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WASHINGTON UNIVERSITY IN ST. LOUIS

School of Engineering & Applied Science

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State Space Analysis of Dominant Structures in Dynamic Social Systems

by

Jeremy B. Sato

A dissertation presented to the Graduate School of Arts and Sciences of Washington University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

August 2016

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For Jess

ABSTRACT OF THE DISSERTATION

State Space Analysis of Dominant Structures in Dynamic Social Systems

by

Jeremy B. Sato

Doctor of Philosophy in Systems Science & Mathematics Washington University in St. Louis, 2016 Professor Joseph A. O'Sullivan, Chair

Many systems involving human relationships are modeled as dynamic systems, as diverse as urban population growth, diffusion of innovations, spread of viruses, and supply chain management. A fundamental assumption is that these systems contain variables which accumulate and deplete over time (people, innovation adoptions, infections, and orders), and whose dynamics are determined by societal rules and human decision making processes. These assumptions allow the system to be formally expressed by ordinary differential equations which are often nonlinear and contain multiple state variables and feedback loops. Analytical methods have been developed to identify the dominant feedback loops which primarily influence the behavior of the system. However, these dominance methods can produce conflicting results and are often performed in the time-domain under specific initial conditions. This thesis takes a state-space approach to dominance analysis and, in the process, re-examines the definition of dominance.

A formal, mathematical definition of dominance is proposed and an analytical procedure is developed and applied to previously studied models. The method produces results consistent with previous analyses and is able explain inconsistencies between other methods. The procedure is then applied in the state-domain and used to identify state-space regions in which certain feedback loops dominate behavior. The procedure is then used to identify the stability properties of equilibria, and a theorem is developed to provide a necessary condition for stability, based on the dominance of balancing (negative) feedback.

Lastly, the method is applied to a problem in public health in which a model of the supply and demand of cancer control services is analyzed. The dominant feedback loops are identified for the purpose of revealing potential sources of health disparities between distinct population segments. The analysis revealed the existence of a tipping point condition associated with a single unstable equilibrium point which influences population health outcomes. Furthermore, trajectories near the unstable equilibrium point are dominated by reinforcing (positive) feedback loops which affect the proportion of people seeking cancer control services. These loops result in either virtuous or vicious cycles, depending on which side of the tipping point the system is operating in the state-space. The methods were then used to identify potential leverage points in the system in which small parameter changes cause significant behavior changes.

Potential avenues for future dominance research are discussed as well as future transdisciplinary research in public health and implementation science.

Chapter 1

Introduction

1.1 Background

Systems typically studied in engineering are governed by laws of physics, which determine how mass or energy is transferred, lost or stored. Laws of physics can be conveniently described in mathematical terms in order to understand how such systems change over time, that is, to understand the system's dynamics [105].

Similarly, mathematical models are also used to investigate a wide range of social system behavior, including the spread of infectious diseases [81], oscillations in factory production and inventory [29], market growth [34], diffusion of innovations [8], urban population dynamics [36], and economic cycles [134]. More recently, models of population dynamics have been used to study problems in public health related to obesity [65], diabetes [75], and cardiovascular diseases [61]. Whereas models of engineering systems are based on conservation laws of physics, models of social systems may be derived from economic or social theories.

One way to model dynamic behavior is to identify and define the states of a system and how they change over time. Typical states in engineering systems include position, velocity, orientation, temperature, and volume. States in social systems might be the inventory of vaccines, the number of staff in an organization, the balance in a savings account, order backlog, and customer demand. States can also be *soft* variables such as service quality, cultural norms, emotional states, memories, beliefs, and perception¹. States are sometimes referred to as state variables, stocks, integrals, accumulations, or levels, depending on the discipline [136, p. 198]. In state-determined dynamic systems, state variables not only define the current state of the system, but also determine how the system changes over time. That is, the changes (or time-derivatives) of state variables depend on the states themselves and form a system of ordinary differential equations (ODEs)².

A simple engineering example of a dynamic system is the spring-mass-damper system. Consider an engineer who wants to understand the dynamics of a mass connected to a spring and damper, so she can design a car suspension system that provides a smooth ride. State variables are chosen to be the car's vertical *position* and *velocity*. The car's vertical *position* determines spring displacement which affects the force exerted by the spring, whereas vertical *velocity* affects the dampening force of the shock absorbers. Finally, the car is subject to Newton's second law of motion: *acceleration* = *net force* \div *mass*. All the forces together cause acceleration, which changes *velocity* and *position*. The causal circle is complete as *velocity* and *position* in return affect net forces through the spring and shock absorber. All these interactions occur continuously in time and determine the system's dynamics, and can be succinctly described by the following system of ordinary differential equations (Equation 1.1).

¹Soft variables can be quantified but may be challenging to directly observe or measure, and are often approximated though latent or indicator variables. For example, love can be measured by observing behaviors. Soft variables have been successfully used across a wide range of models in management and social sciences as discussed in [66] and [136, p. 192].

²An ordinary differential equation (ODE) is an equation containing a function of one independent variable and its derivatives. For dynamic systems, time is the independent variable. An ODE is one type of dynamic model. Other types include partial differential equations, while others are simulations composed of individual agents whose interactions are governed by rules (agent-based models).

$$\dot{x}_1 = x_2$$

 $\dot{x}_2 = \frac{1}{M} \cdot (-Kx_1 - Bx_2 + u(t))$

where

$$x_{1} = \text{car vertical position}$$
(1.1)

$$x_{2} = \text{car vertical velocity}$$

$$M = \text{car mass}$$

$$K = \text{spring coefficient}$$

$$B = \text{damping coefficient of shock absorbers}$$

$$u(t) = \text{external applied force (e.g. from bumps in road)}$$

Equation 1.1 is an example of a linear dynamic model, in that the time derivative of the state variables are linear functions of the state variables. Models of dynamic social systems, which are of primary interest in this thesis, often include nonlinear interactions between state variables. Specifically, the types of models considered in this thesis are dynamic, deterministic (i.e. no random processes), continuous-time, nonlinear ordinary differential equations (see Figure 1.1).

The car suspension model illustrates another feature of state-determined dynamic systems, that of circular causality or causal feedback, in which states determine the system's dynamics, which in turn changes the states (Figure 1.2).

Feedback can be positive, or reinforcing, in which a signal, change, or disturbance is reinforced or amplified through the system, as observed when a microphone gets too close to a speaker resulting in exponential growth of the signal until it reaches the limits of the system.



Figure 1.1: Mathematical models fall into a variety of different forms. The models of interest in this thesis are dynamic, deterministic, continuous-time, nonlinear ODEs.



Figure 1.2: For dynamic systems, ODEs define causal feedback between the states and the dynamics.

Feedback can also be negative, or balancing, as typically designed by control engineers for regulating a system to some desired performance (for example: a thermostat, cruise control, or autopilot)³. Negative feedback can be characterized as a control law that takes measurements of a system, compares measurements against the desired goal, and makes corrections or adjustments to the system until the goal is achieved [41].

 $^{^{3}}$ The terms *positive* and *negative* do not refer to whether or not the feedback is viewed as good or beneficial to a system.

In the thermostat example, Figure 1.3 illustrates how a thermostat uses negative (balancing) feedback to regulate room temperature in the winter. The thermostat measures the gap, or difference, between the desired and measured temperature.



Thermostat as Negative (Balancing) Feedback Control

Figure 1.3: A thermostat is an example of a negative, or balancing, feedback control system.

If the gap is positive, the thermostat activates the furnace, thus increasing room temperature, until the measured temperature is equal to the desired temperature. The room temperature is the state variable, or stock. The arrows flowing into and out of the temperature block represent the processes of heat transfer that increase and decrease room temperature over time. The clouds at the beginning and ends of the arrows represent the model boundaries and are unrestricted heat sources and sinks. The arc arrows represent causal relationships in which a positive sign (+) indicates a positive causal relationship between variables, where the dependent variable (effect) at the head of the arrow tends to move in the same direction as the causal variable. The negative sign (-) indicates a negative causal relationship, in which the dependent variable tends to move opposite of the causal variable. In a mathematical model, these causal relationships are made explicit in the equations.

Positive and negative feedback are also found in natural and social systems such as the human body (e.g. homeostasis is negative feedback regulating body temperature, hormone levels, etc. to a stable condition), ecological systems (e.g. balancing feedback between predator and prey populations), economic systems (e.g. reinforcing feedback of compounding interest which can be viewed as a virtuous cycle if on the receiving end of interest, or a vicious cycle if on the paying end), socio-economic systems (e.g. vicious cycles of poverty, or virtuous cycles of wealth) and business systems (e.g. negative feedback of managing inventory to a desired level).

In the 1950s, Jay Forrester of Massachusetts Institute of Technology (MIT) Sloan School of Management, with a background in servomechanisms⁴, observed that the source of a company's inventory and production oscillations were due not to external market fluctuations, but rather to internal decision-making policies affected by delays and information feedback [29]⁵. This realization, coupled with advances in computers and feedback control theory, led to the development of a language and approach for modeling the dynamics of industrial systems, known as industrial dynamics [29]. This language and approach to modeling was also applied to market growth dynamics [34], urban population dynamics [36], and world population dynamics [92]. The language, modeling conventions, and principles were eventually formalized into the field known as system dynamics (SD).

1.2 The Structure-Behavior Problem

A central premise in SD is that the behavior of a system arises from its own structure, where structure consists of the nonlinear interactions between variables and feedback loops

 $^{^4\}mathrm{A}$ servome chanism is an automatic device using error-sensing negative feedback to correct the performance of a mechanism.

⁵Industrial Dynamics was established against the backdrop of challenges in the fields of Operations Research and Management Science to account for the softer aspects of management such as how decisions are made in the presence of limited, imperfect and delayed information feedback.

[136, p. 107]. This is also referred to as the *endogenous perspective*, in which explanations for behavior look within the system [122]. In Principles of Systems, Jay Forrester quotes Jerome Bruner, "to learn structure...is to learn how things are related" and then goes on to state in his first principle: "...dynamic behavior arises within its internal structure." [35].

This is not to say that external influences are not important (such as the road bumps in the car suspension example, or the sun's radiation which was excluded from the thermostat example), but rather that for social systems, often undesired behavior, such as supply chain oscillations, is a result of how decisions are made in a human feedback system consisting of goals, delays, beliefs, and imperfect information. External influences can actually reveal that a system is internally unstable. In the car suspension example, external forces of a certain frequency may excite the natural modes of the system and lead to unstable oscillations if the suspension is not sufficiently damped. In supply chains with significant delays, a onetime change in orders could result in large inventory fluctuations, resulting in stock-outs and excess inventory [29]. In these cases, the undesired behavior is a consequence of the system's structure.

Researchers in the field of SD will formulate a dynamic hypothesis (expressed formally as a dynamic model) about how causal mechanisms lead to some observed behavior over time [136, pp. 94-95]. Questions are then aimed at uncovering and explaining the endogenous sources of behavior, such as, "What are the causal structures in the system that are primarily responsible for generating the observed behavior? Under what conditions are these causal structures influential?" The aim of these questions is to gain insight into what this thesis calls the *structure-behavior problem*: identifying which structural elements produce the observed behavior over time, and how [121]. These questions aim to identify potential places to intervene in the system, or leverage points, in which small changes can have significant impact and where policies can be designed to achieve an objective. Since models may have hundreds or thousands of loops, answering this question is non-trivial, especially considering that in practice, typically only a few loops have significant influence on the behavior at any given time [52]. Methods have been developed within the field of system dynamics to help address questions such as, "What is the dominant feedback loop?" According to some, "to identify these (dominant) loops...is to identify the fundamental causes of the system's behavior." [51].

1.3 Current Methods and Challenges

For certain classes of systems, structure-behavior relationships can be explicitly defined. For example, many systems in engineering are modeled and analyzed as linear dynamic systems, for which a rich tool-set of methods have been developed. Classical feedback methods in the frequency domain as well as modern control methods in the time and state domain make use of the fact that explicit solutions can be found for linear time-invariant dynamic systems⁶ [12].

For such systems, there exist well-established algorithms for deriving control laws to achieve desired performance. For example, procedures exist for tuning parameters of Proportional, Integral, Derivative (PID) controllers to achieve desired properties [41]. Also, one can formulate control law design as an optimization problem that minimizes error, energy, time, etc. [12].

Many challenges prevent a straightforward application of these methods to the control of social systems, including observability of states, uncertainty of parameters, long time constants in the system, and the extent to which some parameters in the system can be changed or controlled. Another challenge is that dynamic social systems are often modeled as nonlinear dynamic systems. In contrast to linear systems, explicit analytical solutions cannot

⁶Time-invariant systems are such that the parameters in the system do not change with time.

generally be found for nonlinear systems. Early contributions in the field of system dynamics by Michael Goodman [49] and Alan Graham [52] applied basic principles from linear systems theory and feedback control to make general statements about the relationship between structure and behavior in certain classes of oscillatory nonlinear systems. However, the extent to which these principles could be applied to most social systems of interest to SD practitioners was limited due to their dynamic complexity [80]⁷. Nathan Forrester applied techniques from engineering to linearize a system around an operating point and applied methods from linear systems theory [40]. Several other methods have been developed based on these analytical foundations and fall in the category of *dominance analysis* [20]. Today, however, analysis is still mostly performed experimentally where models are investigated through the removal or deactivation of partial structures and simulated to examine the impact, or through sensitivity analysis in which parameters are varied and results observed. The effectiveness of these methods varies depending on the size or dimension of the model and the types of nonlinear relationships.

In the engineering sciences, significant progress has been made in the tools and methods for analyzing and controlling nonlinear systems [82, 72, 73]. However, due to the aforementioned challenges, application of these methods to dynamic problems in social sciences has remained largely unexplored⁸. The following sections provide a brief history and introduction to the line of analytical methods developed in system dynamics for exploring influential or dominant parts of system behavior, as well as methods developed in mathematics and engineering. It then summarizes the methodological challenges that motivate this thesis.

⁷Dynamic complexity is a result of the types of nonlinear interactions between the state variables and feedback loops and the number of state variables or stocks in the systems.

⁸There has been progress applying nonlinear systems theory to social and business sciences in a few domains such as bifurcation analysis, chaos theory [118, 101, 5, 15, 87, 54], and stability theory [83, 148].

1.4 Overview of Dominance Analysis

Dominance analysis became a focused area of research shortly after George Richardson proposed the research problem of detecting dominant structure in nonlinear systems [120]. He states:

Underlying the formal, quantitative methods of system dynamics is the goal of understanding how the feedback structure of a system contributes to its dynamic behavior...there is a conspicuous gap in our literature between intuitive statements about loop dominance and precise statements about how to define and detect dominant structure.

In the following years, new methods emerged for detecting *dominant* parts of system structure. Kampmann and Oliva summarized many of these methods and also described the *structure-behavior problem* as a fundamental pursuit in the field of system dynamics [80], but the authors also raised the more fundamental question of what constitutes an explanation of the link between structure and behavior. Since then, while many methods have been developed in an attempt to identify dominant structure [20], there has been a lack of debate on the topic of what should count as a sufficient explanation for structure-behavior relationships.

Model analysis methods have primarily focused on feedback loops as the principle structure for explaining behavior of nonlinear dynamic systems [121, 26, 136, 99, 78]. Feedback loop methods fall into two broad categories: exploratory/behavioral methods and formal/structural methods [20].

Exploratory/behavioral methods seek to explain which feedback loops cause behavior by systematically deactivating the loops and observing the impact on behavior [26, 111, 69]. The explanation for behavior rests in whether or not the existence of the feedback loop significantly impacts the behavior of a specific variable at a certain time in its trajectory, assuming specific initial conditions. The method, while it identifies if and when a loop determines behavior, does not explain how or why. The desire to understand how and why certain feedback loops are influential motivates the second type of methods: formal/structural methods.

Formal/structural methods construct metrics to measure the relative influence of different feedback loops on variable behavior. The Loop Eigenvalue Elasticity Analysis (LEEA) method, for example, linearizes the model, computes the eigenvalues, and evaluates elasticities of the eigenvalues with respect to model parameters using partial derivatives. These elasticities are combined for each feedback loop to arrive at a single metric of influence [78]. Another method, the Pathway Participation Metric (PPM) method, calculates the influence of different causal pathways on a variable of interest by calculating each pathway's contribution to a metric of behavior based on the first and second time derivatives. Influential loops are then identified based on the pathways [99]. The Loop Impact method [60] is similar to PPM in its search for influential pathways but differs in how influence is measured and detected. All formal/structural methods seek to explain behavior by measuring the relative strength of feedback loops. They reveal how a loop's influence changes over time, but are unable to detect which loops actually determine or cause certain behaviors because the analyses use normalized metrics that are only proxies for system behavior.

In summary, exploratory/behavioral methods seek to explain when and if a loop determines behavior but are unable to explain how, whereas formal/structural methods provide insight into the shifting influence of loops over time, but do not detect if a loop actually determines behavior.

1.5 Overview of Nonlinear Dynamic Systems Analysis and Control

While no generalized theory exists for analytically solving nonlinear dynamic systems, mathematicians and physicists have used methods from analysis, geometry and topology to qualitatively characterize families of solutions (trajectories in state-space) without explicitly solving them [82]. Origins trace back to Poincare's work on celestial mechanics (1899) (with origins in Newtonian mechanics), laying the foundation for future developments by Lyapunov, Birkhoff, Andronov and Pontryagin in local and global analysis of nonlinear ODEs and optimal control [63]. Further developments by van der Pol, Lefschetz, and Levinson on deterministic chaos and periodic orbits were motivated by nonlinear oscillations found in electronic applications such as telecommunication, radios and radar [63]. In the 1950s and 60s, Smale, Moser, Melnikov, and Popov continued to advance theories in stability and manifolds while Lorenz advanced theories in chaos. During this time, the Soviet and US space programs also motivated the development of control theory that could deal with nonlinear differential equations. In more recent history, methods have been developed for feedback regulation and control of nonlinear systems and applied to a variety of industries including aerospace, energy, telecommunications, and process control [72].

The concept behind many of these methods is that trajectories are characterized by the existence and stability properties of alpha and omega limit sets [82]. Alpha limit sets define the origin of trajectories whereas omega limit sets define their destination (solutions to the equation as time goes to negative or positive infinity). Limit sets include the stationary points and limit cycles of the system, if they exist⁹. The quantity, location, and stability properties of these sets characterize the state trajectories. Many of the theorems of existence and stability of stationary points and limit cycles apply only to systems of certain qualities

⁹In dynamic systems, stationary points are also known as equilibrium points which are conditions in which the system is static or not moving

and dimensions, making the direct application to the general nonlinear system challenging or impossible. Therefore, engineers and physicists will typically study specific classes of nonlinear systems that apply to their field, from the viewpoint of stability and control.

1.6 Research Gaps

This thesis addresses three gaps in the current methods for identifying dominant system structure.

Gap 1: Lack of a formal, rigorous definition of *dominance*. Researchers have used the term dominance to represent different concepts, from most influential to solely determines the behavior. Throughout SD literature, there exists at least 18 different definitions of the term *dominance* or *dominant*, making it challenging to critically evaluate and compare results across methods and models. Many of the definitions offered also use terms that lack mathematical rigor (for example, *influential*). Even more challenging, often the term *dominance* is used without any definition provided. Of the 44 articles reviewed that are considered to be focused on dominance analysis methods, only 16 offer definitions for the term. Methods also assume different definitions for *structure* and *behavior* as well. For example, some methods consider the important structural unit to be a closed feedback loop, while others consider causal pathways from one variable to another [97]. Some methods identify instances of no dominant structure and multiple dominant structures, while others always detect a single dominant structure. This is further complicated by the fact that for some models, there may not exist a unique set of loops, and so results may depend on the analyst's choice of loop set [71]. Some methods consider behavior at a single point in time, while others consider global behavior patterns observable over longer time intervals¹⁰ [138].

¹⁰Local behavior patterns can be quantified by the first and second time derivatives, such as exponential growth or decay. Global behavior patterns, such as stable or unstable oscillations, require observation over a time interval.

Some methods consider the behavior of all variables while others focus on a specific variable. The lack of a formal and universally accepted definition of dominance has led to explanations of structure-behavior relationships in some cases that are ambiguous and untestable (and therefore not falsifiable), which can hinder progress in the field. Furthermore, some questions are not being asked, such as how *behavior*, *structure* and *dominant* should be defined in order to best explain structure-behavior relationships.

Gap 2: Dominance methods have not been applied in the state domain. Dominance analysis methods are typically applied in the time domain assuming specific initial conditions, but are not designed to explore how dominance changes across the state space [111, 152]. Furthermore, some methods seem to confound the influence of initial conditions and model structure [132]. State space methods have not been widely used in the system dynamics field to characterize trajectories, let alone to examine loop dominance. Certain questions are not being asked, such as: how does loop dominance shift across state-space regions? Is the dominance of a particular loop limited to a defined subspace? It is not clear to what extent methods using linearization techniques, such as LEEA and Dynamic Decomposition Weights Analysis (DDWA), are suitable for gaining insights across a wide region of the state space.

Gap 3: Application of dominance methods for identifying high-leverage policies and interventions. In engineering, feedback control laws are synthesized in a procedural fashion to achieve desired performance characteristics (for example, rise time, settling time, steady state error, stability margins) [41]. In system dynamics, as well as other methods in systems science (e.g. social network analysis and agent-based modeling), while general principles exist for identifying high leverage places to intervene in a system, implementation is largely experimental and based on expertise and experience [93]. Formal methods do not exist for developing and implementing policy in the form of modifying existing or designing new system structure and feedback mechanisms. While there will never be a one-size-fits-all method for developing such policies for nonlinear systems [80], opportunity may exist to use dominance methods to develop effective policies and interventions that are tailored to certain classes of nonlinear systems studied in the social sciences and in public health [64].

1.7 Research Questions and Aims

Research Question 1. What should constitute an explanation for how structure determines behavior? How can dominance be formally and rigorously defined in a way that resolves existing ambiguities, explains discrepancies in previous dominance analyses, leads to new insights, and advances methods and theory?

Research Aim 1. Propose a rigorous, formal definition of *dominance*. Develop criteria for what constitutes an explanation for structure-behavior relationships. Examine historical and modern definitions and use of the term *dominance*, the current state of dominance analysis, modern concepts of causality, and evaluate implications of the proposed definition for current methods of dominance analysis. Identify resolved discrepancies, new insights, and potential extensions or modifications to current methods. Identify any new insights about the relationship between current methods (e.g. between behavioral methods and structural methods) [26, 99, 78]. Employ definition of dominance to understand the mathematical basis for concepts that have been described in practice such as shadow dominance, shared dominance, multiple dominance, and shifts in dominance [26].

Research Question 2. Can dominance analysis be applied in the state-domain as well as the time-domain? Supporting questions:

• What is the mathematical relationship between state-space methods and loop dominance methods?

- Does the concept of state-space regions of loop dominance make sense? Can such regions or subspaces be determined?
- Can state-space analysis methods be used to understand sensitivity of loop dominance to initial conditions?
- Does this perspective offer new insights into previously analyzed models?
- What questions are best answered by traditional loop dominance methods vs. statespace methods?
- Can the methods be used together? Under what conditions? Can an integrated method be developed?

Research Aim 2. Apply current methods in nonlinear control theory to understand how structural dominance shifts over the state space. Identify the mathematical relationship between state-space methods and structural dominance methods. Identify conditions for when state-space and loop dominance methods provide additional insight when used together. Some analysis questions will likely be best addressed by dominance methods while others by state-space methods, and some by a combination. This research aims to refine the set of questions most suitably addressed by each method, and how each method can be used together.

Using the proposed definition for dominance under Aim 1, develop an integrated state-space dominance method and test against other methods. Design and implement a procedure for performing loop dominance analysis across the state space. To achieve this aim, the new analysis procedure will be applied to simple models which have been studied by other dominance methods, such as the Susceptible-Infectious-Recovered (SIR) model and the Lotka-Volterra Predator-Prey model. The procedure will then be applied to the *cancer control services model*, which uses a generic supply and demand structure that has been used for several problems in public health. **Research Question 3.** How can state-space dominance methods be automated and applied to analyze dynamically complex models in public health, in order to identify potential interventions and policies? Can stability theory from nonlinear controls be used to guide methods for modifying existing feedback structure? For systems that belong to a particular generic structure, such as the supply and demand structure, is it possible to formulate procedures for constructing policies and strategies that have desired performance characteristics? Is there anything to be gained by using state-space and dominance analysis methods as opposed to traditional sensitivity analysis?

Research Aim 3. This aim seeks to use the definitions of dominance, the methods of dominance analysis, and the insights into dominant structure from Aims 1 and 2 to develop principles for identifying potential interventions and policies. Insights from Aim 2 will be used to identify influential structure and formalize structure-behavior relationships for the cancer control services model.

1.8 Summary of Contributions to Systems Science

- A theorem and corollary was developed which provides a necessary condition for stable equilibrium points in terms of feedback loop dominance. This provides a rigorous way to test for stability and formally connects stability to the dominant causal mechanisms in social systems.
- 2. A formal framework was created for evaluating and comparing different types of dominance as they have appeared in SD literature, enabling the comparison between current dominance analysis methods, and leading to a rigorous and formal definition of dominance, as well as the development of a new dominance procedure.
- 3. A procedure was created for identifying state-space regions of feedback dominance which integrates features of both behavioral and structural dominance methods. The

procedure was implemented in software and applied to several models to determine how dominance shifts across the state space, and to detect state-space regions of dominance.

- 4. Relationships were identified between current dominance methods, generating new structural-behavioral insights for previously studied models, and revealing conditions in which different dominance methods produce similar and different results.
- 5. By applying the state-space dominance procedure to the cancer control services model, an approach was developed for how to use dominance, state-space, and sensitivity analysis together to answer analysis questions.
- 6. New tests for model confidence-building were demonstrated using state-space and dominance methods together.

1.9 Summary of Contributions to Transdisciplinary Science and Public Health

- 1. The delivery of cancer control services was framed as a dynamic problem through the formal expression of a dynamic model, making the problem accessible to analysis tools in system dynamics and nonlinear systems theory. This represents a novel approach to analyzing the supply and demand of health services.
- 2. Using methods developed in this thesis, a tipping point was identified in the cancer services model, linked to specific reinforcing feedback loops, which can cause significant disparities between different segments of the population.
- 3. Leverage points were identified for increasing health status and decreasing disparities in the cancer control services model.

- 4. Using the dominance framework developed in this thesis, sources of policy resistance were identified in the cancer control services model.
- 5. The model and analysis methods in this thesis provide a means for analyzing the scale-up of health services as a dynamic systems problem.

1.10 Outline of the Dissertation

Chapter 1: Introduction. The introduction provides the background, research gaps, questions, aims, and a summary of main contributions.

Chapter 2: Dominance systematic literature review (Aim 1). This chapter describes the process for conducting the dominance literature review and presents the results.

Chapter 3: Defining dominance (Aim 1). This chapter develops a formal definition for dominance and discusses implications.

Chapter 4: Applying and testing the definition of dominance (Aim 1). This chapter applies and tests the proposed definition of dominance against current methods of analysis. A new dominance procedure is developed and applied to the logistic growth model and results are presented.

Chapter 5: State space dominance (Aim 2). This chapter applies the results from Aim 1 to develop a procedure for conducting dominance analysis across the state space. The procedure is applied to several well-known models which have been extensively analyzed in the dominance literature in order to compare and contrast results.

Chapter 6: Analysis of the cancer control services model (Aims 2 and 3). This chapter presents the results and conclusions from applying the methodological innovations
from Aim 2 to the cancer control services model. This chapter also presents a theorem relating stability and dominance which is used to analyze the model.

Chapter 7: Summary and conclusions. This chapter presents the main conclusions and contributions and proposes potential areas for future research.

Chapter 2

Systematic Review of Dominance Analysis

2.1 Objective

A fundamental pursuit in the field of system dynamics (SD) is explaining how the structure of a system drives behavior. Over time, a diverse set of practices and tools have emerged for identifying which parts of structure *dominate* behavior, and when. Over the course of these developments, various criteria have been offered for *dominance*. The objective of the systematic literature review is to understand, historically, what has constituted an explanation of behavior and how dominance methods and definitions have evolved¹¹.

The insights and conclusions from the review are then used to arrive at a rigorous and formal definition of *dominance* as it is pertains to structure-behavior relationships in dynamic systems (Research Aim 1). This formal definition provides the mathematical foundation

¹¹A systematic review is a type of literature review often used in medicine to find and analyze studies pertaining to a specific research question. Methods, databases, search terms, and other criteria are documented for repeatability and for conducting statistical analysis. This review on dominance, while differing from those often found in medicine, is considered to be systematic, thorough, and exhaustive for the research question and has been documented for repeatability.

for developing a state-space dominance analysis procedure (Research Aim 2) which is then applied to a model of public health (Research Aim 3).

The review begins with examining how Jay Forrester, founder of the field of SD, and his contemporaries provided explanations of behavior, and then examines how methods of explanation evolved over time, making particular note of the definitions and descriptive phrases for the term *dominance*.

2.2 Literature Search Criteria

The search set includes all literature commonly accepted as belonging to the field of SD, including the primary peer-reviewed journal *System Dynamics Review*, proceedings from the annual International System Dynamics Conference, and the MIT SD literature collection (including over 5,000 memos, course material, research proposals, masters and PhD theses, historical documents, and miscellaneous publications). The search set also included articles in other journals in which SD was used as a primary method. Finally, the review includes classic and modern texts in SD.

Within the search set, the primary criteria for inclusion was the appearance of one or more of the following words: *dominant*, *dominance*, *dominate*, *dominates*, *dominated*, and *dominating*. Earlier literature also revealed that the words *predominant* and *predominate* were used in a similar fashion to *dominant* and *dominate*, therefore the various forms of these words were also included.

All literature considered foundational to the field (from 1956 to 1963) were included and reviewed, regardless of the presence of the term *dominant*, in order to observe the components of explanations for behavior and the words and methods used to describe structure-behavior relationships.

Additional criteria for the inclusion set:

- Excluded instances in which the term *dominant* (and variations) were not used to explain behavior or structure-behavior relationships. Specifically, if the term did not refer to model structure, model behavior, or the relationship between structure and behavior, it was not included. For example, the following search result was excluded from the review: "The so-called Phillips Curve has dominated much of the debate about inflation." (D-3606).
- Excluded instances in which the word *dominant* was used without any context from which its meaning could be inferred.
- The review noted cases in which authors cited and re-used previous uses and definitions of the term *dominant*, however not every re-use or repetition was included, especially when it did not offer any new context or information about its meaning.

2.3 Limitations

There are several limitations of the systematic review. First, while attempts were made to include all literature in the field of SD addressing the concept of dominance, the search may have missed some literature which discussed the concept of dominance without using the word or any of its derivatives. The greatest risk of missed literature occurs during the foundational time period (pre-1964), prior to when the term *dominance* was in wide-spread use. Attempts were made to mitigate this risk by including nearly every literature found during this time period, regardless of whether the word *dominant* or its derivatives appeared. This strategy was effective in identifying early thoughts on structure-behavior explanations. Post 1964, however, the volume of SD literature grew rapidly and therefore it was no longer practical to include every piece of literature. Instead, the keyword search criteria was expanded to include *cause*, *explain*, *explanation*, and *influence*, however this also greatly expanded the inclusion set to the point where the majority of search hits were not relevant to dominance, and so many were excluded. It is possible some of these exclusions may have resulted in missing important literature which, while not using the term *dominance*, would have added value to understanding the nature of explanations.

Second, there is a degree of subjectivity in judging whether or not each use of the term *dominance* applies to behavior-structure relationships.

Third, there may have been misses in the older literature due to quality of scans and text recognition software. For example, at least one instance of this was caught in *Dynamics of Product Growth in a Competitive Market*, Miller, Jr., where the term *dominate* was missed by text recognition software, but discovered through manual inspection.

With the above limitations in mind, the review is considered to be both systematic and exhaustive in that for the foundation documents, nearly all literature was manually inspected for relevance. Furthermore, the search criteria included all possible derivations of the word *dominance*. Precise documentation was kept for all literature included in the review, including the reason for inclusion. The search included textbooks, educational and training literature, journal articles, conference proceedings, memos, research proposals, and masters and PhD theses.

2.4 Organization of Results

From the review, four time periods emerged which provide a helpful way to understand the development of *dominance analysis*.

• Period 1: Foundational system dynamics literature (1956-1963)

- Period 2: Early uses of the term *dominance* (1964-74)
- Period 3: Analytical foundations for *dominance analysis* (1975-1986)
- Period 4: The development of dominance analysis methods (1989 to present)

The following sections present the findings of the systematic review, including the trends and evolution of the use, definitions, and methods of *dominance* which are observed across the four time periods.

2.5 Period 1: Foundational System Dynamics Literature (1956-1963)

The foundation literature begins with Jay Forrester's letter to the MIT faculty research seminar in 1956 which proposed a new approach to modeling industrial systems that later become known as *industrial dynamics* [27], and more generally, *system dynamics*. The inaugural publication of the field came two years later in the *Harvard Business Review* [28] and the first text, *Industrial Dynamics*, was introduced three years after [29]. This period also includes the first documents, called *D-memos*, from 1958-1963 in which the first instances of *predominance* and *dominance* appear in SD literature.

In his earliest writings on system dynamics, Forrester does not use the word *dominant* (or any of its forms) [27, 28, 29]¹². However, system behavior is inextricably linked to structure since the beginning of the field. The first D-memo traces behavior to accumulations, delays and decision making criteria [27].

In *Industrial Dynamics* [29, p. 66], several statements refer to structure-behavior relationships and the role of nonlinearity in determining behavior.

 $^{^{12}}$ In Industrial Dynamics, Forrester uses the word predominate several times, but without definition.

Our social systems are highly nonlinear. It seems likely that such nonlinearities, coupled with the unstable tendencies caused by amplifications and time delays, create the characteristic modes of behavior...

In the written history of SD, the term *dominance* is actually first used by Forrester's contemporaries¹³. The term is used to describe behavior characteristics, and specifically, the amplification or dampening of oscillations [76], the periods in oscillatory systems [9, 149], and frequencies or modes of behavior [24]. For example:

...the second peak in the orders to the manufacturing sector is thirty-eight weeks removed from the initial surge, after the shut-down. This may be a reflection of the natural period, but it is not a dominant characteristic in this case [9, p. 36].

The systems being described were typical of those studied by Forrester in *Industrial Dynamics*, characterized by steady-state oscillations. This use of the term *dominant* is also consistent with how feedback control engineers describe dominant frequencies and dominant modes in classical control theory [105, p. 304]. This language, therefore, is not surprising given Forrester and his colleagues' background in servomechanisms. The term *dominant* or *dominate* is also used to describe structural elements, such as system parameters [18, p. 2].

Interestingly, the term *feedback loop dominance*, of common use in the field today, is never used to describe the industrial systems exhibiting oscillatory modes which were studied in the earliest literature. Rather, the term is first used to describe growth processes which exhibit transient modes, as in Nord and Swanson, 1962, in "Growth of a New Product" [102, p. 37]

¹³The terms *predominance*, *predominate*, *predominant* have definitions similar to that of *dominance*, *dominate*, *dominant* and are used in a similar fashion, and therefore usage of these words are included in the literature review.

Because the forecast uses information which is an integral part of the feedback loop, the loop itself dominates the forecast.

