action on $\beta\mathbb{S}^\infty$ is *not* free (i.e., has a fixed point), then for any compact Hausdorff space X with a $\mathbb{Z}_2$ action and an equivariant map from ΣX to X, the equivariant map from $\beta\mathbb{S}^\infty$ to X produced above actually implies that the $\mathbb{Z}_2$ action on X cannot be free!

Fixed points of maps on Stone-Čech compactifications of normal spaces were studied in Theorem 2.3 of [5], which shows that the compactification $\widetilde{f} : \beta Y \to \beta Y$ of a map $f : Y \to Y$ has a fixed point if and only if Y fails to admit a finite open cover with disjointness properties under f-images. Similarly, from the main result of [52], the antipodal map on the Stone-Čech compactification of the disjoint union of spheres $\mathbb{S}^k$ must have a fixed point, again by considering open covers, and in [40], section 3, the antipodal map is used to construct a zero dimensional space and a fixed point free map whose compactification does admit a fixed point. In general, we find that fixed points are likely to exist on Stone-Čech compactifications, and section 4 of [21] shows that $\beta\mathbb{S}^\infty$ does have a fixed point for the antipodal map.

Theorem 4.1.7. If X is a compact Hausdorff space (possibly infinite dimensional) with a free $\mathbb{Z}_2$ action, then there is no equivariant, continuous map from ΣX to X.

Proof. If an equivariant map from ΣX to X exists, the iteration technique in (4.1.6) yields an equivariant map from $\mathbb{S}^\infty$ to X, which may be lifted to the Stone-Čech compactification as $\widetilde{\psi} : \beta\mathbb{S}^\infty \to X$. The image of a fixed point of $\beta\mathbb{S}^\infty$ is a fixed point of X. $\square$

Remark. The theorem has a converse, as if the action on X is not free, and $x_0 \in X$ is a fixed point, then the constant map that sends any element of ΣX to x_0 is equivariant.

In the topological setting, the image of a fixed point of $\beta\mathbb{S}^\infty$ produces a fixed point of X, contrary to assumption. Because a noncommutative C^*-algebra with unit cannot be a function algebra, in order to apply similar techniques on C^*-algebras, we must see this fixed point from a different perspective, in which covering lemmas are replaced with equivalent facts about functions. A $\mathbb{Z}_2$ action on a C^*-algebra A, generated by $\alpha : A \to A$, gives rise to a grading of A into even elements, $A_0 = \{a \in A : \alpha(a) = a\}$, and odd elements, $A_1 = \{a \in A : \alpha(a) = -a\}$. The action is *saturated* if $\overline{A_0 A A_0^*} = A$ and $\overline{A_1 A A_1^*} = A$, which in particular implies that the right ideal generated by odd elements is full. If $A = C(X)$, then the $\mathbb{Z}_2$ action on A is dual to a $\mathbb{Z}_2$ action on X, and freeness is equivalent to saturation (see [38]).

Consider the C^*-algebra $C(\beta\mathbb{S}^\infty) = C_b(\mathbb{S}^\infty)$ of bounded functions on $\mathbb{S}^\infty$. We should find that the antipodal action on $C_b(\mathbb{S}^\infty)$ is not saturated; indeed, if $f_1, \ldots, f_n \in C_b(\mathbb{S}^\infty)$ are odd functions and $a_1, \ldots, a_n \in C_b(\mathbb{S}^\infty)$ are any functions such that $||f_1a_1 + \ldots + f_na_n - 1|| < 1$, then $f_1a_1+\ldots+f_na_n$ is nowhere-vanishing, and the odd functions $f_1, \ldots, f_n$ have no common zeroes. These may be combined into a single odd function $F : \mathbb{S}^\infty \to \mathbb{C}^n \setminus \{0\} \cong \mathbb{R}^{2n} \setminus \{0\}$, and the restriction of the domain to $\mathbb{S}^{2n}$ would contradict the Borsuk-Ulam theorem. In other words, another way to see that a fixed point of $\beta\mathbb{S}^\infty$ exists is that the number of odd functions required to generate an invertible in $C(\mathbb{S}^k)$ increases without bound as k increases, so an approximation using finitely many odd functions cannot be done in $C_b(\mathbb{S}^\infty)$. If we exploit the setting of C^*-algebras further, we can prove Dąbrowski's conjecture while avoiding the unwieldly Stone-Čech compactification all together.

Lemma 4.1.8. Suppose $\phi : A \to B$ is a unital $*$-homomorphism that is (α, β)-equivariant for $\mathbb{Z}_2$ actions α on A and β on B. Let $\mathfrak{a}$ and $\mathfrak{b}$ denote the associated $\mathbb{Z}_2$ actions on the unreduced suspensions. Then $\Phi = \Sigma\phi : \Sigma A \to \Sigma B$ defined by $\Phi(f)[t] = \phi(f(t))$ is equivariant for $(\mathfrak{a}, \mathfrak{b})$.

Proof. This is a direct check, by evaluating each at t.

$$\mathfrak{b}(\Phi(f))[t] = \beta(\Phi(f)[-t]) = \beta(\phi(f(-t))) = \phi(\alpha(f(-t))) = \phi(\mathfrak{a}(f)[t]) = \Phi(\mathfrak{a}(f))[t]$$

□

Theorem 4.1.9. Suppose A is a unital C^*-algebra with a saturated $\mathbb{Z}_2$ action α, which defines the $\mathbb{Z}_2$ action $\mathfrak{a}$ on ΣA. Then there is no $(\alpha, \mathfrak{a})$-equivariant, unital $*$-homomorphism from A to ΣA.

Proof. Suppose the theorem fails and $\psi : A \to \Sigma A$ is equivariant. Then by the previous lemma, we may repeatedly suspend the map to an equivariant homomorphism from $\Sigma^n A$ to

$\Sigma^{n+1}A$, where iterated unreduced suspensions have the $\mathbb{Z}_2$ actions obtained from repeatedly applying (4.1.1). By composing these maps in a chain, for each k we can form a unital, equivariant $*$-homomorphism from A to $\Sigma^k A$. Now, $\Sigma^k A$ has an equivariant quotient algebra $\Sigma^{k-1}C(\mathbb{Z}_2) = C(\mathbb{S}^{k-1})$ seen by evaluating on the boundary, so for any k we may form an equivariant, unital $*$-homomorphism $\psi_k : A \to C(\mathbb{S}^{k-1})$, where $C(\mathbb{S}^{k-1})$ has the antipodal action.

Since the action on A is saturated, the right ideal generated by odd elements is full, so let $f_1, \ldots, f_n \in A$ be odd elements and let $a_1, \ldots, a_n \in A$ be such that $||f_1 a_1 + \ldots + f_n a_n - 1|| < 1$. Now that n is fixed, consider $\psi_{2n+1} : A \to C(\mathbb{S}^{2n})$, so the images $\psi_{2n+1}(f_1), \ldots, \psi_{2n+1}(f_n)$ are odd complex-valued functions on $\mathbb{S}^{2n}$. Since ψ_{2n+1} is a unital map and all C^*-homomorphisms have operator norm at most 1, we obtain the estimate $||\psi_{2n+1}(f_1)\psi_{2n+1}(a_1) + \ldots + \psi_{2n+1}(f_n)\psi_{2n+1}(a_n) - 1|| < 1$. This implies that the closed ideal generated by $\psi_{2n+1}(f_1), \ldots, \psi_{2n+1}(f_n)$ is full, so these n complex-valued, odd functions on $\mathbb{S}^{2n}$ have no common zeroes, a contradiction of the traditional Borsuk-Ulam theorem. $\square$

Remark. There is no general converse to this theorem. Let A be a noncommutative, simple C^*-algebra, such as an irrational quantum 2-torus, with the trivial action. This action is certainly unsaturated. Nevertheless, there is no unital $*$-homomorphism from A to ΣA, as such a map would produce a maximal ideal from evaluation at $t = 1$.

Based on the discussion in [14], we reach the following additional corollaries, which are also related to questions in [49].

Corollary 4.1.10. The following statements hold.

1. If X is a compact Hausdorff space with a free $\mathbb{Z}_2$ action, then there is no continuous map from the cone ΓX to X whose restriction to $X \subset \Gamma X$ is equivariant (or in particular is the identity map). So, any equivariant map from X to itself is homotopically nontrivial.

2. If A is a unital C^* algebra with a saturated $\mathbb{Z}_2$ action, then there is no unital $*$-homomorphism from A to the cone $\Gamma A = \{f \in C([0,1], A) : f(1) \in \mathbb{C}\}$ which is equivariant on the boundary $t = 0$. So, any equivariant map from A to itself must not be homotopically equivalent in the strong topology to a representation of A into $\mathbb{C}$.

The unreduced suspension of a space X may be written as a join $X * \mathbb{Z}_2$, where the join of two topological spaces X and Y is defined as the following quotient space.

$$X * Y \cong X \times Y \times [0,1]/\sim$$

$$(0, x_0, y) \sim (0, x_1, y) \text{ for all } x_0, x_1 \in X, y \in Y$$

$$(1, x, y_0) \sim (1, x, y_1) \text{ for all } x \in X, y_0, y_1 \in Y$$

If G acts on X freely, then $X * G$ admits a free action induced by $h \cdot (x, g, t) = (h \cdot x, hg, t)$. In [2], the authors ask (for G and X also compact and Hausdorff) if there is ever an equivariant, continuous map from $X * G$ to X. Similar techniques as above settle this conjecture when G has nontrivial torsion; we may appeal to the Stone-Čech compactification of infinite joins, which is considered for finite groups in [21], or we may use $\mathbb{Z}_k$ Borsuk-Ulam theorems, which can be found in [18]. The proofs use the correspondence between free actions on X and saturated actions on $C(X)$ by more general groups than $\mathbb{Z}_2$, as follows (see [38]).

Definition 4.1.11. Suppose a compact abelian group G acts on a unital C^*-algebra A via α. This gives rise to homogeneous subspaces $A_\tau = \{a \in A : \alpha_g(a) = \tau(g)a \text{ for all } g \in G\}$, which are defined for any $\tau \in \widehat{G}$, the Pontryagin dual of G. The action is *saturated* if for each $\tau \in \widehat{G}$, $\overline{A_\tau A A_\tau^*} = A$. When $A = C(X)$ is commutative, the action on $C(X)$ is saturated if and only if the associated action on X is free.

The Pontryagin dual of $\mathbb{Z}_k$ is $\mathbb{Z}_k$, more specifically given by the kth roots of unity, and the homogeneous subspace A_n consists of elements of A such that $\alpha(a) = e^{2\pi i n/k}a$. This property is more visible when $\mathbb{Z}_2$ acts on $A = C(X)$; elements of A_0 are even functions, and elements of A_1 are odd functions.

Lemma 4.1.12 (Rephrasing Classical Results)**.** Let $\mathbb{Z}_k$, $k \geq 2$, act freely on itself by mutliplication. Denote $\mathbb{Z}_k$ by G_0, and let $G_{n+1} = G_n * \mathbb{Z}_k$, which has a free $\mathbb{Z}_k$ action determined recursively as above. Equip $\mathbb{C}$ with the $\mathbb{Z}_k$ rotation action, $z \mapsto e^{2\pi i/k}z$. If $n \geq 2$ and $f_1, \ldots, f_n \in C(G_{2n})$ are equivariant for these actions, then $f_1, \ldots, f_n$ have a common zero.