Then, in his thesis "Dynamics of Product Growth in a Competitive Market," Miller describes the activity of examining feedback loops with respect to explaining behavior [94, p. 7].

This will be done first by tracing through a time history which might be typical of many products, and later by examining the most important feedback loops in the system to determine how the characteristics of these loops influence the behavior.

These are the earliest references for what today is termed *feedback loop dominance*. In the following years, the term *dominant* is used with significantly increased frequency. Also, during the next period the concept of *shifts in dominance* is first introduced.

2.6 Period 2: Early Uses of the Term *Dominance* (1964-74)

The next period examines how the term *dominance* is used as the field of SD begins to take shape. The review examines the main contributions of Forrester and his contemporaries such as Carl Swanson, Dennis Meadows, Donella Meadows, and Michael Goodman, and how the term *dominant* takes form. New phrases emerge, such as *shift in dominance*. This period includes classic publications of the field such as *Market Growth* [34], *Principles of Systems* [35], *Urban Dynamics* [36], *The Limits to Growth* [92], and *Study Notes in System Dynamics* [49]. During this time, while many descriptive phrases and synonyms are used for *dominance* and *shifts in dominance*, a formal definition has yet to be offered. One of the first uses of the word *dominance* by Jay Forrester is to describe growth processes and also indicates a distinction between engineering and social systems [30].

Most of the literature on feedback systems deals with the negative feedback loop. Negative feedback is the form normally encountered in the control of physical systems. Yet, positive feedback dominates in the growth and decline patterns of social systems.

Forrester also notes the transition from growth processes of positive loops to stagnation of negative loops, the first indication of what is now referred to as *shifts in loop dominance* [30]. Then, Swanson discuss how shifts in dominance occur [140].

These loops that dominate the behavior of a variable shift and usually produce different characteristic behavior due to the shift. The mechanisms that shift the dominance are the nonlinearities in the system which change the gain and delay of feedback loops.

Meanwhile, the phrases feedback loop dominance, loop dominance, and shift in dominance do not appear in key word searches outside of SD, during any time period. It appears the concept originates entirely from within SD, and furthermore, it does not appear until the study of growth processes involving the coupling of reinforcing (positive) feedback with balancing (negative) feedback which limits growth¹⁴. Another researcher observes, "As positive feedback is not of interest for engineering design purposes there is little if any literature with which to compare this approach" [107].

Post-1964, references to *loop dominance* and *shifts in dominance* become prolific in the field. Many references to *shifts in dominance* refer specifically to the shift that occurs in growth processes, as mentioned above, in which there is a shift from a reinforcing to balancing loop

¹⁴This is consistent with recent conversations with George Richardson (2016) on the history of feedback loop dominance, who suspects that the concept originated from within the field.

[139, 31, 4, 32, 33, 62, 48, 49]. In the vast majority of references to *shifts in loop dominance* during this period, it is the nonlinear relationships which are implicated as the cause of the shifts [31, 140, 141, 150, 32, 33, 48, 49]. In fact, even before the phrase *shifts in dominance* was in use, nonlinearities were implicated as the source for interesting and counter-intuitive behavior of systems, tracing back to the first D-memo by Forrester [27] and the text *Industrial Dynamics* [29]¹⁵.

As the term *dominance* gained traction, assertions and general principles about *dominance* were proposed based on experience and anecdotal evidence, but without formal proof or rigorous definition. For example:

Even in a complex system only one or a few loops dominate the behavior of a variable of interest over an interval of time... These loops that dominate the behavior of a variable shift and usually produce different characteristic behavior due to the shift [141].

The assertion is also made that linear systems cannot shift in loop dominance and that only nonlinear systems are capable of shifts in dominance [48, 49].

Michael Goodman is the first to offer precise statements about how shifts in loop dominance result in S-shape growth [47, p. 3].

Previous knowledge of positive and negative feedback would indicate that the S-shape phenomenon is a two-stage process beginning with positive feedback and after some time becoming dominated by negative feedback.

Goodman also makes several generalizations about system structure and behavior while employing the term *dominance* in his masters thesis "Elementary SD Structures," directly linking dominance to sigmoid growth which produces an S-shape behavior [48].

¹⁵Email exchanges with George Richardson (2016) on the history of feedback loop dominance also indicate a strong, historical association between nonlinearity and shifts in loop dominance.

In *Study Notes in System Dynamics* [49], Goodman introduces generic feedback structures such as positive and negative feedback loops, relating them to behavior modes such as exponential growth, goal-seeking behavior, and S-shape growth. He also mentions *shifts in loop dominance*, most often in the context of shifts between positive and negative loops and uses this notion as a key concept for explaining how S-shape growth occurs.

While *dominance* is not defined by any of the authors, many phrases are used to describe the term, listed alphabetically in Table 2.1.

achieving	influence	
affect	leading to	
cause to occur	most important factor	
causing	necessary to produce	
control	overtake	
determinant of	powerful	
determine	predominate	
driven by	primary determinant	
encouraging	produce	
governs	responsible for	
important	role in determining	

Table 2.1: Words and phrases used to describe *dominate* or *dominant* in literature from 1964-74.

Some phrases indicate a relative or subjective measure of influence such as *affects*, *encouraging*, *important*, *influence*, *powerful*, and *role in determining*. For example:

As the periods increase, the third order delay plays less and less of a dominant role in determining the behavior of the system [10].

While other phrases indicate an objective causal relationship that is linked to a direct behavioral outcome, such as *cause*, *determine*, *governs*, *necessary to produce*, and *responsible for*. For example: Loop 3, an additional negative feedback loop based on Calhoun's observations, must then be responsible for transferring dominance from Loop 1 to Loop 2, necessary to produce equilibrium [48].

Lastly, other phrases are used in both a relative and objective sense. For example, *determine* vs. *role in determining*. The former seems to be an objective statement of causality between structure and behavior, whereas the latter indicates a relative relationship.

During this time period, the word *dominant* is not only used to describe behavior patterns, feedback loops, and parameters (as in the previous period), but it is now expanded to describe a wider variety of concepts such as dominant pressures, forces, time constants, variables, nonlinear interactions, subsystems, causal mechanisms, inputs, and flows. The word is used to identify whatever aspect of a system is considered important or critical in affecting behavior, but it is not accompanied by formal criteria.

The next time period sees the first proposed definition for *dominance* and the development of analytical and experimental methods for identifying *dominant* structure. This period establishes the mathematical foundations of future work, culminating with the emergence of *dominance analysis* as a sub-discipline within SD.

2.7 Period 3: Analytical Foundations for *Dominance* Analysis (1975-1986)

This time period includes the work of Alan Graham, Nathan Forrester, George Richardson, Alexander Pugh III, Robert Eberlein, and many others who laid the analytical foundation for future model analysis research. The term *dominance* had been used for nearly twenty years without formal definition, which was offered for the first time by Alan Graham in 1977 [52]. Here, a loop dominates the behavior in the sense that if the loop is disconnected or substantially altered, the behavior mode also changes substantially.

Graham's definition influenced future work by Ford, who used a similar definition and test of *dominance*, and Rahmandad, who stressed the importance of counterfactual testing. In Graham's definition, the counterfactual is the objective criteria for dominance in which the *feedback loop* is the structural unit of interest, and behavior is defined by *behavior modes*. Other definitions of dominance during this time period focus on causal links as well as feedback loops as the important elements of structure [38]. Other definitions also appeal to different quantifications of behavior, such as eigenvalues [38], while other definitions do not precisely define behavior [37].

In contrast to Graham's definition for *dominant*, an alternate one using a more relative criteria, is:

...In a feedback structure, a loop that is primarily responsible for model behavior over some time interval is known as a dominant loop [123, p. 285].

Shortly after, Richardson formalizes what is meant by behavior and introduces the concept of *dominant polarity* for first order systems ($\dot{x} = f(x)$) and used the concept to classify behavior as either goal-seeking/convergent if *dominant polarity* = -1, or goal-divergent if *dominant polarity* = +1, where:

$$dominant \ polarity = \operatorname{sgn} \frac{\partial \dot{x}}{\partial x} \tag{2.1}$$

Furthermore, he uses the concept of *dominant polarity* to define *shifts in loop dominance*:

...a shift in loop dominance is said to occur if and when $\partial \dot{x}/\partial x$ changes sign, that is, when the dominant polarity of the system changes [121].

However, Richardson acknowledged that not all shifts in dominance may be associated with changes in dominant polarity, and many of the definitions and results he proposed apply only to first-order systems.

During this time period, many of the previous words are still used to describe dominance, however new phrases are also introduced (Table 2.2). Some phrases indicate a subjective or relative sense of dominance: *accounts for the majority of, active, come into operation, exerting greater pressure, high elasticity,* and *significant.* Other phrases indicate an objective sense of dominance: *accounts for, essential to behavior,* and *generates.* Some words are used in both a subjective and objective manner, for example: *accounts for and accounts for the majority of.*

accounts for	essential to behavior	overpower
accounts for the majority of	exerting greater pressure	overtake
active	exerts increasing control	powerful
becomes active	fundamental cause of behavior	predominate
cause	generates	primarily responsible
cause the behavior	greatest effects on behavior	principally responsible
come into operation	high elasticity	produce
control	important	produce behavior
critical in producing	influential	relative importance
determine behavior	largely responsible for	significant
domination	lead to	significant influence
drives	most important	takes full control

Table 2.2: Words and phrases used to describe *dominate* or *dominant* in literature from 1975-1986.

During this time period, the term *dominance* is also applied to several new structures such as dominant links, effects, factors, components, mechanisms, and constraints. It is also applied to a variety of behavioral concepts such as dominant behavior modes, patterns, trends, and eigenvalues. This usage seemed to both expand and confuse what was meant by the term dominant. The phrase *shifts in dominance* was used by an increasing number of authors to explain sources of system behavior, often without formal explanation or definition. For example:

A nonlinear relationship causes the feedback loop of which it is a part to vary in strength, depending on the state of the system. Linked nonlinear feedback loops thus form patterns of shifting loop dominance- under some conditions one part of the system is very active, and under other conditions another set of relationships takes control and shifts the entire system behavior. A model composed of several feedback loops linked nonlinearly can produce a wide variety of complex behavior patterns [117, p. 33].

In this text, the words *strength*, *active*, *takes control* indicate something occurring mathematically but are not formally defined.

By this point, the sigmoid growth process, or logistics growth (Verhulst equation 2.2 [145]) emerged as the canonical example of *shifts in loop dominance* [48, 49, 88, 144, 124, 121, 37]. Logistic growth corresponds to the simplest of growth processes consisting of one reinforcing and one balancing loop, and nearly all dominance claims, examples, and methods were illustrated using this model.

$$\frac{dP}{dt} = aP - bP^2$$

where : $a \ge 0$ $b \ge 0$ P(t) = population at time t(2.2) This trend continued into the next time period as well, especially with the development of SD training and instructional material [85, 147, 44, 154, 90, 2].

Methodological developments during this time period were derived primarily from linear feedback control theory. In his thesis, Alan Graham offered the first focused attempt to generalize principles for inferring relationships between feedback loops and oscillatory behavior, drawing insights from classical control on how signals propagate through a system [52]. Similarly, Nathan Forrester conducted research on the sensitivity and elasticity of the eigenvalues of a linearized system, laying the foundation for future eigenvalue and eigenvector analysis in the field [38, 39]. Nathan Forrester's work was based on prior eigenvalue sensitivity research in modal control theory with applications to power systems by Perez-Arriaga in 1981 [109], tracing back to methods developed by Porter and Crossley in 1972 [112, p. 53]. Also motivated by linear analysis, Robert Eberlein researched how to simplify and reduce linear dynamic models by retaining specific behavior modes using selective modal analysis, drawing from dominant mode techniques in engineering feedback control [22]. This laid the foundation for future work on model simplification techniques for the purposes of identifying the minimum structure needed to explain behavior.

Richardson and Pugh were the first to describe an analytical procedure specifically for nonlinear systems that did not leverage linear systems theory [123, pp. 268-272]. They describe the *loop knock-out* experimental procedure in which a feedback loop is deactivated (for example, by making the associated time constant sufficiently large) and the results observed through simulation. If the simulated results are significantly different, the feedback loop may be dominant.

With all the methodological advances and the broad use of the term dominance, Richardson observed that the field was in need of more rigorous definitions. He helped establish the line of research called *dominance analysis* which emerged as one of the field's top priorities [120]. In establishing *dominance analysis* as a research focus, Richardson states:

There is a conspicuous gap in our literature between intuitive statements about loop dominance and precise statements about how to define and detect dominant structure.

This time period concludes with Richardson proposing specific research questions aimed at advancing the field of dominance analysis. Among his questions are how to precisely define *dominance*, *structure*, and *behavior*.

To move beyond intuition, two questions need to be answered: What do we mean by a particular pattern of behavior? and What do we mean by principally responsible? Precise answers would raise exciting possibilities. If we can define these terms formally and unambiguously, we might then be able to devise means of detecting, rapidly and with certainty, the dominant structures underlying the patterns of behavior exhibited by a model [120].

With dominance analysis identified as an important research topic for the next decade, attention now shifts to literature in which specific methods are developed and matured. The literature review no longer includes every instance of the term *dominant*, which by this point seems to have reached widespread use, but rather focuses on the *dominance analysis* literature which aims to develop methods for identifying dominant structures.

2.8 Period 4: Dominance Analysis Literature (1989 to Present)

Following Richardson's challenge, many new methods were developed to detect *dominance*, building upon the foundational research from the 70s and 80s, but each offering their own interpretation of *dominance*. At least 14 new definitions for dominance were proposed, making a total of 18 documented definitions (see Appendix A for a list of all definitions of dominance from the literature review). Although, of the 44 articles on dominance methodology during this time period, only a third explicitly define the term. Some new descriptive phrases for dominance also emerged. New phrases which describe *dominance* in a relative/subjective manner were used by researchers who proposed methods using normalized metrics to determine relative dominance. These terms include: *contribute most to*, *contribute positively in the same direction, contribute significantly to*, *having larger magnitudes*, and *mainly contributes*. Other new terms which indicate relative influence include: *largest gain, mainly influences, mainly responsible for, more important, outweighs*, and *stronger*. New phrases describing dominance in an objective manner also emerged such as *explains* and *power of changing*, indicating that the structure generates or determines the behavior.

As more models were evaluated, new phenomena were also observed which also led to new terms. For example, *shadow loop dominance* was introduced to describe situations in which two or more loops are required to be deactivated in order to change behavior [26]. The term *multiple loop dominance* was used to describe situations in which there exists more than one loop which independently affects behavior. A common theme throughout this period, however, is the lack of reference to a single formal definition or criteria for dominance. Each method introduces its own criteria, and new seemingly scientific terms are introduced but without formal and rigorous definition.

2.8.1 Dominance Analysis Methods

Dominance analysis methods can be categorized as either *exploratory methods* or *formal methods*, as recently summarized by Duggan and Oliva [20]. A similar dichotomy was offered earlier by Ford, who distinguished between *behavioral* and *structural* methods [26]. This thesis refers to *exploratory/behavioral* methods as those which use changes in simulated model behavioral as the criteria for dominance. The term *exploratory* refers to the exploration of

model behavior by changing different elements of model structure and then simulating. In contrast, *formal/structural* methods refer to those which analyze the structure (equations) of the model mathematically.

Exploratory/Behavioral Methods

With exploratory or behavioral methods, a structural element is dominant if it determines behavior, which is tested by deactivating the structure and examining how the system is affected. This is a counterfactual method of explanation, and the strength of this approach is the explicit connection between structure and behavior.

Ford's behavioral analysis method. Researchers discovered challenges applying concepts such as *loop polarity* and *dominant polarity* to develop intuition for large and complex models [100], and there was a clear need to establish methods which would scale. Ford introduced a behavioral approach to feedback loop dominance which formalized a routine for performing *loop knock-outs*, as described earlier by Richardson and Pugh. That is, systematically testing (through simulation) the deactivation of each loop in order to evaluate its impact on behavior [26, 111]. In doing so, Ford extended the concept of *dominant polarity* and defined three *atomic behavior patterns (ABP)* to classify behavior as linear (ABP = 0), logarithmic (ABP < 0), or exponential (ABP > 0), where:

atomic behavior pattern (ABP) =
$$\frac{\partial |\dot{x}|}{\partial t}$$
 (2.3)

Similarly, Saleh and Davidsen [129] described Ford's atomic behavior patterns as convergent (ABP < 0), and divergent (ABP > 0), and defined a normalized proxy measure for ABP to indicate convergence/divergence of a variable, calling it *Behavior Pattern Index (BPI)*:

behavior pattern index
$$(BPI) = \frac{\ddot{x}}{\dot{x}}$$
 (2.4)

Ford's definition of dominance built upon the earlier work of Graham, Richardson, and Pugh who defined dominance from a behavioral perspective. According to Ford, "A feedback loop dominates the behavior of a variable during a time interval in a given structure and set of conditions when the loop determines the atomic pattern of that variable's behavior" [26]. One limitation of Ford's method was that it did not specify exactly how to deactivate a feedback loop, which presents challenges when a loop does not contain a unique variable. Methodological extensions, such as the Generalised Loop Deactivation Method (GLDM), provide heuristics for deactivating loops which do not contain a unique control variable [111, 69], however, these extensions may not work in every situation.

Statistical screening. Statistical screening methods are also considered *exploratory/behavioral* in that the criteria for dominance is based on the sensitivity of simulated responses based on changes in model structure [25, 142, 143].

Formal/Structural Methods

A discussion then emerged regarding whether or not methods should identify a single or multiple dominant loops [99, 110, 78]. There was also a desire to understand not just which loop(s) were dominant, but which parts of the system were causing the loop(s) to be dominant. Ford indicated that perhaps both behavioral and structural approaches could be used together. Future research can further validate our procedure, expand our initial investigations of simultaneous dominance and shadow feedback structures and integrate our behavioral perspective with structural approaches to feedback loop dominance analysis [26].

Structural or formal methods (i.e. PPM, LEEA, DDWA, Loop Impact Method) were developed which assess the relative strengths of loops through normalized metrics derived directly from the equations in order to determine which loops and links are most influential. Formal/structural methods identify which structural elements are the most influential in determining behavior and rank orders the elements based on their relative sensitivity.

Loop eigenvalue elasticity analysis (LEEA). [38, 77, 129, 104, 128, 1, 57, 110, 78, 131, 79, 80, 96, 152, 46, 97] In this method, systems are linearized at each point along a variable's trajectory, and the eigenvalues are computed in order to identify the dominant behavior modes of the system. The feedback loops and corresponding links of each loop are identified. The sensitivity of the eigenvalues to each link gain is calculated using partial derivatives. From this, feedback loop sensitivities are calculated and then normalized to produce feedback loop elasticity values. This procedure is conducted for each feedback loop, and the loop with the largest elasticity is determined to be dominant at that local point in the trajectory. This procedure is then iterated along the entire state trajectory.

An important finding is that the number of loops in a maximally connected system with n state variables and p auxiliary variables grows more than factorially $(2^{np} \cdot (n-1)!)$ while the number of independent loops with N links only grows linearly: N - n + 1. The challenge, therefore, is how to identify and choose a suitable independent loop set (ILS), which may not be unique [77]. Different methods using graph theory have been developed to identify a minimum set of loops such as the shortest independent loop set (SILS) and minimal SILS (MSILS) [77, 129, 103, 104, 68, 71]. This remains one of the research challenges today for models in which a minimal SILS does not exist.

Eigenvector or dynamic decomposition weight analysis (DDWA). The dynamic decomposition weights analysis (DDWA) method improves upon LEEA to consider not only the entire system behavior, but that of specific variables of interest [132]. This method extends LEEA and examines sensitivity and elasticity of loops with respect to the eigenvectors of the linearized system [45, 46, 68]. While the eigenvalue and eigenvector based methods have been implemented in software such as *Mathematica* and have been tested against relatively simple models, there are no documented cases of applying these methods to realistic models [115].

One of the challenges for both eigenvalue and eigenvector methods is developing intuitive interpretations for loop elasticities [80, 132]. Additionally, it has been found that eigenvalue elasticities can be misleading and result in missed detections of high-leverage intervention points as well as false positives due to phantom loops in some highly nonlinear models [152, 132, 97].

Pathway participation method (PPM). In this method, local behavior is determined by the variable's first and second time derivative, as shown in Figure 2.1, resulting in nine possible local behavior patterns [99].

Mojtahedzadeh and colleagues use the *BPI* metric for a variable of interest, calling it the *Total Pathway Participation Metric (TPPM)*.

total pathway participation metric
$$(TPPM) = \frac{\partial \dot{x}}{\partial x} = \frac{\ddot{x}}{\dot{x}}$$
 (2.5)

The PPM algorithm evaluates the contribution of each causal pathway to the TPPM and identifies the dominant pathway as that which has the largest contribution. The procedure then iterates on the next most influential variable and continues until either a closed loop or



Figure 2.1: Nine possible local patterns based on the signs of the first and second derivatives.

exogenous variable is identified. The procedure identifies either a dominant feedback loop containing the variable of interest, a dominant pathway from the variable of interest to a feedback loop elsewhere in the system, or a pathway to a dominant exogenous variable.

Loop impact method. Similar to PPM, this method looks at the strength or impact of causal links as opposed to feedback loops and uses a metric similar to the PPM. It addresses one of the critiques of PPM and performs a breadth-first (as opposed to depth-first) search of the set of feedback loops that together have the largest impact and dominate the dynamics [58, 60, 59]. The Loop Impact method considers the cases of $\dot{x} = 0$ and $\ddot{x} = 0$ as transition cases, and simplifies the scheme in Figure 2.1 to the four local modes of behavior in the corners. Dominance is defined as the loop, or minimum combination of loops of like polarity, whose (combined) impact is greater than the sum of all loops of opposite polarity.

A summary of the strengths and limitations of each dominance method discussed, along with their criteria for dominance, is included in Appendix B.

2.8.2 Definitions for System *Structure*

Returning to Richardson's question of how to define *structure*, there have been primarily three ways to define sub-structure for the purposes of explaining model behavior: causal links, causal pathways, and feedback loops. Causal pathways consist of one or more causal links connecting a state variable (stock) to a derivative of a state variable (net flow). Feedback loops consist of one or more causal pathways which form a closed causal loop, as shown in Figure 2.2.



Causal links

Causal pathways = chain of causal links from a state (stock) to a derivative (net flow)

$$x \longrightarrow a_1 \longrightarrow \dots \longrightarrow a_n \longrightarrow \dot{x} = f(a_n(\dots(a_1(x))\dots))$$
$$= f(x)$$

Feedback loops = closed chain of causal pathways



Figure 2.2: Causal links, causal pathways, and feedback loops.

The causal links and feedback loops in an SD model can be succinctly defined as a directed graph in which the nodes represent the variables, and the links or edges represent causal relationships between the variables [77, 103]. Figure 2.3 illustrates how a simple birth and death model can be represented by a graph.



Figure 2.3: Graph representation of system structure [103].

The question of whether causal links, causal pathways, or feedback loops should be used to explain behavior has not been entirely resolved. However, the vast majority of methods focus on feedback loops as the explanatory element of system structure. Even pathway approaches, such as PPM, are constructed such that they are able to identify feedback loops (as a closed chain of causal pathways) as being dominant. Only a few papers focus on variables, parameters, or links as the influential elements of structure [55, 25, 153, 143, 132]. These approaches mainly look at the sensitivity of individual variables and parameters on the system behavior (or proxy measure of behavior). There are notable challenges with identifying feedback loops as the explanatory element of system structure. As noted above, the number of possible feedback loops in a maximally connected system, that is, where every state derivative depends on every variable (represented as a directed cycle within a directed graph), assuming the graph is strongly connected¹⁶ with N links and n state variables and p auxiliary variables grows by $(2^{np} \cdot (n-1)!)$ [77]. While it has been shown that the number of independent loops in such a set only grows linearly (N - n + 1), it has also been shown that for some models, a shortest independent set (SILS) cannot be generated [71], and in other cases it is not unique, therefore introducing subjectivity into how the loops are defined [98]. Another research challenge has been how to test a loop, independent from all other loops, once it has been identified [26]. While methods have been developed to test loops independent from one another, they are not adequate for all systems [111, 69].

Finally, it is interesting to observe that a few researchers have raised the question to what extent the very notion of feedback loops makes sense as explanatory structure [80]. The literature review did not reveal any papers that have specifically dealt with this question, although observations have been made about the limitations of being able to distinguish between the effects of feedback loops which share common links [60].

While progress has been made in defining structure and behavior independent from one another [26], it is clear there still exists multiple competing definitions in use today.

2.8.3 Definitions for *Dominance*

There have been two approaches to defining *dominance*. Exploratory/behavioral methods define dominance with respect to behavior (e.g. a feedback loop is dominant if its deactivation causes a significant change in behavior), whereas formal/structural methods define

¹⁶Strongly connected graph is one in which for any pair of nodes, there exists is a directed path in both directions.

dominance relative to structure (e.g. a feedback loop dominates if its relative influence is greater than that of other loops). The exploratory/behavioral methods detect dominance based on an objective criteria for how structure determines behavior, using phrases such as *determine*, *results in* and *cause*.

Structural methods, on the other hand, use subjective or relative terms for dominance, relating one piece of structure to another, such as *most influence*, *greater than*, and *larger*.

These two approaches to defining dominance lead to different questions and answers. Behavioral methods ask which loops determine behavior, and when. Structural methods ask which loops are more influential than others and how that influence changes over time. Both questions are important for explaining how structure produces behavior. Each captures a different dimension of dominance. The exploratory/behavioral methods employing objective criteria captures the structure-behavior dimension, whereas the formal/structural methods employing relative criteria capture the structure-to-structure dimension.

There is ample support for the position that both behavioral and structural approaches are needed to provide satisfactory explanations of system behavior. For instance, Nathan Forrester, who first introduced the eigenvalue methods to SD, states:

...the elasticities may not be clearly grouped by magnitude, in such cases the cutoff between "dominant" and "secondary" feedback loops is arbitrary [38].

Others using the eigenvalue approach have suggested that the methods could be used well in conjunction with behavioral approach.

In this context primarily the most dominant eigenvalue would be considered, - i.e. the eigenvalue with the most significant contribution, yet one may test the effect

of the second and/or the third and/or any higher order dominant eigenvalue on the behavior of a state [1].

2.8.4 Research Gaps in Dominance Analysis

While early feedback loop methods and definitions were being formalized and tested on small models, the question was raised as to the usefulness of the feedback loop concept for large-scale models [77]. The development of several dominance methods shows promise in this regard, however most have still only been tested on relatively small models. For all the methods, additional testing is required on large-scale nonlinear models [23, 17, 84, 77, 56, 1, 131, 69, 60].

Today, there remains significant limitations of not having a single, formal, and rigorous definition for *dominance*. The systematic review found multiple inconsistencies between methods which are based on different definitions and criteria for detecting *dominance*. These differences have been acknowledged and studied by several authors in the field [57, 110, 78, 80, 97].

It seems possible that in some of these instances, a formal definition of dominance would reveal that the methods being compared are asking different questions about the same model, explaining why they produce different answers. Additionally, the subjective nature of terms used to describe dominance in structure-behavior explanations, such as *highly influential* and *important* makes it challenging to infer precisely how and to what extent certain structures determine behavior.

The lack of a rigorous definition of *dominance* also makes it difficult to test, prove, or falsify assertions that have been made about dominance, such as, "Additive effects cannot dominate an expression; multiplicative effects can" [133]. It is also commonly asserted that

linear systems are incapable of shifting dominance [48, 126, 124, 37, 136]. This statement seems like it could be formally proven or falsified. However, while there have been compelling arguments and examples offered in support of this claim, using the superposition principle in linear systems theory and appealing to metrics such as dominant polarity, the systematic review found no instance of a formal proof of this claim. Furthermore, a counter-example is offered by Guneralp in which a linear system appears to exhibit a shift in dominance [56]. Upon closer inspection, however, the counter-example employs a different criteria for dominance than that used by previous authors. This illustrates the challenge of not having a common, formal definition of dominance.

It is also commonly claimed that in large systems only a few loops dominate [50, 53, 89]. This claim also seems testable, but to do so requires a precise definition of dominance which allows for more than one loop to be dominant. For many methods, such as LEEA, there is not objective criteria for the number of loops that are dominant [57], while other methods only identify a single dominant loop.

Important questions are being asked in the field which are challenged by the fact that a single formal definition of the term *dominance* does not exist. The following lines of research in dominance analysis address important research gaps and would benefit from a precise definition:

- Identifying the proper set of loops to analyze when more than one unique set exists [77, 98, 103, 57, 79, 80, 71].
- The manner in which to deactivate loops in tests for dominance [26, 111, 69].
- The threshold or criteria for establishing whether one, two, or multiple loops are identified as dominant [99, 110, 78].

• Understanding why some methods, in some cases, result in missed detections or false positive (e.g. phantom loops) [152, 132, 97].

Lastly, each dominance method operates in the time-domain and is not designed to detect how dominance changes across the state-space, and thus does not evaluate sensitivity to initial conditions [111, 153]. In fact, often it is advised to set the initial conditions such that the model begins in equilibrium, prior to conducting analysis, which inherently confines the analysis to specific regions of the state-space [123, p. 286].

2.9 Summary

In response to the challenges proposed by Richardson in 1986, the field has made considerable progress in developing methods to detect dominance. Methods have been developed based on deactivation of entire elements of model structure, and others based on marginal changes in model structure. Each have advantages and limitations and provide some explanation for how and why structure drives behavior, especially when combined with the intuition afforded by linear systems theory. However, there remains significant challenges. There are instances when different methods produce different results and some methods are not suitable for addressing certain classes of systems. Additionally, widespread use of dominance tools has still not occurred and there is a lack of real-world examples showing the relative utility of dominance methods over other methods, such as sensitivity analysis [115].

Kampmann and Oliva, 20 years after Richardson's initial challenge, reiterate the need to clearly define terms for the sake of progress in the field.

To the extent that we can both rigorously define and identify such dominant structures, we choose to say that we have found a 'theory' of the observed behavior [80].

The literature review concludes that, in fact, there still exists a need for a formal and rigorous definition of *dominance* to facilitate further advancements in the field¹⁷ (See Appendix C for a concise itemized list of the conclusions). The next chapter proposes a formal and rigorous definition for *dominance*.

¹⁷In conversations with loop dominance researcher Rogelio Oliva, after presenting my research topic at the PhD colloquium at the international system dynamics conference in Cambridge, MA in 2015, he stated there is still a need for robustly defining terms such as *dominance*.

Chapter 3

Defining Dominance

Towards the objective of Research Aim 1, this chapter develops a rigorous and formal definition of *dominance* for describing structure-behavior relationships in dynamic systems, by:

- 1. Considering the findings of the systematic review.
- 2. Considering how *dominance* is defined and used in engineering and mathematics.
- 3. Considering current methods of scientific explanation in the philosophy of science.

The following guiding principles are also considered:

- Do no harm. A new definition of *dominance* should not create additional confusion, rather it should promote understanding about how previous definitions relate to one another, and the implications. More importantly, a new definition should not contradict well-established results or principles in the field but should shed light on known discrepancies and inconsistencies.
- 2. Observing the prevalence of the term *dominance* in casual use, unaccompanied by formal definition, any new definition should also not be inconsistent with its general English meaning.

3. A formal definition should promote scientific progress by providing sufficient rigor such that claims and theories employing the definition are testable and falsifiable. The definition should facilitate the development of precise and testable questions in future research, thus promoting future advances in the field.

3.1 Use of the Term *Dominance* in Mathematics and Engineering

While it appears the phrases *feedback loop dominance* and *structural dominance* originate within SD, the term *dominance* occurs frequently in mathematics, engineering, and the sciences. Just as in SD, the word is sometimes used without formal definition and the meaning is inferred from its context.

In other instances, dominant or dominate carries a precise and formal definition. For example, in linear feedback systems the dominant root of a system's characteristic equation (also called the dominant pole, dominant frequency, dominant eigenvalue, or dominant mode) is the root which lies furthest to the right in the s-plane [105, p. 304]. The dominant pole is thus the one with the greatest real component and therefore the least stable, driving the overall dynamics of the system. For stable systems, the behavior mode associated with the dominant pole will take the longest to die out. For unstable systems, the dominant pole is the most unstable and will dominate the exponential growth in the long-term. The terms dominant mode and dominant frequency also appear in earlier system dynamics literature in reference to oscillatory systems, before the term was applied to feedback loops. Eberlein, in his thesis on model simplification through modal analysis, states:

Historically, the tendency has been to retain the "dominant," or least stable, modes, since these are normally the modes responsible for determining the overall character of the dynamic response [23].

In the eigenvalue dominance analysis methods, such as LEEA, part of the method requires identifying the dominant eigenvalues, however more than one eigenvalue may be considered dominant [38, 77].

This definition of dominance is defined in a relative fashion in that it compares the location of the eigenvalues relative to each other and identifies the dominant one as that which is furthest to the right. In other words, the definition requires that there will always be a dominant eigenvalue. If two or more dominant eigenvalues are clustered relatively close together and are collectively the furthest to the right in the s-plane, they may all be considered dominant, implying a somewhat subjective nature to the criteria for dominance.

However, there is also an objective and behavioral aspect to the definition in that the dominant eigenvalue is the limiting factor for how long it takes for a system response to decay, therefore it objectively drives the overall dynamic behavior of the system.

With the concept of dominant eigenvalue established in feedback control engineering, it was perhaps a natural progression within the field of SD, with its emphasis on structural explanations for behavior, to then look for the elements of structure with the greatest influence on the eigenvalues. This problem of finding eigenvalue sensitivity coefficients originated within linear systems theory [16], but SD then took the concept of influential parameters and extrapolated to influential links and feedback loops, calling them dominant. The relative nature of the term *dominance* seemed to be retained in that one could compare the sensitivities (or elasticities) of the different loops, however the objective/behavioral aspect seems to now be one step removed from actual behavior, in that loops influence eigenvalues, and eigenvalues influence behavior. To what extent, based on this relationship, can one particular loop be identified as dominant? To what extent can the effects of each individual loop be isolated and determined?

In other dominance-related expressions in mathematics and engineering (e.g. diagonal dominance and dominating series), dominance is determined by comparing the absolute values of the terms or components, and thus has a structure-relative nature. However, these concepts also have an objective aspect as well. In control theory, diagonal dominance imposes an objective, well-defined criteria for matrices. In mathematics, given two or more series, it is possible that none may dominate over another. In other words, there is an objective standard for identifying dominance.

It seems a formal definition of dominance should accommodate both a structure-relative aspect as well as a behavior-objective aspect. On one hand, structural dominance seems to be established by comparing two or more structural elements to each other (structurerelative) but it may also be the case that no structure elements dominate the behavior of a system, according to an objective/behavioral test.

3.2 Consideration of Methods of Scientific Explanation

There have been lengthy discourses within the philosophy of science on topics such as causality and what constitutes scientific explanation. This thesis considers the topic of scientific explanation as it pertains to two distinct but related activities. The first activity is modeling, and more specifically, the process by which dynamic processes are formally characterized by mathematical relationships. The second activity is analysis: the process by which a formal mathematical model is examined in order to explain behavior.

First, modeling. Pragmatically, engineers are concerned with constructing causal dynamic models in which equations are not merely statements of mathematical equivalence but reflect

causal processes. A causal model is such that the response (effect) to some input (cause) does not occur prior to that input (cause) in time [108, pp. 274-275]. This distinction is important because while it is possible (and not uncommon) to construct non-causal models to represent an ideal system (either for instructional purposes or to understand the limits of a system), engineers are primarily concerned with causal systems which are physically realizable¹⁸.

Similarly, dynamic models of social systems, as those typically constructed within SD, are considered to be causal mathematical models and have therefore been likened to scientific theories [7]. In such models, if an independent variable appears in the equation of a dependent variable, it presumes that a causal relationship exists. Richardson, in his thesis on the evolution of the feedback concept in social sciences [124], discusses how some have questioned the notion of causality in social sciences, but concludes that, "even if we do not know what causality means in social reality we can be precise about what it means in models of reality"¹⁹. J.A. Bell and J.F. Bell, in discussing different views of scientific knowledge, claim that (with respect to the refutationist view of knowledge), "Causal models are important because they are refutable." [117, p. 20]. Donella Meadows states, "The primary assumption of the system dynamics paradigm is that the persistent dynamic tendencies of any complex system arise from it causal structure..." [117, p. 31]. This position also seems consistent with contemporary thought-leaders in the field such as Kampmann, who describes the activity of system dynamics as theory building [80]. Under this position, models of dynamic social systems are also suitable for providing explanations [7], which leads to the next activity: analysis.

Some have argued that not only are system dynamics models suitable for providing explanation and understanding, but that is actually their primary purpose, and so they must [146].

 $^{^{18}}$ For example, the *ideal filter* is a mathematical construct that perfectly filters signals falling within a certain range of frequencies, but is actually non-causal (also called anticipatory) and thus impossible to physically realize.

¹⁹Interested readers are referred to Richardson's citations of the work by Herbert Simon.
Therefore, given a model, using analysis to generate understanding about the relationship between structure and behavior is of primary importance. Practically speaking, this perspective is motivated by evidence that people often misinterpret and fail to properly explain even simple cause-and-effect relationships in the presence of time delays and feedback (for example, exponential growth) [135, 19], and thus formal models can help improve understanding and intuition.

If, therefore, the goal is to generate understanding and explanations for behavior, what principles should be considered in this undertaking? And, does this have any bearing on how to think about and define dominance?

Authors have examined various scientific philosophies such as logical positivism, critical rationalism, relativism, and internal realism to understand the suitability of dynamic systems models for scientific inquiry [146]. One principle that appears to be a common theme is that of internal realism, or, closely related, the concept of face validity (also called structure validity, representational validity, or internal validity) [74, 6].

Explaining and understanding the behavior of a system requires the identification of some mechanism or structure that, from the standpoint that guides the SD model-building process, may be considered as the one which brings about the behavior [146].

This principle requires a model to include the most realistic content possible and to use variables and variable names that directly correspond to ideas and concepts as they are known by observers and users in the real system. This was also a key insight by Repenning, reflecting on his failure to provide clear model-based explanations to his audience [119].

Historical critique of dominance methods provides further evidence that explanations of behavior are unsatisfying if resting solely on mathematical artifacts not anchored in realworld concepts. This has been one of the critiques of eigenvalue elasticity analyses [80]. Another practice that often appears in model-based explanations is the use of counterfactuals. A counterfactual tests the conditional statement that had a cause in question not occurred, the effect would not have occurred. Some conclude that all major attribution theories are based on counterfactual information (attributed to MG Lipe by Rahmandad and colleagues [116]). While there has been criticism of the counterfactual theory of causation, there have also been arguments for its use and evidence of its role in explaining behavior [13].

In practice, counterfactuals have been used throughout SD to provide explanations. The earliest definition of dominance employs a counterfactual argument [52]. Subsequent definitions of dominance also utilize counterfactual tests [26]. Finally, the most recent work by a thought-leader in the field concludes:

...doing the counterfactual tests is indispensable for building our understanding, convincing a larger audience and being honest about the results [114].

In summary, two key components of successful scientific explanations utilizing mathematical causal models is internal-realism and counterfactual testing.

3.3 Definition of Behavior

The systematic review concluded that a rigorous definition of *dominance* requires *behavior* and *structure* to be defined independent of each other.

First, behavior is defined for a single state variable (stock) of interest. While some authors have sought to explain the behavior of all system variables through the use of a single metric [129], summary function [153], or the system eigenvalues [78], there have been challenges interpreting these metrics, and there remains a desire to understand behavior of individual

variables. The majority of dominance methods have therefore sought to explain the behavior of a particular variable of interest. This does not imply that system-wide behavior is unimportant. On the contrary, the relationship between different variables (as captured in phase portraits in the state space) is critical in nonlinear systems analysis. However, data used to validate a model or compare results is typically based on individual variable behavior over time. Being able to explain individual variable behavior also aligns with the principle that explanations should correspond to real world concepts. If a model has a high degree of internal realism, the individual model variables will correspond to real world concepts while aggregate functions of variables may not. Starting at the level of the individual variable, the relationships between two or more variables and system-wide behaviors can be constructed and explained. It is worth noting that the dynamic decomposition weights analysis (DDWA) method extended from the loop eigenvalue elasticity analysis (LEEA) method in order to provide insights for individual variables as opposed to system-wide behavior [132].

Second, behavior can be defined for a specific operating point (either a point in state-space, or, equivalently, a point in time along a state trajectory). Behavior is then defined over state-space regions and time-intervals consisting of a connected set of points which have the same behavior. Nearly every loop dominance method (behavioral and structural) follows this approach, quantifying behavior at both specific points as well as time-intervals in a simulation. Methods are then iterated at multiple points along a variable's trajectory to identify global behavior patterns.

Third, behavior patterns are defined by the signs of the variable's first and second time derivative as shown in Figure 3.1, following the tradition of Ford, Saleh, Davidsen, Mojtahedzadeh, and many others. Zero values of the first and second derivatives indicate transitions between behavior patterns.

More complex behavior patterns such as S-shape growth and oscillations are identified by sequences of these four local patterns.



Figure 3.1: Four behavior modes based on the signs of the first and second time derivatives.

Fourth, and finally, the second time derivative is considered to be the primary measure of local behavior and shifts in behavior²⁰. Many authors have discussed the importance of examining the second time derivative [26, 129, 130, 99, 110, 111, 96, 69, 60]. Most recently, Hayward and Boswell motivate the use of the second time derivative based on Newton's law in physics, which states that *acceleration* of an object is proportional to the *netforces* acting upon the object [60]:

$$acceleration = \frac{1}{mass} \times net \ force$$
 (3.1)

Since mass is non-negative,

²⁰Some have referred to the second time derivative as *curvature* ([129, 60]). However, curvature carries a different, distinct mathematical definition, so this term is not used.

$$\operatorname{sgn}(acceleration) = \operatorname{sgn}(\ddot{x}) = \operatorname{sgn}(net \ force)$$
(3.2)

The systematic review of dominance also revealed that the word *force* has frequently been used to describe internal structural influences on behavior. For example, in *Study Notes in System Dynamics*,

...exponential growth occurs only as long as the growth forces within the system can dominate retarding forces. However, as system variables grow, "negative" or controlling feedback forces must eventually overtake the positive feedback forces as determinants of sys behavior [49, p. 28].

Similar instances of the word *force* are found throughout SD literature. While the meaning of the term *force* here is different than the term *force* in Newtonian mechanics, there are interesting similarities, as has been discussed recently by Hayward, a physicist, who argues feedback loops can be interpreted as applying stabilizing or destabilizing forces on the system [59]. The chain rule of differentiation, in fact, allows one to decompose the second derivative, which can be thought of as acceleration, into a sum of contributions from different causal structures, in the same way net force can be decomposed into its distinct sources (as commonly done using free-body diagrams, such as the one in Figure 3.2 which applies to the car example in Chapter 1). Therefore, the sign of the second derivative indicates the orientation of the net force that results from the different forces (from causal mechanisms) acting upon the variable of interest.

All dominance methods which use the second derivative (i.e. Atomic Behavior Pattern, Behavior Pattern Index, and Total Pathway Participation Metric), also use the first derivative, by taking the ratio of the second derivative to the first derivative. This ratio, however, is



Figure 3.2: Free-body diagram of the individual forces acting upon the mass of the car in the spring-mass-damper system.

an unnecessary normalization. The motivation for this ratio traces back to Richardson's metric of *dominant polarity* [121] (Equation 3.3), which can also be expressed as the ratio of the second derivative to the first derivative by the chain rule of differentiation, as noted in Chapter 2.

dominant polarity = sgn
$$\frac{\partial \dot{x}}{\partial x}$$
 = sgn $\frac{\ddot{x}}{\dot{x}}$ (3.3)

This metric was motivated by understanding how, in minor feedback loops containing x, how fast the derivative of x changes with respect to x. In minor loops where changes in x lead to bigger changes in x, dominant polarity is positive and the minor loop is said to be goal-divergent (positive feedback), otherwise it is goal-seeking/convergent (negative feedback). However, the dominant polarity metric, when calculated via the chain rule for

major feedback loops (loops containing more than one state variable) is not a true indicator of convergent or divergent behavior in nonlinear systems [100].

Distinguishing between convergent and divergent behavior can be determined simply by noting the signs of the first and second derivatives and does not require taking the ratio of the two. The actual contribution of each pathway or loop to divergence or convergence is determined by their contributions to the second derivative, not the first. In the metrics discussed (BPI and TPPM), the first derivative is simply divided into the influence metrics for each causal pathway/loop and thus does not change their relative influence, and thus provides no additional information about dominance. Furthermore, metrics such as BPI and TPPM which divide by the first derivative are undefined when the first derivative is zero.

The proposed definition of behavior, i.e. the second derivative, is independent of how structure is defined. It also has face validity in that it directly corresponds to behavior over time for a single model variable and does not rely on a normalized metric that is a proxy for behavior, or abstract constructs such as eigenvalue elasticities which lack straightforward interpretation.

The sign and magnitude of the second derivative is well-defined and can also can also be calculated for twice differentiable systems in a straightforward fashion so that it can be used in both an objective/behavioral sense (i.e. counterfactual tests can be performed to determine if the sign changes) as well as a subjective/relative structure-to-structure sense (by comparing the contributions to the second derivative from each structural element).

Finally, the second derivative is sufficient for distinguishing between convergent and divergent behavior, given the sign of the first derivative is known, and indicates inflection points (zero values) which seem to be associated with shifts in dominance observed in S-shape growth.

3.4 Definition of Structure

The fundamental explanatory element of system structure is defined as the causal pathway from one state variable to the derivative of the state variable whose behavior is being examined (*note*: these state variables can be the same, in which case the causal pathway also defines a minor feedback loop). There may be more than one causal pathway from one state variable to the derivative of another, distinguished by different auxiliary variables, as shown in Figure 3.3.



Figure 3.3: Causal pathways are composed of one or more causal links which may include auxiliary variables.

This deserves some explanation given the overwhelming emphasis on feedback loops since the foundation of SD. Forrester states that information feedback is the first and most important foundation for industrial dynamics [29, p. 14]. In his reflections after the first decade, he states that, "the feedback loop is seen as the basic structural element of systems" [33]. In his text summarizing the principles of systems, Forrester states that, "...feedback loop is the basic unit of which systems are composed." [35, p. 2-39]. Even more, this is also seen as a philosophical distinction of the field:

The SD approach... takes the philosophical position that feedback structures are responsible for the changes we experience over time [123].

This position has also been cited by other authors in the field as a defense for looking at feedback structures [46].

Graham equates finding the dominant loops as equivalent to identifying the fundamental causes of the system's behavior [51]. In Richardson's thesis "The Evolution of the Feedback Concept in American Social Science" [124], he states:

The "cause" of an arms race is viewed not as a given event or even a given sequence of events, but as a feedback structure dominated by self-reinforcing positive loops, within which events take place. The causal view in this thread is summarized in the assertion that "patterns of dynamic behavior are consequences of feedback structure."

Furthermore, in course material developed by the MIT SD Group for the Guided Study Program in SD (1999), feedback loops are used as the primary unit of explanation of behavior. Finally, the systematic review finds that the vast majority of loop dominance methods consider feedback loops as the structural element of interest, while only a few consider causal links or pathways as the structural element of interest.

In light of this evidence, one may justifiably wonder whether or not, at least within the SD tradition, the question of whether or not feedback loops are the most basic unit of structure is even up for debate. However, the literature review also reveals there has consistently been more than one aspect to explanations of behavior, which are not mutually exclusive, but rather different perspectives of the same explanation. For example, in *Industrial Dynamics*, Forrester, after stating that the most important foundation is the concept of information-feedback, continues, "...the interactions between system components can be more important than the components themselves [29, p. 14]." Later he states, "...information-feedback systems owe their behavior to three characteristics: structure, delays, and amplification. The structure of a system tells how the parts are related to one another." Common to both of these statements is the important role of *interactions* and *relationships* between elements

of structure. Throughout the text, such interactions are a central theme in what causes behavior. The systematic review found that the terms *nonlinear relationships* and *nonlinear interactions* arise frequently throughout SD literature (going as far back as Forrester's original D-memo in 1956) and nonlinearities are consistently implicated as the cause for shifts in loop dominance. The evidence from the review suggests that it is the nonlinear interaction between the feedback loops that gives rise to complex system behavior.

This also makes intuitive sense as different feedback loops (described as directed cycles in a graph) intersect with one another at one or more common nodes or directed arcs, which are the intersections of two or more causal pathways. The nonlinear relationships which have been described to be fundamental to the rise of system behavior is represented in the rate equations by the nonlinear combination of two or more causal pathways which are themselves, functions of state variables.

There is also a pragmatic argument for considering causal pathways as the explanatory element of system structure. Suppose a method identifies a single feedback loop as that which dominates behavior. Does it then follow that all the pathways within the loop are of equal importance in influencing the behavior, or might some be more influential than others? For the purposes of identifying high leverage places to intervene in the system, one must identify specific places of intervention, and thus practically speaking, specific causal pathways. It would be helpful to understand the relative sensitivity of the pathways that make-up the dominant loop. In fact, this is precisely how all formal/structural dominant methods are constructed, by first identifying the relative influence of the individual links, and then determining the dominant loops [77, 99]. The PPM method considers causal pathways as the building blocks of feedback loops. Eigenvalue methods (such as LEEA) compute sensitivities and elasticities at the link level first, and then are synthesized at the loop level.

Consider also the case in which a method identifies multiple dominant loops. A natural question might be whether or not there are certain causal pathways these dominant loops

have in common. There is also evidence to suggest that in some cases, causal pathways may be better suited than feedback loops to explain some transitional dynamics, where behavior is shifting from one mode to another [96].

Therefore, by first considering causal pathways as the basic element of system structure, inferences can then be made about the role of feedback loops containing the pathway. If there is only one such loop, this exercise is trivial. However, if more than one loop shares a common pathway, the next question is to what extent can their separate influences be distinguished.

Causal pathways are distinct from causal links in that they connect a derivative (net flow) to a state variable (possibly through one or more auxiliary variables), whereas causal links may trace back to a single auxiliary variable. The reason to consider causal pathways as fundamental elements of structure (as opposed to causal links) is the state-determined nature of systems [136, p. 202]. The concept of states (or stocks or accumulations) is fundamental to system dynamics, and is fundamental to dynamic systems theory in general. It is more fundamental than the concept of feedback loops since feedback loops by definition must consist of at least one state variable [35]. The systematic review also revealed the important role of states and found multiple references of systems being dominated by the state.

In dynamic systems theory, it is the concept of *state*, and specifically, properties built upon the concept of state that allow linear and nonlinear dynamic systems to share many common properties (e.g. existence and uniqueness of solutions and the semigroup property²¹) [12, 82]. The state captures the memory or history of the system. Apart from outside influence, the

 $^{^{21}}$ The semigroup property implies that the solution (trajectory) does not depend on future inputs and that any future state can be computed from any past state (initial condition) [12].

future trajectory of the system depends only on the structure of the system (equations) and on the current state, and not how it arrived at that $state^{22}$.

Another argument against considering feedback loops as the explanatory structural element is that there have been several challenges associated with using feedback loops to explain behavior, as seen in the systematic review. Some authors have raised the question of whether or not the feedback loop concept is useful in large-scale models, which has motivated research to find algorithms for identifying independent loop sets and methods for testing each loop individually [77]. Currently, it has been shown that independent loop sets may not always be unique, thereby introducing subjectivity in the analysis process [97]. Additionally, it may not always be possible to isolate the effects of a single loop that shares all its links with other loops [69]. These challenges demonstrate the difficulty that can arise in isolating the effects of a particular feedback loop from another. Finally, it is worth noting that Kampmann and Oliva [80], in discussing some of the limitations of LEEA method, admit that loops are more of a derived and relative concept rather than a fundamental building block²³.

In summary, the proposed definition of structure aligns with the state-determined view of systems and emphasizes the role of nonlinear coupling of states through their causal pathways in producing system behavior (Figure 3.3). The causal pathways to state variables, and their interactions, are seen as the important piece of structure. For dynamic systems, causal pathway interactions often result in feedback loops involving more than one stock variable, otherwise, the systems can be decoupled and analyzed as separate systems. Therefore, this perspective is not in conflict with those that consider feedback loops as the fundamental

 $^{^{22}}$ As a secondary observation from the literature review, the concept of state seems to have taken a lesser role in the SD tradition, however both state and structure (equations) play a necessary role in determining behavior, neither alone is sufficient. Perhaps the maxim, "structure drives behavior" would be better modified: "structure and state drives behavior."

²³There is an interesting circular relationship between how dynamic hypotheses are initially developed (by postulating certain balancing and reinforcing feedback loops), and how subsequent explanations of simulated behavior are sought using those same feedback loops [46]. While not a central argument to this thesis, this may also explain why the majority of dominance methods consider feedback loops as the fundamental explanatory element of structure.

structure, but presents an alternate view, and one that may provide more rigorous and formal structure-behavior insights.

We conclude by also observing how this perspective seems to align well with a view offered by Forrester in his original text:

They [real systems of importance] are unstable, tending toward increasing amplitudes of oscillation that are contained by a continuously shifting balance of forces among the system nonlinearities. Our social systems are highly nonlinear and most of the time are operating against limitations of overemployment, politically unacceptable employment, money shortage, pressures to overcome inflation or recession, or inadequacy of capital equipment. It seems likely that such nonlinearities, coupled with the unstable tendencies caused by amplifications and time delays, create the characteristic modes of behavior that we see in free-enterprise economic systems [29].

This passage employs the term *forces* within a system, which has conceptual ties to acceleration. It implicates nonlinear relationships as a cause of behavior. Also limiting the behavior of the system are a list of factors that appear to be stocks (state variables), and thus the states are an important cause of behavior. Nonlinear interactions coupled with amplifications (gains) and delays (which are stocks or states), is what creates the behavior modes. In other words, the nonlinear interaction of states, each affected by different gains (from causal pathways). Finally, an interesting observation is that the term *feedback loop* is not mentioned in this passage.

3.5 Criteria for Dominance

With *behavior* and *structure* defined independent of one another, this section now defines the criteria for structure dominating behavior and proposes a formal definition. Both behavioral/objective and structural/relative aspects of the term *dominance* are addressed and are depicted visually in Figure 3.4.



Figure 3.4: Two dimensions or aspects of the criteria for dominance.

- 1. Behavioral/objective criteria for structure determining behavior. This criteria addresses the objective nature of the term *dominance*, in which dominance indicates that structure in some way causes or determines behavior.
- 2. Structural/relative criteria for structure dominating over other structures. This criteria addresses the relative nature of the term *dominance*, in which dominance indicates one structure being dominant over (or relative to) another element of structure.

3.5.1 Behavioral/Objective Criteria for Dominance

Dominance, according to the behavioral/objective criteria, implies that structure in some way leads to, gives rise to, determines, produces, generates, or causes behavior. These words, however, require greater precision. A few definitions from the literature review provide help:

A feedback loop dominates the behavior of a variable during a time interval in a given structure and set of conditions when the loop determines the atomic pattern of that variable's behavior [26].

This criteria for dominance uses a counterfactual test, as previously discussed. Several other definitions of dominance also use the counterfactual as the criteria for dominance, tracing back to the first definition by Graham [52]. Counterfactuals test for *necessary conditions* or *necessary causes* and have been a fundamental method of determining causality, both within and outside of SD.

Necessary Causes

A structure S is a *necessary cause* for behavior B if the existence of S is necessary in order to produce the behavior B. This can be written in any of the following logically equivalent forms:

> S is necessary for B B implies S $B \rightarrow S$ if B then S

To test a necessary cause, one can test a logically equivalent form of the above statement by changing the direction of causality (the arrow) and negating each of the components [113, ch. 3]. The result is:

not S implies not B $\bar{S} \to \bar{B}$ if not S then not B

Or, in other words, if the structure S is deactivated, the same behavior B should not be observed. This is precisely the counterfactual test described above, thus behavioral dominance methods such as Ford's lead to the detection of necessary structure. For the proposed definition of structure and behavior, a test for necessary causes would be to deactivate a causal pathway to the variable of interest at a specific point in time, and observe if the sign of the second time derivative changes. If it does, this implies that the causal pathway is a necessary cause, otherwise it is not a necessary cause.

Sufficient Causes

The logical dual of a necessary cause is a sufficient cause²⁴. Whereas a necessary cause indicates that the cause is required or necessary for an effect to occur, a sufficient cause is one such that its existence is sufficient to produce the effect. If the cause S occurs, then the effect B will always occur, however B does not require S. This can be written in any one of the logically equivalent forms:

Structure S is sufficient for Behavior B

S implies B

²⁴In logic, operators or relations $N(\cdot)$ and $S(\cdot)$ are duals if $N(\bar{X}, \bar{Y})$ is logically equivalent to $\bar{S}(X, Y)$, where X are Y are the components being related/operated on, such as X = cause, Y = effect [113, p. 80].

$S \to B$ if S then B

Using the proposed definition of structure and behavior, if a causal pathway to a variable of interest is a sufficient cause, it implies that the pathway alone is sufficient to determine the sign of the second derivative, regardless of the contributions from any other causal pathway. An example from the literature review of *dominance* describing a sufficient condition or cause is:

Exponential growth is generated by the dominance of positive feedback loops over the equilibrating tendencies of negative feedback loops [3].

In this example, what is being stated is that the dominance of positive feedback loops is sufficient for generating exponential growth, thus structure implies behavior, which is a statement about sufficient causes. At least one dominance method, The Loop Impact method, focuses on identifying sufficient causes for determining dominance [60].

A cause may be necessary, but not sufficient. Likewise, a cause may be sufficient but not necessary. Finally, causes may be both necessary and sufficient, or they can be neither. The systematic review found instances in which the term *dominance* corresponded with both necessary and sufficient conditions. For example,

Wherever there is a dominant positive feedback loop of this form, exponential growth will be observed. Wherever exponential growth is observed there must be a positive feedback loop of this type [91].

In this statement, the first sentence describes a sufficient condition (structure implies behavior), whereas the second sentence describes a necessary condition (behavior implies structure). Additionally, causes that are neither necessary nor sufficient may still contribute to the effect and be considered important or highly influential. This leads to the second criteria for dominance, that of structure relative to other structure.

3.5.2 Structural/Relative Criteria for Dominance and Contributory Causes

In some fields (such as clinical fields), it has been useful to consider partial or contributory causes which are unnecessary and insufficient, but which requires that the cause precedes the effect and that altering the cause alters the effect [125]. This type of cause may relate to the dominance detected by formal/structural methods which use a structural/relative definition of dominance, based on comparing the values of a normalized metric for different elements of structure. These methods do not determine whether or not the elements are necessary or sufficient, however they do quantify the relative influence of each element of structure. Applying this aspect of dominance to the proposed definition of structure and behavior, if one causal pathway contributes more to the second derivative than another causal pathway, it may be considered dominant over that pathway in a structure-relative sense.

Lastly, there are structural elements that do not contribute to the effect. They are neither sufficient, necessary, nor contributory. They may either be neutral or contrary to the observed behavior. The five types of causes discussed are summarized in Table 3.1.

Type	Description	Necessary?	Sufficient?	Logic Statement
1	necessary cause	yes	no	effect \rightarrow cause no cause \rightarrow no effect
2	sufficient cause	no	yes	cause \rightarrow effect no effect \rightarrow no cause
3	necessary and sufficient cause	yes	yes	$\begin{array}{c} \text{cause} \leftrightarrow \text{effect} \\ \text{no cause} \leftrightarrow \text{no effect} \end{array}$
4	contributory cause	no	no	N/A
5	none of the above	no	no	N/A

Table 3.1: Five types of causes.

Applying the five types of causes to the proposed definitions of structure and behavior, leads to the following different types of causal pathways, along with their tests, as shown in Table 3.2.

Type	Description	Necessary?	Sufficient?	Test
1	necessary pathway	yes	no	$\operatorname{sgn} \ddot{x} \to \operatorname{pathway}$ no pathway $\to -\operatorname{sgn} \ddot{x}$
2	sufficient pathway	no	yes	pathway $\rightarrow \text{sgn}\ddot{x}$ $-\text{sgn}\ddot{x} \rightarrow \text{no pathway}$
3	necessary and sufficient pathway	yes	yes	pathway $\leftrightarrow \operatorname{sgn} \ddot{x}$ no pathway $\leftrightarrow -\operatorname{sgn} \ddot{x}$
4	contributory pathway	no	no	$\operatorname{sgn} \operatorname{pathway} = \operatorname{sgn} \ddot{x}$
5	none of the above	no	no	$\operatorname{sgn} \operatorname{pathway} = -\operatorname{sgn} \ddot{x}$

Table 3.2: Five types of causal pathways.

3.6 The Dominance Framework

Consider that the behavior (second derivative) of a variable of interest at a specific point in time can be affected by multiple causal pathways, where each pathway can be classified as one of the five types in Table 3.2. This creates a framework for thinking about dominant structure-behavior relationships, where at any given time, the behavior of a variable may be subject to one or more necessary causes, one or more sufficient causes, a single necessary and sufficient cause, or lastly, only contributory causes. This dominance framework for describing different combinations of causes is illustrated in Figure 3.5.

An example of multiple necessary pathways, point (0, n) in Figure 3.5, is when a system is in dynamic equilibrium and thus all forces are in perfect balance resulting in zero velocity and acceleration. If any force were to change, the system would no longer be in equilibrium and the sign of acceleration would change. Therefore, at the moment of equilibrium, each causal pathway exerting a force is a necessary pathway for maintaining equilibrium, and likewise no force by itself is sufficient to maintain equilibrium.

Dominance Framework



Figure 3.5: Dominance framework depicting the number of necessary and sufficient pathways that determine the behavior of a variable of interest.

An example of multiple sufficient pathways, point (m, 0) in Figure 3.5, is when all the causal pathways are pushing the behavior's acceleration in the same direction. Each causal pathway alone would be sufficient to produce an acceleration (since none are opposed), and likewise none are necessary because of the presence of the others.

An example of a single necessary and sufficient pathway, point (1, 1) in Figure 3.5, is when there are only two pathways, each opposing each other and pushing on the variable of interest in opposite directions. Whichever pathway is exerting the greatest force would, at that time, be both necessary and sufficient for determining the sign of the second derivative.

Lastly, an example of only contributory causes, point (0, 0) in Figure 3.5, in which none are necessary nor sufficient, is when the sign of acceleration is determined by the sum of many small contributions from a large number of causal pathways that overwhelm the opposing forces with sufficient margin. The removal of any one pathway does not change the overall behavior (and hence no pathway is necessary), but no pathway on its own is sufficient to overcome the opposing forces (and hence no pathway is sufficient).

3.7 Formal Definition of *Dominance*

Using the above dominance framework, the following precise definition for *determine* is proposed:

A structure *determines* behavior if and only if the structure is both necessary and sufficient for producing the behavior.

Applying this to the proposed definition of structure and behavior, results in the following proposed definition of *dominance*:

Given a specific variable of interest and point in time along its trajectory, a causal pathway to that variable is *dominant* if and only if it is both necessary and sufficient for determining the sign of the variable's second derivative.

By implication, therefore, it can also be said that if a causal pathway is dominant, it is both necessary and sufficient for determining the dominant polarity as well. In the system dynamics field, a variable is said to be dominated by reinforcing loops if the polarity is positive and balancing loops if negative. This definition provides a rigorous criteria for identifying which loop, specifically, is dominant. **Claim:** For a behavior of interest at a specific point in time, if there exists at least one necessary pathway and at least one sufficient pathway, those pathways are one and the same (that is, a single pathway is both necessary and sufficient).

Proof:

Suppose pathway A is sufficient and pathway B is necessary.

Sufficiency of A: $A \rightarrow behavior$

Necessity of B: behavior $\rightarrow B$

Therefore, $A \rightarrow behavior \rightarrow B$, or $A \rightarrow B$

Since no conditions of existence are assumed between different elements of structure, that is, causal pathways A and B are evaluated separately, this only holds if A and B are the same pathway. \Box

Claim: By similar argument, there can be at most one pathway that is both necessary and sufficient. Suppose there is more than one pathway that is necessary and sufficient (for example, A and B). Since A and B are both necessary, neither of them alone can be sufficient. Likewise, if A and B are both sufficient, neither can be necessary. \Box

3.8 Discussion

The dominance framework offers a lens through which the various descriptions and definitions of dominance can be interpreted. Chapter 4 further develops and tests this definition against previously studied models and methods. Examining the frequency of each case in Figure 3.5 in previously studied models could shed insight into the discrepancies between different methods. It seems plausible that if a model contains a single necessary and sufficient structure, that regardless of whether methods identify necessary, sufficient, or contributory causes, there would be a high likelihood they would identify the same dominant structure. This would also explain how despite their different approaches, dominance methods are able to produce similar insights (Mojtahedzadeh, 2008). This would also explain why in some cases they might lead to different insights and conclusions. For instance, if a model contains multiple loops which are found to be influential (as in the case of shadow dominance, shared dominance, and multiple dominance), perhaps these are cases in which there is no single necessary and sufficient structure, and that the system is operating outside the point (1, 1) in Figure 3.5. Chapter 4 investigates this hypothesis, examining models for which different methods have produced slightly different answers.

On a similar note, the systematic review revealed that phrases used to describe *dominance* fall into two categories: objective with respect to behavior, or relative with respect to structure. For relative terms used such as *important*, *influence*, and *powerful*, they are not objective in the sense that these phrases alone do not convey whether the cause may be necessary or sufficient, so it is possible they fall into the class of contributing causes, point (0, 0) in the framework. Many of these words are used by formal/structural methods in which dominance is based on the relative value of a normalized metric, and in a sense indicates the level of contribution. In fact, the word *contribution* did not appear until these types of methods were introduced (e.g. PPM, LEEA, etc).

Examining the objective phrases used to describe dominance, some appear to indicate necessity but not sufficiency, while others sufficiency but not necessity, while others seem to indicate both necessity and sufficiency.

Terms that convey sufficiency but perhaps not necessity would be *achieving*, *produce*, *generates*, and *lead to*. These terms do not eliminate the possibility that other factors may also achieve, produce, generate, or lead to the same behavior. Thus, these terms may imply

sufficiency but not necessity (associated with points along the horizontal access in Figure 3.5).

Likewise, objective words which give a sense of necessity but perhaps not sufficiency, include: determine behavior, necessary to produce, critical in producing, essential to behavior, critical in determining. These words indicate that if the cause were not there, the effect would be different, but they do not imply that the cause is the only necessary cause or that it alone is sufficient to create the behavior. Thus, these words convey a sense of necessity, but not sufficiency (associated with points along the vertical axis in Figure 3.5.

Finally, other words for dominance which convey both necessity and sufficiency include: causing, control, govern, responsible for, accounts for, takes full control, associated with point (1, 1) in Figure 3.5). Perhaps thinking about necessary, sufficient, and contributory causes will facilitate using the most appropriate adjective when explaining structure-behavior relationships, whether the term dominant is used or not. This dominance framework can help increase precision in how structure-behavior explanations are provided and how dominant structure is identified. It can also serve as a foundation for understanding and developing new methods of dominance analysis. The proposed dominance framework and definition should facilitate mathematical explanation of observed phenomena such as shadow dominance, shared dominance, multiple dominance and shifts in dominance. Towards this goal, Chapter 4 tests the proposed dominance framework and definition against simple models which have been well-studied by other methods of dominance analysis.

Chapter 4

Applying and Testing the Definition of Dominance

This chapter begins with a concise summary of the proposed definitions for *behavior*, *structure*, and *dominance* from Chapter 3. Using these definitions and the *dominance framework* from Chapter 3, a procedure for identifying dominant structure is developed and tested against several forms of the logistic growth model. The procedure is compared with other dominance methods, and relationships are identified between each method. The chapter concludes with a summary of insights and conclusions from Research Aim 1, on the proposed definition and procedure for determining dominance.

4.1 Definitions and Implications

Consider the dynamic system described by the following n^{th} -order ordinary differential equation (ODE):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

where,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$
(4.1)

where each f_i is differentiable in **x** and in u_i

Towards the goal of identifying elements of system structure which dominate behavior, the following definitions are proposed for *structure*, *behavior*, and *dominance* (see Chapter 3 for details):

Behavior. Behavior is defined as the sign of the second time derivative of a state variable of interest, $x_i(t)$, evaluated at a specific point t_0 along the state's trajectory.

behavior of
$$x_j(t)$$
 at time $t_0 = \operatorname{sgn} \ddot{x}_j(t_0)$

Structure. The explanatory elements of system *structure* are the immediate causal pathways from state variables (x_1, x_2, \ldots, x_n) to the first time derivative of the state variable of interest $\dot{x}_j(t) = f_j(\mathbf{x}, u_j)$, as shown in Figure 4.1. A causal pathway $p_{ijk}(x_i)$ is a scalar function representing a causal process which maps a *cause*, state x_i , to an *effect*, state derivative \dot{x}_j . As discussed in the previous chapter, there may exist more than one causal pathway (representing different causal mechanisms) from x_i to \dot{x}_j . In such cases, p_{ijk} is the k^{th} pathway from x_i to \dot{x}_j .



Figure 4.1: Immediate causal pathways from state variables (x_1, x_2, \ldots, x_n) to the derivative of the state variable of interest (\dot{x}_j) .

Dominant. Given a state variable $x_j(t)$ whose behavior is of interest, and point t_0 along its trajectory, causal pathway p_{ijk} is *dominant* if and only if it is both necessary and sufficient for determining sgn $\ddot{x}_j(t_0)$.

4.1.1 Decomposing *Behavior* into Pathway Contributions

From Equation 4.1, the dynamics of \dot{x}_j can be succinctly written as a function of the state variables:

$$\dot{x}_j = f_j(x_1, x_2, \dots, x_n, u_j)$$
(4.2)

However, in order to distinguish between multiple causal pathways from a common state variable, the dynamics can alternatively be expressed as a function of the pathways depicted in Figure 4.1, as shown in Equation 5.2.

$$\dot{x}_j = f_j(p_{1ja}, p_{1jb}, \dots, p_{ijk}, \dots, p_{nja}, p_{njb}, \dots, p_{njm}, u_j)$$
(4.3)

From Equation 5.2, \ddot{x}_j is derived using the chain rule of differentiation, and expressed as a sum of contributions from each immediate causal pathway p_{ijk} :

$$\ddot{x}_j = \frac{\partial f_j}{\partial p_{1ja}} \dot{p}_{1ja} + \frac{\partial f_j}{\partial p_{1jb}} \dot{p}_{1jb} + \dots + \frac{\partial f_j}{\partial p_{ijk}} \dot{p}_{ijk} + \dots + \frac{\partial f_j}{\partial p_{njm}} \dot{p}_{njm} + \frac{\partial f_j}{\partial u_j} \dot{u}_j \qquad (4.4)$$

In Equation 5.3, the term

$$\frac{\partial f_j}{\partial p_{ijk}} \dot{p}_{ijk}$$

quantifies the contribution of causal pathway p_{ijk} to the second derivative (\ddot{x}_j) . By analogy to Newton's second law of motion, motivated in Chapter 3, this term can also be conceptualized as the force exerted by p_{ijk} on the variable x_j , causing an acceleration. The first factor of this term (the partial derivative) represents the gain of p_{ijk} , which is the change in f_j with respect to changes in p_{ijk} . The second factor is the rate of change of p_{ijk} . The second derivative, or acceleration, of the variable of interest (\ddot{x}_j) is expressed as the sum of the force contributions from each immediate pathway, and is visually depicted using a free body diagram as shown in Figure 4.2.