Proof. If $f_1, \ldots, f_n \in C(G_{2n})$ do not have a common zero, then we may may scale them to form a single map $F : G_{2n} \to \mathbb{S}^{2n-1}$. If $\mathbb{S}^{2n-1}$ is given the same type of coordinate rotation action $(z_1, \ldots, z_n) \mapsto (e^{2\pi i/k}z_1, \ldots, e^{2\pi i/k}z_n)$, F is also equivariant. This contradicts the main theorem in [18], which is written in more modern language as Theorem 6.2.6 in [28]. Specifically, G_{2n} is $(2n-1)$-connected (see [28], Definition 4.3.1 and Proposition 4.4.3) and $\mathbb{S}^{2n-1}$ has dimension $2n-1$, so there are no equivariant maps between them for free actions of $\mathbb{Z}_k$. $\square$

Theorem 4.1.13. Suppose X and G are compact and Hausdorff, where G is a topological group acting freely on X and G has a nontrivial torsion element. Then there is no equivariant, continuous map from $X * G$ to X, and any equivariant, continuous map from X to itself is homotopically nontrivial.

Proof. Suppose the theorem fails. Since G has nontrivial torsion, choose a subgroup $\mathbb{Z}_k \leq G$ for $k \geq 2$. Then $\mathbb{Z}_k$ acts freely on X, and by restriction, there is also a continuous, equivariant map $\phi : X * \mathbb{Z}_k \to X$. We may iterate this map on higher joins, and compose these new maps to form an equivariant map from $(X * \mathbb{Z}_k) * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ to X for any number of joins. These restrict to equivariant maps from $\mathbb{Z}_k * \cdots * \mathbb{Z}_k$ to X. Dualizing these maps, we reach equivariant, unital $*$-homomorphisms $\psi_n : C(X) \to C(\mathbb{Z}_k * \cdots * \mathbb{Z}_k)$. The action of $\mathbb{Z}_k$ on $C(X)$ is saturated, so the ideal generated by elements in the $e^{2\pi i/k}$ eigenspace of the action is full. By examining ψ_n for n large enough, we reach a contradiction of the previous lemma, just as in the $\mathbb{Z}_2$ case. $\square$

The infinite join of a finite group G was considered by J. Milnor in [30] and [31] as a universal G-bundle, and the issue of fixed points on the Stone-Čech compactification of G-bundles was once again considered in [21]. However, one may easily write a C^*-algebraic variant of the above theorem using its connectivity-based proof, which is the advantage of an approach using connectivity instead of the Stone-Čech compactification (an equally valid approach in the topological setting). Unfortunately, the (non)existence of equivariant maps from $X * G$ to X when G is torsion-free is still open, as both Stone-Čech techniques in the literature and general Borsuk-Ulam theorems do not seem applicable in that case. Instead of considering $X * G$ for any torsion-free compact group, we may restrict to the join with the closed subgroup generated by a single nontrivial element in G, reducing the problem to the case when G is compact, abelian, and torsion-free. Such groups are fairly exotic, such as $G = \widehat{(\mathbb{Q}, \tau)}$ where τ is the discrete topology, but if there is a counterexample (unlikely) to be found, then it can be found among those groups. A general classification of such groups can be found in [23].

Question 4.1.14. Let G be a compact, Hausdorff, abelian, torsion-free, nontrivial group. Let G act (freely) on itself by multiplication and extend this action to iterated joins. Any such action is free, so the associated action α on $A = C(G * \cdots * G)$ is saturated, and consequently the closed ideal generated by any A_τ is full. Does the number of A_τ elements required to approximate the multiplicative identity 1 remain bounded as the number of iterated joins increases? This would be of particular interest for the evaluation homomorphisms at points $q \in (\mathbb{Q}, \tau) = \widehat{\widehat{(\mathbb{Q}, \tau)}}$, and similarly for related groups.

4.2 Anticommutation in Sphere Relations

In [14], Dąbrowski asked if there was a form of noncommutative unreduced suspension of C^*-algebras that introduces non-central elements, based primarily on the noncommutative double unreduced suspension in [24]. With this in mind, note that the presentation of even θ-spheres as unreduced suspensions of odd θ-spheres includes a self-adjoint, central coordinate x which, when viewed in terms of generators and relations alone, appears to be somewhat artificial. For example, the following universal C^*-algebras replace this with anticommutation.

Definition 4.2.1. Let ω be an $n \times n$-dimensional parameter matrix. Then let $\mathfrak{R}^{2n}_\omega$ denote the universal, unital C^*-algebra generated by normal elements $z_1, \ldots, z_n$ and a self-adjoint element x satisfying the following relations.

$$z_k z_j = \omega_{jk} z_j z_k \qquad\qquad x z_j = -z_j x \qquad\qquad z_1 z_1^* + \ldots + z_n z_n^* + x^2 = 1$$

We also use the convention that $\mathfrak{R}^{2n}$ with no subsript indicates $\mathfrak{R}^{2n}_\omega$ when ω has 1 in every entry, so the generators $z_1, \ldots, z_n$ commute with each other and anticommute with x.

The relation algebras $\mathfrak{R}^{2n}_\omega$ appear to be different from the mirror quantum spheres of [24]. Moreover, unlike the θ-deformed even spheres, the lowest dimension noncommutative relation algebra is $\mathfrak{R}^2$, generated by anticommuting z_1 and x, whereas the only θ-deformed sphere of dimension 2 is commutative. However, $\mathfrak{R}^{2n}_\omega$ is still a quotient of a higher dimension θ-deformed sphere $C(\mathbb{S}^{2n+1}_\rho)$, where ρ has the minor ω in the upper left corner, and its final row and column indicate anticommutation. Specifically, $\mathfrak{R}^{2n}_\omega \cong C(\mathbb{S}^{2n+1}_\rho)/\langle z_{n+1} - z^*_{n+1} \rangle$. To avoid switching notation, we indicate this $C(\mathbb{S}^{2n+1}_\rho)$ as $\mathfrak{R}^{2n+1}_\omega$; whether the dimension is even or odd, algebras $\mathfrak{R}^k_\omega$ have anticommutation properties in the final generator.

Definition 4.2.2. If ω is an $n \times n$-dimensional parameter matrix, then let $\mathfrak{R}^{2n+1}_\omega$ denote $C(\mathbb{S}^{2n+1}_\rho)$, where $\rho_{jk} = \omega_{jk}$ for $j, k \in \{1, \ldots, n\}$, $\rho_{n+1,n+1} = 1$, and $\rho_{j,n+1} = \rho_{n+1,j} = -1$ for $j \in \{1, \ldots, n\}$.

A quotient of $\mathfrak{R}^{2n}_\omega$ that removes the coordinate x is a θ-deformed odd sphere: $\mathfrak{R}^{2n}_\omega / \langle x \rangle \cong$

$C(\mathbb{S}_\omega^{2n-1})$. In order to realize $\mathfrak{R}_\omega^k$ as an algebra of functions, we must use the crossed product, but otherwise the procedure is very similar to previous computations.

Lemma 4.2.3. If ρ is an $(n+1)\times(n+1)$ parameter matrix with $\rho_{j,n+1} = \rho_{n+1,j} = -1$ for all $j \neq n+1$, and ω is the top left $n \times n$ minor of ρ, then

$$C(\mathbb{T}_\rho^{n+1}) \cong \{f \in C(\mathbb{S}^1, C(\mathbb{T}_\omega^n) \rtimes_\alpha \mathbb{Z}_2) : \widehat{\alpha}(f(-e^{i\theta})) = f(e^{i\theta})\}$$

where α is the antipodal action on $C(\mathbb{T}_\omega^n)$ and $\widehat{\alpha}$ is the dual action on $C(\mathbb{T}_\omega^n) \rtimes_\alpha \mathbb{Z}_2$ (meaning $\widehat{\alpha}(a + b\delta) = a - b\delta$).

Proof. The restriction $\widehat{\alpha}(f(-e^{i\theta})) = f(e^{i\theta})$ means that if $f(e^{i\theta})$ is decomposed as $g(e^{i\theta}) + h(e^{i\theta})\delta$ in $C(\mathbb{T}_\omega^n) \rtimes_\alpha \mathbb{Z}_2$, then $g(-e^{i\theta}) = g(e^{i\theta})$ and $h(-e^{i\theta}) = -h(e^{i\theta})$. A homomorphism from $C(\mathbb{T}_\rho^{n+1})$ to the function algebra is defined as follows.

$$\begin{aligned} U_1 &\mapsto f_1(e^{i\theta}) = U_1 \\ &\vdots \\ U_n &\mapsto f_n(e^{i\theta}) = U_n \\ U_{n+1} &\mapsto f_{n+1}(e^{i\theta}) = e^{i\theta}\delta \end{aligned}$$

As usual, this homomorphism is guaranteed to exist because $f_1, \ldots, f_{n+1}$ satisfy the necessary relations; the group element δ anticommutes with each U_j, which matches the demands of the parameter matrix ρ. The usual trace τ_{n+1} on $C(\mathbb{T}_\rho^{n+1})$ corresponds to the trace $g + h\delta \mapsto \frac{1}{2\pi}\int_0^{2\pi} \tau_n(g(e^{i\theta}))\,d\theta$, so as in Lemma 3.4.1, the homomorphism is injective. Surjectivity is also established in a similar way to Lemma 3.4.1; a dense subspace of the function algebra consists of sums $\sum\limits_{\text{finite}} g_p(e^{i\theta})U_1^{p_1}\cdots U_n^{p_n} + \sum\limits_{\text{finite}} h_p(e^{i\theta})U_1^{p_1}\cdots U_n^{p_n}\delta$ where g_p and h_p are continuous and piecewise linear, $g_p(-e^{i\theta}) = g_p(e^{i\theta})$, and $h_p(-e^{i\theta}) = -h_p(e^{i\theta})$. The functions g_p and h_p have uniformly convergent Fourier series, so finite sums of these functions are in the range. An injective homomoprhism between C^*-algebras has closed range, so the map is an isomorphism. $\square$

Unsurprisingly, this idea extends to relation algebras in the following lemma.

Lemma 4.2.4. If ω is an $n \times n$ parameter matrix, then

$$\mathfrak{R}_\omega^{2n+1} \cong \{f \in C(\overline{\mathbb{D}}, C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2) : \widehat{\alpha}(f(-w)) = f(w), \text{ and } w \in \partial\mathbb{D} \text{ implies } f(w) \in \mathbb{C} + \mathbb{C}\delta\}$$

where $\widehat{\alpha}$ is the dual action to the antipodal action α on $C(\mathbb{S}_\omega^{2n-1})$.