4.1.2 Illustration: Pathway Decomposition for the Logistic Growth Model

The logistic growth (or population growth) model, also known as the Verhulst equation as discovered by Pierre-François Verhulst in 1838 (see [121]) is a common model for examining



Figure 4.2: Free body diagram illustrating the contribution of each pathway to the acceleration of the variable of interest.

shifting influence between reinforcing and balancing feedback loops. Figure 4.3 shows a *stock* and *flow diagram* of the logistic growth model.



Figure 4.3: Stock and flow diagram of population growth.

The center box variable, *population*, is the single stock (state variable). The double-lined arrows with valves flowing into and out of population are the flow variables (components of the state derivative), associated with *births* and *deaths*. The clouds represent unconstrained

population sources and sinks. R1 is the reinforcing feedback loop associated with the birth process, in which births add to the population, further increasing the birth rate. B1 is the balancing feedback loop associated with deaths, in which deaths decrease the population, which subsequently slows the death rate. B2 and B3 are the balancing loops associated with the constraints of a fixed environment with a *carrying capacity* (maximum population size the environment can sustain based on finite resources). The ratio of the current *population* size to its *carrying capacity* affects the *birth fraction* and *death fraction* (also known as the birth and death fractional rates), which represents the fraction of births and deaths per unit of time. Normal birth fraction and normal death fraction are the fractional birth and death rates in an unconstrained environment. The dynamics of logistic growth are governed by the following equations:

$$\frac{d}{dt} population = births - deaths$$
$$\dot{P} = b \cdot \left(1 - \frac{P}{C}\right) \cdot P - d \cdot \left(\frac{P}{C}\right) \cdot P$$

where

$$P = population$$
(4.5)

$$b = normal birth fraction$$

$$d = normal death fraction$$

$$C = carrying capacity$$

Through a simple change of variables, system 4.5 can be re-written as a function of the four distinct pathways from P to \dot{P} , representing feedback loops R1, B1, B2, and B3:

$$\dot{P} = p_{114}(P) \cdot p_{111}(P) + p_{113}(P) \cdot p_{112}(P)$$
where
$$R1: p_{111}(P) = P$$

$$B1: p_{112}(P) = -P$$

$$B2: p_{113}(P) = d \cdot \left(\frac{P}{C}\right)$$

$$B3: p_{114}(P) = b \cdot \left(1 - \frac{P}{C}\right)$$
(4.6)

However, logistic equation 4.5 is also sometimes expressed in the following equivalent form [121, 136]:

$$\dot{P} = \alpha P - \beta P^{2}$$
where
$$\alpha = b$$

$$\beta = \frac{(b+d)}{C}$$
(4.7)

This alternate form illustrates how for $\alpha \gg \beta$, when P is relatively small, the reinforcing growth of *births* drives the dynamics \dot{P} , but as P gets larger and reaches its carrying capacity C, the squared term increases and slows the rate of growth \dot{P} . System 4.7, while mathematically equivalent to 4.5, can be written as a function of two pathways representing one reinforcing loop and one balancing loop.

$$\dot{P} = p_{111}(P) + p_{112}(P)$$
where
$$R1: p_{111}(P) = bP$$

$$B1: p_{112}(P) = -\left(\frac{b+d}{C}\right) \cdot P^2$$
(4.8)

This first-order example illustrates how even for simple first-order models, there can be multiple ways to define causal pathways. Systems 4.6 and 4.8, while mathematically equivalent, represent two different sets of causal mechanisms. The choice of pathways depends on the causal mechanisms and the correspondence between model variables and real-world processes. The logistic equation has been applied to the fields of biology, chemistry, demography, ecology, economics, sociology, and political science. In each instance, variable and pathway decomposition depends on the phenomena being described. While pathway selection does not change the behavior of the model, as will be shown, it does change how explanations are developed in terms of structure *dominating* behavior. Since different pathway decompositions represent different variable transformations of the system, the chain rule of differentiation allows the modeler to choose the most appropriate decomposition of the second derivative into contributing forces or causal mechanisms.

4.1.3 Identifying Necessary and Sufficient Pathways

With the second derivative of the variable of interest expressed as a function of causal pathways (Equation 5.3), dominance is evaluated by examining each pathway and its effect on the sign of the second derivative. From Chapter 3, it was observed that testing for necessary and sufficient conditions requires pathways to be isolated and their effects somehow removed or deactivated, independent from one another. This raises the question, What does independent removal or deactivation look like mathematically?

Pathway Removal Versus Deactivation

The practice of isolating and deactivating partial model structures (while leaving the rest of the model intact) and examining the impact on behavior through simulation, has a long history in system dynamics (SD) [114]. For the purpose of identifying dominant structure, there are two primary ways of isolating and deactivating structure. The first is complete removal, and the second is deactivation by holding the partial structure constant [26, 111, 69].

To illustrate the difference between these approaches, consider a second-order system with state variables x and y, in which y is the variable whose behavior is of interest and pathway x to y is to be removed or deactivated, as shown in the upper left corner of Figure 4.4.

This system has two feedback loops: L1 which is a minor feedback loop from y to itself, and L2, a major feedback loop going through both y and x. The intent behind deactivating pathway x to y is to test the influence of feedback loop L2 on \ddot{y} . The bottom left diagram in Figure 4.4 shows an equivalent representation of the causal structure using the decomposition of \ddot{y} , highlighting the role of both the states and their derivatives. The middle column of the figure represents the deactivation approach. The pathway is deactivated by holding the value of the pathway constant, which requires holding x constant (accomplished by setting \dot{x} to zero). The result (lower middle figure) is that the impact of the dynamics of the pathway are eliminated (hence L2 is eliminated), while the pathway itself remains constant and still appears in the equation for \ddot{y} , as a gain applied to feedback loop L1. The right column represents the approach of fully removing the pathway from x to y. Under this approach, the pathway is completely removed from the equation of \dot{y} , and hence neither x nor \dot{x} affect \ddot{y} .



Figure 4.4: Illustration of pathway deactivation versus pathway removal in a second-order system.

For this thesis, the deactivation approach (middle column) is used to test for necessary and sufficient pathways. The primary reason is because the full removal method appears to test two effects simultaneously, which is undesirable. It tests the effect of both the existence of a pathway as well as its dynamics. This is evident in the lower right diagram of Figure 4.4 in which both the effects of x and \dot{x} are eliminated, whereas in the deactivation method (lower middle diagram), only the effect of \dot{x} is eliminated. In other words, the deactivation approach tests the effect of the *dynamics* of the pathway (generated by feedback loop L2), but not its *existence*. This is especially preferable if pathway x to y is an integral aspect of the theory for y (for example, is required to make the equation \dot{y} logically and dimensionally consistent). Through deactivation, one may desire to know if x can be replaced by a parameter or

exogenous variable, or if it being a state variable within a feedback loop is required in order for y to exhibit certain behaviors.

Furthermore, to test the full removal of a pathway goes beyond the question of dominant structure, and to one of model simplification. To test the full removal of pathway x to yrequires the formulation of an alternate, dimensionally consistent theory about the causal mechanisms affecting \dot{y} . There is no general approach for removing a variable from an equation - the method of removal depends on the nature of the equation and requires thoughtful consideration from the modeler. In some cases, when pathways are nonlinearly coupled, it may not be possible to remove one pathway without affecting other pathways, which violates the principle that the influence of different elements of structure should be isolated and tested independently [114]. In some cases, it may not be possible to reformulate a coherent rate equation without the causal pathway of interest²⁵. In contrast, the theory (equation) of \dot{y} remains unaltered and dimensionally consistent if the pathway is deactivated and held constant, but not removed. This also results in a stronger test in which only the dynamics of the pathway are being isolated and tested, as opposed to both dynamics and existence.

There are two important additional observations about the deactivation of pathways, made evident in Figure 4.4. First, with respect to model *behavior*, is that deactivation only impacts \ddot{y} , maintaining the smoothness of trajectories, whereas full removal impacts both \ddot{y} and \dot{y} at the same time, resulting in non-smooth trajectories at the time of removal. This also illustrates how, when conducting pathway deactivation, the second derivative of y alone is sufficient for determining dominance and shifts in dominance, providing further support for the proposed definition of *behavior*.

²⁵For example, in the Lotka-Volterra predator prey model, the causal pathways of the major balancing feedback loop between predators and prey cannot be removed without also removing other minor feedback loops as well (due to their nonlinear coupling), in order to maintain a dimensionally consistent and coherent set of equations.

Second, with respect to model *structure*, suppose there exists an additional feedback loop, L3, through state variable z, between y and \dot{x} , as shown in Figure 4.5. Also, suppose that



Figure 4.5: Inability to distinguish between loops containing a common pathway.

after deactivating and testing the pathway from x to y, it is shown to be necessary for determining the sign of \ddot{y} . A natural question might be which feedback loop, L2 or L3, is the most influential loop influencing y at that time, since both contain the necessary pathway from x to y. Figure 4.5 shows that, with respect to \ddot{y} , this question is unanswerable. At the time of testing, the influence of loops L2 and L3 cannot be distinguished from each other. Their effect is mediated through \dot{x} , and thus one additional simulation time step is required to observe their impact on \ddot{y}^{-26} . A similar observation was also made by Hayward and Boswell in their discussion of the limitations of the Loop Impact method [60]. It would be more appropriate to ask about the relative influence between loops L2 and L3 with respect to \ddot{x} , instead of \ddot{y} . Because the systems are state-determined, and states accumulate (integrate) the effects of feedback loops from history to the present, the influence of loops sharing the same immediate pathway to the variable of interest cannot be distinguished from each other. Their influence, rather, unfolds as the simulation progresses over time and their impacts observed on the pathway they have in common. This lends further support to considering pathways, rather than feedback loops, as the explanatory element of system structure.

 $^{^{26}}$ Deactivating a pathway *n* state variables removed from the variable whose behavior is of interest requires *n* simulation time steps before an effect is observed. This fact makes some loop deactivation methods inadequate for the proposed definition of *dominance*.
Tests for Necessary and Sufficient Pathways

Chapter 3, Table 3.2 summarized the tests for necessary, sufficient, and contributory pathways.

Type	Description	Necessary?	Sufficient?	Test
1	necessary pathway	yes	no	$\operatorname{sgn} \ddot{x} \to \operatorname{pathway}$
				no pathway $\rightarrow -\operatorname{sgn}\ddot{x}$
2	sufficient pathway	no	yes	pathway $\rightarrow \operatorname{sgn} \ddot{x}$
				$-\operatorname{sgn} \ddot{x} \to \operatorname{no pathway}$
3	necessary and	yes	yes	pathway $\leftrightarrow \operatorname{sgn} \ddot{x}$
	sufficient pathway			no pathway $\leftrightarrow - \operatorname{sgn} \ddot{x}$
4	contributory pathway	no	no	$\operatorname{sgn} \operatorname{pathway} = \operatorname{sgn} \ddot{x}$
5	none of the above	no	no	$\operatorname{sgn} \operatorname{pathway} = -\operatorname{sgn} \ddot{x}$

Table 4.1: Five types of causal pathways.

Equation 5.3 expresses behavior \ddot{x}_j as a sum of force contributions (F_{ijk}) from each pathway

$$\ddot{x}_j = F_{1ja} + F_{1jb} + \ldots + F_{ijk} + \ldots + F_{njm} + F_{u_j}$$

where:

$$F_{ijk} = \frac{\partial f_j}{\partial p_{ijk}} \dot{p}_{ijk}$$

Pathway p_{ijk} is necessary if its deactivation changes the sign of \ddot{x}_j . Deactivating pathway p_{ijk} is accomplished by setting $\dot{p}_{ijk} = 0$ in Equation 5.3 at the time of deactivation, which zeros out the force contribution F_{ijk} . Thus, p_{ijk} is a necessary pathway if the following inequality holds:

$$\operatorname{sgn} \ddot{x}_j \neq \operatorname{sgn} \left(\ddot{x}_j - F_{ijk} \right) \tag{4.9}$$

Pathway p_{ijk} is sufficient if it alone guarantees the sign of the \ddot{x}_j . The first condition for this to hold is that the sign of F_{ijk} must be the same as the sign of \ddot{x}_j :

$$\operatorname{sgn} \ddot{x}_j = \operatorname{sgn} F_{ijk}$$
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Secondly, sufficiency of p_{ijk} requires that sgn \ddot{x}_j be unaffected by any combination of other pathways. Equivalently, disproving the sufficiency of p_{ijk} requires finding some combination of other active pathways which results in a sign change of \ddot{x}_j . Because pathway contributions to \ddot{x}_j are purely additive, this is equivalent to the absolute value of F_{ijk} being greater than the absolute value of the sum of all opposing pathway contributions (i.e., pathways whose contributions are opposite in sign of F_{ijk}). Therefore, p_{ijk} is a sufficient pathway if the following inequality holds:

$$|F_{ijk}| > \left| \sum_{\left\{ \operatorname{sgn} F_{ljm} \neq \operatorname{sgn} F_{ijk} \right\}} F_{ljm} \right|$$
(4.10)

Lastly, if pathway p_{ijk} is neither sufficient nor necessary, but contributes in the same direction as the observed behavior,

ī.

$$\operatorname{sgn} \ddot{x}_j = \operatorname{sgn} F_{ijk}$$

then it is called a *contributory* pathway.

4.1.4 Illustration: Dominance Framework Applied to a Linear Model

The above criteria for necessary, sufficient, and contributory pathways creates a *dominance* framework which is applied to a first-order linear model containing multiple pathways, in order to illustrate the different possible combinations of pathway types. Consider the following first-order system (Figure 4.6 and Equation 4.11) with state variable x, governed by three linear inflows and a single outflow.



Figure 4.6: Stock and flow diagram of first-order linear model with four causal pathways.

$$\dot{x} = \alpha_1 x + \alpha_2 x + \alpha_3 x - \beta x$$
where,
(4.11)

 $\alpha_1, \alpha_2, \alpha_3, \beta \ge 0$

In this model, the four pathways could be aggregated into a single pathway with fractional rate $(\alpha_1 + \alpha_2 + \alpha_3 - \beta)$, however, assume each pathway represents a unique causal process and the desire is to understand the relative influence of each pathway on the behavior of x. The pathways are identified as:

path 1 :
$$p_{111}(x) = \alpha_1 x$$

path 2 : $p_{112}(x) = \alpha_2 x$
path 3 : $p_{113}(x) = \alpha_3 x$
path 4 : $p_{114}(x) = -\beta x$

The second derivative of x is expressed as the sum of each pathway contribution:

$$\ddot{x} = \alpha_1 \dot{x} + \alpha_2 \dot{x} + \alpha_3 \dot{x} - \beta \dot{x} \tag{4.12}$$

The relative contributions of each pathway are solely determined by the parameter values which are constant. Therefore, there can be no shifts in dominance over time. However, postulating different sets of values for the parameters illustrates the different possible outcomes with respect to necessary, sufficient, and contributory pathways. Without loss of generality, assume at the time of interest t_0 , $\dot{x}(t_0) = 1$. Equation (4.12) reduces to:

$$\ddot{x}(t_0) = \alpha_1 + \alpha_2 + \alpha_3 - \beta \tag{4.13}$$

Six sets of parameter values in Table 4.2 illustrate the possible outcomes of necessary, sufficient, and contributory pathways, corresponding to different points on the dominance framework from Chapter 3 (Figure 3.5).

	α_1	α_2	α_{3}	β
case 1	3	3	3	5
case 2	6	3	3	5
case 3	6	6	1	5
case 4	4	2	2	5
case 5	6	1	1	5
case 6	3	3	1	5

Table 4.2: Six sets of parameter values in the first-order linear model.

Using a visual representation similar to the free body diagram from Chapter 3 (Figure 3.2), the results of each case are displayed in Figure 4.7, in which $\ddot{x}(t_0)$ is represented by the first vertical bar, labeled *sum*, and the subsequent vertical bars are the individual contributions of each pathway.

Figure 4.8 illustrates how the six cases map onto the dominance framework introduced in Chapter 3.



Figure 4.7: Six cases: combinations of necessary, sufficient, and contributory pathways.

Case 1: No necessary or sufficient pathways. Multiple pathways (1, 2, and 3), each exerting a force smaller than opposing pathway 4 (and thus insufficient), collectively exert a force greater than opposing pathway 4 with margin such that no pathway is necessary.



Figure 4.8: Six cases mapped onto the dominance framework.

Pathways 1, 2, and 3 are therefore considered *contributory* pathways. An analogy is a game of *tug of war* in which on one side of the rope is a team of many weaker individuals who collectively greatly over-power a single opposing strong person. **claim:** A minimum of four pathways are required for a system to be in this condition. **proof:** Suppose there are only three pathways, one opposing the other two. The one opposing pathway cannot exert greater strength than the other two together, otherwise it would be necessary and sufficient. Of the remaining two pathways, if one is not necessary, then the other must be sufficient for determining the sign, which contradicts the claim that no pathways are necessary or sufficient.

Case 2: One sufficient and no necessary pathways. Pathway 1 is sufficient (has magnitude greater than that of opposing pathway 4), but is not necessary, in that pathways 2 and 3, together, also exert a greater force than opposing pathway 4. As in case 1, pathways 2 and 3 are neither necessary nor sufficient. Using the same *tug of war* analogy in case 1, this case occurs when one of the weaker individuals on the winning team is replaced by a

strong person. **claim:** A minimum of four pathways are required for a system to be in this condition. **proof:** Suppose there are only three pathways, one opposing the other two. The one opposing pathway cannot exert greater strength than the other two together, otherwise it would be necessary and sufficient. Of the remaining two pathways, only one of them is not sufficient. The other then is not only sufficient, but must also be necessary, otherwise its removal would cause the sign to change, which contradicts the claim that none are necessary.

Case 3: Two sufficient and no necessary pathways. Pathways 1 and 2 are both sufficient (their force magnitudes are each greater than the opposing force from pathway 4), and therefore neither alone are necessary since their individual deactivation does not change the sign. Pathway 3, as in the previous cases, is neither sufficient nor necessary, but contributes in the same direction as the observed behavior. **claim:** A minimum of two pathways are required for a system to be in this condition. **proof:** By definition, a minimum of two pathways are required for there to exist two sufficient pathways. To illustrate how only two pathways are required, consider the current example of Case 3 but with pathways 3 and 4 completely removed. Pathways 1 and 2 would remain individually sufficient for producing the sign, but each alone unnecessary since they are both contributing in the same direction with no opposing force.

Case 4: One necessary and no sufficient pathway. Pathway 1 is necessary (deactivating it results in a change in sign), but it alone is not sufficient (its force magnitude is less than that of the opposing pathway 4). Pathways 2 and 3, as in Cases 1 and 2, are contributory, but neither necessary nor sufficient. In this case, consider the analogy of a sports team with a single all-star player whose talent is required for the team to win, but who alone does not make up a team and therefore is insufficient. The remaining team members contribute to the team's success but are neither sufficient nor necessary because they are easily replaced. claim: A minimum of four pathways are required for a system to be in this condition. proof: Suppose there are only three pathways. At least one pathway must exert a force

in an opposite direction from the other two, otherwise all pathways would be sufficient. Consider one pathway exerting an opposite force from the two others. Of the remaining two, if one is necessary but not sufficient, then the other must also be necessary, which contradicts the assumption that only one is necessary.

Case 5: One necessary and sufficient pathway. Pathway 1 is both necessary and sufficient. If pathway 1 is deactivated, $\ddot{x}(t_0)$ changes signs, so it is necessary. The absolute value of pathway 1's contribution is greater than that of all opposing paths, so pathway 1 is also sufficient. This is the only case in which a pathway meets the established criteria for *dominance*. Chapter 3 also demonstrated that there can be at most one necessary and sufficient pathway at a point in time, and that if a system has a single necessary pathway and a single sufficient pathway, they must be one and the same. **claim:** A minimum of one pathway is required for a system to be in this condition. **proof.** Consider this example but with pathways 2, 3, and 4 removed. Pathway 1 would remain necessary and sufficient for determining the behavior since its removal would change the sign to zero, and since there are no opposing pathways.

Case 6: Two necessary and no sufficient pathways. Pathways 1 and 2 are both necessary in that their individual deactivations result in a change in sign, but they are also individually insufficient in that their force magnitudes are less than the opposing force magnitude from pathway 4. To use another sports analogy, consider a team with no back-up players and in which every member of the team is necessary to play their position. claim: A minimum of three pathways are required for a system to be in this condition. proof. Consider a system with only two pathways. As demonstrated earlier, if both contribute in the same direction then both are sufficient and neither necessary. If they contribute in opposite directions, then the one with larger magnitude will be necessary and sufficient. Therefore, this case is not possible with only two pathways. For a three pathway system,

consider the current example but with pathway 3 removed. Pathways 1 and 2 would still both be necessary but not individually sufficient.

4.2 Pathway Force Decomposition Procedure

The above process for identifying necessary, sufficient, and contributory pathways, derived from the proposed definitions of *structure*, *behavior*, and *dominance*, is now summarized in the following steps, and is referred to as the pathway force decomposition (PFD) procedure.

- 1. Select state variable whose behavior is of interest.
- Specify initial conditions and time horizon of interest, over which to perform dominance analysis²⁷.
- 3. Specify immediate pathways to the state variable of interest, as functions of other state variables. Express the derivative of the state variable of interest as a function of the pathways.
- 4. Using the chain rule, express the second derivative of the behavior of interest as a sum of contributions from each pathway, by computing each partial derivative.
- 5. For each time step within the time interval of interest, calculate the force contributions of each pathway and test each pathway for necessary, sufficient and contributory conditions.

²⁷While the next chapter develops state-space dominance procedures, the procedure outlined here assumes specific initial conditions to facilitate direct comparison with current methods which also assume specific initial conditions.

4.3 Application to Logistic Growth Model

The PFD procedure is applied to the logistic growth model, introduced earlier in the chapter, which exhibits transient S-shape behavior. This model is chosen due to its association with general growth processes from which the term *feedback loop dominance* originally emerged. The logistic growth model is also the canonical example of *shifts in loop dominance* and is therefore an important test for a formal definition of dominance. Three forms of the logistic model are analyzed and results compared with current methods.

4.3.1 Logistic Model Form 1: Reinforcing Loop and Constraining Loop

Richardson analyzed the two-loop logistic growth model (Figure 4.9) in which R1 is reinforcing growth, and B1 is the constraining effect from carrying capacity C on the fractional growth rate α [121] (also, see [136, p. 296]).



Figure 4.9: Stock and flow diagram of the logistic growth model (form 1).

Step 1: Behavior of interest. Since population P is the only state variable, it is the variable of interest.

Step 2: Initial conditions and time horizon. The population begins with a single member, P(0) = 1. The model is analyzed from t = 0 to t = 100.

Step 3: Express \dot{P} as a function of its immediate pathways.

$$\dot{P} = p_{111} \cdot p_{112}$$

 $p_{111} = \alpha P$ (4.14)
 $p_{112} = 1 - \frac{P}{C}$

Pathway p_{111} is the pathway from P through auxiliary variable *unconstrained growth*, and represents the reinforcing growth feedback loop R1. Pathway p_{112} is the pathway from Pthrough auxiliary variable *constraining factor*, and represents the balancing feedback loop B1 from carrying capacity C which constrains growth.

Step 4. Express \ddot{P} as a function of pathway force contributions.

$$\dot{P} = F_{111} + F_{112}$$

$$F_{111} = \frac{\partial \dot{P}}{\partial p_{111}} \dot{p}_{111} = \left(1 - \frac{P}{C}\right) \cdot \alpha \dot{P}$$

$$F_{112} = \frac{\partial \dot{P}}{\partial p_{112}} \dot{p}_{112} = \alpha P \cdot \left(\frac{-\dot{P}}{C}\right)$$
(4.15)

Step 5. For each time step, calculate force contributions and identify necessary, sufficient, and contributory pathways. Figure 4.10 shows the results of the simulation and analysis. The top graph is the behavior over time of state variable P. The middle graph plots \ddot{P} (*Net Force*) along with the force decomposition of Path 1 (p_{111}) and Path 2 (p_{112}). The bottom graph shows which paths are necessary and which are sufficient for determining the sign of \ddot{P} .



Figure 4.10: Simulation results for logistic growth model (form 1).

Results

P exhibits the expected S-shape growth: divergent exponential growth followed by convergent goal-seeking growth. The inflection point ($\ddot{P} = 0$) occurs around time t = 46, when Preaches half its carrying capacity (50). Path 1 (associated with reinforcing feedback) always exerts a positive force, while Path 2 (associated with the balancing feedback loop) always exerts a negative force. From t = 0 to the inflection point t = 46, the force exerted by Path 1 is greater in magnitude than the force exerted by Path 2, and thus is both necessary and sufficient for creating positive acceleration (and by definition is *dominant*). After t = 46, Path 2 exerts a greater force than Path 1 and is necessary and sufficient for creating negative acceleration (deceleration), and therefore *dominance shifts* to Path 2. As time continues, both forces decrease as P approaches equilibrium.

Excursion

Consider the case in which the population begins above its carrying capacity (Figure 4.11). P exhibits exponential decay as it approaches its carrying capacity, C. The net force is



Figure 4.11: Simulation results for logistic growth model (form 1) (P > C).

always positive, slowing the decline. Path 2 (representing the balancing loop associated with the carrying capacity) exhibits the greatest force, however both Paths 1 and 2 exhibit positive forces and so both are sufficient (neither are necessary) for generating the observed behavior. Therefore, neither pathway is dominant. The result that Path 2 is sufficient is intuitive since it is associated with the balancing feedback loop and has a negative closedloop gain. To understand, however, how Path 1 (associated with the reinforcing growth loop) is sufficient for generating this behavior, observe that if P > C, Path 2 is negative and therefore, when deactivated, switches the polarity of the feedback loop associated with Path 1. This illustrates how nonlinear coupling of pathways can change the polarity of feedback loops and lead to shifts, not just in loop dominance, but the polarity of feedback loops in different regions of the state space.

Discussion and Comparison with Other Methods

The results from applying the PFD procedure agree with the conclusions from Richardson [121] and Sterman [136] who use the concept of *dominant polarity* to determine loop dominance for first-order two-loop systems, where dominant polarity is positive if population is less than half the carrying capacity (and thus the reinforcing loop dominates), and dominant polarity is negative if population is greater than one half (and thus the balancing loop dominates). The results also agree with numerous examples from SD instructional literature which informally introduce the concept of *shifts in dominance* based on the logistics model [21, 90]. However, while applying the concept of *dominant polarity* to identify loop dominance is straightforward when P < C, for P > C, Richardson and Sterman are silent on the specific roles of the two feedback loops. Other literature on logistic growth is also silent on loop dominance when P > C. The PFD procedure provides a formal and rigorous answer to this question.

Mojtahedzadeh performs dominance analysis on the logistic growth model, in the form of the Susceptible-Infectious (SI) epidemic model, using the Pathway Participation Method (PPM) [97]. The SI model uses the following interpretation of variables which make-up the two pathways in system 4.14:

$$population (P) = infectious (I)$$

$$\alpha = contact \, rate \, (c) \times infectivity \, (i)$$

carrying capacity (C) = population size (N)

Contagion Reinforcing Loop :
$$p_{111} = c i I$$

Depletion Balancing Loop : $p_{112} = \left(1 - \frac{I}{N}\right)$

PPM, which uses the total pathway participation metric (TPPM) to determine dominance, concludes that the contagion loop (p_{111}) is dominant in the first phase of S-shape growth, followed by the dominance of the depletion loop (p_{112}) in the second phase. For this two-loop model, because PPM identifies the loop which contributes most to (\ddot{I}/\dot{I}) as the dominant loop, and each loop always has opposite contributions, the one with largest magnitude measure is also guaranteed to be both necessary and sufficient, and so the results agree with the PFD procedure. The case in which population starts above the carrying capacity does not apply to the SI model, since the infectious population can never be greater than the total population, so this case was not analyzed by PPM.

Similarly, Kampmann and Oliva [79] apply PPM to a simple diffusion model of *adopters* (A) and *potential adopters* (N - A) having the same mathematical and causal structure as the SI model, in which:

$$population\left(P\right) = adopters\left(A\right)$$

 $\alpha = contact \, rate \, (c) \times adoption \, fraction \, (i)$

carrying capacity(C) = population size(N)

Word of mouth Reinforcing Loop : $p_{111} = c i A$ Market Saturation Balancing Loop : $p_{112} = \left(1 - \frac{A}{N}\right)$ The PPM results agree with the PFD results and identify the *word of mouth* reinforcing loop as dominant in the first phase, and the *market saturation* balancing loop as dominant in the second phase. Kampmann and Oliva also apply loop eigenvalue elasticity analysis (LEEA) to the same diffusion model, using influence metrics based on the derivative of eigenvalues of the linearized system with respect to loop parameters. LEEA also identifies the *word of mouth* loop as dominant in the first phase, followed by the dominance of the *market saturation* loop, agreeing with both PPM and the PFD procedure.

Taylor and colleagues [143] test their experimental statistical screening procedure on the same diffusion model, which identifies highly influential elements of structure by measuring the correlation between structural and behavioral changes. *Initial adopters* is identified as most influential in the beginning because it defines the initial state. Shortly after, *adoption fraction* and *contact rate* both have the highest correlation coefficients, which agrees with the PFD procedure in that both contribute to the gain of the causal pathway associated with the reinforcing loop. In the last phase, *initial potential adopters* has the highest correlation coefficient which also agrees with the PFD procedure, in that it restricts the system through the balancing *market saturation* loop.

PFD results are now compared to Ford's behavioral method of loop deactivation. First, consider the case in which population P starts below the carrying capacity C. Loops B1 and R1 (Figure 4.9) are each individually deactivated once during the divergent growth phase (t = 10) and once again during the convergent growth phase (t = 50). The resulting simulated responses are shown in Figure 4.12.

During the divergent growth phase, deactivating B1 does not change the atomic behavior pattern (exponential), and thus, according to Ford's method, is not dominant. Deactivating R1 does change the atomic behavior pattern from exponential to logarithmic, and thus R1is dominant. In the second phase, deactivating B1 changes the atomic behavior pattern from logarithmic to exponential, and thus B1 is dominant. Deactivating R1 does not change



Figure 4.12: Ford's behavioral test for dominance applied to the logistic growth model (form 1).

the atomic behavior pattern, and is not dominant. Because Ford's criteria for dominance employs a counterfactual test, it identifies necessary pathways. In this model, since the dominant pathways are both necessary and sufficient, and there are no pathways which are necessary but not sufficient, Ford's test identifies the same dominant structure as the PFD procedure.

Now consider the case in which P > C. The model exhibits only one behavior pattern (logarithmic) and loops B1 and R1 are individually deactivated at time (t = 10), as shown in Figure 4.13. Deactivating R1 results in no noticeable change in behavior, and thus R1 is not dominant. Deactivating B1 causes a noticeable change but does not change the atomic behavior pattern, and thus B1 is also not dominant. However, deactivating R1 and B1simultaneously changes the atomic behavior pattern from logarithmic to linear, indicating what Ford calls a *shadow dominance* condition. However, because their are only two loops, deactivating both loops always results in a change in atomic behavior pattern unless the model is in equilibrium, therefore, Ford's test for shadow dominance may be degenerate in this case. None-the-less, Ford's test agrees with the PFD procedure in not identifying any dominant loops, since Ford's test identifies necessary pathways, and in this case, both



Figure 4.13: Ford's behavioral test for dominance applied to the logistic growth model (form 1) (P > C).

loops are found by PFD to be sufficient and not necessary. This also indicates a potential relationship between *shadow dominance* and cases in which there exists one or more sufficient pathways and no necessary pathways. In examining Ford's method, it appears that a pair of shadow loops are identified when either loop is sufficient for changing the behavior, in the absence of the other, which indeed indicates a sufficient condition. He describes this as a different type of dominance. He also admits that his algorithm is more difficult as the number of shadow feedback structures increase (i.e. the number of sufficient pathways increase).

One observation is that if the two sufficient loops are considered as a set, then together they are both sufficient and necessary for creating the observed behavior, and thus can be considered as a dominant set, meeting the established criteria of dominance. This expands the definition of structure to include not just single pathways, but sets of pathways. Under this expanded definition, B1 and B2 would be identified by the PFD procedure as a dominant pair of pathways. This directly corresponds to Ford's results which identifies B1 and B2 as a pair of shadow feedback structures which are together dominant.

4.3.2 Logistic Model Form 2: A Second Balancing Loop (Deaths)

A three-loop version of logistic growth is analyzed and compared against the Loop Impact method [60]. This version (Figure 4.14), adds a second balancing feedback loop B2 representing population decline through deaths based on a constant fractional death rate b. Accordingly, a third pathway, p_{113} , representing balancing loop B2, is added to the decomposition of P:



Figure 4.14: Stock and flow diagram of the logistic growth model (form 2).

$$\dot{P} = p_{111} \cdot p_{112} + p_{113}$$

$$p_{111} = \alpha P$$

$$p_{112} = 1 - \frac{P}{C}$$

$$p_{113} = -b P$$
(4.16)

$$\dot{P} = F_{111} + F_{112} + F_{113}$$

$$F_{111} = \left(1 - \frac{P}{C}\right) \cdot \alpha \dot{P}$$

$$F_{112} = \alpha P \cdot \left(\frac{-\dot{P}}{C}\right)$$

$$F_{113} = -b \dot{P}$$
(4.17)

The simulation begins from an initial population of 1. Figure 4.15 shows the results of the simulation and force decomposition analysis.



Figure 4.15: Simulation results for logistic growth model (form 2).

Results

As before, the inflection point occurs when P = C/2. Path 1 (R1) always exerts positive force, while Path 2 (B1) and Path 3 (B2) always exert negative force. There are three distinct phases of dominance. Before the inflection point (phase 1), Path 1 is necessary and sufficient (dominant). After the inflection point, there is a brief period in which Paths 2 and 3 are both necessary, and neither sufficient, to produce deceleration. Then, Path 2 becomes necessary and sufficient and dominates for the remainder of the trajectory.

Phases 1 and 3 agree with the Loop Impact method, identifying R1 and B1 as dominant, respectively. In phase 2, the Loop Impact method identifies B1 and B2 as dominant *together*, or as a set, and concludes that in this phase there is no single dominant pathway or loop. The PFD method also concludes that in phase 2, there is no single dominant pathway (i.e. no pathway is both necessary and sufficient). However, if Path 1 and Path 2 are considered as a set, then as a set they are necessary and sufficient in phase 2, and meet the criteria for dominance. Thus, if the definition of dominance is expanded to include as elements of structure sets of pathways, then the results are the same as in the Loop Impact method.