Proof. A homomorphism from $\mathfrak{R}_\omega^{2n+1}$ to the function algebra is defined as follows.

$$\begin{aligned} z_1 &\mapsto f_1(w) = \sqrt{1-|w|^2} z_1 \\ &\vdots \\ z_n &\mapsto f_n(w) = \sqrt{1-|w|^2} z_n \\ z_{n+1} &\mapsto f_{n+1}(w) = w\delta \end{aligned}$$

The inverse is formed from viewing $C(\mathbb{S}_\omega^{2n-1})$ as a function algebra into $C(\mathbb{T}_\omega^n)$ and noting that the antipodal action on $C(\mathbb{S}_\omega^{2n-1})$ is the pointwise antipodal action on $C(\mathbb{T}_\omega^n)$, so $C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2$ is a function algebra into $C(\mathbb{T}_\omega^n) \rtimes_\alpha \mathbb{Z}_2$. Besides this change, the details are the same as the proof of Lemma 3.4.2. $\square$

Now that the C^*-algebra $\mathfrak{R}_\omega^{2n+1}$ is an algebra of functions, its quotient $\mathfrak{R}_\omega^{2n} \cong \mathfrak{R}_\omega^{2n+1}/\langle z_{n+1} - z_{n+1}^*\rangle$ consists of functions on a smaller domain, where $z_{n+1} - z_{n+1}^*$ vanishes. The argument is essentially identical to that for (3.4.3).

$$\mathfrak{R}_\omega^{2n} \cong \{f \in C([-1,1], C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2) : \widehat{\alpha}(f(-t)) = f(t), \text{ and for } t \in \{\pm 1\}, f(t) \in \mathbb{C} + \mathbb{C}\delta\}$$

We are careful to label the coordinate from $[-1,1]$ as t (instead of x), as the generator x, given as $x(t) = t\delta$, also includes a group element term. On the other hand, use of the crossed product makes the domain $[-1, 1]$ redundant. An element $f = g + h\delta \in \mathfrak{R}^{2n}_\omega$ has $g(-t) = g(t)$ and $h(-t) = -h(t)$, so we may restrict to the domain $[0, 1]$ without losing any information, noting that $h(0) = 0$, so $f(0) \in C(\mathbb{S}^{2n-1}_\omega)$.

$$\mathfrak{R}^{2n}_\omega \cong \{f \in C([0,1], C(\mathbb{S}^{2n-1}_\omega) \rtimes_\alpha \mathbb{Z}_2) : f(0) \in C(\mathbb{S}^{2n-1}_\omega) \text{ and } f(1) \in \mathbb{C} + \mathbb{C}\delta\} \tag{4.2.5}$$

In the above expression, if the antipodal action α is replaced by the trivial action, one recovers the θ-deformed sphere $C(\mathbb{S}^{2n}_\omega)$, where the crossed product is artificially encoding even and odd components of functions on $[-1, 1]$. Further, the function algebra on the right hand side is sensible for any $\mathbb{Z}_2$ action, and this suggests an approach to forming "noncommutative" unreduced suspensions of C^*-algebras, which will be investigated in the next section. Computations of the K-theory of $\mathfrak{R}^{2n}_\omega$ will be aided by the following matrix expansion map.

Definition 4.2.6. The *expansion map* $E_\alpha : C(\mathbb{S}^{2n-1}_\omega) \rtimes_\alpha \mathbb{Z}_2 \to M_2(C(\mathbb{S}^{2n-1}_\omega))$ is the injective, unital $*$-homomorphism defined by the following rule.

$$f + g\delta \mapsto \begin{bmatrix} f & g \\ \alpha(g) & \alpha(f) \end{bmatrix}$$

See [54], section 2.5 for a similar map. The dual action $\widehat{\alpha}(f + g\delta) = f - g\delta$ is seen in $M_2(C(\mathbb{S}^{2n-1}_\omega))$ as negating the off-diagonal, and the extension of α to the crossed product as $\alpha(f + g\delta) = \alpha(f) + \alpha(g)\delta$ comes from applying α entrywise. The map E_α is certainly not surjective, as all matrices in its range satisfy a very visible symmetry condition.

Proposition 4.2.7. If $M = \begin{bmatrix} f & g \\ \alpha(g) & \alpha(f) \end{bmatrix} \in \operatorname{Ran}(E_\alpha)$, then M satisfies the symmetry condition $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* \alpha(M) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. That is, if $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\alpha_U(M) = M$.

The antipodal action α is saturated, which gives vital information about the crossed product. In particular, K-theory of the crossed product $C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to the K-theory of the fixed point subalgebra $C(\mathbb{S}_\omega^{2n-1})^\alpha$ (see (1.5.15)), the algebra of even elements. This is useful when considering the ideal J of $\mathfrak{R}_\omega^{2n}$ consisting of functions which vanish at the endpoints 0 and 1, so $J \cong S(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2)$ and $K_i(J) \cong K_i(S(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2))) \cong K_{1-i}(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2) \cong K_{1-i}(C(\mathbb{S}_\omega^{2n-1})^\alpha)$. The fixed point subalgebra $C(\mathbb{S}_\omega^{2n-1})^\alpha$ is a Rieffel deformation of $C(\mathbb{RP}^{2n-1}) \cong C(\mathbb{S}^{2n-1})^\alpha$ and therefore has identical K-theory (K-theory of real projective space is computed in [1], Proposition 2.7.7). Boundary information is contained in the quotient $\mathfrak{R}_\omega^{2n}/J \cong C(\mathbb{S}_\omega^{2n-1}) \oplus (\mathbb{C} \oplus \mathbb{C}\delta)$, and K-theory respects direct sums, so the six term exact sequence yields

$$\begin{tikzcd} K_0(S(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2)) \arrow[r, "\gamma"] & K_0(\mathfrak{R}_\omega^{2n}) \arrow[r, "\upsilon"] & K_0(C(\mathbb{S}_\omega^{2n-1})) \oplus K_0(\mathbb{C} + \mathbb{C}\delta) \arrow[d, "\sigma"] \\ K_1(C(\mathbb{S}_\omega^{2n-1})) \oplus K_1(\mathbb{C} + \mathbb{C}\delta) \arrow[u, "\eta"] & K_1(\mathfrak{R}_\omega^{2n}) \arrow[l, "\lambda"'] & K_1(S(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2)) \arrow[l, "\kappa"'] \end{tikzcd} \tag{4.2.8}$$

and will be used to partially describe the isomorphism classes of the two unspecified K-groups. The following version of the same sequence includes the known isomorphism classes.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\gamma"] & K_0(\mathfrak{R}_\omega^{2n}) \arrow[r, "\upsilon"] & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\sigma"] \\ \mathbb{Z} \arrow[u, "\eta"] & K_1(\mathfrak{R}_\omega^{2n}) \arrow[l, "\lambda"'] & \mathbb{Z} \oplus \mathbb{Z}_{2^{n-1}} \arrow[l, "\kappa"'] \end{tikzcd} \tag{4.2.9}$$

The subalgebra $\mathbb{C} + \mathbb{C}\delta \leq C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2$ is two dimensional and isomorphic to a continuous function algebra on two points, $C(\{\pm 1\})$, so an evaluation of δ at 1 or -1 can be used on this subalgebra (however, an evaluation does not make sense on the entire crossed product). Moreover, $K_0(C(\mathbb{S}_\omega^{2n-1}) \oplus (\mathbb{C} + \mathbb{C}\delta)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is essentially rank data in three instances: a rank for $K_0(C(\mathbb{S}_\omega^{2n-1})) \cong \mathbb{Z}$ and two ranks for $K_0(C(\{\pm 1\})) \cong \mathbb{Z} \oplus \mathbb{Z}$ based on evaluations $\delta = \pm 1$. However, the map υ above is not surjective by the following argument. First, the projections $\dfrac{1 \pm \delta}{2} \in \mathbb{C} + \mathbb{C}\delta$ that generate its K_0 group are sent under E_α to rank one projections $\begin{bmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{bmatrix}$ in $M_2(C(\mathbb{S}_\omega^{2n-1}))$, so in $K_0(M_2(C(\mathbb{S}_\omega^{2n-1}))) \cong K_0(C(\mathbb{S}_\omega^{2n-1})) \cong \mathbb{Z}$ these are the generator 1. Next, if we consider $C(\mathbb{S}_\omega^{2n-1})$, whose K_0 group is again cyclic and generated by the trivial projection 1, the image $E_\alpha(1) = I_2$ has doubled in rank, and the same will happen to any projection over $C(\mathbb{S}_\omega^{2n-1})$ when E_α is applied. Now, any matrix over $\mathfrak{R}_\omega^{2n}$ is a path of matrices over $C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2$, so the K_0-class of the images remains constant along the path. Based on the previous computations, if $P \in M_k(\mathfrak{R}_\omega^{2n})$ is viewed as a path into $M_k(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2)$, then there is the following restriction on the ranks of $P(t)$.

$$\begin{aligned}
2 \cdot \mathrm{Rank}_{C(\mathbb{S}_\omega^{2n-1})}(P(0)) &= \mathrm{Rank}_{C(\mathbb{S}_\omega^{2n-1})}(E_\alpha(P(0))) \\
&= \mathrm{Rank}_{C(\mathbb{S}_\omega^{2n-1})}(E_\alpha(P(1))) \\
&= \mathrm{Rank}_{\delta=1}(P(1)) + \mathrm{Rank}_{\delta=-1}(P(1))
\end{aligned}$$

Note that the ranks $\mathrm{Rank}_{\delta=\pm 1}$ are only defined for projections over $\mathbb{C} + \mathbb{C}\delta$; in general we cannot evaluate δ to a number on the entire crossed product $C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2$, as δ is a symbol encoding noncommutativity information. Regardless, the above equation limits the range of υ in (4.2.9).

$$(l, m, n) \in \mathrm{Ran}(\upsilon) \implies 2l = m + n \tag{4.2.10}$$

The first clear projection in $\mathrm{Ran}(\upsilon)$ is the identity element, which produces the predictable ranks $(1, 1, 1)$. The second is based on adjusting coefficients from a projection over θ-

deformed even spheres. It is easiest to write this projection in terms of the generators $z_1, \ldots, z_n, x$, using the formal $*$-monomial matrix $Z_\omega(n)$ described in [33] and in Section 3.2.

$$P = \frac{1}{2} I_{2^n} + \frac{1}{2} \begin{bmatrix} xI_{2^{n-1}} & Z_\omega(n) \\ Z_\omega(n)^* & xI_{2^{n-1}} \end{bmatrix} \tag{4.2.11}$$

When viewed as a path, the projection P has $P(0) = \frac{1}{2} \begin{bmatrix} I_{2^{n-1}} & Z_\omega(n) \\ Z_\omega(n)^* & I_{2^{n-1}} \end{bmatrix}$, which has rank 2^{n-1} in $K_0(C(\mathbb{S}_\omega^{2n-1}))$, and $P(1) = \frac{1+\delta}{2} I_{2^n}$, which has rank 2^n at $\delta = 1$ and rank 0 at $\delta = -1$. Therefore, P produces the element $(2^{n-1}, 2^n, 0) \in \mathrm{Ran}(\upsilon)$. The tuples $(1,1,1)$ and $(2^{n-1}, 2^n, 0)$ are independent, so their span has free rank 2, but for large n the second tuple is not in reduced form, meaning we cannot tell if $2l = m + n$ completely characterizes elements of $\mathrm{Ran}(\upsilon)$ (that is, if $\mathrm{Ran}(\upsilon)$ is a full or proper subset of $\mathrm{span}_{\mathbb{Z}}\{(1,1,1),(1,2,0)\}$).

$$\mathrm{span}_{\mathbb{Z}}\{(1,1,1), 2^{n-1}(1,2,0)\} \leq \mathrm{Ran}(\upsilon) \leq \mathrm{span}_{\mathbb{Z}}\{(1,1,1),(1,2,0)\}$$

The above containments show that $\mathrm{Ran}(\upsilon)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, so $K_0(\mathfrak{R}_\omega^{2n})$ has free rank at least 2 and is not generated by trivial projections alone. An examination of the maps η and γ will show that $K_0(\mathfrak{R}_\omega^{2n})$ is actually isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. First, note that $K_1(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2) \cong K_1(C(\mathbb{RP}^{2n-1})) \cong \mathbb{Z}$ is generated by $Z_\omega(n)$, which is again seen through E_α. The expansion $E_\alpha(Z_\omega(n))$ is of index 2 because it is unitarily equivalent to $Z_\omega(n) \oplus Z_\omega(n)$, and by Proposition 4.2.7, any matrix in $\mathrm{Ran}(E_\alpha)$ satisfies $\alpha_U(M) = M$ for an order 2 unitary over $\mathbb{C}$. By Theorem 3.3.10, any invertible matrix in $\mathrm{Ran}(E_\alpha)$ will correspond to an even integer in $K_1(C(\mathbb{S}_\omega^{2n-1}))$, so an index 2 matrix in $K_1(\mathrm{Ran}(E_\alpha)) \cong K_1(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}$ is a generator. Since E_α is an isomorphism between $C(\mathbb{S}_\omega^{2n-1} \rtimes_\alpha \mathbb{Z}_2)$ and $\mathrm{Ran}(E_\alpha)$, this implies that $K_1(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2)$ is generated by $Z_\omega(n)$. The connecting map η between $K_1(C(\mathbb{S}_\omega^{2n-1}) \oplus (\mathbb{C} + \mathbb{C}\delta)) \cong K_1(C(\mathbb{S}_\omega^{2n-1}))$ and $K_0(S(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2))$ from the six term sequence (see [3], V.1.2.12 for the general form of the connecting map $K_1(A/J) \to K_0(J)$) mimics the form of the isomorphism $K_1(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2) \to K_0(S(C(\mathbb{S}_\omega^{2n-1} \rtimes_\alpha \mathbb{Z}_2)))$ of Bott periodicity (see [41], 6.1.2), but now that we know $K_1(C(\mathbb{S}_\omega^{2n-1}) \rtimes_\alpha \mathbb{Z}_2)$ is generated by a unitary over $C(\mathbb{S}_\omega^{2n-1})$ alone, we can conclude that η is surjective. Therefore γ is the zero map and υ is injective, so $K_0(\mathfrak{R}_\omega^{2n})$ is isomorphic to $\mathrm{Ran}(\upsilon) \cong \mathbb{Z} \oplus \mathbb{Z}$. This means an analogous question to Theorem 3.4.4 can be asked on this group. Even though the K_0 groups of $\mathfrak{R}_\omega^{2n}$ and $C(\mathbb{S}_\omega^{2n})$ are abstractly isomorphic, the generators are different in that K_0 data of $\mathfrak{R}_\omega^{2n}$ is contained in the possibly distinct ranks $\mathrm{Rank}_{\delta=1}$ and $\mathrm{Rank}_{\delta=-1}$, whereas $K_0(C(\mathbb{S}_\omega^{2n}))$ inherits its K-theory from $K_0(C(\mathbb{S}^{2n}))$, which has one summand for rank and another for a nontrivial vector bundle. Despite all of this, the form of the generator P in (4.2.11) is just a slight sign change from a projection over $C(\mathbb{S}_\omega^{2n})$ seen in the proof of Corollary 3.4.5!

The group $K_1(\mathfrak{R}_\omega^{2n})$ has not been completely determined, but we can still obtain some information. Since η is a nontrivial homomorphism between $\mathbb{Z}$ and $\mathbb{Z}$, it is injective, so λ is the zero map. The images of σ and κ are not clear from this computation, but it is clear from $\mathrm{Ran}(\upsilon)$ that $\mathrm{Ran}(\sigma)$ includes a copy of $\mathbb{Z}$, and therefore $\mathrm{Ran}(\kappa) \cong K_1(\mathfrak{R}_\omega^{2n})$ is a torsion group (possibly trivial). These results are summarized in the following proposition.

Proposition 4.2.12. The group $K_0(\mathfrak{R}_\omega^{2n})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and if matrix projections $P \in M_k(\mathfrak{R}_\omega^{2n})$ are considered as paths from (4.2.5), the K_0 data is obtained from the two ranks $\mathrm{Rank}_{\delta=\pm 1}(P(1))$. On the other hand, $K_1(\mathfrak{R}_\omega^{2n})$ is either trivial or a torsion group.

It is sensible to define a universal algebra that allows x to anticommute with some z_j and commute with other z_j, but the K-theory computations are not as obvious, as the corresponding isomorphism to a path algebra such as (4.2.5) would include a crossed product by an action that is not saturated. On the other hand, it is possible that sufficient information could be obtained by θ-deforming the algebra so that $z_1, \ldots, z_n$ all commute with each other, so now certain z_j are central, as they also commute with x. Presence of central generators suggests that these algebras might arise from an even number of unreduced suspensions of

$\mathfrak{R}^{2k}$, where $k < n$ is the number of z_j which anticommute with x. However, I will not attempt to carry this pursuit out here, as later Borsuk-Ulam results concern only $\mathfrak{R}^{2n}_\omega$ or may be proved in such a way that does not benefit from those ideas.

4.3 Noncommutative Suspensions and $\mathbb{Z}_k$-Joins

Dąbrowski's Conjecture 4.1.2 concerns the unreduced suspension ΣA, which has A as a quotient, but it also includes a self-adjoint, central element from the function $\mathrm{Id}(t) = t$. Consequently, our proof of the conjecture reduced to the classical case of the Borsuk-Ulam theorem by examining endpoint data for iterated unreduced suspensions. The relation algebra $\mathfrak{R}^{2n}_\omega$ of the previous section is the result of adjoining a self-adjoint coordinate that anticommutes with the generators of $C(\mathbb{S}^{2n-1}_\omega)$, and it was isomorphic to the path algebra

$$\Sigma^\alpha C(\mathbb{S}^{2n-1}_\omega) := \{f \in C([0,1], C(\mathbb{S}^{2n-1}_\omega) \rtimes_\alpha \mathbb{Z}_2) : f(0) \in C(\mathbb{S}^{2n-1}_\omega), f(\pm 1) \in \mathbb{C} + \mathbb{C}\delta\}$$

where α is the antipodal action on $C(\mathbb{S}^{2n-1}_\omega)$. In a sense, this is a noncommutative unreduced suspension of $C(\mathbb{S}^{2n-1}_\omega)$, and the idea can be generalized to any unital C^*-algebra.

Definition 4.3.1. If A is a unital C^*-algebra with a $\mathbb{Z}_2$ action β, then the *noncommutative unreduced suspension* of (A, β) is defined as follows.

$$\Sigma^\beta A := \{f \in C([0,1], A \rtimes_\beta \mathbb{Z}_2) : f(0) \in A, f(\pm 1) \in \mathbb{C} + \mathbb{C}\delta\}$$

Remark. This noncommutative unreduced suspension does not appear to relate to the noncommutative double unreduced suspension in [24].

When β is the trivial action, Σ^{triv} is isomorphic to the usual unreduced suspension, where the crossed product encodes homogeneity classes of functions. Specifically, $\Sigma A \cong \Sigma^{\text{triv}} A$ via $f(t) \mapsto \frac{f(t)+f(-t)}{2} + \frac{f(t)-f(-t)}{2}\delta$, where we note that while $f(t)$ is defined on $[-1,1]$, the image functions are only defined on $[0, 1]$. To form the inverse, start with an element $h+k\delta \in \Sigma^{\mathrm{triv}} A$, let H denote the even extension of h, and let K denote the odd extension of k. Then the inverse image of $h + k\delta$ is $H + K$.

For the usual unreduced suspension, any $\mathbb{Z}_2$ action on A could be extended to ΣA in a natural way, by applying the action pointwise and composing with $t \mapsto -t$. For $\Sigma^\beta A$, we may extend an action γ only if it commutes with β, and in this case we apply γ pointwise on $A \rtimes_\beta \mathbb{Z}_2$ and compose with $\widehat{\beta}$, which negates the group element δ. The new action is denoted $\mathfrak{c}$.

$$\mathfrak{c} : f(t) + g(t)\delta \mapsto \gamma(f(t)) - \gamma(f(t))\delta$$

Removing the component $t \mapsto -t$ (which relied on the domain being $[-1, 1]$) suggests that this idea could be applied more generally for a larger class of actions, and indeed this idea extends naturally to the join by finite cyclic groups. If X is a compact Hausdorff space on which a topological group G acts, then the noncommutative join of [2] is motivated by the following identification.

$$\begin{aligned} C(X * G) &\cong \{f \in C([0, 1], C(X \times G)) : f(0) \text{ is } G\text{-independent}, f(1) \text{ is } X\text{-independent}\} \\ &\cong \{f \in C([0, 1], C(X) \otimes C(G)) : f(0) \in C(X), f(1) \in C(G)\} \end{aligned}$$

A different noncommutative join for finite cyclic groups may be developed from realizing $C(X) \otimes C(\mathbb{Z}_k)$ as a trivial crossed product.

$$C(X * \mathbb{Z}_k) \cong \{f \in C([0, 1], C(X) \rtimes_{\mathrm{triv}} \mathbb{Z}_k) : f(0) \in C(X), f(1) \in C^*(\mathbb{Z}_k)\}$$

Definition 4.3.2. Let A be a C^*-algebra with a $\mathbb{Z}_k$ action β. Then the *noncommutative join* $A *_\beta \mathbb{Z}_k$ is defined as follows.

$$A *_\beta \mathbb{Z}_k := \{f \in C([0, 1], A \rtimes_\beta \mathbb{Z}_k) : f(0) \in A, f(1) \in C^*(\mathbb{Z}_k)\}$$

An action γ of $\mathbb{Z}_k$ on A which commutes with β may be extended to $A *_\beta \mathbb{Z}_k$ by applying γ pointwise and also applying the dual action of β. Since the Pontryagin dual of $\mathbb{Z}_k$ is isomorphic to $\mathbb{Z}_k$ in a non-canonical way, we fix a primitive kth root of unity $\omega = e^{2\pi i/k}$. An element of $A * \mathbb{Z}_k$ may be viewed as $f_0 + f_1\delta + \ldots + f_{k-1}\delta^{k-1}$ where $f_1, \ldots, f_n$ are functions from $[0,1]$ into A. Then we extend γ to $A *_\alpha \mathbb{Z}_k$ as $\mathfrak{c}$, just as in the $\mathbb{Z}_2$ case.

$$\mathfrak{c}(f_0 + f_1\delta + \ldots + f_{k-1}\delta^{k-1})(t) := \gamma(f_0(t)) + \omega\gamma(f_1(t))\delta + \ldots + \omega^{k-1}\gamma(f_{k-1}(t))\delta^{k-1}$$

The isomorphism $C(\mathbb{Z}_k) \cong C^*(\mathbb{Z}_k)$ may be written in such a way that the dual action of $\widehat{\mathbb{Z}_k} \cong \mathbb{Z}_k$ on the group C^*-algebra is equivalent to the translation action of $\mathbb{Z}_k$ on its algebra of functions. We ask the following question in analogy with [2] and [14].