The Loop Impact method, by construction, finds the minimum combination of loops of like polarity (i.e forces contributing in same direction) whose combined impact is greater than the sum of all loops of opposite polarity [60]. This criteria, by construction, identifies either a single sufficient loop²⁸, or a minimum set of loops which are sufficient. Thus, it is expected that the PFD procedure, which identifies dominant structures as those which are both necessary and sufficient, would agree with the Loop Impact method, however the Loop Impact method also classifies loops as dominant which are sufficient and not necessary, and thus identifies dominance more frequently than the PFD procedure.

Next, Ford's behavioral method is applied to this model. In Ford's procedure, locations of deactivation are based on changes in behavior patterns. Therefore, in this form of the model, deactivating in two locations is insufficient for identifying the three phases of dominance discussed earlier, which is a shortcoming of Ford's method ²⁹. The dominance transition

²⁸It is not clear how the Loop Impact method works when there are multiple single sufficient loops.

²⁹Extensions to Ford's method include automated testing at every point along the trajectory and not just at behavior mode transition points, and thus address this shortfall in Ford's method [111].

from phase 2 to phase 3 is not associated with a behavior mode change. Regardless, Ford's deactivation method is applied to each loop in each of the three phases (t = 5, t = 10, t = 15) to see what insights are discovered. Figure 4.16 shows the results of Ford's procedure.



Figure 4.16: Ford's behavioral test for dominance applied to the logistic growth model (form 2).

In phase 1, deactivation of R1 is the only case which causes an immediate behavior mode change (exponential to logarithmic), and thus R1 is dominant. In phase 2, deactivating B1 results in a behavior mode change, as does deactivating B2, so both are found to be individually dominant. Ford refers to this as *simultaneous multiple loop dominance*. In phase 3, deactivation of B1 is the only case resulting in a behavior mode change, and thus B1 is dominant.

Ford's behavioral method agrees with PFD and the Loop Impact method in phases 1 and 3. In phase 2, it differs from both PFD and Loop Impact in that it identifies *B*1 and *B*2 as individually dominant. This is because, as observed earlier, Ford's criteria for dominance is a necessary condition (not sufficient), thus the PFD procedure explains why Ford's method produces a different result. Any time loops are necessary (regardless of whether or not they are also sufficient), Ford's method will identify them as dominant, which leads to the potential identification of multiple dominant loops.

4.3.3 Logistic Model Form 3: Alternate Two Loop Version

Figure 4.17 shows an alternative two-pathway representation of the logistic model (compare with form 1, Figure 4.9). Whereas in form 1, pathways represent the reinforcing growth loop



Figure 4.17: Stock and flow diagram of logistic growth (form 3).

and the loop constraining the fractional growth rate, in this form, pathways represent the reinforcing growth loop and the balancing death loop:

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$$\dot{P} = p_{111} + p_{112}$$

$$p_{111} = \alpha P \qquad (4.18)$$

$$p_{112} = \left(\frac{-\alpha}{C}\right) \cdot P^2$$

$$\ddot{P} = F_{111} + F_{112}$$

$$F_{111} = \alpha \dot{P} \qquad (4.19)$$

$$F_{112} = \frac{-2 \alpha P \dot{P}}{C}$$

Note that Equations 4.18 and 4.19 are equivalent to Equations 4.14 and 5.8, respectively, but have different pathway decompositions. Both forms are often found in literature on sigmoid growth, as observed earlier. For P < C, the simulation and analysis results are shown in Figure 4.18.

Results

The results are the same as in form 1 (P < C), in which Path 1 (R1) is necessary and sufficient (dominant) in the first phase of exponential growth, and Path 2 (B1) is necessary and sufficient (dominant) for logarithmic growth in the second phase.



Figure 4.18: Simulation results of logistic growth (form 3).

Next, the results for condition (P > C) are shown in Figure 4.19.



Figure 4.19: Simulation results of logistic growth (form 3) (P > C).

These results differ from form 1 (compare with Figure 4.11). Whereas in form 1, both paths were sufficient for producing the exponential decay behavior, in form 3, the balancing loop (path 2) is necessary and sufficient for producing the behavior while the reinforcing loop (path 1) has an opposing force. In form 1, it was noted that the balancing loop changed the polarity of the reinforcing loop to behave like a balancing loop, so both were sufficient for generating the behavior. In form 3, the balancing and reinforcing loops are added together instead of multiplied, and therefore always represent positive and negative forces strictly associated with births and deaths. The necessary and sufficient pathways for each case are summarized in Tables 4.3 and 4.4.

Loop	$\mathrm{P} < rac{\mathrm{C}}{2}$	$rac{\mathbf{C}}{2} < \mathbf{P} < \mathbf{C}$	$\mathbf{P} > \mathbf{C}$
R1	N & S		S
B1		N & S	S

Table 4.3: Dominance of loops in logistic equation form 1.

Loop	$\mathbf{P} < rac{\mathbf{C}}{2}$	$rac{\mathrm{C}}{2} < \mathrm{P} < \mathrm{C}$	$\mathbf{P} > \mathbf{C}$
R1	N & S		
B1		N & S	N & S

Table 4.4: Dominance of loops in logistic equation form 3.

In summary, while the pathway decomposition does not change the dynamic behavior of the system, this example illustrates how pathway definition and decomposition affects the explanation for how structure determines behavior, and the nature of dominant structure.

4.4 Conclusions

4.4.1 Procedure for Identifying Dominant Structure

Using the proposed definitions for *behavior*, *structure*, and *dominance*, behavior is formally expressed as a sum of contributions from individual elements of structure (pathways). There may be more than one way to decompose behavior, depending on how causal mechanisms are defined. The dominance criteria requires deactivation of pathways to identify necessary and sufficient conditions and it was shown that examining the second derivative is sufficient for detecting dominance and shifts in dominance.

Furthermore, it was shown that immediate pathways (as opposed to feedback loops) are adequate for identifying necessary and sufficient structure. Concise mathematical tests were proposed for detecting necessary and sufficient pathways. A procedure, pathway force decomposition (PFD), was developed for identifying necessary and sufficient pathways and was applied to several forms of the logistic model. The PFD procedure has qualities similar to exploratory/behavioral methods of dominance analysis in that the criteria for dominance is anchored in a behavioral criteria (sign change of the second derivative).

The PFD procedure also has qualities similar to formal/structural methods in which the criteria for dominance is developed from the structure (equations) of the model. Thus, the PFD procedure may be considered both a behavioral and a formal/structural dominance method. One reason for this is that the criteria for dominance has behavioral/objective qualities in that dominance requires an objective behavior change, as well as structure-relative qualities in that the metric of force contribution can be directly compared for different pathways. Therefore, the PFD procedure captures both the behavioral-objective dimension of dominance as well as the structural-relative dimension.

When tested against the logistic growth model, the PFD procedure showed that the loops commonly associated as dominant satisfied both necessary and sufficient conditions, providing further support for the proposed definition of dominance. The procedure also illustrated how pathway choice affects structure-behavior explanations.

4.4.2 Relationship Between Dominance Methods

All methods of dominance analysis have been applied to the logistic growth model and produce consistent results, but for different reasons.

Ford's behavioral test employs a counterfactual criteria and identifies necessary structure as dominant. Conversely, the Loop Impact method, by construction, identifies sufficient elements of structure as dominant. Where the structure is both sufficient and necessary, Ford's behavioral method, the Loop Impact method, and PFD identify the same dominant structure. Where structure is necessary but not sufficient, or sufficient but not necessary, the methods will not identify the same dominant structure. PFD applies Ford's behavioral criterion analytically when it identifies necessary structure, and thus can also be viewed as an automated, analytical version of Ford's method. PFD also improves upon Ford's method in that it directly identifies multiple shadow feedback structures analytically. PFD is also easily applied at each time step along a trajectory and does not require manual deactivation of feedback structure. This is important because, as observed, not all shifts in dominance are associated with changes in behavior modes. Additionally, manual deactivation and testing of models can be a tedious exercise and clutter up the model with switch variables. In Ford's method, deactivation points must be manually identified, and multiple combinations of loops deactivated simultaneously in the case of shadow structure. By using a completely analytical procedure, these challenges are avoided.

Likewise, the Loop Impact method has many similarities to PFD, but uses the Impact metric which is the acceleration contribution divided by the first derivative of the variable of interest. Thus, it uses the same metric as the PPM method.

PPM results agree with PFD and the behavioral methods when the dominant loop also happens to have the largest TPPM contribution, as in the case of the logistics model. In fact, when there only exists a single loop contributing in the direction of the observed behavior, PPM will always agree with the behavioral method and the PFD. TPPM is simply the second derivative divided by the first derivative, and thus is very similar to the proposed definition of behavior. However, because PPM relies on a relative metric for determining dominance, it will always identify a single dominant loop. Whenever there is a change of loop with the largest TPPM, a shift in loop dominance will be identified, thus it is expected that PPM identifies dominance and shifts in dominance more frequently than behavioral-based methods such as Ford's procedure and the PFD procedure. The PFD procedure improves upon the PPM method in that it precisely determines each pathway's contribution to the observed behavior, as opposed to a normalized proxy measure of behavior. The PFD procedure also allows for the possibility that multiple loops or no loops dominate, and it is well-defined when the derivative of the variable of interest is equal to zero.

For similar reasons as for PPM, LEEA (also based on a normalized influence metric), agrees with the results of PFD and the behavioral methods in the case of the logistic model since there is only one reinforcing and one balancing loop. Like PPM, LEEA will also always identify a single dominant loop and detects shifts in dominance more frequently than in behavioral methods. One way to interpret the results of PPM and LEEA is that they identify highly influential structures which are contributory and *potentially* dominant, and are good candidates for subsequent testing for dominance [38].

Ford's behavioral method identifies cases of *shadow loop dominance*, which others have also described as *shared dominance* [111]. It also describes situations of *multiple loops dominating simultaneously*. The phenomenon of *shadow loop dominance* is associated with multiple sufficient and unnecessary pathways, and the phenomenon of *multiple loop dominance* is associated with multiple necessary and insufficient pathways.

4.4.3 Implications of Dominance Framework for Policy Design

Expanded Definition of Dominance

Ford's method considers pairs of shadow feedback structures (i.e. sets of sufficient pathways/loops) as dominant together. Similarly, the Loop Impact method considers sets of pathways/loops which are collectively sufficient to be a dominant set. One observation is that multiple sufficient pathways may form a set that is both sufficient and necessary. Likewise, multiple necessary pathways may form a set that is both necessary and sufficient. Expanding the definition of dominance to consider not just single pathways which are necessary and sufficient, but sets of pathways which together are necessary and sufficient, permits a broader use of the term.

Therefore, the following modified definition is proposed:

Given a state variable $x_j(t)$ whose behavior is of interest, and point t_0 along its trajectory, a set of causal pathways are *dominant* if and only if the set is both necessary and sufficient for determining sgn $\ddot{x}_j(t_0)$.

System Robustness and Fragility

Ford's method and the Loop Impact method above illustrate three possible compositions of dominant sets:

- 1. A dominant set contains a single necessary and sufficient pathway.
- 2. A dominant set contains some combination of necessary and contributory pathways, but no sufficient pathways.
- 3. A dominant set contains some combination of sufficient and contributory pathways, but no necessary pathways.

The robustness or fragility of a variable's behavior at a given time depends on the number of necessary and sufficient pathways to that variable at that time. Consider Figure 4.20, which expands upon the dominance framework introduced earlier.

Figure 4.20 shows the typical case of *dominance* associated with a single necessary and sufficient pathway. It also shows that *shadow feedback* or *shared dominance* occurs when there exists one or more sufficient and no necessary pathways. Similarly, *multiple loop*



Figure 4.20: System robustness depending on the number of necessary and sufficient pathways.

dominance or simultaneous dominance occurs when there exists two or more necessary and no sufficient pathways. A system with no sufficient or necessary pathways contains multiple contributing pathways. The robustness or resilience of a system increases as the number of necessary pathways decrease and as the number of sufficient pathways increase. Here, the terms *robustness* or *resilience* are used to describe the likelihood of system behavior change when there are changes in causal pathways.

To illustrate, consider a four-pathway system such as the linear example in Figure 4.6, which can exist in any one of the six cases in the dominance framework. In each of the six cases, path 4 opposes the observed behavior, while the other three paths contribute to the observed behavior. The magnitudes of the three supporting pathways determined the number of necessary and sufficient pathways. Suppose some of the supporting paths are disabled in order to change in the behavior mode, but it is not known which pathway should be deactivated. If one of the three paths is disabled at random, what is the probability that the behavior mode changes? In the case of zero necessary pathways (cases 1, 2 and 3), there is zero probability of behavior change. This is because no single pathway is necessary, or critical, for determining the behavior mode. In the case of one necessary pathway (cases 4 and 5), there is a one-third chance of changing the behavior mode by randomly deactivating one of the three supporting pathways. In the case of two necessary pathways (case 6), there is a two-thirds probability of changing the behavior. If all three supporting pathways are necessary (for example, each has a magnitude 2), then the probability of changing the behavior is 1. Therefore, as the number of necessary pathways increase, the behavior becomes more susceptible to change and conversely as the number of pathways decrease, the behavior becomes more resilient to change.

Now consider if two pathways are deactivated at random. If there are zero sufficient pathways (cases 1, 4, and 6), the probability of behavior change is 1 (a single non-sufficient pathway remains after deactivating two, and thus the behavior changes). If there is one sufficient pathway (cases 2 and 5), the probability of behavior change is two-thirds. If there are two sufficient pathways (case 3), there is a one-third chance of behavior change. Finally, if all three supporting pathways are sufficient (for example, each has magnitude 6), deactivating any two pathways has a zero probability of changing the behavior. Therefore, the likelihood of behavior change decreases as the number of sufficient pathways increase.

Necessary causes are analogous to *critical processes*, *vulnerabilities*, or *single points of failure* of a system, and thus decreasing the number of necessary causes decreases the susceptibility to change. Likewise, sufficient causes are analogous to redundant mechanisms, and thus

increasing the number of sufficient causes increases the redundancy of the system. This illustration is summarized in Figure 4.21.

Probability of behavior change (2 random pathway deactivations) Probability of behavior change (1 random pathway deactivation)		Sufficient Pathways = 0	Sufficient Pathways = 1	Sufficient Pathways = 2	Sufficient Pathways = 3
		3/3	2/3	1/3	0/3
Necessary Pathways = 0	0/3	2-path prob: 1 1-path prob: 0	2-path prob: 2/3 1-path prob: 0	2-path prob: 1/3 1-path prob: 0	2-path prob: 0 1-path prob: 0
Necessary Pathways = 1	1/3	2-path prob: 1 1-path prob: 1/3	2-path prob: 2/3 1-path prob: 1/3	N/A	N/A
Necessary Pathways = 2	2/3	2-path prob: 1 1-path prob: 2/3	N/A	N/A	N/A
Necessary Pathways = 3	3/3	2-path prob: 1 1-path prob: 1	N/A	N/A	N/A

Figure 4.21: Probability of behavior change from deactivating one or two behavior-supporting pathways.

Design Implications for Robust Systems

General principles for making a system more or less robust can be inferred from the dominance framework. First, robustness can be increased by decreasing the necessary pathways. This is accomplished by adding new supporting causal pathways, increasing the force of the weaker contributing pathways, or by reducing the force of the opposing pathways. Second, robustness can be increased by increasing the sufficient pathways. This is accomplished by increasing the number of individual pathways which are stronger than the opposing pathways, either by increasing the force of supporting pathways, adding new strong supporting pathways, or reducing the strength of opposing pathways. This also raises new questions of how to identify the best way to intervene in a system that is operating in one of these six states. Also, can these insights be used to develop design heuristics for sustainable policies and interventions? Can they be used to better understand policy resistance observed in real systems? These questions will be explored in Chapter 6 as dominance methods are applied to a problem in public health.

4.5 Summary

Research Question 1 asks, "What should constitute an explanation for how structure determines behavior? How can dominance be formally and rigorously defined in a way that resolves existing ambiguities, explains discrepancies in previous dominance analyses, leads to new insights, and advances methods and theory?"

Towards answering these questions and satisfying Research Aim 1, a rigorous and formal definition of *dominance* was proposed in Chapter 3 and then tested against a simple model alongside other methods. In the process, some discrepancies between current methods were addressed, and ambiguities associated with shadow loop dominance, multiple loop dominance, and pathway/loop representation were also addressed. Additionally, based on the tests, the definition of dominance was expanded to include not just single pathways, but sets of pathways. Based on the proposed definition of dominance and the PFD procedure, new insights were offered regarding the relationship between dominance and system robustness, policy resistance, and leverage points.

To fully address the question of whether or not the proposed definition of dominance facilitates fundamental advances and new insights, the next chapter applies the definition and PFD procedure to the state domain (according to Research Question 2 and the objectives of Research Aim 2). State-space explanations have already been introduced for the logistic growth model. Observe that Tables 4.3 and 4.4 summarize dominance based on state-space relationships, not time relationships. This becomes more challenging in higher-order systems. Phaff observes, "The fact that the eliminated structure did not play a large role in generating the reference run, does not mean that ...there is no region in state space where it does not play a role [111]". The PFD procedure, in fact, is not restricted to the time domain, and because it expresses behavior as a function of pathways which are themselves functions of states, it is easily applied in the state-domain. In the next chapter, this is conducted analytically for several models which have been extensively analyzed by other methods, including the Susceptible-Infectious-Recovered (SIR) model, the Lotka-Volterra Predator Prey model, and the yeast cell growth model.

Chapter 5

State-space Dominance

The focus of this chapter is on Research Aim 2, the development of a state-space approach to performing dominance analysis. It begins with a brief overview of the state-space perspective of dynamic systems, and then develops a procedure for identifying dominant state-space regions. The procedure is tested against several models and results are compared with other methods. The chapter concludes with a summary of insights.

5.1 Introduction to Research Aim 2

As discussed in earlier chapters, a fundamental property of the type of systems addressed in this thesis is that they are *state-determined*. Forrester observed similarities between ideas conveyed in system dynamics (SD) and those in the *state variable approach* in engineering feedback systems [33], quoting DeRusso et al. (1965),

The state variable approach ...provides a unifying basis for thinking about linear and nonlinear problems...the state of the network is related to the memory of the network..., the state of a system separates the future from the past, so that the state contains all the relevant information concerning the past history of the system required to determine the response for any input...the manner in which a system reaches a present state does not affect the future output.
While linear and nonlinear systems share these common properties, an important difference is that linear systems can be decomposed as the sum of the *zero-state response* (ZSR) and *zero-input response* (ZIR). The zero state is an equilibrium point, thus if it begins at the origin, absent of external input, it will stay at the origin. The ZSR is the response of a system starting at the origin, perturbed by an external input and therefore reflects only the dynamics associated with the external input. Conversely, the ZIR is the behavior in the absence of external forces, and the response is only determined by where the system begins. The fact that linear system behavior is always the sum of its ZSR and ZIR allows one to distinguish between the effects of exogenous and endogenous forces, regardless of where the system begins.

The same principle, however, does not hold for nonlinear systems. Accordingly, it is often recommended that one initialize nonlinear systems in an equilibrium state before testing its response to external inputs ([29, p. 166] and [136, p. 716]). This is analogous to characterizing the system's ZSR³⁰. Equally important, however, is to explore the model's ZIR. That is, How does the model respond in various regions of the state space in the absence of external forces? This question perhaps is more insightful for identifying endogenous explanations of behavioral.

That said, it is somewhat surprising then that methods for exploring state space have not gained as wide of attention in SD, with a few notable exceptions. Graham uses the phase plane to explain origins of oscillations in a spring-mass-damper system and an employmentbacklog system [52]. Wang suggests the potential for using Lyapunov stability theory in conjunction with eigenvalue sensitivity analysis, however this line of research was never further explored [148]. Davidsen and Guneralp also use phase-plane techniques for describing behavior [17, 56]. Zhang demonstrates challenges of eigenvalue-based methods in identifying

 $^{^{30}\}mathrm{An}$ equilibrium point of a nonlinear system, through a change of variables, can always be mapped to the origin, or zero-state.

dominant structure for some models with a large number of states [152]. He suggests examining sensitivity to state-space rather than the eigenspace, but this has not been further developed. In 2010, Zhang uses summary functions defining a scalar field over the state space which summarizes the global state of the system [153]. He examines the sensitivity of the summary function with respect to model parameters. As an observation, Lyapunov functions are a type of summary function, however not all summary functions are Lyapunov. Zhang's approach has not been extended to Lyapunov stability theory.

In general, state-space methods have not enjoyed widespread use in the SD community, and certainly not in dominance research literature. Current dominance methods assume specific initial conditions and sensitivity to the initial conditions is not typically explored. The purpose, then, of the second research question of this thesis is to understand how dominance methods can be applied in the state-domain. How does dominance shift across state space? Are there state space regions of dominant structure? How do these relate to equilibrium points of the system or limit cycles? Can a formal mathematical relationship be defined between state space and dominance methods? Between transient analysis and steady-state stability analysis? If successful, will this result in new structure-behavior insights for previously studied models? This chapter begins to answer these questions and applies and compare state-space analysis techniques with dominance analysis methods for two classes of models: those exhibiting transient growth dynamics and those exhibiting oscillatory dynamics.

5.2 Overview of State-Space Approach

The main idea behind state-space methods for nonlinear systems is to characterize certain qualities of the family of solutions (trajectories) over the state space, without necessarily solving for the solutions (trajectories) [82]. Thus, methods are sometimes referred to as qualitative. Methods leverage results from geometry and topology and have traditionally focused on identifying the source and destination of trajectories (alpha and omega limit sets) and their stability properties. Figure 5.1 shows a phase plane representation of the dynamics of a second order system, illustrating a single stable equilibrium point and its region of attraction, and an unstable limit cycle separating the stable and unstable state spaces. These artifacts characterize the nature of all trajectories of the system.



Figure 5.1: Example of two-dimensional state space with a single stable equilibrium point and an unstable limit cycle. The limit cycle divides the state space into stable and unstable subspaces.

The following approach for characterizing trajectories in state space will be applied to models and compared with results from dominance methods:

- 1. Identify all equilibrium points and limit cycles.
- 2. Identify stability properties of equilibria and limit cycles.
- 3. Characterize trajectories between equilibria and limit cycles.
- 4. Identify/estimate regions of attraction for stable equilibria and limit cycles.

- 5. Identify stable and unstable state spaces.
- 6. Identify state space regions of loop dominance, and the borders of regions which represents shifts in dominance across the state space.
- 7. Identify bifurcations (changes in model parameters which produce a change in the number and/or stability of equilibria and limit cycles).

5.3 A State-Space Approach to Dominance

From Chapter 4, the dynamics \dot{x}_j is a function of the state variables and external input u_j :

$$\dot{x}_j = f_j(x_1, x_2, \dots, x_n, u_j)$$
(5.1)

The dynamics can also be expressed as a function of causal pathways, which are themselves functions of state variables. Causal pathways are auxiliary variables which represent the causal mechanisms and which map the state variables to the dynamics of interest (See Chapter 4, Figure 4.1). Pathway $p_{ijk}(x_i)$ is the k^{th} pathway from state x_i to the dynamics \dot{x}_{j} .

$$\dot{x}_j = f_j(p_{1ja}(x_1), p_{1jb}(x_1), \dots, p_{ijk}(x_i), \dots, p_{njm}(x_n), u_j)$$
(5.2)

From (5.2), the behavior of interest (\ddot{x}_j) is derived using the chain rule which decomposes the contributions from each pathway, and is also expressed as a function of the state variables (represented by state vector \mathbf{x}).

$$\ddot{x}_{j} = \frac{\partial f_{j}}{\partial p_{1ja}}(\mathbf{x}) \, \dot{p}_{1ja}(\mathbf{x}) + \frac{\partial f_{j}}{\partial p_{1jb}}(\mathbf{x}) \, \dot{p}_{1jb}(\mathbf{x}) + \dots + \frac{\partial f_{j}}{\partial p_{ijk}}(\mathbf{x}) \, \dot{p}_{ijk}(\mathbf{x}) + \dots + \frac{\partial f_{j}}{\partial p_{njm}}(\mathbf{x}) \, \dot{p}_{njm}(\mathbf{x}) + \frac{\partial f_{j}}{\partial u_{j}} \, \dot{u}_{j}$$
(5.3)

The states (\mathbf{x}) represent the degrees-of-freedom of the system and can be assigned arbitrarily, independent from one another. Thus, analyzing (5.3) over (\mathbf{x}) characterizes how behavior changes over the state space. In the last chapter, formal tests were defined for identifying contributory, sufficient, and necessary pathways of (5.3), which in this chapter will be conducted as a function of (\mathbf{x}) instead of time (t). Figure 5.2 illustrates the difference between conducting this analysis over time for a single trajectory versus conducting the analysis over the state space.



Figure 5.2: Dominance analysis applied to state-space regions versus initial conditiondependent trajectories.

This procedure can be conducted analytically for sufficiently simple models, or numerically for more complex models in higher dimension. The result is a perspective of dominance based on state-space versus a single time trajectory. Where a system lies in state space determines which causal pathways are dominant and explains how the same system may behave differently depending on where it begins.

This approach also describes changes in the contributory, necessary, sufficient, and dominant pathways as trajectories cross between different regions of the state space. Applying this approach to regions surrounding limit sets will then allow us to understand the dominant characteristics of causal pathways near equilibrium points and limit cycles in order to understand how specific mechanisms (balancing feedback loops, reinforcing feedback loops) are attracting or repelling the system to/from the limit sets. We investigate if it is possible to draw mathematical relationships between the stability properties of equilibrium points and the dominance properties of causal pathways.

5.4 Testing State-Space Approach with Simple Models

5.4.1 Logistic Growth

The logistic growth model is revisited from a state-space perspective.

$$\dot{P} = f(P) = \alpha P - \beta P^2$$

where

$$P = population$$

$$\alpha = b$$

$$\beta = \frac{(b+d)}{C}$$

$$b = normal \ birth \ fraction > 0$$

$$d = normal \ death \ fraction \ge 0$$

$$C = carrying \ capacity > 0$$
(5.4)

The equilibrium points are found by setting $\dot{P} = 0$, which gives:

$$0 = (\alpha - \beta P) \cdot P$$
(stationary points) $P^* = 0, \frac{\alpha}{\beta}$
(5.5)

One way to determine stability of the equilibrium points is to examine the signs of the eigenvalues of the Jacobian of (5.4) $(\partial f/\partial P)$ evaluated at the equilibrium points P^* .

$$\frac{\partial f}{\partial P}(P^*) = \alpha - 2\beta P^*$$

for $P^* = 0$, eigenvalue $\lambda = \alpha > 0$ (unstable) (5.6)
for $P^* = \frac{\alpha}{\beta}$, eigenvalue $\lambda = -\alpha < 0$ (asymptotically stable)

For first-order systems, stability properties of the equilibrium points can also be inferred through visual inspection of the phase portrait (Figure 5.3). The phase portrait shows that



Figure 5.3: Phase portrait of the logistic growth model.

for P < C = 20, $\dot{P} > 0$, and thus trajectories move to the right and $P^* = 0$ is unstable. For P > C, $\dot{P} < 0$, and trajectories move to the left and thus $P^* = \alpha/\beta = C$ is stable, with \mathbb{R}^+ as the region of attraction. Thus, the analysis concludes that the origin is an unstable

equilibrium point and that for any non-zero population, the population will eventually reach its carrying capacity in the steady-state.

State-Space Dominance Analysis

Analysis of the equilibria depends only on the model parameters and is not affected by the choice of causal pathways. But for dominance analysis, causal pathways must first be defined. Chapter 4 showed several ways in which pathways could be defined for logistic growth. Here, forms 1 and 3 are evaluated. Form 1 is defined by the following pathways:

$$\dot{P} = p_{111} \cdot p_{112}$$
(reinforcing growth) $p_{111} = \alpha P$
(carrying capacity constraint) $p_{112} = 1 - \left(\frac{\beta}{\alpha}\right) \cdot P$
(5.7)

The behavior \ddot{P} is then decomposed into pathway contributions:

$$\ddot{P} = F_{111} + F_{112}$$

$$F_{111} = \frac{\partial \dot{P}}{\partial p_{111}} \dot{p}_{111} = \left(1 - \left(\frac{\beta}{\alpha}\right) \cdot P\right) \cdot \alpha \dot{P} = \alpha \dot{P} - \beta P \dot{P} \qquad (5.8)$$

$$F_{112} = \frac{\partial \dot{P}}{\partial p_{112}} \dot{p}_{112} = \alpha P \cdot \left(\frac{-\beta}{\alpha}\right) \cdot \dot{P} = -\beta P \dot{P}$$

Since there are only two pathways, only two possibilities exist: either the pathways contribute in same direction (sgn $F_{111} = \text{sgn } F_{112}$) and therefore both are sufficient or they contribute in opposite directions and the one with the largest magnitude contribution is necessary and sufficient (dominant). Consider the first case. What regions of state space do the pathways contribute in the same direction and is sgn \ddot{P} positive or negative? From (5.8), $\operatorname{sgn} F_{111} = \operatorname{sgn} F_{112}$ if

$$\begin{aligned} |\alpha \dot{P}| &< \left| -\beta P \dot{P} \right| \\ \alpha &< \beta P \\ P &> \frac{\alpha}{\beta} = \left(\frac{b}{b+d} \right) \cdot C \end{aligned}$$

Therefore, for P satisfying this condition, both pathways are sufficient (and neither are necessary) and it can be shown $\dot{P} < 0$ and $\ddot{P} > 0$. In the second case, the pathways contribute in opposite directions. The state space in which p_{111} is necessary and sufficient (dominates) requires $|F_{111} < F_{112}|$, which holds if:

$$\left| -\beta P\dot{P} \right| < |\alpha\dot{P}| < 2\left| -\beta P\dot{P} \right|$$
$$\left(\frac{1}{2}\right) \left(\frac{b}{b+d}\right) \cdot C < P < \left(\frac{b}{b+d}\right) \cdot C$$

in which case $\dot{P} > 0$ and $\ddot{P} < 0$. Else, p_{112} dominates when:

$$P < \left(\frac{1}{2}\right) \left(\frac{b}{b+d}\right) \cdot C$$

in which case $\dot{P} > 0$ and $\ddot{P} > 0$.

Form 3 of the logistic model is defined by the following pathways:

$$\dot{P} = p_{111} + p_{112}$$
(reinforcing growth) $p_{111} = \alpha P$
(5.9)

(carrying capacity constraint and decline)
$$p_{112} = -\beta P^2$$

 \ddot{P} is decomposed into the following pathway contributions:

$$\dot{P} = F_{111} + F_{112}$$

$$F_{111} = \frac{\partial \dot{P}}{\partial p_{111}} \dot{p}_{111} = \alpha \dot{P}$$

$$F_{112} = \frac{\partial \dot{P}}{\partial p_{112}} \dot{p}_{112} = -2\beta P \dot{P}$$
(5.10)

Since α , β , and P are non-negative, sgn $F_{111} \neq$ sgn F_{112} and therefore the pathways always oppose each other. The one with the largest magnitude force contribution is necessary and sufficient (dominant). Pathway p_{111} dominates when $|F_{111}| > |F_{112}|$, which occurs when $P < \alpha/(2\beta)$. Else, pathway p_{112} dominates. Figure 5.4 summarizes the state-space regions of dominance associated with the two forms of the logistic model, applied to the specific case in which b = 2, d = 0, C = 20.



Figure 5.4: State-space regions of dominance for two forms of the logistic growth model

Observations and Discussion

The analytical state-space dominance results validate the simulated time-domain results in Chapter 4 and provide a more comprehensive picture of dominance applying to all initial conditions. The dominance characteristics of the pathways are determined solely by the value of the state P. One observation is that in both forms of the logistic model, the state-space region encompassing the unstable equilibrium point ($P^* = 0$) is dominated by the pathway belonging to the reinforcing feedback loop, while the stable equilibrium point ($P^* = \frac{\alpha}{\beta}$) is encompassed by a state-space region in which causal pathways associated with balancing feedback loops are sufficient and/or dominant (in form 1, for P > C, p_{111} acts as a balancing loop, not a reinforcing loop). In both forms, a shift in dominance occurs in the state-space region between the unstable and stable equilibrium point (as a result of a shift in loop dominance) is mathematically related to the steady-state behavior (stable equilibrium point) in that the shift occurs at precisely one-half the value of the stable equilibrium point.

The two forms of the logistic model both have a single reinforcing loop and a single balancing loop, however in form 1 they are multiplied, and in form 3 they are added. The different representation leads to different explanations for how structure determines behavior. For P < C, the dominance characteristics of the reinforcing and balancing loops are the same. However for P > C, in form 1 (the multiplicative case), the reinforcing loop switches polarity and acts as a balancing loop, and thus both are sufficient for explaining the behavior. Whereas in form 2 (the additive case), the reinforcing loop never switches polarity and always behaves as a reinforcing loop, and thus only the pathway associated with the balancing loop is dominant.

No new insights were produced by this analysis. The logistic model is perhaps one the simplest of nonlinear models which has been extensively analyzed for nearly two hundred years and serves as an important test case for which any new dominance methods and definitions should be first applied. The analysis agrees with previous results by others in the field, and illustrates the possibility for identifying relationships between state space and dominance methods. We now move on to additional models with more pathways and dimensions to further develop these relationships.

5.4.2 Bass Diffusion

Closely related to logistic growth is the Bass diffusion model which also exhibits S-shape growth and is used to understand the dynamics of innovation diffusion [8]. The diffusion model is similar to the SI structure (logistic growth) but also includes a pathway representing the effects of advertising, allowing for growth even when the initial state is zero. This model is examined to illustrate how model parameters affect the state space regions of dominance as well as the sequence of shifts in dominance. The following representation of the model is derived from Sterman [136].



Figure 5.5: Stock and flow diagram of Bass diffusion model [136].

 $\dot{A} = AR = adoption\ from\ advertising + adoption\ from\ word\ of\ mouth$

$$= aP + \frac{ciAP}{N}$$

where

$$P = N - A$$

$$a, c, i, N > 0$$

$$0 \le P(0) \le N$$
(5.11)

The equilibrium points of (5.11) are:

$$A^* = N, \ -\frac{aN}{ci} \tag{5.12}$$

The eigenvalues for the Jacobian of (5.11) evaluated at the equilibrium points (5.12) determine stability:

$$A^* = N; \ \lambda = -a - ci < 0 \text{ (asymptotically stable)}$$

$$A^* = -\frac{aN}{ci}; \ \lambda = a + ci > 0 \text{ (unstable)}$$
(5.13)

The phase portrait is shown in Figure 5.6.

As shown, unlike in the logistic model, for a > 0, A = 0 is not an equilibrium point and for all non-negative values of A(0), trajectories eventually reach the only non-negative, and asymptotically stable, equilibrium point N. The region of attraction is A > -aN/(ci). Note that if a = 0 (no advertising), this model is the same as the logistic model. Also, if a > ci, there is no inflection point and the behavior is purely goal-seeking growth. Whereas in the logistic model, the inflection point occurs at N/2, in the Bass model, the inflection point occurs earlier due to advertising effectiveness a.



Figure 5.6: Phase portrait of the Bass diffusion model.