Question 4.3.3. Suppose A is a unital C^*-algebra with two commuting $\mathbb{Z}_k$ actions β and γ. If β is unsaturated and γ is saturated, must there not exist any unital, $(\gamma, \mathfrak{c})$-equivariant $*$-homomorphisms from A to $A *_\beta \mathbb{Z}_k$?

When β is the trivial action, the techniques of Theorem 4.1.13 give a positive answer, but we are interested in the case where β is nontrivial. Any odd θ-deformed sphere $C(\mathbb{S}^{2n-1}_\rho)$ admits $\mathbb{Z}_k$ actions which rotate the coordinates $z_1, \ldots, z_n$, so the generator for an action β could map z_i to $\omega_i z_i$ where the ω_i are kth roots of unity, not all of which need to be primitive. If γ denotes another such action where the roots are primitive, then γ is a saturated action, β and γ commute, and γ extends to an action $\mathfrak{c}$ on $C(\mathbb{S}^{2n-1}_\rho) *_\alpha \mathbb{Z}_k$.

Example 4.3.4. Let $C(\mathbb{S}^{2n-1}_\rho)$ be equipped with coordinate rotations β and γ, as above. Then there is no $(\gamma, \mathfrak{c})$-equivariant, unital $*$-homomorphism from $C(\mathbb{S}^{2n-1}_\rho)$ to the noncommutative join $C(\mathbb{S}^{2n-1}_\rho) *_\beta \mathbb{Z}_k$.

Proof. If such a homomorphism existed, then evaluate at $t = 0$ to form a γ-equivariant map ψ from $C(\mathbb{S}^{2n-1}_\rho)$ to itself. The image of the K_1 generator $Z_\rho(n)$ under ψ is a matrix over

$C(\mathbb{S}_\rho^{2n-1})$ which is trivial in $K_1(C(\mathbb{S}_\rho^{2n-1}) \rtimes_\beta \mathbb{Z}_k)$ by following a path from $t=0$ to $t=1$. The endpoint $t=1$ corresponds to an invertible matrix with entries in $C^*(\mathbb{Z}_k)$, which is K_1-trivial. However, $K_1(C(\mathbb{S}_\rho^{2n-1}))$ maps injectively into $K_1(C(\mathbb{S}_\rho^{2n-1}) \rtimes_\beta \mathbb{Z}_k)$, so $\psi(Z_\rho(n))$ is trivial in $K_1(C(\mathbb{S}_\rho^{2n-1}))$ as well. It follows that ψ is a K_1-trivial map, contradicting Theorem 3.3.11. □

In the above example, a homomorphism to the join did not exist because it would trivialize a certain algebraic invariant, K_1. However, the proof method of Theorem 4.1.13 and Theorem 4.1.9 is to iterate a map through higher suspensions, a technique which is not applicable to Question 4.3.3 by design. In particular, we require β to be unsaturated to rule out some simple counterexamples. If $\gamma = \beta$ were saturated, one could attempt to prove a version of Question 4.3.3 by using iteration techniques, which would suggest examining iterated joins of $C^*(\mathbb{Z}_2)$ by $\mathbb{Z}_2$ in analogy with the commutative case. This method will surely fail.

Example 4.3.5. Let B_n be the universal, unital C^*-algebra generated by self-adjoint elements $x_1, \ldots, x_{n+1}$ which pairwise anticommute and satisfy $x_1^2 + \ldots + x_{n+1}^2 = 1$. Equip B_n with a $\mathbb{Z}_2$ action β that negates each generator. Then there is an odd, self-adjoint, unitary element $x_1 + \ldots + x_{n+1}$, and therefore there is also an equivariant homomorphism from $B_0 = C(\mathbb{S}^0)$ to B_n. Composing the equivariant map $B_0 \to B_1$ with the map $B_1 \to B_0 *_\beta \mathbb{Z}_2$ guaranteed by the universal property gives an equivariant map $B_0 \to B_0 *_\beta \mathbb{Z}_2 = \Sigma^\beta B_0$.

The assumption that β and γ have different saturation properties removes the above pathological case, which is similar to the counterexample used in Theorem 2.4.8 for purely element-based Borsuk-Ulam theorems on $C(\mathbb{S}_\rho^{2n-1})$. Question 4.3.3 (in the $\mathbb{Z}_2$ case) is an attempt at removing the commutative spheres lurking in the background of any equivariant map $A \to \Sigma A$ found when resolving Dąbrowski's conjecture, in pursuit of an algebraic invariant that must behave nontrivially.

4.4 A K_0 Borsuk-Ulam Theorem

The algebra $\mathfrak{R}^{2n}_\omega$ may be obtained as the noncommutative suspension $\Sigma^\alpha C(\mathbb{S}^{2n-1}_\omega)$ of $C(\mathbb{S}^{2n-1}_\omega)$, where α is the antipodal action on $C(\mathbb{S}^{2n-1}_\omega)$. The antipodal action on $\mathfrak{R}^{2n}_\omega$, which negates each generator and will be denoted $\mathfrak{a}$, comes from the composition of α and $\widehat{\alpha}$ pointwise on $\Sigma^\alpha C(\mathbb{S}^{2n-1}_\omega)$. This action behaves well with respect to the projection

$$P_\omega(n) = \frac{1}{2} I_{2^n} + \frac{1}{2} \begin{bmatrix} x I_{2^{n-1}} & Z_\omega(n) \\ Z_\omega(n)^* & x I_{2^{n-1}} \end{bmatrix}$$

in that $\mathfrak{a}(P_\omega(n)) = I - P_\omega(n)$. Based on the results for θ-deformed spheres, we can ask if $2^n \times 2^n$ projections satisfying this identity are always nontrivial in K_0. This is evident for $P_\omega(n)$ from the ranks $\mathrm{Rank}_{\delta=\pm 1}$, which take different values, as the endpoint evaluation of $P_\omega(n)$ at $t = 1$ is $\frac{1+\delta}{2} I_{2^n}$. This nontriviality does hold in general, and the proof requires study of a non-free $\mathbb{Z}_2$ action on commutative spheres $C(\mathbb{S}^k)$.

Theorem 4.4.1. Let the commutative sphere $C(\mathbb{S}^k)$ be equipped with a $\mathbb{Z}_2$ action γ that fixes a distinguished real-valued coordinate h but negates the remaining coordinates $x_1, \ldots, x_k$, and let U be a unitary matrix over $C(\mathbb{S}^k)$ with $\gamma(U) = U^*$ and $U|_{h=\pm 1} = I$. Then the following hold.

1. If U represents a trivial element in $K_1(C(\mathbb{S}^k))$, then there is a path connecting $U \oplus I$ to I within the unitaries that satisfy $\gamma(M) = M^*$ and $M|_{h=\pm 1} = I$.

2. If k is odd, then U corresponds to an even integer in $K_1(C(\mathbb{S}^k)) \cong \mathbb{Z}$.

Proof. We proceed by induction, noting that the only applicable unitaries over $C(\mathbb{S}^0)$ are the identity matrices, so the claim at dimension 0 is automatically satisfied. If the claim holds for k, then suppose U is a unitary matrix over $C(\mathbb{S}^{k+1})$ with $\gamma(U) = U^*$ and $U|_{h=\pm 1} = I$.

Let $\mathbb{S}^k$ denote the equator in $\mathbb{S}^{k+1}$ determined by $x_k = 0$ (x_k is a coordinate negated by γ), so that the restriction $\widetilde{\gamma}$ of γ to $C(\mathbb{S}^k)$ is an action of the same type: it fixes h and negates $x_1, \ldots, x_{k-1}$. Also note that the fixed points $h = \pm 1$ are in this equator $\mathbb{S}^k$. To use the inductive hypothesis, first form a path of unitaries that "stretches" the equator data of U. Realize $C(\mathbb{S}^{k+1})$ as the unreduced suspension $\Sigma C(\mathbb{S}^k)$, with x_k representing the path coordinate in $[-1, 1]$. Then $U = U(x_k)$ represents a path of unitary matrices over $C(\mathbb{S}^k)$, and we form a continuous path U_t as follows.

$$U_t(x_k) = \begin{cases} U\left(\frac{-2}{t-2}(x_k+1)-1\right) & \text{if} \quad -1 \leq x_k \leq \frac{-t}{2} \\ U(0) & \text{if} \quad \frac{-t}{2} \leq x_k \leq \frac{t}{2} \\ U\left(\frac{-2}{t-2}(x_k-1)+1\right) & \text{if} \quad \frac{t}{2} \leq x_k \leq 1 \end{cases}$$

This path connects $U = U_0$ to $V = U_1$, which has its equator data repeated on a band neighborhood $-\frac{1}{2} \leq x_k \leq \frac{1}{2}$. The path U_t also satisfies $\gamma(U_t) = U_t^*$ and still assigns the identity on $h = \pm 1$, as these points are in the equator $x_k = 0$. The equator function $U(0) = V(0)$ is trivial in $K_1(C(\mathbb{S}^k))$ because $\mathbb{S}^k$ sits inside a contractible subset of $\mathbb{S}^{k+1}$; it also satisfies $\widetilde{\gamma}(U(0)) = U(0)^*$ and sends the fixed points at $h = \pm 1$ to the identity matrix. By the inductive hypothesis, there is a path W_t connecting $W_0 = U(0) \oplus I$ to $W_1 = I$ within the unitaries over $C(\mathbb{S}^k)$ that also satisfy $\widetilde{\gamma}(W_t) = W_t^*$ and assign the fixed points $h = \pm 1$ to the identity matrix. Apply this path to the equator of $V \oplus I$ while maintaining the same values on the path for $|x_k| \geq \frac{1}{2}$.

$$V_t(x_k) = \begin{cases} W_{\phi_t(x_k)} & \text{if} \quad |x_k| \leq \frac{1}{2} \\ V(x_k) \oplus I & \text{if} \quad \frac{1}{2} \leq |x_k| \leq 1 \end{cases}$$

$$\phi_t(x_k) = \begin{cases} -2|x_k| + t & \text{if} \quad |x_k| \leq \frac{t}{2} \\ 0 & \text{if} \quad \frac{t}{2} \leq |x_k| \leq \frac{1}{2} \end{cases}$$

The path V_t connects $V_0 = V \oplus I$ to V_1, where the unitaries V_t still satisfy $\gamma(V_t) = V_t^*$, and by the inductive assumption, the fixed points $h = \pm 1$ of γ are still always assigned the identity. Because V_1 also assigns the identity matrix on the entire equator, V_1 is the commuting product of two unitary matrices F and G, where F assigns the identity matrix for $x_k \geq 0$, G assigns the identity matrix for $x_k \leq 0$, and $\gamma(F^*) = G$. How to proceeed is slightly different based on the parity of $k+1$.