State-Space Dominance Analysis

Behavior of A is decomposed into the following pathways:

$$\dot{A} = p_{111} + p_{112} \cdot p_{113}$$

$$(B1) \ p_{111} = a(N - A)$$

$$(B2) \ p_{112} = \frac{(N - A)}{N}$$

$$(R1) \ p_{113} = ciA$$

$$(5.14)$$

$$\ddot{A} = F_{111} + F_{112} + F_{113}$$
(B1) $F_{111} = -a\dot{A}$
(B2) $F_{112} = -\frac{ciA\dot{A}}{N}$
(B1) $F_{113} = ci\dot{A} - \frac{ciA\dot{A}}{N}$

Pathway p_{113} is necessary and sufficient when $|F_{113}| > |F_{111}| + |F_{112}|$ which occurs when A < (ci - a)N/(2ci). Else, if neither p_{111} nor p_{112} are sufficient, then both are necessary. p_{111} is sufficient when $|F_{111}| > |F_{113}|$ which occurs when A > N(1 - a/(ci)). p_{112} is sufficient when $|F_{112}| > |F_{113}|$ which occurs when A > N/2. Therefore, the necessity and sufficiency of each pathway depends on the value of A relative to the terms a, ci, and N, as summarized in figure 5.7.

Figure 5.7 shows there exists five possible dominance shift sequences, depending on the relationship between parameters a and ci. These shifts move the system across different regimes in the dominance framework and are summarized in Figure 5.8. Therefore, the dominance characteristics of the Bass diffusion model are completely determined and described by the model parameters and where the model is operating in state-space.

Observations and Discussion

The pathway force decomposition of the Bass model shows that the model has the same causal mechanisms and forces as the logistic model (pathways p_{113} and p_{112} causing forces F_{113} and F_{112} , associated with loops R1 and B2, respectively), but adds a new causal process p_{111} , the effect of advertising, causing force F_{111} associated with loop B1. Adding B1 causes an earlier transition from exponential to goal-seeking growth, and a more robust behavior pattern in the second phase. Unlike the logistic model, there does not exist a single unique sequence of dominance shifts. The sequences of shifts, rather, depend on the relationship between the advertising effectiveness a and the product of contact rate and adoption fraction ci. As advertising effectiveness decreases, as expected, the model behaves more like the logistic model. Conversely, as advertising effectiveness increases, the phase of exponential growth in which R1 is dominant contracts in state space. Also, the state-space region in which B2 (word-of-mouth) dominates decreases while the region in which B1 (advertising) dominates increases. In the dominance framework, the system behavior in the second phase



Figure 5.7: State space dominance analysis results for the Bass diffusion model, for different parameter values.



Figure 5.8: Dominance shifts in parameter and state space for the Bass diffusion model.

of goal-seeking growth becomes more robust as advertising effectiveness a increases and as the state A increases, which occurs as both B1 and B2 become sufficient.

Because there are only three pathways, the system can only operate in the three points A, B, and C in the dominance framework (Figure 5.8), in which C is the most robust condition associated with two sufficient pathways, or *shadow loops*, B is the case in which a single pathway dominates, and A is the least robust in which two pathways are necessary or critical. The Bass diffusion model transitions between each of these conditions in the dominance framework, depending on the parameter values. Similar to how changes in model parameters can lead to changes in the number and stability of equilibrium points of nonlinear systems (also known as bifurcations), the Bass model demonstrates how changes in parameters can

change the dominance shift sequences. Specifically, the relationship between a and ci/2 causes a sort of *dominance bifurcation* in which for a < ci/2, B2 dominates in a region of the goal-seeking state space, and for a > ci/2, B1 dominates in a region of the goal-seeking state space.

While all dominance methods have been applied to the simple logistic form of the diffusion model (a = 0), no results were found to compare against for the general version introduced by Bass³¹. This state space analysis is the first known complete dominance analysis of this structure. We can reason about how the results would compare with Ford's method in that Ford's method associates necessary conditions with dominance and multiple sufficient conditions with shadow dominance. The Loop Impact method identifies the smallest set of loops sufficient for determining the sign of \ddot{A}/\dot{A} as dominant. For this first-order system, this is equivalent to finding the smallest set of loops sufficient for determining the sign of \ddot{A} , and thus is comparable to the PFD procedure. Since both PPM and LEEA always identify a single dominant loop, their results will necessarily disagree with PFD in the state-space regions in which PFD does not identify any dominant structure which are the regions in which B1 and B2 are both necessary and not sufficient, and in which B1 and B2 are both sufficient and not necessary.

As noted by Sterman [136], the original formulation of the model by Bass does not appeal to feedback loops, but rather to a hazard function defining the probability of adoption at time t. The parameters of the model are estimated through regression based on historical data for the purposes of forecasting the sales peak, which according to the model, occurs when the number of adopters reaches (ci-a)N/(2ci). In the state space dominance analysis performed

 $^{^{31}}$ Several papers claim to have tested dominance methods against what is described as the *Bass model*, however upon inspection the model being tested is actually the logistic growth model and not the model as originally formulated by Bass which includes both adoption through diffusion as well as adoption from external advertising.

here, this corresponds to the point in which the reinforcing loop R1 (associated with wordof-mouth) loses dominance. Dominance then shifts to the market saturation balancing loops of B2 alone or B1 and B2 as a sufficient pair of individually necessary loops.

5.4.3 SIR

Attention now turns to a well-known second-order nonlinear model exhibiting transient growth dynamics, the Susceptible-Infectious-Recovered (SIR) epidemic model describing the spread of acute infectious diseases, developed by Kermack and McKendrick in 1927 [81]. This model is used to explore the concept of regions and boundaries of state space dominance in models of two dimensions, and the relationship between the dominance regions of each state variable. The following representation is derived from Sterman [136].



Figure 5.9: Stock and flow diagram of SIR model.

$$\dot{S} = -IR = -\left(\frac{ci}{N}\right)SI$$
$$\dot{I} = IR - RR = \left(\frac{ci}{N}\right)SI - \left(\frac{1}{d}\right)I$$
where
$$c, i, d, N > 0$$
(5.16)

$$S(0), I(0), R(0) \ge 0$$

 $S(0) + I(0) \le N$

The equilibrium points of (5.16) are $\{I^* = 0\}$, an infinite number of non-isolated equilibria. Linearizing the system about the equilibrium points results in eigenvalues on the imaginary axis, and thus the equilibrium points are non-hyperbolic and linearization cannot be used to determine stability. Rather, geometric reasoning will be used to characterize trajectories and stability. Consider the phase portrait of solution trajectories as shown in Figure 5.10.

The phase plot indicates that the trajectories are bounded within the viable region of the state space $\{S, I \ge 0; S + I \le N\}$. The lower region is bounded by the equilibrium points $\{I^* = 0\}$ and no trajectories can escape the upper right or left boundaries. The epidemic tipping point (TP) occurs when the *infection rate* (*IR*) equals the *recovery rate* (*RR*) (when $\dot{I} = 0$), which occurs when S = N/(cid) = 30. For S < 30, infections decline until the system reaches a disease-free equilibrium. For S > 30, infections increases (epidemic), before it declines. Thus, equilibrium points $\{I^* = 0, S^* > 30\}$ are clearly unstable ($\dot{I} > 0$ and thus trajectories are repelled from the equilibrium points), whereas for equilibrium points $\{I^* = 0, S^* < 30\}$, $\dot{I} < 0$ and trajectories approach the equilibrium points which are stable but non-attractive.



Figure 5.10: Phase plot of SIR model trajectories.

State-Space Dominance Analysis

The behavior of S is decomposed into the following pathways:

$$\dot{S} = p_{111} \cdot p_{211}$$

 $(B1) \ p_{111} = -\left(\frac{ci}{N}\right)S$ (5.17)
 $(R1) \ p_{211} = I$

$$S = F_{111} + F_{211}$$

$$(B1) F_{111} = -\left(\frac{ci}{N}\right) I\dot{S}$$

$$(R1) F_{211} = -\left(\frac{ci}{N}\right) S\dot{I}$$

$$(5.18)$$

For S < TP = N/(cid), F_{111} , $F_{211} > 0$ and $\ddot{S} > 0$, and thus B1 and R1 are both sufficient. Else, for TP < S < I + TP, $F_{111} > 0$, $F_{211} < 0$ and $\ddot{S} > 0$, thus B1 is necessary and

••

sufficient. Else, S > I + TP, $F_{111} > 0$, $F_{211} < 0$ and $\ddot{S} < 0$, thus R1 is necessary and sufficient. Figure 5.11 illustrates the geometric boundaries of the three state-space regions of dominance for state variable S, as well as an example trajectory which transitions between the three regions (bold curve). Taking the example trajectory in Figure 5.11 which begins



Figure 5.11: Three state-space regions of dominance for susceptible population in SIR model.

with one infected individual and 99 susceptible individuals, the shifts in dominance for the susceptible population is also shown in the time-domain for comparison (Figure 5.12).



Figure 5.12: Dominance results for susceptible population in the time domain for an example trajectory of the SIR model.

Dominance is now analyzed for the second state variable in the SIR model: the infectious population I. The behavior of I is decomposed into the following pathways

$$\dot{I} = p_{121} \cdot p_{221} + p_{222}$$
(B1) $p_{121} = \left(\frac{ci}{N}\right) S$
(R1) $p_{221} = I$
(B2) $p_{222} = -\left(\frac{1}{d}\right) I$
(5.19)

$$\ddot{I} = F_{121} + F_{221} + F_{222}$$

$$(B1) \ F_{121} = \left(\frac{ci}{N}\right) I\dot{S}$$

$$(R1) \ F_{221} = \left(\frac{ci}{N}\right) S\dot{I}$$

$$(B2) \ F_{222} = -\left(\frac{1}{d}\right) \dot{I}$$

$$(5.20)$$

B1 always exerts a negative force on I ($F_{121} < 0$), whereas R1 and B2 always exert forces in opposite directions and switch orientations (polarities) depending on whether S is above or below the tipping point (TP = N/cid). For S > TP, $F_{221} > 0$ and $F_{222} < 0$ and for S < TP, the signs switch. I has positive acceleration ($\ddot{I} > 0$) when the following inequality holds

$$I < \frac{(S - TP)^2}{S}$$

In this case, when S > TP, R1 is necessary and sufficient for producing the positive acceleration. Otherwise, when S < TP, B2 is necessary and sufficient.

For the case in which I has negative acceleration and the above inequality does not hold, then for S > TP, B2 is never sufficient. B1 is necessary and sufficient when TP < S < I + TP, otherwise B1 and B2 are both necessary to produce the negative acceleration. For S < TP, R1 is never sufficient to produce negative acceleration and B1 is necessary and sufficient when

$$I > \frac{TP^2}{S} - TP$$

Otherwise, B1 and R1 are both necessary to produce negative acceleration. The resulting five regions of state-space dominance for I are shown in Figure 5.13.

The dominance results for the example trajectory are also shown in the time domain for comparison (Figure 5.14).

The dominance analysis concludes that for each state variable in the SIR model, there exists multiple, connected regions in which different causal pathways (and thus feedback loops) possess unique necessary and sufficient properties. Therefore, the concept of state-space regions of dominance applies to this second-order model. The geometric regions, when combined



Figure 5.13: Five state-space regions of dominance for infectious population in the SIR model.



Figure 5.14: Dominance results for infectious population in the time domain for an example trajectory of the SIR model.

with the phase plot defining the trajectories, also defines the sequence of shifts in dominance for all possible trajectories and initial conditions of the model. This was illustrated by taking an example trajectory of the model and analyzing the shifts in dominance in time as the trajectory crossed through the various state-space regions of dominance. For the SIR model, it appears that all trajectories beginning from an unstable equilibrium point, will eventually transition through all three regions of dominance in S and all five regions of dominance in I. It also appears that each feedback loop in the model, in some region of the state-space, is necessary and sufficient (dominant) for determining the behavior of the state variable it influences. There are also regions in which there exists multiple sufficient loops and regions in which there exists multiple necessary loops, illustrating how trajectories move through various points of the dominance framework and various levels of behavior robustness. The regions of dominance are different for S and I, which is to be expected given that the set of causal pathways affecting each variable are different. Taken together, there are six distinct regions of state-space in which dominance properties change for both state variables.

There are, however, also regions of state space where both states share similar dominance properties. For S > TP, the reinforcing loop R1 dominates the behavior of both S and I in the first phase. This region in which the reinforcing loop is dominant also encompasses the set of unstable equilibrium points, similar to the Bass and logistic model. There is also overlap between the regions in which B1 dominates both S and I. The stable equilibrium points are encompassed by regions of dominance in which balancing loops are sufficient, also similar to the Bass and logistic model. Finally, it is worth noting that some loops change polarity and at times contribute to acceleration, and at other times contribute to deceleration. All instances of dominance and shifts in dominance can be completely and rigorously defined by the relationship between the state variables and model parameters.

Observations and Discussion

The results are consistent with analysis by Hayward and Boswell who use the Loop Impact method, using the same parameter values, to evaluate loop dominance on variable I [60]. The loops in their representation of the model are defined slightly differently, but they identify R1, B1 and B2 as dominant in approximately the same phases as in this analysis. The two transitional phases between dominance, however, are described in their analysis as other loops "providing assistance". The state-space analysis reveals this is actually a condition of multiple necessary loops, which dominate when taken together as a set. Hayward and Boswell also refer to the situation when a loop changes polarity as changes is flow dominance. In this analysis, this occurs when the force contribution of a pathway changes sign.

In Sterman's analysis of the SIR model [136, pp. 303-309], an epidemic is associated with R1 dominating over B1 and B2 when a single infective individual arrives in a community, which occurs if the average infection rate exceeds the recovery rate (that is, the system is past the tipping point TP). Otherwise, if R1 is weaker than the balancing loops B1 and B2, an epidemic will not occur (in this case, it can be shown TP is actually greater than N and thus the entire state-space region is left of the tipping point). The state space analysis is consistent with Sterman's analysis, and further specifies that at the introduction of a single infectious individual, the system is operating in a neighborhood around the I = 0 axis, in which case either R1 dominates if S > TP or B2 dominates if S < TP. The loop B1 does not come into play.

This state space dominance analysis is the first known instance of mapping the shifts in dominance between each feedback loop across the entire state space of the SIR model. It is consistent with prior dominance analysis in the field and further specifies which specific loops dominate and how, and the nature of the dominance shifts across the state space.

5.4.4 Linear Harmonic Oscillator

The previous models all exhibit transient dynamics with no limit cycles. Attention now shifts to models containing limit cycles to examine how state-space dominance methods apply. Oscillations have been a source of confusion with respect to claims about dominance, and some methods of dominance have been found to be inadequate for explaining shifts in dominance for oscillatory modes [121, 79].

The simplest model that produces sustained oscillations is the second-order linear harmonic oscillator. This model has been used to counter the claim that linear models cannot shift dominance [130]. This claim and counter-claim will be evaluated using the proposed formal definition of dominance. A stock and flow diagram of the model is shown in Figure 5.15.



Figure 5.15: Stock and flow diagram of harmonic oscillator.

$$\begin{aligned} \dot{x} &= y\\ \dot{y} &= -x \end{aligned} \tag{5.21}$$

The system has a single stable equilibrium point at the origin $(\lambda = \pm i)$ and an infinite number of non-isolated stable orbits centered around the origin. The phase portrait and example system response over time is shown in Figure 5.16.



Figure 5.16: Phase plot and example behavior over time for the harmonic oscillator.

State-Space Dominance Analysis

Two pathways make-up a single balancing feedback loop which drives the system. By definition, therefore, the single balancing feedback loop is trivially always necessary and sufficient (dominant) for determining the acceleration of x and y, even as the acceleration changes sign. The polarity or orientation of the force generated by the loop switches over time, which is evident in the explicit representation of the acceleration of x and y, and illustrated in Figure 5.17.

$$\begin{aligned} \ddot{x} &= -x \\ \ddot{y} &= -y \end{aligned} \tag{5.22}$$

Observations and Discussion

The force exerted by the balancing loop pushes the variables opposite the sign of their current state. Detecting shifts in dominance using Richardson's definition of dominant polarity of major feedback loops which is based on the product of the gains of the Jacobian of the



Figure 5.17: State-space regions of behavior modes of the linear harmonic oscillator.

system (closed loop gain), linear time-invariant systems will never shift dominance since the gains are constant [121]. Mojtahedzadeh and Richardson later modified the definition of dominant polarity to consider what they call implicit loops which can shift in polarity [100]. The implicit loops take into consideration the values of the flows or first time derivatives, which in this system change between positive and negative values, thus feedback loops can change in dominant polarity. This is seen in the state space analysis in that the force exerted by the balancing loop changes between positive and negative. If one then determines that shifts in dominance occur anytime the polarity of the loop changes (or the orientation of the force contribution to acceleration), then of course linear systems can shift in dominance. The proposed definition of dominance is based on which structures are necessary and sufficient for determining the second derivative. Not all changes in second derivative are associated with shifts from one structure to another, as evident in this model. In this case, the same structure is responsible for changing the behavior pattern, thus this is not considered a shift in structural dominance, just a shift in behavior mode.

This also corresponds to the traditional definition of dominant modes or eigenvalues. For linear systems, the eigenvalues of the system do not change, and thus the dominant eigenvalues (the least stable) do not change. Thus, the sensitivity of the loops with respect to the eigenvalues do not change and the dominant loops do not change. For this system the fact that the eigenvalues are on the imaginary axis is what creates the oscillatory behavior. In nonlinear oscillatory systems, this is not necessarily the case and the eigenvalues of the linearized system may change and the loops may shift in dominance, as we shall see in the next example.

5.4.5 Lotka-Volterra Predator Prey Model

The Lotka-Volterra Predator Prey Model is one of the simplest nonlinear models which contains limit cycles and is well studied in engineering, math, and science texts on differential equations. It has also been evaluated by several different loop dominance methods and therefore is a suitable model for comparing the state-space approach.



Figure 5.18: Stock and flow diagram of Lotka-Volterra predator prey model.

R1 represents prey natural births. B1 is prey deaths based on prey population. R2 is predator births based on predator population. B2 is predator deaths. B3 is the balancing

loop representing prey deaths from predation and predator births based on prey population.

$$\dot{x} = ax - bxy$$

$$\dot{y} = -cy + dxy$$

where

$$x = prey population$$

$$y = predator population$$

$$a = prey birth fraction$$

$$b = fraction of prey deaths per predator$$

$$c = prey death fraction$$

$$d = fraction of predator births per prey$$

$$a, b, c, d > 0$$

$$x(0), y(0) \ge 0$$

(5.23)

The equilibrium points of (5.23) are (0, 0) and $\left(\frac{c}{d}, \frac{a}{b}\right)$ which have the following stability properties:

$$p_1^* = (0, 0); \ \lambda_1 = a, \ \lambda_2 = -c \text{ (unstable)}$$

$$p_2^* = \left(\frac{c}{d}, \frac{a}{b}\right); \ \lambda = \pm i\sqrt{ac} \text{ (stability cannot be determined by linearization)}$$
(5.24)

The phase portrait of solution trajectories (Figure 5.19) depicts the stable and unstable subspaces of (0, 0), which are the vertical and horizontal axes, respectively. It also illustrates that equilibrium point $\left(\frac{c}{d}, \frac{a}{b}\right)$ is at the center of an infinite number of non-isolated, neutrally stable limit cycles, and is thus stable but not asymptotically stable. This system is also not structurally stable in that small changes in parameter values alter the equilibrium points and family of limit cycles [137, pp.189-190].

Figure 5.20 shows the behavior over time for one of the orbits of this system.



Figure 5.19: Phase plot of Lotka-Voltera model.



Figure 5.20: Example trajectories of predator and prey populations in the Lotka-Volterra model.

State-Space Dominance Analysis

The behavior of x is decomposed into the following pathways:

$$x = p_{111} + p_{112} \cdot p_{211}$$
(R1) $p_{111} = ax$
(B1) $p_{112} = x$
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(B3) $p_{211} = -by$

$$\ddot{x} = F_{111} + F_{112} + F_{211}$$
(R1) $F_{111} = a\dot{x} = a(ax - bxy)$
(B1) $F_{112} = -by\dot{x} = -by(ax - bxy)$
(B3) $F_{211} = -bx\dot{y} = -bx(dxy - cy)$

From (5.26), the sign of the force contributions F_{111} and F_{112} , associated with loops R1 and B1 respectively, depends on the relationship between y and (a/b). The sign of the force contribution F_{211} associated with loop B3 depends on the relationship between x and (c/d), as summarized in Figure 5.21.



Figure 5.21: Changing polarities of feedback loops affecting prey population over four regions of state space.

The figure illustrates how the polarities or orientations of the force contributions from each feedback loop change signs across four distinct phases of the orbits (regions in state-space). Each loop in fact can contribute to positive or negative acceleration, depending on the region. The general approach for identifying necessary and sufficient conditions for each loop is to evaluate, algebraically, the relationship between the magnitudes of each force contribution in determining the sign of \ddot{x} in each of the four regions. For example, in the lower left quadrant, since B1 is the only negative loop, when its magnitude is greater than the sum of R1 and B3, it is necessary and sufficient (dominant) in creating negative acceleration. Otherwise, \ddot{x} is positive and either R1 and B3 are both necessary, or one is sufficient and not the other, or both are sufficient for producing positive acceleration. This analysis is conducted in each of the four quadrants.

The first result is that prey has positive acceleration $(\ddot{x} > 0)$ when the following inequality holds:

$$x < \frac{(a-by)^2}{bdy} + \frac{c}{d} \tag{5.27}$$

Then, evaluating the sufficient conditions for each loop in determining sgn \ddot{x} , in each quadrant, results in seven state-space regions, as shown in Figure 5.22.

Each region is designated by a letter followed by a plus or minus sign, indicating whether \ddot{x} is positive or negative in that region. The dominant loops in each region and the shifts in dominance are identified in Figure 5.23.

All orbits, regardless of the initial conditions pass through each of the seven regions, since the region boundaries all intersect at the single stable equilibrium point at the center of the orbits, and extend radially outward. Shifts in dominance occur along each boundary, which is when the necessary and sufficient conditions of the loops change. Each loop is dominant in some region of the state-space. There are also regions in which multiple loops are necessary and in which multiple are sufficient. The regions of positive acceleration map to the robust regimes in the dominance framework (multiple sufficient conditions) whereas the regions of negative acceleration map to the less robust regimes (multiple necessary conditions).



Figure 5.22: Seven state-space regions of dominance for prey population in the Lotka-Volterra model.



Figure 5.23: Dominance and shifts in dominance for the prey population in the Lotka-Volterra model.
The state-space dominance of the predator population y is now examined. Similarly, the behavior of y is decomposed into the following pathways:

$$\dot{y} = p_{221} + p_{222} \cdot p_{121}$$
(B2) $p_{221} = -cy$
(B2) $p_{222} = y$
(B3) $p_{121} = dx$

$$\ddot{y} = F_{221} + F_{222} + F_{121}$$
(B2) $F_{221} = -c\dot{y} = -c(dxy - cy)$
(B2) $F_{222} = dx\dot{y} = dx(dxy - cy)$
(B3) $F_{121} = dy\dot{x} = dy(ax - bxy)$
(5.29)

Equation (5.29) is equivalent to the pathway force decomposition for x (5.26), using the following variable and parameter changes:

$$y \leftrightarrow x$$
$$b \leftrightarrow d$$
$$a \leftrightarrow c$$

Thus, the loop force contributions affecting y are similar to those affecting x. B3 has a similar affect on y as it does for x. B2 acts upon y similar to how R1 acts upon on x, and R2 is similar to B1. Similar to x, the orientation of the loops affecting y change in each quadrant as shown in Figure 5.24.

Predators have a positive acceleration $(\ddot{y} > 0)$ when the following inequality holds:

$$y < \frac{(c-dx)^2}{bdx} + \frac{a}{b} \tag{5.30}$$



Figure 5.24: Changing polarities of feedback loops affecting predator population over four regions of state space.

Figure 5.25 shows the seven state-space regions of dominance for y. Note the similarity to Figure 5.22 for x.

The dominant loops in each region and the shifts in dominance are identified in Figure 5.26.

Similar to x, each loop affecting y is dominant in some region in the state-space. Also similar to x, regions in which there exists multiple sufficient loops occur when acceleration is positive, and regions in which there exists multiple necessary loops occur when acceleration is negative.

Combining the dominance regions in x with those in y results in 13 distinct state-space regions in which different loops dominate x and y (Figure 5.27). Note the intersection of regions for x and y in the upper left quadrant, illustrating how different orbits may experience



Figure 5.25: Seven state-space regions of dominance for predator population in the Lotka-Volterra model.



Figure 5.26: Dominance and shifts in dominance for the predator population in the Lotka-Volterra model.

a different sequence in dominance shifts in x and in y depending on the initial conditions, for that one area of the state-space.



Figure 5.27: 13 distinct state-space regions of dominance for both predators and prey in the Lotka-Volterra model.

Finally, to illustrate how dominance shifts over time for a single particular orbit, consider the orbit indicated by the bold line in the above figures, which begins with 2 prey and 1 predator per acre. Figure 5.28 shows the trajectory over time for a single cycle, along with the pathway force decompositions, net force, and the necessary and sufficient loops.

This specific example uses the same parameter settings as used by others who have analyzed the model using different dominance methods, and will be used for comparison in the following section.



Figure 5.28: Example trajectory, pathway force decomposition, and dominance for one cycle of the prey population.

Observations and Discussion

In Richardson's analysis of the Lotka-Volterra model, he calculates the dominant polarity of the minor feedback loops for x and y (which are the diagonal elements of the Jacobian), and uses these values to suggest that prey is dominated by the births reinforcing loop R1 when y < a/b and by the deaths balancing loop B1 when y > a/b [121]. Similarly, predators are dominated by R2 when x > c/d and by B2 when x < c/d. As Richardson acknowledges, this analysis does not utilize the off-diagonal terms of the Jacobian which make-up the fifth feedback loop B3. The state-space analysis reveals that indeed the loops identified by Richardson as dominant do in fact dominate within the half-spaces he indicated, however the precise boundaries are explicitly identified as sub-regions within each half-space, and furthermore the loops indicated by Richardson as dominant are also found to be necessary in sub regions of the opposite half-space as well, in every instance. For example, R1 is a necessary loop within a sub-region of y > a/b. The dominance of B3 which is not explicitly determined by Richardson, but acknowledged to play an important role in generating oscillations, is found by the state-space methods to be uniquely dominant during the peaks of predator and prey populations and a sufficient contributor in the valleys.

Guneral evaluates dominance of the prey population using Loop Eigenvalue Elasticity Analysis (LEEA) and also compares it to the analysis by Mojtahedzadeh who used the Pathway Participation Method (PPM) [57, 95]. Both analyses identify R1 as initially dominant, but LEEA then finds R^2 dominant whereas in this analysis, since R^2 does not directly impact prey x, it cannot be dominant. However, R^2 does indirectly impact x through B3, and this analysis finds B3 as necessary and then uniquely dominant around the same time (until t = 7). LEEA and this method both find B1 dominant during the same middle time period. LEEA then finds B2 dominant next, which in this analysis indirectly affects x through B3 which is found to be sufficient during the same time period. B1 and R1 are also found to be sufficient during this period which does seem to correspond with the loop elasticity metrics from LEEA. In the final phase, R_1 is identified as dominant by both methods. Some of the differences are attributed to the fact that LEEA looks at the sensitivity and elasticity of all loops on the variable of interest, whereas this method only distinguishes the impact of loops which directly affect the behavior. Also, LEEA evaluates the elasticity of loops relative to the envelope and frequency of oscillations, whereas this method evaluates the contribution to the second derivative. However, despite these differences, the results are consistent.

The PPM analysis mostly agrees with both LEEA and this method but underplays the role of B3, the major balancing loop that connects the predator and prey systems. LEEA identifies B3 as completely responsible for the oscillations. Similarly, in this method, B3 is found to

be necessary and/or sufficient in determining behavior during the peaks and the valleys of the oscillations.

Finally, we look at Hayward's analysis using the Loop Impact method [58]. Although his specific parameters are different than the ones used above, the state space analysis shows that the general structure and shifts in dominance should be consistent regardless of the initial conditions. Thus, we can use the state-space analysis to compare against his specific instance of the model. The Loop Impact method agrees with this analysis in every instance except when there are multiple sufficient loops. Because the loop impact method uses sufficiency as the criteria for dominance, when there is a single necessary and sufficient loop, the results will be the same. The loop impact method will also identify a set of multiple necessary loops which are together sufficient and thus dominant. The only condition in which the loop impact method results in ambiguities is when there are multiple sufficient loops. In which case, the loop picker algorithm will pick the sufficient set with the minimum number of loops. If there are multiple minimum sets (for example, in this case in which there are two individually sufficient loops), the algorithm picks the one with the largest impact value. In this case, during the valley of the oscillation, B3 has the largest impact, and so it is identified as dominant. However, the pathway force decomposition analysis reveals that during the valley, B1 and R1 also have consecutive periods of sufficiency alongside B3 in determining the behavior, and thus detects a case of shadow dominance or shared dominance between B1 and B3, and then between B3 and R1. These cases are not identified by the loop impact method.

A final observation from the state space dominance method is that the sign of acceleration only changes when there exists one or more necessary loops. The sign never changes from a state-space region in which there exists multiple sufficient loops. This indicates that behavior changes may only occur when the system is in a somewhat fragile state. If true, one implication of this is that systems are not likely to change behavior mode (acceleration) while the system is in a robust regime in the dominance framework.

5.4.6 Yeast Model

The final model of this chapter is the yeast cell growth model. This model exhibits S-shape rise and collapse behavior, similar to the behavior seen in the supply and demand of health services model which will be examined in the following chapter. It also happens to be one the most analyzed simple models (nonlinear with degree less than three) which has been evaluated by all the current loop dominance methods [129, 127, 57, 110, 96, 70, 60]. While the model structure and behavior are relatively simple, this model illustrates how different definitions and methods of dominance have produced to a wide variety of different and sometimes conflicting explanations for behavior. This is particularly troubling in light of the fact that most realistic models of social systems contain a significantly larger number of state variables and causal mechanisms and at the same time are not subject to nearly the same level of analytical scrutiny from such a wide variety of methods as this simple model. The goal of this analysis is to see if the proposed definition and method for identifying time and state-space regions of dominance can shed light on the discrepancies between the methods.

The yeast model has two states variables: yeast cells(C) and alcohol(A) (Figure 5.29).

Cells multiply and create alcohol which in return stunts cell growth and accelerates cell death. The model has two minor feedback loops (R1 : births and B1 : deaths) and two



Figure 5.29: Stock and flow diagram of yeast cell growth model.

major feedback loops (B2: alcohol stunting births and B3: alcohol increasing deaths).

$$\dot{C} = cell \, births - cell \, deaths$$
$$= \left(\frac{C}{\tau_d}\right) \cdot g(A) - \left(\frac{C}{\tau_l}\right) \cdot h(A)$$
$$\dot{A} = \alpha \, C$$

where

$$\tau_{d} = \text{cell division time} > 0$$

$$\tau_{l} = \text{cell lifetime} > 0$$

$$g(A) = \text{effect of alcohol on births (negative linear function)}$$

$$h(A) = \text{effect of alcohol on deaths (positive exponential function)}$$

$$\alpha = \text{alcohol generated per cell} > 0$$
(5.31)

The equilibrium points of (5.31) are $\{C = 0\}$, a set of non-isolated equilibria. Similar to the SIR model, this set is divided into unstable and stable equilibria representing the

unique origins and destinations of trajectories. The separation between unstable and stable equilibria occurs at $\dot{C} = 0$ which occurs at $g \cdot \tau_l = h \cdot \tau_d$ (analogous to the tipping point in the SIR model), and is also the maximum point of C. For $g \cdot \tau_l > h \cdot \tau_d$, the associated equilibria are unstable and $\dot{C} > 0$ (cell growth). For $g \cdot \tau_l < h \cdot \tau_d$, the associated equilibria are stable and $\dot{C} < 0$ (cell decline).

State-Space Dominance Analysis

The number of cells C is the state variable whose behavior is of interest and is decomposed into the following four pathways representing the contributions from each of the four feedback loops:

$$\dot{C} = p_{111} \cdot p_{211} + p_{112} \cdot p_{212}$$

$$(R1) \ p_{111} = \frac{C}{\tau_d}$$

$$(B1) \ p_{112} = -\frac{C}{\tau_l}$$

$$(B2) \ p_{211} = g(A)$$

$$(B3) \ p_{212} = h(A)$$

$$\ddot{C} = F_{111} + F_{112} + F_{211} + F_{212}$$

$$(R1) \ F_{111} = \left(\left(\frac{g(A)}{\tau_d} \right)^2 - \frac{g(A) \ h(A)}{\tau_d \ \tau_l} \right) C$$

$$(B1) \ F_{112} = \left(\left(\frac{h(A)}{\tau_l} \right)^2 - \frac{g(A) \ h(A)}{\tau_d \ \tau_l} \right) C$$

$$(B2) \ F_{211} = \frac{\dot{g}(A)}{\tau_d} C$$

$$(B3) \ F_{212} = \frac{-\dot{h}(A)}{\tau_l} C$$

The following parameter values and function definitions are used which correspond to those used in previous dominance analyses of this model:

$$\tau_d = 15$$

$$\tau_l = 30$$

$$\alpha = .01$$

$$g(A) = -.1A + 1.1$$

$$h(A) = e^{A - 11}$$

Even for these relatively simple functions g(A) and h(A), analytically deriving the inequality expressions for the necessity and sufficiency of each of the four pathways and combinations of pathways in state-space (A, C) (as done for the previous models) becomes a significantly more challenging and complex task. Imagine, then, how much more difficult, if not impossible, this becomes for models of higher dimension with many more causal pathways (as in the case of the public health models which are analyzed in this thesis).

Yet, the state-space analyses conducted for the previous models strongly suggests that statespace regions of dominance should in fact exist, even if closed-form analytical expressions cannot be derived for their boundaries. Therefore, instead of solving for the state-space boundaries algebraically, the pathway force decomposition (PFD) procedure from Chapter 4 is extended to analyze the four pathway contributions of (5.33) (and subsets of contributions) across the state space (A, C) in order to discover the dominance region boundaries (see Appendix D for Matlab code). This extension is referred to as the state-space PFD procedure (SSPFD).

Applying the procedure to the Yeast model reveals six state-space regions of dominance for C which overlay the phase plot in Figure 5.30. There also exists two relatively small regions which are not as distinguishable in the figure.



Figure 5.30: State-space regions of dominance for the Yeast model.

The letters represent the six primary dominance regions which are sequentially visited by each trajectory. The sign following the letter indicates if C is accelerating or decelerating in that region, indicating that trajectories begin with accelerated growth followed by an inflection point and decelerated growth until reaching a maximum, after which they exponentially decline and transition to exponential decay. The blue line is a reference trajectory beginning with a single cell (and no alcohol) and is used for comparison to other methods. The dominance properties of each region are summarized in Figure 5.31.

The reference trajectory beginning at (0, 1) is also analyzed using PFD in the time-domain to illustrate the temporal aspects of behavior, force contributions, and dominance shifts and to facilitate a direct comparison with other methods (Figure 5.32).



Figure 5.31: Summary of dominance and shifts in dominance across each state-space region of the Yeast model.



Figure 5.32: Behavior, force contributions, and dominance of the reference trajectory for the Yeast model.

In the beginning, when little alcohol is present, the reinforcing birth process R1 is dominant in the first phase, producing accelerating cell growth, until alcohol reaches a level in which the stunting of new cell development through B2 causes growth to decelerate at the first inflection point. B2 remains dominant through this phase as the birth process R1 continues to decline, until R1 becomes small enough that the effect of alcohol on deaths B3, while not as strong as B2, becomes stronger than R1 and so it too is sufficient to produce deceleration, thus B2 and B3 become a dominant pair causing the yeast cells to reach their peak and begin to decline. As cells begin to decline from their peak, alcohol levels have a greater decelerating effect on cells through deaths (B3) than through births (B2), while the natural death process actually has an accelerating effect on cells (by decreasing the deaths as cell population decreases) and thus B3 becomes solely dominant in creating exponential decline. Eventually cells drop to levels in which the lack of remaining cells becomes the dominating influence and slows down decline (B1 is dominant), changing the behavior to exponential decay (second inflection point). As decay continues and cell levels approach the stable equilibrium of zero, the influence of alcohol on births and deaths becomes negligible, to the point that even the natural birth process R1 exerts a greater force and thus the influence of R1, while nearly zero, would even in the absence of B1 be sufficient in slowing decline, and thus even though the force of B1 is significantly greater than any other force at this point, R1 and B1 are both sufficient to produce the behavior pattern in the final phase. It would be appropriate, however, that given the magnitude difference between R1 and B1 in the final phase, to conclude that B1 is definitively dominant.

Two interesting phenomena are observed in the bottom graph of Figure 5.32. For a moment, B3 appears to have a brief role around time t = 51. Similarly, B1 appears to have a brief role around time t = 65. Associated with time t = 51, the state-space graph 5.30 also reveals a small sliver of a region within the left-most boundary. Figure 5.33 provides a closer-look at what is occurring.



Figure 5.33: Brief regions of dominance during the transition between accelerating and decelerating growth (inflection point) in the Yeast model.

Coming into the inflection point, the force of R1 which is dominant in producing positive acceleration, is nearly canceled by the opposing force of B2. The magnitude of R1 continues to decline as the magnitude of B2 continues to increase. Meanwhile, the other forces from balancing loops B1 and B3 are nearly zero but just slightly negative. At the point of inflection (t = 50.85), the positive force of R1 exactly cancels out with the sum of the negative forces of B1, B2, and B3, and thus all balancing negative forces are at that moment necessary (critical) in producing a net zero force. This is also a very fleeting or fragile state, as also indicated by being associated with multiple necessary conditions in the upper-left corner of the dominance framework (Figure 5.34), and as soon as the system passes the inflection point at (t = 50.86) since the balancing loop forces are not all equal, the weakest one is no longer necessary (B1 is the weakest and thus drops out), and shortly after, the next weakest is no longer necessary (B3) at time (t = 50.91) leaving the remaining balancing loop B2 as alone sufficient for causing the decelerated growth, and thus is both necessary and sufficient (dominant).



Figure 5.34: Transitions across the dominance framework for the Yeast model.

This example illustrates how behavior transitions of a state variable x at inflection points $(\ddot{x} = 0)$ in which x is affected by multiple pathways, will always occur on the vertical axis of the dominance framework where there are multiple necessary pathways (specifically, all the pathways associated with the opposing force are by definition necessary at the point of inflection). If one of these pathways exerts significantly greater force than the others, then this will only be a brief state, as this example shows. If all the pathways exert slightly different forces, then the system will march sequentially down the vertical axis of the dominance framework as the number of necessary pathways decrease and the system gains robustness in its behavior pattern. This example may also indicate that when variables affected by multiple pathways change behavior patterns (most commonly associated with inflection points), this is associated with the system being in its most fragile state from a behavior perspective, which also makes intuitive sense. In the examples seen thus far, no transitions occur when the system is in a robust state with multiple sufficient pathways or zero necessary pathways.

The other phenomenon occurs at time t = 65 and is associated with the peak where $\dot{C} = 0$. This corresponds to the point in which R1 exerts zero influence, and therefore any balancing loop which decelerates growth at that moment in time is sufficient, even B1 which is nearly zero. This represents the opposite situation as described above, where for a brief moment of transition the system is in its most robust state with three sufficient pathways (associated with the lower right corner of the dominance framework). The behavior is robust because there are no forces that oppose the observed deceleration. How long the system stays in this state depends on the magnitude of the supporting forces and how quickly the opposing forces increase in magnitude.

The two cases described above correspond to the examples that were also given in Chapter 3 when first introducing the dominance framework, associated with multiple necessary and multiple sufficient conditions.

Comparison With Other Methods

The results are now compared with previous studies which evaluate the Yeast model using loop eigenvalue elasticity analysis (LEEA) [129, 127, 57, 110]; Ford's behavioral approach [110]; the Pathway participation metric (PPM) method [96]; a sensitivity analysis approach [70]; and finally, the Loop Impact method [60].

LEEA. Dominance is identified by comparing the relative magnitudes of the loop eigenvalue elasticities, and is evaluated for discrete time intervals which are defined based on the behavior of the eigenvalues which reflect system-wide behavior and not just the behavior of C, and therefore results in time phases which do not exactly line-up with the four phases based on atomic behavior pattern of C. In the first accelerating growth phase, LEEA identifies both R1 and B2 as the most influential and R1 as primarily responsible for the behavior. This is consistent with PFD which reveals that the forces contributions of R1 and B2 grow more than the other loops, but in opposition to one another, with R1 being slightly larger than B2, and is thus dominant. In the decelerating growth phase, LEEA identifies B2 as most influential. The momentary necessity of B1 and B3 and the inflection point are not detected by LEEA since LEEA bases influence on normalized metrics and not conditions of necessity or sufficiency. Also, LEEA identifies the shift in dominance from phase 1 to phase 2 significantly earlier than the PFD method, since it is based on eigenvalues of the whole system whereas the PFD method is focused on the behavior of the variable of interest only, C. In the exponential decline phase, LEEA identifies B3 and B1 as most influential. The identification of B3 is consistent with PFD, and the identification of B1 is also consistent with PFD given that it becomes necessary and sufficient towards the end of the third phase as defined evaluated by LEEA. However, LEEA does not identify the sufficiency of B2 at the beginning of this phase. PFD shows that the force contribution of B2 is relatively weak when compared to that of B1 at the beginning of the phase, even though it is sufficient, and since LEEA's criteria for dominance is based on normalized influence metrics, not sufficiency, it would not identify B2 as influential. In the last phase of exponential decay, LEEA identifies B1 as the most influential, also agreeing with PFD. LEEA does not identify the sufficiency of R1 at the tail end of the phase due to its relatively weak force contribution, as evident by the PFD analysis. In summary, in each phase, the loop identified by LEEA as most influential corresponds to the loop identified by PFD as necessary and sufficient (dominant), however LEEA does not identify the cases in which there are multiple necessary or sufficient pathways, due to the reasons mentioned. There also seems to be some correlation between the force contribution of the pathways and the eigenvalue elasticities of the pathways, since the PFD analysis largely supported the findings of LEEA.

Ford's behavioral approach. Ford's method divides the behavior into four phases based on the atomic behavior pattern, which changes at the two inflection points and at the maximum value of cells, and identifies dominance based on changes in behavior pattern when structure is deactivated. In the first accelerating growth phase R1 is identified as dominant, consistent with PFD. In the decelerating growth phase B2 is identified as dominant. The momentary necessity of B1 and B3 were not identified, either due to the fact that the deactivation may have been applied just after their brief period of dominance (note that Ford's method is only tested once in each behavior phase), or that the effects were indistinguishable based on visual inspection of the deactivation. Also, because Ford's method is only applied once in each behavior period, typically near the beginning, it also in this phase missed the transition to the multiple sufficiency of B2 and B3 at the end of the phase. In the exponential decline phase, no single loop was identified as dominant and B2 and B3 were identified as a pair of shadow loops which dominated together, which is consistent with PFD which identifies B2 and B3 as both sufficient. However, Ford's method does not identify the momentary sufficiency of B1, likely for similar reasons why it failed to identify the momentary necessity of B1 and B3 in the second phase. It also does not identify the transition at the end of the third phase in which B3 alone is dominant. In the last phase of exponential decay, B1 is identified as dominant, consistent with PFD. Just as in previous phases, Ford's method does not identify the transition to the shadow structure of R1 and B1 at the end of the phase. To conclude, the results of Ford's method can be explained by identifying, in PFD, cases of necessary conditions (in which Ford ascribes as dominance), and cases of multiple sufficient conditions (in which Ford ascribes as shadow structure). It also reveals why Ford's method does not identify changes in dominance within each behavior phase.

PPM. PPM establishes dominance based on an iterative algorithm which identifies the feedback loop with the largest total pathway participation metric (TPPM) relative to the behavior of interest (C). In the first accelerating growth phase, PPM identifies R1 as dominant, consistent with PFD. In the decelerating growth phase, PPM identifies B2 as dominant, consistent with PFD. PPM does not identify the momentary necessity of B1 and B3 at the beginning of the phase nor does it identify the dual sufficiency of B2 and B3 at the end of the phase, since the algorithm is based on the magnitude of a normalized metric, and not based on necessity or sufficiency. In the exponential decline phase, PPM identifies B3 as dominant, consistent with PFD. It does not identify the sufficiency of B2 at the beginning

of this phase, for the same reason as previously mentioned. In the last phase of exponential decay, PPM identifies B1 as dominant, consistent with PFD. Unlike PFD, PPM does not identify the sufficiency of R1 at the tail end of the phase since the magnitude of the force contribution is significantly weaker than that of B1 and PPM does not identify sufficient but relatively weak structures.

Sensitivity analysis. Sensitivity analysis is performed by varying the values of the variables affecting the variable C associated with each loop, and evaluating the deviation from the baseline. In the first accelerating growth phase, R1 is identified as the most sensitive with a high likelihood of being dominant, and B2 the second most sensitive, whereas B1 and B3are insensitive and are likely not dominant. In the decelerating growth phase, B1, B3 and R1 are not very sensitive and are likely not dominant. B2 is the most sensitive and is likely to be dominant. In the exponential decline phase, B3 is the most sensitive and is likely to be dominant, and R_1 is completely insensitive and is not dominant. B1 and B2 are moderately sensitive. In the last phase of exponential decay, R1 and B2 are insensitive, and B1 is the most sensitive and likely to be dominant. B3 also appears to be sensitive and may be dominant in the last phase. In summary, the sensitivity analysis does not definitely identify dominant structures, but identifies structures which are likely to be dominant. The results can mostly be explained by inspecting the magnitudes of each pathway's force contributions. In each phase, the loop identified as potentially dominant with sensitivity analysis happens to correspond to the loop whose pathway has the largest magnitude force contribution. This makes intuitive sense, and the dominance characteristics of PFD align with what we expect based on sensitivity analysis. The sensitivity analysis however does not identify which loops determine the behavior and the nature in which they determine the behavior.

Loop Impact. The loop impact method identifies dominant loops based on the loop impact metric associated with the immediate pathway coming into the variable of interest. The pathway or set of pathways whose loop impact is larger than all opposing pathways is the one that is determined to be dominant. In the first accelerating growth phase, R_1 is identified as dominant, consistent with PFD. In the decelerating growth phase, the loop impact method identifies the brief period in which all the balancing loops together dominate, followed by the dominance of B2 alone. The loop impact method revealed that B1 and B3were very weak during their brief period of dominance. This result is consistent with PFD which also identifies all the balancing loops as necessary at the inflection point, and therefore, taken as a set, are necessary and sufficient and thus dominant for that brief moment. The results are consistent since the loop impact method determines dominance based on sufficient conditions, and not necessary conditions, thus it will identify the smallest set which is collectively sufficient. However, for some reason the loop impact method did not identify the brief period in which only B2 and B3 dominated as a set, right after B1, B2, and B3dominated as a set. This could be due to the computation time step of the simulation or loop picker algorithm. Also, while Hayward recognizes that both B2 and B3 are sufficient for determining the behavior at the end of the second phase just before the peak, the loop picker algorithm identifies B_2 as dominant when its impact metric is greater than that of B3, and B3 as dominant when its impact metrics is greater than that of B2. The PFD procedure identifies the force contributions of both B2 and B3, and recognizes that both are sufficient at the end of phase 2, and at the beginning of phase 3. In the exponential decline phase, the loop impact method identifies B3 as dominant. Again, if multiple loops are sufficient, the loop picker algorithm picks the loop with the greatest impact, and so only B3is identified even though both B2 and B3 are sufficient in determining the behavior at the beginning of the phase. The loop picker does not identify the brief period at the beginning of phase 3 in which all three balancing loops are sufficient, due to the fact that it only picks the sufficient loop with the greatest impact. In the last phase of exponential decay, B1 is identified as dominant, which is consistent with PFD. For the same reasons as state above, it does not identify that R1 is also sufficient at the tail end, due to the fact that its impact is significantly less than that of B1.

5.5 Conclusions

The purpose of Research Aim 2 was to understand if dominance methods could be applied to the state domain, similar to how they have been applied to the time domain, and to discover if state space regions of dominance existed and were well-defined. The goal was then to develop and apply a state-space dominance method to analyze shifts in dominance for previously-studied models, and to compare the results against existing methods to see if any new insights were formed, or inconsistencies explained. Additionally, the goal included identifying relationships between dominance and equilibrium points and their stability, and the relationship between transient and steady-state behavior.

In this chapter, we saw the advantage of using phase plots to understand global behavior of the system. By removing time, one can see the family of trajectories from birth to death, and the relationship between transient and steady-state behavior. The examples used in this chapter were of sufficient size such that two-dimensional phase plots sufficed, however for larger systems, different methods of visualizing the state-space are required, which could include taking a series of two- or three-dimensional slices through the state-space.

This chapter demonstrated that indeed, for first- and second-order nonlinear systems, that state space regions of dominance exist and are well-defined. Shifts in dominance as trajectories move through the regions are also well-defined. One advantage to analyzing dominance in the state-space is that general structural-behavioral relationships can be identified and described for a family of trajectories, and not just a single trajectory with a specific initial condition. This provides greater system-wide insights into the relationship between dominant structures and behavior. The pathway force decomposition (PFD) method which was developed and applied in the time domain in the previous chapter was extended to the state-domain in this chapter. It was applied algebraically for sufficiently simple models, and numerically for a more complicated model. The PFD proved useful for explaining discrepancies between current dominance methods for a variety of models exhibiting different behavior characteristics. One advantage of PFD is that it includes both structural and behavioral aspects of dominance and thus is able to explain why in some cases methods seem to produce the same results, and why in other cases conflicting results. The models evaluated in this chapter also reveal that when dominance methods identify dominant structure, they are almost always satisfy both necessary and sufficient conditions for determining behavior, lending further support to the proposed definition of dominance, and also explaining why despite their differences, often the methods produce similar results. Where methods differ the most is when systems are operating outside the necessary and sufficient point in the dominance framework.

Another finding was that for the models analyzed, stable equilibria are located within statespace regions in which dominant loops are balancing or negative, while unstable equilibria lie within regions in which positive or reinforcing loops are dominant. This finding is also consistent with intuitive statements about balancing loops as goal-seeking, and reinforcing loops as goal-divergent. For regions between unstable and stable equilibria (i.e.the transient behavior), the nature of the dominance regions varied from model to model. It seems that the dominance regions explain which feedback loops are responsible for determining the behavioral aspects of the transient behavior, as well as which feedback loops are responsible for attracting the system to its steady-state values (if they exist).

Finally, the analysis revealed how systems traverse through the dominance framework and become more robust and less robust over the course of shifts in dominance. It seems that behavior mode changes associated with inflection points occur when the system is in a fragile state of at least one necessary pathway.

5.6 Summary

This chapter completes the objectives of Research Aim 2 and establishes an analytical framework for dominance as well as a procedure for determining state-space regions of dominance. Thus far, however, the models analyzed have been relatively small in dimension (only one or two states). In the next chapter, according to the goals of Research Aim 3, the state-space dominance methods are applied to a more complex model of public health for the purposes of understanding sources of health disparities and how to develop sustainable and equitable policies and interventions.

Chapter 6

Analysis of the Cancer Control Services Model

The focus of this chapter is on Research Aim 3 in which the dominance methods developed and tested in previous chapters are now used to evaluate and identify influential structure in the cancer control services model. The chapter begins with an overview of the model and then describes the analysis and policy implications. A theorem relating stability and dominance is presented which is then used to assess the model. The chapter concludes with a summary of insights on the model and on the state-space dominance method.

6.1 Model Overview

In recent work at the Brown School Social System Design Lab, problems pertaining to health services have examined the dynamic interaction between supply and demand of services. Such problems include how to sustain the downward trends in maternal and newborn mortality rates in Latin America, or understanding disparities in mental health services. For some systems, service capacity changes slower than demand. In others, demand changes slower due to cultural factors such as acceptability of services. While in others, access may be the limiting factor.

A static view of health services might assume that supply and demand are in equilibrium. However, in conversations with at least one client, it seems some systems are demandconstrained while others supply-constrained, causing them to behave differently. Delays in the system also affect the interaction between supply and demand, leading to non-intuitive behavior. Dynamic models of the relationship between supply and demand of services have been shown to produce a variety of different behaviors, including overshoot and collapse, oscillations, gradual rise, or gradual decline. It is possible for two distinct population segments to exhibit different behaviors even if they are part of the same system, depending on their conditions. Differences in the health status between distinct population segments are described as *health disparities*, and can change over timer, which is one motivation for looking at the health services as a dynamic problem.

Levin and Roberts considered the dynamics of health care delivery systems and the relationships between patients and service providers as a consequence of the balancing and reinforcing loops in the system [86]. In a similar way, at the SSDL, a generic simulation structure has emerged in which balancing and reinforcing loops which exist within and between supply and demand lead to a variety of different behaviors. This chapter examines one particular implementation of the structure for cancer control services and uses state-space dominance methods to analyze sources of disparities. One example of a disparity applicable to this model is the difference between African-American and white women in breast cancer diagnosis, treatment, and survival.

6.1.1 Problem Framing and Analysis Questions

The cancer control services model is an early concept model which can be used to identify future research questions and explore potential causes of health disparities. The purpose is not to make precise predictions of future states, but to investigate the range of behaviors which result from the relationship between supply and demand, and their sources. Therefore, identifying the exact values of parameters is less important than identifying which parameters are the most sensitive, along with the conditions in which they are sensitive. For such models, the qualitative characteristics of the simulation are of primary focus (for example, the number of equilibria, divergent vs. convergent growth, oscillations vs. no oscillations). The purpose of the model fits well with nonlinear state-space methods which also focus on the qualitative aspects of trajectories more than individual solutions.

Illustrative Problem

Consider the following situation. Despite lower incidence rates, black women are 40% more likely to die from breast cancer than white women. One possible contributing factor is the widening gap between black and white women in breast cancer diagnosis and treatment. The widening gap may be attributed to social, economic, environmental, as well as geographical differences in these population segments. Such factors may affect the likelihood that women seek and are able to obtain diagnosis and treatment services ³².

 $^{^{32}}$ According to the community report by Williams and Zoellner [151], in the St. Louis region, a white paper released in 2014 by the St. Louis Susan G. Komen foundation and Washington University noted that over 50% of African-American women diagnosed with breast cancer in the area do not start treatment. In attempting to understand this issue, a study was conducted to examine social and environmental factors that cause women with suspicious mammograms not to seek treatment

Model Purpose and Analysis Questions

Health disparities (differences in health status among distinct population segments) may be systemic and avoidable and linked to differences in the reach and quality of health services. The cancer control services model is used to investigate how supply and demand of services affects the reach of services (number of patients), the quality experienced by the average patient, and how these factors in return affect both demand and supply³³.

Reach is defined as the number of people receiving services (patients) within a defined population segment, which may be less than the number who actually need the services. For example, some black women in St. Louis may decide not to seek diagnosis or treatment services for a variety of social or personal factors which have been shared in group model building sessions. Additionally, for those who do seek services, not all may be able to access or afford the services.

Average service quality experienced by the average patient is defined on a relative scale from zero (equivalent to receiving no care), to a maximum value of one (the highest possible care given the current state of practice). Quality encompasses diagnosis and treatment effectiveness, timeliness of services, safety, and patient-centeredness (respecting patient choices, culture, social context, and specific needs) [11].

The following questions are used to scope the model analysis.

- 1. Questions about the behavior of the system
 - (a) What are the primary drivers of metrics *reach* and *quality*?
 - (b) Is it possible for one metric to increase while the other decreases?

³³In addition to being applicable to understanding disparities in breast cancer treatment delays, this modeling approach could potentially be applied to other cancer problems including the project on social determinants of health, obesity, and non-Hodgkin Lymphoma [106], as well as colorectal cancer screening social marketing strategies.

- (c) Under what conditions will two population segments experience qualitatively different trends for *reach* and *quality*?
- (d) Is it possible for disparities to continue or increase between population segments even as *reach* or *quality* improve for both?³⁴. In public health, this is known as the *inequality paradox* [42].
- 2. Questions about policy and intervention
 - (a) Where are effective places to intervene in the system? Which parameters have the greatest leverage in changing behavior?
 - (b) How does the system respond to a sudden increase in demand for services? How might the responses be different across population segments?
 - (c) How can scale-up of services be sustained in response to an increase in demand?

6.1.2 Model Description

The model is for a single population segment in need of a specific type of cancer control services. For example, the model is suitable for representing a population of African-American women in St. Louis in need of breast cancer diagnosis and treatment services.

The model consists of five sub-components:

- 1. Proportion seeking services
- 2. Proportion able to get services
- 3. Reach of services (number of patients)

 $^{^{34}}$ For example, between 2000-2010 rates of death from breast cancer decreased for both black and white women in St. Louis. However, disparities between racial groups still exist, as cited by the Community Report [151].

- 4. Cancer control service capacity
- 5. Service quality

Proportion Seeking Services

Proportion seeking services is a state-variable (stock) defined as the proportion of the segment needing cancer control services who actively seek the services. Because stocks are an aggregation of homogeneous elements, it can also be interpreted as the likelihood or probability that a person chosen at random within the segment would seek services. In the real world, the proportion seeking services may depend on factors such as fear, stress, anxiety, beliefs, knowledge about the disease, and social support³⁵.

Proportion seeking services is governed by a Bass diffusion process (introduced in Chapter 5), with the addition of a de-adoption process as shown in Figure 6.1 [136].

$$changing mind about seeking services = external adoption + internal adoption \qquad (6.1)$$

 $external adoption = proportion not seeking services \cdot external adoption fractional rate$

(6.2)

$$internal adoption = proportion not seeking services \cdot$$

$$proportion seeking services \cdot$$
(6.3)

$internal \, adoption \, fractional \, rate$

³⁵For example, in group model building sessions, some shared that cancer of any kind is viewed as a "death sentence" because most of the people they knew who had the disease have died. The fear and belief that once you are diagnosed you are bound to die has caused many women to live in denial about having cancer. Women also shared that community supports, such as counseling services and breast cancer support groups, as well as family support and trust, are powerful factors in ensuring that a woman follows up with her suspicious mammogram, starts treatment, and finishes it [151].



Figure 6.1: Stock and flow diagram of model sub-component on proportion seeking services

$$deadoption = proportion \ seeking \ services \ \cdot \ deadoption \ fractional \ rate \tag{6.4}$$

The Bass diffusion model is widely used and adapted for the spread of ideas, fads, rumors, and innovations. It can also be used to understand the gradual change of cultural norms and ideas over time. For this model, *Adoption* describes the process by which people make the decision to seek cancer control services, which occurs through external means (*external adoption*) such as advertising and cancer education programs, or internal means (*internal adoption*) such as influence by word-of-mouth between seekers and non-seekers. The fractional rates associated with internal and external adoption have units of fraction of people per year.

The reinforcing loop R1 represents the process by which adoption grows through word-ofmouth. As the proportion seeking services increases, balancing loops B1 and B2 cause a slowing of growth due to depletion of the remaining people who do not seek services. The de-adoption process represents those who formerly sought services (or who would have been willing to seek services) deciding to no longer seek services. They return to the stock of nonseekers. Balancing loop B3 slows the depletion of *proportion seeking services*. As will be described, *quality* of services and the proportion of people currently getting services affects internal adoption and de-adoption fractional rates.

Proportion Able to Get Services

Proportion able to get services is the second state-variable, defined as the proportion of the those seeking cancer control services who are able to get the services. The ability to get services would depend upon the ability to afford, access, and obtain the services. It can also be interpreted as the likelihood or probability that a person who is seeking services has the ability to obtain services and may depend on factors such as access to transportation, job flexibility, and the ability to afford the services³⁶.

Proportion able to get services is governed by a first-order goal-seeking process with an average adjustment time (AT), as shown in Figure 6.2.

$$\frac{effect \, of \, service \, ratio \dots - proportion \, able \, to \, get \, services}{AT \, ability \, to \, get \, services}$$

$$(6.5)$$

The goal (steady-state value) of *proportion able to get services* is a monotonically increasing function of the ratio of *service capacity* to *patients* (ratio of supply to demand), which assumes that as the ratio increases, availability and/or affordability also increase. For example, as capacity increases, timeliness of services increase, resulting in reduced waiting times for

³⁶For example, access to insurance and affordability of out-of-pocket payments can enable low-income women to obtain early medical care for cancer. Group model building participants also discussed that the majority of women in the community work in the service sector and are paid minimum wage and cannot afford medical care [151].



Figure 6.2: Stock and flow diagram of model segment on *proportion of population able to get* cancer services.

diagnosis and treatment services, and overall greater access. As shown in Figure 6.3, as the ratio goes to zero (service capacity goes to zero), no one is able to get services.



Figure 6.3: Graph of effect of service ratio on proportion able to get services

As the ratio goes to positive infinite, the system will never be supply-constrained and *proportion able to get services* eventually levels-off, asymptotically approaching a steady-state value less than or equal to one. It is possible that the maximum value may be less than one due to reasons unrelated to the *ratio of service capacity to patients* (such as lack of insurance, etc.). The graph in Figure 6.3 illustrates one of several possible types of functions to describe a causal relationship between two auxiliary variables. Figure 6.4 shows the graphs of four types of relationship functions used in this model. Figure 6.3 is of type g3.



Figure 6.4: Four types of relationships between auxiliary variables used in the model.

The steepness of the graphs in Figure 6.4 is determined by a sensitivity factor represented by the term λ . For example, the auxiliary variable *effect of service ratio*... which has relationship type g3, has a steepness determined by: $\lambda = 2$ (see Figure 6.3). This sensitivity factor is represented by the variable *Sensitivity of ability to get services to service ratio* in the stock and flow diagram in Figure 6.2.

Number of Patients

Number of patients, or reach of services, is an auxiliary variable. As shown in Figure 6.5 and Equations (6.6) and (6.7), number of patients is equal to the number of people in need of services multiplied by the proportion who receive services. The proportion who receive services is proportion seeking services multiplied by proportion able to get services.



Figure 6.5: Diagram of number of patients



 $proportion\ who\ receive\ services\ =\ proportion\ seeking\ services\ \cdot\ proportion\ able\ to\ get\ services$

(6.7)

The multiplication of *proportion seeking services* and *proportion able to get services* suggests that the variables are either statistically independent, or one is conditional upon the other ³⁷. Since they are likely not independent (for example, affordability may affect both one's ability to get services and whether or not they seek services), *proportion able to get services* is more precisely defined as the proportion of those seeking services who are able to get services. This can also be thought of as the conditional probability that someone is able to get services, given they are seeking services.

 ${}^{37}P(A \text{ and } B) = P(A) \cdot P(B|A)$, where P(B|A) = P(B) if A and B are independent.

Cancer Control Service Capacity

Service capacity, the third state-variable, is the number of patients for which a system can provide services at a standard level of care without additional resources or efficiency improvements. The actual number of patients receiving services at any time may be greater or less than the service capacity. The ratio of *service capacity* to *patients* affects the quality of the services 38 .

Service capacity is governed by a first-order goal-seeking process, with an average adjustment time (AT), as shown in Figure 6.6.



Figure 6.6: Stock and flow diagram of model segment on *cancer control service capacity*.

$$service \ capacity \ adjustments \ = \ \frac{service \ capacity \ shortfall}{AT \ service \ capacity} \tag{6.8}$$

³⁸For example, if the number of patients far exceeds the service capacity, average quality decreases, reflected by increased waiting times between appointments (that is, a decrease in the timeliness of the service for the average patient).
The service capacity is continuously adjusted based the number of patients. That is, over time, supply will adjust higher or lower in an attempt to match demand. AT determines how quickly service capacity adjusts. In the case of increasing capacity, AT can be thought of as the reciprocal of the annual investment rate for capacity.

Quality

Average service quality is an auxiliary variable of type g3 function³⁹ of ratio of service capacity to patients, as shown in Figures 6.7 and 6.8, in which the ratio of service capacity to patients is defined by Equation (6.10).



Figure 6.7: Diagram of average service quality

³⁹See Figure 6.4

$$ratio of service capacity to patients = \frac{service capacity}{patients} (for patients > 0)$$
(6.10)

The ratio is well-defined for patients > 0 and approaches positive infinite as patients approaches zero.



Figure 6.8: Graph of average service quality

Average service quality is quantified on a relative scale from zero (no care) to one (the highest possible care) and reflects the care experience for the average patient. Quality includes factors such as diagnosis and treatment effectiveness, patient-centeredness, safety, and service timeliness [11]. The graph in Figure 6.8 indicates that as the *ratio of service capacity to patients* increases, there are more resources available per patient (as the ratio goes to positive infinite, quality goes to the maximum value of one). As the ratio decreases, there are less resources available per patient which reduces availability, timeliness, or attentiveness of services (as the ratio goes to zero, quality also goes to zero)⁴⁰.

 $^{^{40}}$ It was pointed out by a dissertation committee member that if the ratio becomes too high, the experience and skill level of service providers decreases which adversely affects quality. This effect is not reflected in the graph in Figure 6.8, but would be represented by its own relationship function since it describes a different causal mechanism that is not in the current model.

Effect of Quality on De-adoption Fractional Rate

Average service quality and the proportion who receive services together impact the deadoption fractional rate, as shown in Figures 6.9 and 6.10.



Figure 6.9: Diagram of effect of quality on de-adoption fractional rate



Figure 6.10: Graph of *de-adoption fractional rate*

The product of Average service quality and proportion who receive services represents the cumulative impact of the cancer control services on the population, which affects perception of the service effectiveness, which inversely affects the fractional rate at which people deadopt (decide not to seek the services). The multiplication of the terms implies that either factor can limit the overall impact. The product also represents the average level of care being received by the entire segment of people in need of services, since a quality of zero is associated with no care, and since those not getting services are also not receiving care. This is evident in the following equation

average level of care for entire population in need = proportion who receive services $\times Q+$ $(1 - proportion who receive services) \times 0$ $= proportion who receive services \times Q$ (6.11)

Effect of Quality on Internal Adoption Fractional Rate

Similarly, the product of *average service quality* and *proportion who receive services* impacts *internal adoption fractional rate*, as shown in Figures 6.11 and 6.12.



Figure 6.11: Diagram of effect of quality on internal adoption fractional rate



Figure 6.12: Graph of internal adoption fractional rate

Again, the product represents the cumulative impact of the services and directly affects the fractional rate at which people convince others to seek cancer control services. This implies that the likelihood of someone becoming an *adopter* (deciding to seek services) depends on both the quality and the reach of the services, either one being a limiting factor.

Effect of Quality on Adjustment Time (AT) of Service Capacity

Average service quality also affects adjustment time of service capacity, as shown in Figures 6.13 and 6.14.

As average service quality decreases, the perceived need to grow capacity through investment, hiring, innovation, etc. increases, which increases the fractional investment rate, or equivalently, decreases the adjustment time. Conversely, as average service quality increases, the perceived need to invest in capacity decreases, which decreases the growth rate, or equivalently, lengthens the adjustment time of service capacity. There are also likely to be minimum and maximum adjustment times for organizations which bound how quickly or slowly service capacity can change.



Figure 6.13: Diagram of effect of quality on adjustment time of service capacity



Figure 6.14: Graph of adjustment time of service capacity

Complete Model Diagram

The entire model is comprised of the five components and consists of three state variables (two on the demand-side and one on the supply-side), as shown in Figure 6.15.

The interaction between supply and demand is captured through multiple balancing and reinforcing feedback loops. Reinforcing loops R1 and R2 are virtuous cycles when more people receiving services increases population outcomes and perceived effectiveness of the services,



Figure 6.15: Stock and flow diagram of cancer control services model

leading to more people who seek services. Reinforcing loops are generally responsible for the initial growth experienced by organizations. However, reinforcing loops can also be vicious cycles in that if the proportion of people receiving services decreases, the population outcomes and perceived effectiveness of services will decrease, further decreasing the likelihood that people will seek services. In this way, reinforcing loops which can cause the initial growth in demand can also be responsible for a collapse in demand.

Balancing loops B1 and B2 suppress the growth in demand typically caused by reinforcing loops such as R1 and R2 in that as demand increases and supply lags to catch-up, average service quality begins to erode which consequently slows the growth in demand. This is commonly referred to as *limits to growth* as growth reaches a natural *carrying capacity* based on the resources currently available. The only way to sustain growth in the long run is to increase the service capacity so that quality can be maintained. This adjustment of supply to keep up with demand is captured by balancing loop B3. If B3 is sufficiently fast, it can lead to sustained growth, depicted by reinforcing loop R3, until the saturation level is reached when the potential demand has been exhausted (see depletion balancing loops in Figure 6.1).

It is the interaction of these balancing and reinforcing loops, determined by the parameters and time constants of the model, along with the initial conditions of the state values, that lead to interesting and sometimes counterintuitive behavior, as each loop exerts a force on the system that changes over time.

Summary of Model Equations and Parameter Values

The cancer control services model is defined by the following equations

$$\dot{P}_{S} = f_{1} = (1 - P_{S}) a + P_{S} (1 - P_{S}) w - P_{S} d$$
$$\dot{P}_{A} = f_{2} = \frac{P_{A}^{*} - P_{A}}{\tau_{A}}$$
$$\dot{S} = f_{3} = \frac{N P_{S} P_{A} - S}{\tau_{S}}$$

where

 $P_S =$ proportion seeking services

 $P_A =$ proportion able to get services

S =service capacity

 $a \ge 0$ (advertising or external adoption FR)

$$w = g_1(QP_SP_A, \lambda_w, w_{min}, w_{max}) \text{ (word of mouth or internal adoption FR)}$$
(6.12)

$$d = g_2(QP_SP_A, \lambda_d, d_{min}, d_{max}) \text{ (deadoption FR)}$$
(6.12)

$$Q = g_3 \left(\frac{S}{NP_SP_A}, \lambda_Q, Q_{min}, Q_{max}\right) \text{ (for } P_S, P_A > 0) \text{ (average service quality)}$$

$$= 1 \text{ (for } P_S = 0 \text{ or } P_A = 0)$$
(for $P_S, P_A > 0$) (goal of P_A)

$$= 1 \text{ (for } P_S = 0 \text{ or } P_A = 0)$$
(for $P_S, P_A > 0$) (goal of P_A)

$$= 1 \text{ (for } P_S = 0 \text{ or } P_A = 0)$$

$$\tau_A > 0 \text{ (AT ability to get services)}$$

$$\tau_S = g_1(Q, \lambda_{\tau_S}, \tau_{S,min}, \tau_{S,max}) \text{ (AT service capacity)}$$

N > 0 (people in need of services)

The baseline parameter values are listed in Table 6.1.