If $k+1$ is even, then $K_1(C(\mathbb{S}^{k+1}))$ is the trivial group, so let F_t denote a path of unitaries connecting $F_0 = F \oplus I$ to I. Moreover, we can insist that F_t always assigns the identity matrix on points with $x_k \geq 0$, as this region is contractible to a point and $K_1(C(\mathbb{S}^{k+1} \setminus \{\text{pt}\}))$ is also trivial. Then the commuting product $Q_t = F_t \cdot \gamma(F_t^*)$ forms a path of unitaries that connects $V_1 \oplus I$ to I, satisfies $\gamma(Q_t) = Q_t^*$, and assigns the identity matrix at (at least) $h = \pm 1$. Tracing back all of the paths used so far shows that $U \oplus I$ is connected to I within the same set of restricted unitaries.

If $k+1$ is odd, then $K_1(C(\mathbb{S}^{k+1})) \cong \mathbb{Z}$, and as γ and the adjoint are both orientation-reversing, F and $G = \gamma(F^*)$ represent the same element in $K_1(C(\mathbb{S}^{k+1}))$. The product $V_1 = F \cdot \gamma(F^*)$ shows that the class of V_1 (and therefore of U) in K_1 is $2[F]_{K_1}$, an even integer. If this integer is zero, then once again, $K_1(C(\mathbb{S}^{k+1} \setminus \{\text{pt}\}))$ is isomorphic to $K_1(C(\mathbb{S}^{k+1}))$, so just as in the previous paragraph, the trivial element $F \oplus I$ can be connected to I in a path of unitaries F_t that always assign the identity on points with $x_k \geq 0$. Finally, the commuting product $Q_t = F_t \cdot \gamma(F_t^*)$ connects $V_1 \oplus I$ to I, satisfies $\gamma(Q_t) = Q_t^*$, and always send the points $h = \pm 1$ to the identity matrix. A composition of paths establishes the same for U and completes the induction. $\square$

In the above theorem, the set of matrices M satisfying $\gamma(M) = M^*$ is not a C^*-subalgebra of the matrix algebra over $C(\mathbb{S}^k)$, as the adjoint operation reverses the order of multiplication. As such, some of the ideas in the proof are motivated by the six term exact sequence, but cannot be implemented this way. Mention of function values at the fixed points $h = \pm 1$

appears to be unnecessary on first glance, but this is crucial even at low dimensions.

Example 4.4.2. Let $C(\mathbb{S}^1)$ be generated by the complex coordinate $z = x + iy$ with a $\mathbb{Z}_2$ action γ that negates y but fixes x. Then $\gamma(z) = z^*$, but z is the generator of $K_1(C(\mathbb{S}^1))$, associated to the odd integer 1. The fixed points are $x = 1$ and $x = -1$, and z assigns these points to ± 1. The previous theorem does not apply, as it only considers matrices which assign the identity matrix on the fixed points of γ. The same scenario happens in any $C(\mathbb{S}^{2n-1})$ for the standard K_1 generator $Z(n)$.

Theorem 4.4.1 concerns an action γ that is not free, but it will be instrumental in defining an invariant for unitaries that satisfy $\alpha(U) = U^*$, where α is the antipodal map on $C(\mathbb{S}^{2n-1})$. In turn, this invariant will show that the relation algebras $\mathfrak{R}^{2n}_\omega$ have a Borsuk-Ulam property for projections in K_0, by first proving the case when $z_1, \ldots, z_n$ all commute with each other.

Definition 4.4.3. Fix an isomorphism $C(\mathbb{S}^k) \cong \Sigma C(\mathbb{S}^{k-1})$, with the path coordinate denoted by h. If $M \in M_p(C(\mathbb{S}^k))$ is such that $M(0) \in M_p(\mathbb{C})$ (i.e., it has constant entries), then let $M^+ \in M_p(C(\mathbb{S}^k)) \cong M_p(\Sigma C(\mathbb{S}^{k-1}))$ be defined as follows for $h \in [-1, 1]$.

$$M^+(h) = M\left(\frac{h+1}{2}\right)$$

The matrix M^+ encodes information about M only for points $h \geq 0$. Note in particular that M is also only defined in reference to a particular, fixed coordinate h and a suppressed isomorphism with an unreduced suspension. We do not expect that any M^+ that is definable with respect to multiple coordinate choices has any uniqueness properties of any kind.

Theorem 4.4.4. Suppose $U_0, U_1 \in U_k(C(\mathbb{S}^{2n-1}))$ are unitary matrices which assign an equator $h = 0$ to the identity matrix and satisfy $\alpha(U_j) = U_j^*$. If there is a path of unitary matrices U_t connecting U_0 and U_1 that satisfy $\alpha(U_t) = U_t^*$ (but have no restriction on the equator), then the classes of U_0^+ and U_1^+ in $K_1(C(\mathbb{S}^{2n-1})) \cong \mathbb{Z}$ differ by an even integer.

Proof. Fix an isomorphism $C(\mathbb{S}^{2n-1}) \cong \Sigma C(\mathbb{S}^{2n-2})$, with the path coordinate specified by h, and let γ be a $\mathbb{Z}_2$ action on $C(\mathbb{S}^{2n-1})$ that fixes h but applies the antipodal map pointwise on $C(\mathbb{S}^{2n-2})$. Define $V_t \in U_k(C(\mathbb{S}^{2n-1}))$ as follows.

$$V_t(h) = \begin{cases} U_{-h}(0) & \text{if } \; -1 \leq h \leq -t \\ U_t(\frac{h-1}{1+t}+1) & \text{if } \; -t \leq h \leq 1 \end{cases}$$

Note that $V_t(-1) = U_1(0)$ is always the identity matrix and $V_t(1) = U_t(1)$ is a constant matrix, meeting the necessary boundary conditions to define a unitary over the sphere. Further, V_1 is equal to U_1^+, and V_0 contains data of U_0^+ with equator information of U_t.

$$V_0(h) = \begin{cases} U_{-h}(0) & \text{if } \; -1 \leq h \leq 0 \\ U_0(h) & \text{if } \; 0 \leq h \leq 1 \end{cases}$$

Since $U_0(0)$ is the identity matrix, this formula shows that V_0 can be written as a commuting product of two unitaries $F \cdot G$, as follows.

$$F(h) = \begin{cases} I & \text{if } \; -1 \leq h \leq 0 \\ U_0(h) & \text{if } \; 0 \leq h \leq 1 \end{cases} \qquad \Longrightarrow \qquad [F]_{K_1} = [U_0^+]_{K_1}$$

$$G(h) = \begin{cases} U_{-h}(0) & \text{if } \; -1 \leq h \leq 0 \\ I & \text{if } \; 0 \leq h \leq 1 \end{cases}$$

Now, the path matrices U_t satisfy $\alpha(U_t) = U_t^*$, and the matrix G contains equator data from U_t, so it satisfies $\gamma(G) = G^*$ where γ negates every coordinate in $C(\mathbb{S}^{2n-1})$ except h. That is, γ applies the antipodal map of $C(\mathbb{S}^{2n-2})$ pointwise on $\Sigma C(\mathbb{S}^{2n-2})$. G also assigns the fixed points of γ, $h = \pm 1$, to the identity matrix, so the class of G in $K_1(C(\mathbb{S}^{2n-1}))$ is an even integer by Theorem 4.4.1. The formulas $V_0 = F \cdot G$ and $[F]_{K_1} = [U_0^+]_{K_1}$ imply that

$[V_0]_{K_1}$ is an even integer away from $[U_0^+]_{K_1}$, and $[V_0]_{K_1}$ is the same as $[V_1]_{K_1} = [U_1^+]_{K_1}$. $\square$

The above theorem defines a $\mathbb{Z}_2$ invariant for any unitary over $C(\mathbb{S}^{2n-1})$ satisfying $\alpha(U) = U^*$ which assigns the identity on a specified equator $h = 0$: $[U^+]_{K_1}$ mod 2. This invariant is preserved by paths U_t satisfying $\alpha(U_t) = U_t^*$ where U_0 and U_1 assign the identity on the equator. It is important to note that only the endpoints of the path have specified equator data; if the equators of U_t were also the identity, U_t^+ would be a well-defined path and the K_1 classes of U_0^+ and U_1^+ would be equal, with no need for modding by 2. Some examples of unitary paths with $\alpha(U_t) = U_t^*$ will be necessary for later calculations.

Example 4.4.5. Fix an isomorphism $C(\mathbb{S}^{2n-1}) \cong \Sigma C(\mathbb{S}^{2n-2})$ with path coordinate $x_1 = \mathrm{Re}(z_1)$, and consider the standard K_1 generator $Z(n) \in U_{2^{n-1}}(C(\mathbb{S}^{2n-1}))$. Define a unitary $V \in U_{2^{n-1}}(C(\mathbb{S}^{2n-1}))$ as follows.

$$V(x_1) = -Z(n)[2|x_1| - 1]$$

Now, V has $V(\pm 1) = -Z(n)[1] = -I$ and $V(0) = -Z(n)[-1] = I$, and the antipodal map α on $C(\mathbb{S}^{2n-1})$ comes from applying $x_1 \mapsto -x_1$ and the pointwise antipodal map $\widetilde{\alpha}$ on $C(\mathbb{S}^{2n-2})$.

$$\alpha(V)[x_1] = \widetilde{\alpha}(V(-x_1)) = \widetilde{\alpha}(V(x_1)) = \widetilde{\alpha}(-Z(n)[2|x_1| - 1])$$

The application of $\widetilde{\alpha}$ to $-Z(n)[2|x_1|-1]$ negates $y_1 = \mathrm{Im}(z_1)$ and $z_2, \ldots, z_n$, which produces $(-Z(n)[2|x_1|-1])^*$ by an inductive argument, so $\widetilde{\alpha}(V(x_1)) = V(x_1)^*$ for each x_1. Finally, the above equation implies that $\alpha(V) = V^*$. Since $V(0) = I$ and $V^+ = -Z(n)$, it follows that $[V^+]_{K_1} = 1$. Moreover, V is connected to $-I_{2^{n-1}}$ within the unitaries satisfying $\alpha(M) = M^*$ by considering the following path V_t.

$$V_t(x_1) = V((1-t)|x_1| + t)$$

Now, $V_0(x_1) = V(|x_1|) = V(x_1)$, so $V_0 = V$, and $V_1(x_1) = V(1) = -I_{2^{n-1}}$.

$$\begin{aligned}
\alpha(V_t)[x_1] &= \widetilde{\alpha}(V_t(-x_1)) \\
&= \widetilde{\alpha}(V_t(x_1)) \\
&= \widetilde{\alpha}(V((1-t)|x_1| + t)) \\
&= V((1-t)|x_1| + t)^* \\
&= V_t(x_1)^*
\end{aligned}$$

Finally, V is connected to $-I_{2^{n-1}}$ within the unitary matrices satisfying $\alpha(M) = M^*$.

Example 4.4.6. Fix an isomorphism $C(\mathbb{S}^{2n-1}) \cong \Sigma C(\mathbb{S}^{2n-2})$ with path coordinate x_1. Suppose $U_0 \in U_k(C(\mathbb{S}^{2n-1}))$ is anti self-adjoint ($U_0^* = -U_0$) and odd ($\alpha(U_0) = -U_0$), which implies that $\alpha(U_0) = U_0^*$. Define a path U_t as follows.

$$U_t(x_1) = \begin{cases}
\widetilde{\alpha}(U_0(\frac{2}{2-t}(-x_1 - 1) + 1)^*) & \text{if} \quad -1 \leq x_1 \leq -\frac{t}{2} \\
(t + 2x_1)I + \sqrt{1 - (t + 2x_1)^2}\, \widetilde{\alpha}(U_0(0)^*) & \text{if} \quad -\frac{t}{2} \leq x_1 \leq 0 \\
(t - 2x_1)I + \sqrt{1 - (t - 2x_1)^2}\, U_0(0) & \text{if} \quad 0 \leq x_1 \leq \frac{t}{2} \\
U_0(\frac{2}{2-t}(x_1 - 1) + 1) & \text{if} \quad \frac{t}{2} \leq x_1 \leq 1
\end{cases}$$

Note that $U_0(0)$ is anti self-adjoint and odd under the antipodal action $\widetilde{\alpha}$ on $C(\mathbb{S}^{2n-2})$, so U_t is well-defined and unitary with $\alpha(U_t) = U_t^*$. Let $U_1 = W_0$ and define another path of unitaries W_t which have $\alpha(W_t^*) = W_t$ and also assign the equator to I.

$$W_t(x_1) = \begin{cases}
\widetilde{\alpha}(U_0(-2x_1 - 1)^*) & \text{if} \quad -1 \leq x_1 \leq -\frac{1+t}{2} \\
(1 + \frac{2}{1+t}x_1)I + \sqrt{1 - (1 + \frac{2}{1+t}x_1)^2}\, \widetilde{\alpha}(U_0(t)^*) & \text{if} \quad -\frac{1+t}{2} \leq x_1 \leq 0 \\
(1 - \frac{2}{1+t}x_1)I + \sqrt{1 - (1 - \frac{2}{1+t}x_1)^2}\, U_0(t) & \text{if} \quad 0 \leq x_1 \leq \frac{1+t}{2} \\
U_0(2x_1 - 1) & \text{if} \quad \frac{1+t}{2} \leq x_1 \leq 1
\end{cases}$$

Note that the formula defining W_t produces a unitary matrix because each $U_0(t)$ is anti self-adjoint, as is each $\widetilde{\alpha}(U_0(t)^*)$. Finally, $W_1(0) = I$, and because $U_0(1)$ has scalar entries, $W_1(x_1)$ has scalar entries for each x_1, which implies that $[W_1^+]_{K_0} = 0$.

Consider the relation algebra $\mathfrak{R}^{2n} \cong \Sigma^\alpha C(\mathbb{S}^{2n-1})$, in which $z_1, \ldots, z_n$ commute with each other. Because the antipodal map $\mathfrak{a}$ is implemented as $\alpha\widehat{\alpha}$ pointwise on $C(\mathbb{S}^{2n-1}) \rtimes_\alpha \mathbb{Z}_2$, an element or matrix $F(t) + G(t)\delta$ over $\mathfrak{R}^{2n}_\rho$ is odd if and only if each $F(t)$ is odd in $C(\mathbb{S}^{2n-1})$ and each $G(t)$ is even in $C(\mathbb{S}^{2n-1})$, which implies that $G(t)$ commutes with δ and $F(t)$ anticommutes with δ. An odd matrix $F(t) + G(t)\delta$ is then self-adjoint if and only if $F(t)$ and $G(t)$ are self-adjoint. If $F(t)+G(t)\delta$ is also unitary, then it satisfies $(F(t)+G(t)\delta)^2 = 1$, which places tight restrictions on F and G.

$$\begin{aligned}(F(t) + G(t)\delta)^2 &= F(t)^2 + (G(t)\delta)^2 + F(t)G(t)\delta + G(t)\delta F(t) \\ &= (F(t)^2 + G(t)^2) + (F(t)G(t) - G(t)F(t))\delta \\ &= 1 + 0\delta\end{aligned} \tag{4.4.7}$$

This implies that the self-adjoint elements $F(t)$ and $G(t)$ must commute and satisfy $F(t)^2 + G(t)^2 = 1$. In other words, $U(t) = G(t) + iF(t)$ is a unitary matrix over $C(\mathbb{S}^{2n-1})$ for each t, and because $G(t)$ is even and $F(t)$ is odd, $\alpha(U(t)) = U(t)^*$.

Theorem 4.4.8. Let α denote the antipodal action on the commutative sphere $C(\mathbb{S}^{2n-1})$ and consider the path algebra $\mathfrak{R}^{2n} = \Sigma^\alpha C(\mathbb{S}^{2n-1})$ with antipodal action $\mathfrak{a} = \alpha\widehat{\alpha}$. If $P \in M_{(2m+1)2^n}(\mathfrak{R}^{2n})$ is a projection with $\mathfrak{a}(P) = I - P$, then $[P]_{K_0}$ is not in the subgroup generated by trivial bundles.

Proof. Write $P = \frac{1}{2}I + \frac{1}{2}B$ where B is an odd, self-adjoint unitary matrix. By the calculation (4.4.7), $B(t) = F(t) + G(t)\delta$ produces a path of unitaries $U(t) = G(t) + iF(t) \in U_{(2m+1)2^n}(C(\mathbb{S}^{2n-1}))$ with $\alpha(U(t)) = U(t)^*$. Since $B(1)$ is a matrix over $\mathbb{C} + \mathbb{C}\delta$, but $F(1)$ must be odd, it follows that $F(1) = 0$ and $B(1) = G(1)\delta$ where $G(1)$ is a self-adjoint matrix with constant entries. Moreover, the boundary conditions of $\Sigma^\alpha C(\mathbb{S}^{2n-1})$ imply that

$G(0) = 0$, so $U(0) = iF(0)$ is an anti self-adjoint odd unitary. If $[P]_{K_0}$ is in the subgroup generated by trivial bundles, then the ranks $\mathrm{Rank}_{\delta=\pm 1}(P(1))$ are equal, so $U(1) = G(1)$ is a self-adjoint, unitary matrix over $\mathbb{C}$ whose eigenspaces for eigenvalues ± 1 are of equal dimension. It follows that $U(1)$ may be connected within the self-adjoint, unitary matrices over $\mathbb{C}$ to $I_{(2m+1)2^{n-1}} \oplus -I_{(2m+1)2^{n-1}}$, and the matrices in this path satisfy $\alpha(M) = M = M^*$.

Because $U(0)$ is an anti self-adjoint, odd, unitary matrix, by Example 4.4.6, $U(0)$ is connected via a path of unitaries satisfying $\alpha(M) = M^*$ to a matrix W with identity on the equator and $[W^+]_{K_1} \cong 0 \bmod 2$. Similarly, since $U(1)$ is connected via a path of unitaries satisfying $\alpha(M) = M^*$ to $I_{(2m+1)2^{n-1}} \oplus -I_{(2m+1)2^{n-1}}$, repeated use of Example 4.4.5 for $(2m+1)$ summands of $-I_{2^{n-1}}$ shows $U(1)$ is also connected to a matrix V such that V assigns the identity on the equator and $[V^+]_{K_1} \cong (2m+1) \cong 1 \bmod 2$. This contradicts Theorem 4.4.4, as V and W assign the identity on the equator and are connected via a path of unitaries satisfying $\alpha(M) = M^*$ (with no assumption on the path's equator data), even though their invariants $[V^+]_{K_1} \bmod 2$ and $[W^+]_{K_1} \bmod 2$ are different. $\square$

All that remains is to remove the assumption that $z_1, \ldots, z_n$ commute with each other. First, note that the expansion map $E_\alpha : C(\mathbb{S}^{2n-1}_\omega) \rtimes_\alpha \mathbb{Z}_2 \to M_2(C(\mathbb{S}^{2n-1}_\omega))$ shows that the relation algebra $\mathfrak{R}^{2n}_\omega = \Sigma^\alpha C(\mathbb{S}^{2n-1}_\omega)$ embeds into $M_2(C(\mathbb{S}^{2n}_\omega))$, as follows.

$$\begin{aligned}
\Sigma^\alpha C(\mathbb{S}^{2n-1}_\omega) &= \{f \in C([0,1], C(\mathbb{S}^{2n-1}_\omega) \rtimes_\alpha \mathbb{Z}_2) : f(0) \in C(\mathbb{S}^{2n-1}_\omega), f(1) \in \mathbb{C} + \mathbb{C}\delta\} \\
&\cong \{f \in C([-1,1], C(\mathbb{S}^{2n-1}_\omega) \rtimes_\alpha \mathbb{Z}_2) : f(-1) \in C(\mathbb{S}^{2n-1}_\omega), f(1) \in \mathbb{C} + \mathbb{C}\delta\} \\
&\cong \{f \in C([-1,1], M_2(C(\mathbb{S}^{2n-1}_\omega))) : f(t) = \begin{bmatrix} g(t) & h(t) \\ \alpha(h(t)) & \alpha(g(t)) \end{bmatrix} \text{ for all } t, \\
&\qquad h(-1) = 0, \text{and } g(1), h(1) \in \mathbb{C}\} \\
&\leq M_2(\Sigma C(\mathbb{S}^{2n-1}_\omega)) = M_2(C(\mathbb{S}^{2n}_\omega))
\end{aligned}$$

Denote this subalgebra of $M_2(C(\mathbb{S}^{2n}_\omega))$ isomorphic to $\mathfrak{R}^{2n}_\omega$ as B_ω. The antipodal action $\mathfrak{a}$ on B_ω is realized as $\begin{bmatrix} g(t) & h(t) \\ \alpha(h(t)) & \alpha(g(t)) \end{bmatrix} \mapsto \begin{bmatrix} \alpha(g(t)) & -\alpha(h(t)) \\ -h(t) & g(t) \end{bmatrix}$, which is entrywise application of α and conjugation by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The spheres $C(\mathbb{S}^{2n}_\omega)$ are θ-deformations of $C(\mathbb{S}^{2n})$, so if f and g are fixed (matrices of) smooth elements, the following continuity properties apply (as in (2.3.2) and (2.3.3)). Here $\mathcal{C}_\omega$ denotes the collection of parameter matrices ρ which differ from ω only in one pair of conjugate entries, (i,j) and (j,i).

$$||f \cdot_\omega g - f \cdot_\rho g||_\rho \to 0 \text{ as } \rho \to \omega \text{ within } \mathcal{C}_\omega$$

$$||f||_\rho \to ||f||_\omega \text{ as } \rho \to \omega \text{ within } \mathcal{C}_\omega$$

Any smooth approximations in $M_2(C(\mathbb{S}^{2n}_\omega))$ to an element of B_ω can be made without leaving B_ω. Note that the pointwise action α on $\Sigma C(\mathbb{S}^{2n-1}_\omega) = C(\mathbb{S}^{2n}_\omega)$ is an action γ on $C(\mathbb{S}^{2n}_\omega)$ which fixes x but negates every generator z_i, so γ commutes with the rotation action of $\mathbb{R}^n$ defining the Rieffel deformation. Moreover, because the antipodal map on B_ω is implemented with an entrywise action and a conjugation, a matrix function $\begin{bmatrix} g(t) & h(t) \\ \alpha(h(t)) & \alpha(g(t)) \end{bmatrix} = \begin{bmatrix} g & h \\ \gamma(h) & \gamma(g) \end{bmatrix}$ in B_ω is odd if and only if $\gamma(g) = -g$ and $\gamma(h) = h$. Together, these observations imply that smooth approximations to B_ω elements may be made in B_ω while preserving homogeneity properties in the antipodal action. Also, as the adjoint operations of Rieffel deformations all have the same effect on smooth elements, smooth approximations can also be made to preserve self-adjointness.

Theorem 4.4.9. Suppose $p \in \mathbb{Z}_{\geq 0}$ and $P \in M_{(2p+1)2^n}(\mathfrak{R}^{2n}_\omega)$ is a projection with $\mathfrak{a}(P) = I - P$. Then $[P]_{K_0}$ is not in the subgroup generated by the trivial projections.

Proof. Suppose the theorem fails for $P \in M_{(2p+1)2^n}(\mathfrak{R}^{2n}_\omega)$, so there is a path of projections connecting $P \oplus 0 \oplus I$ to a trivial bundle $I \oplus 0$. If this path is viewed under the isomorphism $\mathfrak{R}^{2n}_\omega \cong B_\omega \leq M_2(C(\mathbb{S}^{2n}_\omega))$, then any matrix over B_ω can be approximated by matrices with smooth entries, and this approximation can be done without leaving B_ω. Moreover, if the original matrix is odd or self-adjoint, these properties can be preserved in the approximation. Any smooth matrix over $B_\omega \leq M_2(C(\mathbb{S}^{2n-1}_\omega))$ may then be viewed as a matrix over $B_\rho \leq M_2(C(\mathbb{S}^{2n-1}_\rho))$ for any ρ.

The path of projections connecting $P \oplus 0 \oplus I$ to $I \oplus 0$ is uniformly continuous, so let $P_0, \ldots, P_k$ be finitely elements of this path with the following properties.

$$P_0 = P \oplus 0 \oplus I \qquad ||P_m - P_{m-1}||_\omega < \varepsilon \qquad P_k = I \oplus 0$$

Fix $0 < \varepsilon < \frac{1}{13}$ and let Q be a self-adjoint smooth approximation to P such that $\mathfrak{a}(Q) = I - Q$ and $||Q \cdot_\omega Q - Q||_\omega < \varepsilon$, and let $Q_0 = Q \oplus 0 \oplus I$, so that $||Q_0 \cdot_\omega Q_0 - Q_0||_\omega < \varepsilon$. The matrix $P_k = I \oplus 0$ is already smooth, so let $Q_k = P_k$. Finally, choose smooth approximations $Q_1, \ldots, Q_{k-1}$ of $P_1, \ldots, P_{k-1}$ so that these restrictions hold for each m.

$$Q_0 = Q \oplus 0 \oplus I \qquad Q_m = Q_m^* \qquad ||Q_m \cdot_\omega Q_m - Q_m||_\omega < \varepsilon \qquad ||Q_m - Q_{m-1}||_\omega < \varepsilon \qquad Q_k = I \oplus 0$$

If ρ is another parameter matrix differing from ω only in a prescribed pair of conjugate entries, then if ρ is close enough to ω, similar inequalities hold.

$$Q_0 = Q \oplus 0 \oplus I \qquad Q_m = Q_m^* \qquad ||Q_m \cdot_\rho Q_m - Q_m||_\rho < \varepsilon \qquad ||Q_m - Q_{m-1}||_\rho < \varepsilon \qquad Q_k = I \oplus 0$$

Let R_t, $t \in [0,1]$, denote the piecewise linear path connecting $Q_0, \ldots, Q_k$, so each R_t is self-adjoint and has $||R_t||_\rho \leq \max\{||Q_m||_\rho : m \in \{0, \ldots, k\}\} \leq 1 + \varepsilon$. Further, for any t there is an m such that $||R_t - Q_m||_\rho < \varepsilon$, so liberal use of the triangle inequality and properties of

Q_m shows that $||R_t \cdot_\rho R_t - R_t||_\rho < 3\varepsilon + 2\varepsilon^2$.

$$R_0 = Q \oplus 0 \oplus I \qquad R_t = R_t^* \qquad ||R_t \cdot_\rho R_t - R_t||_\rho < 3\varepsilon + 2\varepsilon^2 \qquad R_1 = I \oplus 0$$

Write $R_t = \frac{1}{2}I + \frac{1}{2}F_t$, so F_t is self-adjoint and $||F_t \cdot_\rho F_t - 1||_\rho < 12\varepsilon + 8\varepsilon^2 < 1$. This implies that F_t is invertible, so normalize F_t to a self-adjoint unitary (under $\cdot_\rho$) to produce $F_t \cdot_\rho |F_t|^{-1_\rho}$ and the projection $\widetilde{R_t} = \frac{1}{2}I + \frac{1}{2}F_t \cdot_\rho |F_t|^{-1_\rho}$. Note that this normalization process does nothing to R_1 and respects the summands of R_0: $\widetilde{R_0} = \widetilde{Q} \oplus 0 \oplus I$ where $\widetilde{Q}$ is a projection obtained from Q in the $(2p+1)2^n$-dimensional matrix algebra by similar means. Moreover, since $Q = \frac{1}{2}I + \frac{1}{2}F$ satisfies $\mathfrak{a}(Q) = I - Q$, F is a self-adjoint odd invertible, so $F \cdot_\rho |F|^{-1_\rho}$ is a self-adjoint odd unitary, and $\mathfrak{a}(\widetilde{Q}) = I - \widetilde{Q}$. Finally, the path $\widetilde{R_t}$ is a path of projections in $B_\rho \cong \mathfrak{R}^{2n}_\rho$ connecting $\widetilde{Q} \oplus 0 \oplus I$ to $I \oplus 0$, and $\widetilde{Q} \in M_{(2p+1)2^n}(B_\rho) = M_{(2p+1)2^n}(\mathfrak{R}^{2n}_\rho)$ is in the K_0 subgroup generated by trivial projections. This means that assuming the theorem fails for a single parameter matrix ω implies that the theorem fails for all ρ that are sufficiently close to ω and differ from ρ in a prescribed pair of conjugate entries. We may select ρ such that this pair of conjugate entries consists of roots of unity of odd order, and then repeat the argument starting with ρ and another pair of conjugate entries. In finitely many iterations, it will follow that the theorem fails for a parameter matrix μ which has odd order roots of unity in every entry. This is a contradiction by the following argument.

Suppose μ is a parameter matrix with an odd order root of unity in each entry such that the theorem fails for a projection $P \in M_{(2p+1)2^n}(\mathfrak{R}^{2n}_\mu)$, and consider $\mathfrak{R}^{2n} = \Sigma^\alpha C(\mathbb{S}^{2n-1})$. By Lemma 2.3.4 there are unitary matrices $V_1, \ldots, V_n \in U_{2q+1}(\mathbb{C})$ such that $V_k V_j = \mu_{jk} V_j V_k$, so define $B : \mathfrak{R}^{2n}_\mu \mapsto M_{2q+1}(\mathfrak{R}^{2n})$ as follows.

$$B : z_j \mapsto z_j V_j \qquad\qquad B : x \mapsto x I_{2q+1}$$

The $*$-homomorphism B exists because the desired images satisfy the relations defining $\mathfrak{R}^{2n}_\mu$, as all noncommutativity information among $z_1, \ldots, z_n$ is pushed to the matrices V_j. Further, B is equivariant for the antipodal map, as the odd generators are sent to matrices with odd entries. So, if $P \in M_{(2p+1)2^n}(\mathfrak{R}^{2n}_\mu)$ satisfies $\mathfrak{a}(P) = I - P$ and $[P]_{K_0}$ is in the subgroup generated by trivial projections, then the same properties apply to $B(P) \in M_{(2q+1)(2p+1)2^n}(\mathfrak{R}^{2n})$, contradicting Theorem 4.4.8. $\square$

Corollary 4.4.10. Suppose $\phi : \mathfrak{R}^{2n}_\omega \to \mathfrak{R}^{2n}_\rho$ is a unital $*$-homomorphism that is equivariant for the antipodal map. Then the induced map on $K_0 \cong \mathbb{Z} \oplus \mathbb{Z}$ is nontrivial on the component that is not generated by trivial projections.

Proof. The image $E(P_\omega(n))$ is a $2^n \times 2^n$ projection that satisfies $\alpha(E(P_\omega(n))) = I - E(P_\omega(n))$ and is therefore not in the K_0 subgroup generated by trivial projections. $\square$

Remark. Because the θ-deformed spheres have the same nontriviality statement in K_0, the domain or codomain could be replaced by some $C(\mathbb{S}^{2n}_\eta)$.

Finally, the anticommutation relation algebras $\mathfrak{R}^{2n}_\omega$ satisfy a noncommutative Borsuk-Ulam theorem analogous to that of $C(\mathbb{S}^{2n}_\omega)$. A curious aspect of these proofs is that the theorem on relation algebras pushed the problem down one dimension to discuss unitaries satisfying $\alpha(M) = M^*$ over $C(\mathbb{S}^{2n-1}_\omega)$, whereas the argument for $C(\mathbb{S}^{2n}_\omega)$ worked by pushing up one dimension and focusing on odd unitaries. In both cases, the alternative method of proof appears to have "insurmountable" road blocks, where one switch of sign devastates an entire argument. It is not clear to me if this indicates key differences in structure or just a technical problem; after all, the first algebra embeds into a matrix algebra over the second. Further, the noncommutative unreduced suspensions $\Sigma^\beta C(\mathbb{S}^{2n-1})$ for various $\mathbb{Z}_2$ actions β that negate some, but not all, generators z_j produce a similar family of universal algebras, and the results of Section 4.2 up to and including (4.2.5) can be generalized to this case, but the computation of K-theory is not quite as clear. The action β is not saturated, and saturation was a crucial point for K_0 computations of $\mathfrak{R}^{2n}_\omega$ (although there is an analogous projection to $P_\omega(n)$, which might be a candidate nontrivial element), and preliminary computations suggest that they might not satsify the same type of K_0 Borsuk-Ulam theorem for projections of certain dimensions. Along with the almost antithetical problem of saturation in Example 4.3.5 and Question 4.3.3, this shows that the exact Borsuk-Ulam properties that noncommutative suspension preserves are not yet clear – a story, perhaps, for another time.

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