External adoption is not included in the baseline model (that is, a = 0) to keep the focus on endogenous sources of behavior, however external adoption through means such as advertising or education are evaluated as a potential interventions under policy analysis. There are

a	0 [1/year]
λ_w	7
w_{min}	$.01 \ [1/year]$
w_{max}	1 [1/year]
λ_d	7
d_{min}	$.01 \ [1/year]$
d_{max}	.3 [1/year]
λ_Q	2
Q_{min}	0
Q_{max}	1
λ_{P_A}	2
$P_{A,min}$	0
$P_{A,max}$	1
$ au_A$.5 [years]
$\lambda_{ au_S}$	3
$ au_{S,min}$.1 [years]
$ au_{S,max}$	5 [years]
N	$10,\overline{000}$ [people]

Table 6.1: Baseline parameter values for cancer control services model.

also no time delays corresponding to perception and measurement, although these would be candidate additions to the model⁴¹. The number of people in need of services is fixed over the time horizon in the baseline analysis.

Metrics

The metrics are listed and defined in Table 6.2.

reach (number of patients)	NP_SP_A [people]
average service quality	Q [no units]
performance	$NP_SP_A \times Q$ [people]

Table 6.2: Key metrics for the cancer control services model.

⁴¹Potential sources of delays on the demand-side include the population's perception of quality based on actual quality, and on the supply-side, measurement delays for quality, capacity, and patients

Reach is the total number of patients which is a fraction of the total number needing services N. Average service quality is on a scale from zero to one. Total Performance is reach multiplied by average service quality. The final metric, health disparities, is not at the model-level but provides a way to measure the difference in outcomes between two or more instantiations of the model for different population segments. Disparities are defined as the relative difference between any of the model metrics experienced by different population segments. For negligible interactions between the supply and demand of services for each segment, the models for each segment can be run independently. However, if significant interactions exist, such as in a capacity-constrained system in which two segments compete for the same limited supply of services, then the models would be coupled.

6.2 Baseline Run and Sensitivity Analysis

The model is simulated using the following baseline initial conditions

$$P_S(0) = .8$$

 $P_A(0) = .8$ (6.13)
 $S(0) = NP_S(0)P_A(0)$

Simulation results are shown in Figure 6.16, which shows the behavior over a twenty-year time period for each state variable and metric *average service quality*, *patients*, and *performance* (quality \times patients).

The baseline run indicates at least one steady-state condition exists and that the baseline initial conditions are relatively close to the steady-state. This suggests at least one stable equilibrium. A simple sensitivity analysis around the baseline consists of three runs where each state's initial condition is changed, one at a time. A fourth run keeps the initial



Figure 6.16: Baseline simulation run of cancer services model.

conditions the same, but changes a single model parameter (sensitivity of quality to the ratio of capacity to patients).

Run 2, shown in Figure 6.17, lowers the initial proportion of people seeking services from 0.8 to 0.5, keeping all other parameters and initial conditions the same.

Beginning with a lower proportion of people seeking services, Run 2 exhibits a qualitatively different response than the baseline, in that the proportion of people seeking services collapses to zero, while the proportion of those seeking who are able to get services goes to one. Eventually, the number of patients goes to zero, followed by service capacity. This indicates that the model has a second steady-state (i.e. a second stable equilibrium point) attracting trajectories which start in a different region of the state-space. In Run 2, with proportion seeking services declining, the number of patients declines and thus the ratio of service capacity to patients increases. A greater ratio results in a greater ability to obtain services for those still seeking services, as well as higher service quality since there is more capacity



Figure 6.17: Cancer services model run 2.

per patient⁴². In summary, total patients goes to zero (and therefore performance goes to zero), while at the same time the ability to get services and average service quality for the diminishing number of patients increases. This result warns against choosing a single metric, such as quality or ability to get services, to evaluate overall performance.

As patients reach zero, the ratio of capacity to patients approaches positive infinite (driving quality to one and proportion with access to one), however as capacity (which lags patients) reaches zero, the ratio becomes undefined, making the model non-differentiable. There are multiple ways to resolve this issue. For this model, when supply and demand equal zero, quality and the goal of proportion able to get services is defined to be one, which agrees with the limit of the trajectories for S > 0.

Run 3, shown in Figure 6.18, lowers the initial proportion of people who are able to get services from 0.8 to 0.5, keeping all other parameters and initial conditions the same.

⁴²This model does not take into account the potential for the skills and experience level of service providers to diminish if the demand is too low.



Figure 6.18: Cancer services model run 3.

Run 3 shows that starting with a lower proportion who are able to get services does not cause that proportion to go lower (as in the case with proportion seeking services in Run 2), but the opposite occurs: the value rises just as it did in run 2. It does, however, cause the proportion seeking to collapse to zero, and thus the total reach and performance, just as in Run 2.

Run 4, shown in Figure 6.19, lowers the initial capacity to be half of the initial patients.

Run 4 shows that reducing the initial capacity also causes the proportion of people seeking services to go to zero, as in the previous two runs. A potential reason, from visual inspection of the model, is that with less capacity, average quality is reduced which causes people to eventually stop seeking services. However, alternative explanations are possible since proportion seeking services is connected to both itself and other model components through nonlinear relationships.



Figure 6.19: Cancer services model run 4.

Run 5, shown in Figure 6.20, keeps all initial conditions the same and lowers the sensitivity of quality with respect to the ratio of capacity to patients. This has the effect of requiring the ratio to be higher in order to achieve the same level of quality as in the baseline case.

The change in run 5 also causes P_S to go to zero, just as in Runs 2, 3, and 4, indicating that the behavior is sensitive to this parameter. One possible explanation is that decreasing the parameter value increases the size of the region of attraction associated with an asymptotically stable equilibrium point in which $P_S = 0$. This would explain how a trajectory starting in the same place can gravitate towards different steady-state conditions for different parameter values.

There are twelve additional parameters and each are tested individually while holding all other parameters and initial conditions constant. The results are shown in Table 6.3. A parameter is determined to be sensitive if a change within logical bounds produces a qualitatively different behavior (for example, P_S gravitating towards one steady-state value versus



Figure 6.20: Cancer services model run 5.

another). The (+) and (-) indicate the direction of the parameter change which leads to the qualitative change in behavior.

sensitive	insensitive
$w_{max}(-)$	a
$d_{min}(+)$	λ_w
$d_{max}(+)$	w_{min}
$\lambda_Q(-)$	λ_d
$\lambda_{P_A}(-)$	$ au_A$
	$\lambda_{ au_S}$
	$ au_{S,min}$
	$ au_{S,max}$

Table 6.3: Sensitive parameters in baseline case.

A parameter may not have the same level of sensitivity in all regions of the state-space. The sensitivity results in Table 6.3 only apply to specific initial conditions. Performing the same sensitivity tests using different initial conditions (Run 2: $P_S = 0.5$), results in a different set of sensitive parameters, as shown in Table 6.4.

sensitive	insensitive
a(+)	w_{max}
$\lambda_w(-)$	d_{min}
$d_{max}(-)$	λ_Q
$w_{min}(+)$	λ_d
	$ au_A$
	$\lambda_{ au_S}$
	$ au_{S,min}$
	$ au_{S,max}$
	λ_{P_A}

Table 6.4: Sensitive parameters in second case.

Approximately one third of the parameters are found to be sensitive in each of the cases, with only one parameter (d_{max}) sensitive in both. This may indicate which parameters should be estimated with greater precision in the model, or, which parameters might be looked at more closely for policies and interventions. For example, the sensitive parameters which seem to have an impact mostly pertain to fractional rates of the Bass diffusion process governing P_s . However, caution is warranted before directly applying these results since only two points in the state-space were considered, and each parameter was tested in isolation. Ideally, a full sensitivity analysis would sample multiple regions of state-space and also consider pairwise and higher-order combinations of parameter changes. It is easy to see how even a modest exploration of each potential combination can become computationally expensive. For example, a full factorial experiment of the model containing 13 parameters and 3 state variables results in 16 degrees of freedom. Testing every combination of parameter and initial condition at three levels each (low, medium, and high) results in $3^{16} \approx 43$ million simulation runs. Additionally, this relatively modest design includes only one interior point of the statespace. More points would be needed to understand sensitivity to initial conditions. This is one area in which state-space methods which analyze the nature of trajectories across state-space may help provide insight without having to perform an exhaustive sensitivity analysis, or in a way that could guide a more focused sensitivity analysis by reducing the size of the parameter- and state-space of interest.

To summarize, sensitivity analysis indicates two sets of steady-state values which depend on the parameters and initial conditions. P_S seems to be more sensitive to initial conditions and parameter values than the other two state variables. However, it is not immediately evident which mechanisms cause P_S to collapse in Runs 2, 3, 4, and 5, and why the collapse occurs even when its initial value is unchanged (runs 3, 4 and 5). There are multiple reinforcing and balancing loops connected to P_S in which any one (or combination) might be the reason for the collapse or the pull towards a non-zero steady-state value. These questions are further explored using state-space and dominance analysis in the next sections.

6.3 State-Space Analysis

The system trajectories are characterized in state-space by identifying the equilibrium points (EPs) and limit cycles (if they exist), their stability properties, and by sampling the trajectories in different planes of the state-space.

6.3.1 Equilibrium Points

The EPs of system (6.12) are found by setting $\dot{\mathbf{f}} = \mathbf{0}$ where $\mathbf{f} = [f_1 f_2 f_3]$, which gives

$$S^{*} = NP_{S}^{*}P_{A}^{*}$$

$$P_{A}^{*} = P_{A,max} - (P_{A,max} - P_{A,min})e^{-\lambda_{P_{A}}} \text{ (for } P_{S}^{*} > 0)$$

$$= 1 \text{ (for } P_{S}^{*} = 0)$$

$$P_{S}^{*} = 0, \ 1 - \frac{d}{w} \text{ (for } a = 0)$$

$$d_{max} - \left(\frac{(d_{max} - d_{min})}{\frac{-\lambda_{d}\left(Q^{*}P_{S}^{*}P_{A}^{*} - \frac{1}{2}\right)}{1 + e}\right)}$$

$$= 0, \ 1 - \frac{d_{max} - \left(\frac{(w_{max} - w_{min})}{\frac{-\lambda_{w}\left(Q^{*}P_{S}^{*}P_{A}^{*} - \frac{1}{2}\right)}{\frac{1 + e}{2}}\right)}{(6.14)}$$

where

$$Q^* = Q_{max} - (Q_{max} - Q_{min})e^{-\lambda_Q}$$

Solving (6.14) results in one or more EPs. The first EP (EP_1) is associated with $P_S^* = 0$ in which $S_* = 0$ and $P_A^* = 1$. Additional EPs, if they exist, are associated with $P_S^* > 0$. The strategy for determining these EP(s) is to first solve for P_A^* and Q^* which have straight-forward closed-form expressions (6.14). Then, non-zero solution(s) to P_S^* are found by numerically solving the equation

$$P_S = 1 - \frac{d(P_S P_A^* Q^*)}{w(P_S P_A^* Q^*)}$$
(6.15)

For the parameter values in Table 6.1, Equation (6.15) has two solutions $(P_{S,1}^* \text{ and } P_{S,2}^*)$, as shown in Figure 6.21, implying the existence of two additional EPs $(EP_2 \text{ and } EP_3)$.

Finally, the two sets of values for S^* (associated with EP_2 and EP_3) derive from $P^*_{S,1}$ and $P^*_{S,2}$.



Figure 6.21: Equilibrium points for P_S .

Therefore, the cancer services model has three EPs in the state-space $[P_S P_A S]^T$

$$EP_{1} = \begin{bmatrix} 0\\1\\0 \end{bmatrix}$$

$$EP_{2} = \begin{bmatrix} .633\\.8647\\.5473.3 \end{bmatrix}$$

$$EP_{3} = \begin{bmatrix} .9015\\.8647\\.7795 \end{bmatrix}$$
(6.16)

This is consistent with the results of the sensitivity analysis which indicate the existence of two distinct steady-state values for P_S , which appear to be very near the values $P_{S,1}^* = .633$ and $P_{S,2}^* = .9015$. Therefore, it is suspected that the first and third EPs of the system are stable. The stability properties of the three EPs are now formally assessed.

6.3.2 Stability of Equilibrium Points and Regions of Attraction

The approach used in Chapter 5 for relatively small models was to linearize the system and evaluate the eigenvalues of the Jacobian matrix at the equilibrium points. For hyperbolic equilibrium points (no eigenvalues on the imaginary axis), local stability is determined from the signs of the real components of the eigenvalues. However, for this model (and many others of social systems), there is not a straightforward method for finding the eigenvalues in closed-form. While a numeric approach is possible, another option is to use Lyapunov stability theory [82, ch. 4]. The idea is to find a non-negative scalar function (Lyapunov function) defined over a state-space region around the equilibrium which also decreases along the trajectories in that region. Often Lyapunov functions represent the energy of physical systems, but they can take on many forms. They are also useful for approximating the region of attraction of asymptotically stable EPs, proving that an EP is globally stable, or proving that trajectories are bounded within some region. The following well-known theorem will be used to analyze the stability of the model.

Lyapunov's Stability Theorem [82, pp. 112-114]:

Consider the autonomous system $\dot{x} = f(x)$ where $f: D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n . Let x = 0 be an equilibrium point for $\dot{x} = f(x)$ (that is, f(x) = 0) and $D \subset R^n$ be a domain containing x = 0. Let $V: D \to R$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \le 0 \text{ in } D$$
(6.17)

Then, x = 0 is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\}$$
 (6.18)

Without loss of generality, the equilibrium point of interest is considered to be the origin⁴³. Another way of stating the theorem is: the origin is stable if there exists a continuously differentiable positive definite function V such that \dot{V} is negative semi-definite, and is asymptotically stable if \dot{V} is negative definite.

A common form of Lyapunov function takes the form $V(x) = x^T P x$, where P is a real symmetric matrix. V is positive definite (positive semi-definite) if and only if the eigenvalues of P are all positive (non-negative). Consider for example the first-order linear system $\dot{x} = \alpha x$ with equilibrium point at the origin, and the Lyapunov function $V = \frac{1}{2}x^2$. Then, $\dot{V} = \alpha x^2 \leq 0$ (stable) $\leftrightarrow \alpha \leq 0$, and $\dot{V} < 0$ (asymptotically stable) $\leftrightarrow \alpha < 0$.

This illustrates the relationship between stability and loop dominance for first-order linear systems. In Chapter 4 it was demonstrated that for a first-order linear system $\dot{x} = \alpha x$ containing *n* distinct feedback loops through auxiliary variables, we have $\alpha = \sum_{i=1}^{n} \alpha_i$ where α_i is the gain associated with the *i*th feedback loop. The result is that the *dominant polarity* of *x* is negative $\left(\frac{\partial \dot{x}}{\partial x} < 0\right)$ if and only if $\alpha < 0$. From the definition of loop dominance in Chapter 3, and the Lyapunov stability result above, we can claim that first-order linear dynamic systems are asymptotically stable if and only if they are dominated by balancing feedback loop(s).

Lyapunov's theorem and the above illustration are used to analyze the stability of the EPs of the model (6.12). First, each state variable is examined in isolation to gain a better understanding of the decoupled dynamics. Observe that the equations for \dot{P}_A and \dot{S} , when considered in isolation, have a linear, first-order goal-seeking structure of the form $\dot{x} = \frac{G-x}{\tau}$, $\tau > 0$, with equilibrium point $x^* = G^{44}$. For now, assume the equations are

⁴³A system with a non-zero equilibrium point can be equivalently written as a system with an equilibrium point at the origin through a suitable change of variables.

 $^{^{44}}$ In system dynamics, G often represents the goal associated with a balancing feedback loop.

uncoupled and that $\tau > 0$ is constant. The following change of variables transfers the equilibrium point to the origin

$$y = x - G \tag{6.19}$$

The new equivalent system is

$$\dot{y} = -\frac{1}{\tau}y \tag{6.20}$$

By the earlier example, since $-\frac{1}{\tau} < 0$, the point y = 0 is asymptotically stable implying the EP $x^* = G$ is asymptotically stable. This is another illustration of asymptotically stable equilibrium points associated with dominant balancing feedback loops for linear systems.

Lastly, consider the decoupled version of the equation for \dot{P}_S in which a, w and d are taken to be non-negative constants

$$\dot{P}_S = (1 - P_S)a + P_S(1 - P_S)w - P_Sd \tag{6.21}$$

For the baseline model, a = 0 (no adoption through external means such as advertising), which gives

$$\dot{P}_S = P_S(1 - P_S)w - P_Sd$$
$$= (w - d)P_S - wP_S^2$$
(6.22)

with equilibrium points

$$P_S^* = 0, \ 1 - \frac{d}{w}$$

Using Lyapunov function $V(x) = \frac{1}{2}x^2$,

$$\dot{V}(P_S) = \frac{\partial V}{\partial P_S} \dot{P}_S = P_S((w-d)P_S - wP_S^2)$$
$$= P_S^2(w-d - wP_S)$$
$$< 0 \text{ on } D = \left\{ P_S > 1 - \frac{d}{w} \right\}$$
(6.23)

where

$$0 \in D \leftrightarrow w < d$$

Therefore, the origin is asymptotically stable when w < d. This makes intuitive sense for the Bass diffusion process in that if the de-adoption fractional rate is greater than the adoption fractional rate, the number of adopters will approach zero in the steady-state. Having established the condition for asymptotic stability of the origin, it is natural to then wonder the size of the region of attraction.

The domain D of a Lyapunov function is not necessarily an estimate of the region of attraction since trajectories are not required to remain forever in D and \dot{V} is not required to be negative semi-definite outside of D [82, p. 317]. However, if D or a subset of D is positively invariant (every trajectory starting in the set remains in the set for all future time), as in this case, then it is an estimate of the region of attraction. This is illustrated by the phase portrait in Figure 6.22. $D = \left\{ P_S > 1 - \frac{d}{w} \right\}$ is an estimate of the region of attraction for the origin and, in this case, happens to be the exact region of attraction.

To analyze the stability of the second EP $\left(P_S^* = 1 - \frac{d}{w}\right)$, the following change of variables are used

$$y = P_S - \left(1 - \frac{d}{w}\right) \tag{6.24}$$



Figure 6.22: Phase portrait of P_S showing region of attraction D (gray bar) for the origin equilibrium point.

which gives the following system with EP at the origin.

$$\dot{y} = -wy^2 - (w - d)y \tag{6.25}$$

Using $V(y) = \frac{1}{2}y^2$,

$$V(y) = y^{2}(d - w - wy)$$

< 0 on $D = \left\{ y > \frac{d}{w} - 1 \right\}$ (6.26)

where

 $0\in D\leftrightarrow w>d$

Translating back to the original coordinate system gives

$$D = \{P_S > 0\}$$
where
$$d$$
(6.27)

$$P_S^* = 1 - \frac{d}{w} \in D \leftrightarrow w > d$$

Therefore, $1 - \frac{d}{w}$ is asymptotically stable when w > d. Additionally, the trajectories are positively invariant on domain $D = \{P_S > 0\}$, as illustrated by the phase portrait in Figure 6.23. Therefore, D is also an estimate of the region of attraction of $1 - \frac{d}{w}$ and, in this case, is the exact region of attraction.



Figure 6.23: Phase portrait of P_S showing region of attraction D (gray bar) for the non-zero equilibrium point.

Just as with the two first-order linear models analyzed before, a relationship is suspected to exist between stability and loop dominance for the Bass diffusion process. The balancing feedback loops for this model (B2 and B3) and reinforcing loop (R1) are shown in the stock and flow diagram in Figure 6.1. Feedback loop B1 is not analyzed since a = 0. To perform the loop dominance analysis, Equation (6.22) is expressed in terms of the causal pathways associated with each feedback loop (R1, B2, and B3)

$$\dot{P}_S = R1 \cdot B2 + B3$$

where

$$R1 = P_S$$

$$B2 = w(1 - P_S)$$

$$B3 = -dP_S$$

$$(6.28)$$

The pathway force decomposition is

$$\ddot{P}_{S} = F_{R1} + F_{B2} + F_{B3}$$
where
$$F_{R1} = \frac{\partial \dot{P}_{S}}{\partial R1} \dot{R}1 = w(1 - P_{S}) \dot{P}_{S}$$

$$F_{B2} = \frac{\partial \dot{P}_{S}}{\partial B2} \dot{B}2 = -w P_{S} \dot{P}_{S}$$

$$F_{B3} = \frac{\partial \dot{P}_{S}}{\partial B3} \dot{B}3 = -d \dot{P}_{S}$$
(6.29)

The dominant polarity is negative (implying that balancing loops are dominant) when $\operatorname{sgn}(\dot{P}_S) \neq \operatorname{sgn}(\dot{P}_S)$, which is when

$$w(1 - P_S) - wP_S - d < 0$$

$$P_S > \frac{w - d}{2w}$$
(6.30)

Specifically, B2 is sufficient when $P_S > \frac{1}{2}$, B3 is sufficient when $P_S > 1 - \frac{d}{w}$, and B2 and B3 are both necessary for $\frac{w-d}{2w} < P_S < \min\left(\frac{1}{2}, 1 - \frac{d}{w}\right)$. Comparing these results with the Lyapunov stability results, the origin is found to be asymptotically stable when w < d, which is contained in the state-space region in which balancing loop B3 is necessary and sufficient (dominant). Likewise, the EP $1 - \frac{d}{w}$ is asymptotically stable when w > d, which is contained in the region in which B2 and B3 are dominant. Therefore, in both cases, the stable equilibrium point lies in a state-space region in which balancing loops are dominant.

In summary, the stability and dominance characteristics for each of the three state variables, when considered in isolation with decoupled dynamics, indicate a plausible relationship between stability and the dominance of balancing loops. Motivated by these three examples for first-order linear and nonlinear systems, we now seek to prove the existence of a formal relationship between stability and dominance which applies to general n^{th} -order nonlinear dynamic systems.

Theorem 6.1

Consider the autonomous system $\dot{x} = f(x)$ in which $f: D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into \mathbb{R}^n . Let x = 0 be an isolated equilibrium point for $\dot{x} = f(x)$ (that is, f(x) = 0 and $\exists \epsilon > 0$ such that $f(y) \neq 0$ for $0 < ||y|| < \epsilon$). If x = 0 is stable, then one or more balancing feedback loops are dominant over a subset of an arbitrarily small state-space region containing x = 0.⁴⁵

Proof: For this analysis, assume that x = 0 is an isolated stable equilibrium point. Then, by definition of stability, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon$, $\forall t \ge 0$ [82, ch. 4]. Therefore, consider the trajectories x(t) such that $||x(0)|| < \delta(\epsilon)$.

⁴⁵Without loss of generality, the origin is assumed to be the equilibrium point. A non-origin equilibrium point can be transferred to the origin through a suitable change of variables.

Also assume, towards contradiction, that there exists a dimension x_i which is not dominated by a balancing loop anywhere inside the arbitrarily small region $||x(t)|| < \epsilon$. Then, by definition of dominance (Chapter 3), the *dominant polarity* of x_i is positive $(\operatorname{sgn} \dot{x}_i(t) = \operatorname{sgn} \ddot{x}_i(t))$ everywhere in the region $||x(t)|| < \epsilon$.

The trajectory $x_i(t)$ is now evaluated for two possible cases in which the initial slope is either positive or negative (since x = 0 is an isolated EP, we can assume the initial slope is not zero). For the first case, $\dot{x}_i(0) = f_i(0) > 0$. Then $\ddot{x}_i(t) > 0$, $\forall t \ge 0$, and thus $\dot{x}_i(t) = f_i(t) > f_i(0), \forall t \ge 0$. The trajectory $x_i(t)$ is now evaluated

$$x_{i}(t) = \int_{0}^{t} f_{i}(s)ds + x_{i}(0)$$

> $\int_{0}^{t} f_{i}(0)ds + x_{i}(0) = f_{i}(0) \cdot t + x_{i}(0)$
> ϵ for $t > \frac{\epsilon - x_{i}(0)}{f_{i}(0)}$
 $\Rightarrow ||x(t)|| > \epsilon$ for $t > \frac{\epsilon - x_{i}(0)}{f_{i}(0)}$ (6.31)

which contradicts the assumption that x = 0 is stable $\Rightarrow \Leftarrow$.

Now consider the case in which the initial slope is negative $\dot{x}_i(0) = f_i(0) < 0$. Then $\ddot{x}_i(t) < 0, \ \forall t \ge 0$, and thus $\dot{x}_i(t) = f_i(t) < f_i(0), \ \forall t \ge 0$.

$$\begin{aligned} x_{i}(t) &= \int_{0}^{t} f_{i}(s)ds + x_{i}(0) \\ &< \int_{0}^{t} f_{i}(0)ds + x_{i}(0) = f_{i}(0) \cdot t + x_{i}(0) \\ &< -\epsilon \text{ for } t > \frac{-\epsilon - x_{i}(0)}{f_{i}(0)} \end{aligned}$$
(6.32)
$$\Rightarrow \|x(t)\| > \epsilon \text{ for } t > \frac{-\epsilon - x_{i}(0)}{f_{i}(0)}$$

which contradicts the assumption that x = 0 is stable $\Rightarrow \Leftarrow$. This completes the proof. \Box

This result is a necessary (but not sufficient) condition of stability, and can be used to help locate stable equilibria and identify the associated dominant balancing loops. The contrapositive of this theorem (used in the proof by contradiction) is a sufficient (but not necessary) criteria for determining if an EP is unstable. This will be used to help determine the stability of the EPs in the cancer control services model.

Having gained intuition by determining the stability of the three states as if they were isolated, we now seek to analyze the stability for the actual system in which the variables are coupled. Lyapunov theory provides a means of analyzing the stability of the EPs, the boundary of trajectories, and an estimate for the region of attraction. The dominant loops around the EPs are then identified, with Theorem 6.1 implying that dominant balancing loops should exist around the stable EPs. We are interested in which loops, specifically, are dominant around the stable and unstable EPs.

As stated in the previous section, the EPs for system (6.12), given by (6.14), are

$$EP_{1} = \begin{bmatrix} 0\\1\\0 \end{bmatrix}$$

$$EP_{2} = \begin{bmatrix} P_{S,1}^{*}\\P_{A}^{*}\\NP_{S,1}^{*}P_{A}^{*} \end{bmatrix}$$

$$EP_{3} = \begin{bmatrix} P_{S,2}^{*}\\P_{A}^{*}\\NP_{S,2}^{*}P_{A}^{*} \end{bmatrix}$$
(6.33)

For EP_1 , the following change of variables are used

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} P_S \\ P_A - 1 \\ S \end{bmatrix}$$
(6.34)

Lyapunov function $V(y) = y^T P y$ is used, where

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$
(6.35)

which gives

$$\begin{split} \dot{V}(y) &= \frac{\partial V}{\partial y} f(y) \\ &= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} y_1(1-y_1)w(x(y)) - y_1d(x(y)) \\ \frac{P_A^*(x(y)) - y_2 - 1}{\tau_A} \\ \frac{Ny_1(y_2+1) - y_3}{\tau_S(x(y))} \end{bmatrix} \\ &= y_1^2(w(x(y)) - d(x(y)) - w(x(y))y_1) + \frac{y_2(P_A^*(x(y)) - 1) - y_2^2}{\tau_A} + \frac{y_3(Ny_1(y_2+1)) - y_3^2}{\tau_S(x(y))} \end{split}$$

$$(6.36)$$

The first of the three terms of \dot{V} is strictly negative on $D = \left\{ y_1 > 1 - \frac{d(x(y))}{w(x(y))} \right\}$ where $\{y = 0\} \in D \leftrightarrow w(x(y)) < d(x(y)).$

The second term is non-positive on $D = \{y_2 \ge P_A^*(x(y)) - 1\}$ where $\{y = 0\} \in D \leftrightarrow P_A^*(x(y)) \le 1$.

The third term is non-positive on $D = \{y_3 \ge Ny_1(y_2 + 1)\}$ which contains $\{y = 0\}$.

Translating back to the original coordinate system, the following domain D containing EP_1 exists over which \dot{V} is strictly negative

$$D = \left\{ P_S > 1 - \frac{d(x)}{w(x)}, \ w(x) < d(x), \ P_A \ge P_A^*(x), \ S \ge NP_S P_A \right\}$$
(6.37)

Therefore, EP_1 is asymptotically stable if the state-space region $\{w(x) < d(x)\}$ is non-empty. Furthermore, if the region is non-empty and the trajectories are positively invariant on D, then D is also an estimate of the region of attraction. Analysis of the other EPs will show that indeed, D is non-empty and thus EP_1 is asymptotically stable for the cancer control services model.

For EP_2 and EP_3 , the following change of variables is used

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} P_S - P_S^* \\ P_A - P_A^* \\ S - S^* \end{bmatrix}$$
(6.38)

which gives the following system where $d = d(x(y)), w = w(x(y)), \tau_S = \tau_S(x(y)).$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} w(y_1 + P_S^*)(1 - y_1 - P_S^*) - d(y_1 + P_S^*) \\ \frac{P_A^*(x(y)) - y_2 - P_A^*}{\tau_A} \\ \frac{N(y_1 + P_S^*)(y_2 + P_A^*) - y_3 - S^*}{\tau_S} \end{bmatrix}$$
(6.39)

Using the same Lyapunov function as before, $V(y) = y^T P y$, gives

$$\dot{V}(y) = wy_1(y_1 + P_S^*)(1 - y_1 - P_S^*) - dy_1(y_1 + P_S^*) + \frac{y_2(P_A^*(x(y)) - P_A^*) - y_2^2}{\tau_A} + \frac{y_3(N(y_1 + P_S^*)(y_2 + P_A^*) - S^*) - y_3^2}{\tau_S}$$
(6.40)

The second part of \dot{V} associated with y_2 , that is $(y_2 \cdot \dot{y}_2)$ is non-positive on $D = \{y_2 \ge P_A^*(x(y)) - P_A^*\}$ which contains y = 0 when $P_A^*(x(y)) \le P_A^*$. In the original coordinates, this is the domain $D = \{P_A \ge P_A^*(x(y))\}$, which is non-empty. The third part of \dot{V} associated with y_3 , that is $(y_3 \cdot \dot{y}_3)$ is non-positive on $D = \{y_3 \ge N(y_1 + P_S^*)(y_2 + P_A^*) - S^*\}$ which contains y = 0. In the original coordinates, this is the domain $D = \{S \ge NP_SP_A\}$ which is also nonempty.

Therefore, whether or not \dot{V} is negative definite rests on the first part of \dot{V} associated with y_1 , that is $(y_1 \cdot \dot{y}_1)$. For the EP associated with $P_{S,1}^*$ (EP_2) the time-derivative of the corresponding Lyapunov function \dot{V}_1 is shown in Figure 6.24 (in the original coordinate system) as a function of P_S .



Figure 6.24: Derivative of Lyapunov function for equilibrium point EP_2

From the graph in Figure 6.24, it is apparent that there does not exist a domain D containing the equilibrium point EP_2 associated with $P_{S,1}^*$ over which $\dot{V}_1 < 0$. While this does not prove that EP_2 is unstable (Lyapunov Stability Theorem specifies a sufficient but not necessary condition for stability), in that it is possible there could be some other Lyapunov function that satisfies the theorem, using the common Lyapunov function clearly does not work. It is suspected that EP_2 is unstable. Here, Theorem 6.1 is a convenient complement to the Lyapunov stability theorem in that the Lyapunov stability theorem provides sufficient conditions for stability, but not instability, while Theorem 6.1 provides sufficient conditions for instability, and a necessary condition for stability. This will be used in the next section to identify the stability of EP_2 .

For the third equilibrium point EP_3 associated with $P_{S,2}^*$, the time-derivative of the corresponding Lyapunov function \dot{V}_2 is shown in Figure 6.25.

From the graph in Figure 6.25, a domain D exists (gray horizontal bar) which contains the equilibrium point EP_3 associated with $P_{S,2}^*$ and over which $\dot{V}_2 < 0$. Therefore, by Lyapunov's stability theorem, EP_3 is asymptotically stable. If trajectories are positively invariant on D, then D is also an estimate of the region of attraction for $P_{S,2}^*$.



Figure 6.25: Derivative of Lyapunov function for EP_3

It was initially expected, based on the above analysis of the Bass diffusion structure, that the state-space $\{d(x) < w(x)\}$ would be contained within the region of attraction of $P_{S,2}^*$, however based on the Lyapunov stability results, this does not seem to be the case. Figure 6.26 shows that for $P_S > .435$, d < w which also happens to contain $P_{S,1}^* = .633$.



Figure 6.26: word-of-mouth (w) and de-adoption (d) as a function of P_S .

This implies that for the cancer services model, ensuring that the de-adoption fractional rate is less than the adoption fractional rate does not guarantee P_S from going to zero, as it does in the first-order system. Figure 6.26 also confirms that the regions $\{w(x) < d(x)\}$ and $\{w(x) > d(x)\}$ both exist, and thus the domain D used in the analysis of EP_1 exists, and thus EP_1 is asymptotically stable.

To summarize, the first and third equilibrium points of the cancer control services model are formally determined to be asymptotically stable using the Lyapunov stability theorem, which is consistent with the sensitivity analysis which revealed two steady-state conditions. The second equilibrium point is suspected to be unstable which pushes trajectories towards one stable EP or the other. Lyapunov theory also identified conditions in which the EPs are stable and hinted at estimates of the region of attraction for the EPs. Next, the trajectories around EP_1 and EP_3 will be characterized. We also seek to formally determine the stability of EP_2 , and identify the feedback loops which dominate around each EP.

6.3.3 Phase Plots of Trajectories

As a confidence-building test, it is useful to inspect the values of the derivatives of each state variable at the logical boundaries of the state-space. For the cancer control services model, each state variable should be bounded below by zero. P_S and P_A should be bounded above by 1. While S may not have a predetermined upper bound, S should not be allowed to grow unbounded.

Evaluation of model equations 6.12 confirms that \dot{P}_S and \dot{P}_A are non-negative when their state is 0, and non-positive when their state is 1. \dot{S} is non-negative when S is 0, and strictly negative when S is greater than the number of patients, which is upper bounded by a positive constant. Therefore, the model's trajectories are bounded within the logical constraints of the state-space.

A sample of two-dimensional phase plots are also used to qualitatively characterize the solution trajectories. For each of the three dimensions, three planes are sampled at the
minimum, middle, and maximum values, resulting in nine two-dimensional phase plots, as depicted in Figure 6.27.



Figure 6.27: Depiction of the nine planes by which the phase plot is sampled (three planes parallel to each of the three dimensions).

The nine plots are shown in Figure 6.28. The actual solution trajectories do not necessarily lie within these planes and are not necessarily parallel to the planes, as is the case with second-order systems. Therefore, while the phase plots indicate the orientation of the solution vectors for the two dimensions of each plot, they should not be confused with the actual trajectories through three-dimensional space.

The upper-middle, upper-right, and lower-left phase plots of Figure 6.28 indicate that trajectories are attractive to $EP_1 = [P_S = 0; P_A = 1; S = 0]$ which is consistent with the finding that EP_1 is asymptotically stable. EP_2 and EP_3 lie within the interior of the state-space and are not completely evident from the phase plots, however, one can see evidence of divergence and convergence of trajectories which indicate the existence of EPs, especially in the middle-left, lower-left, central, and lower-middle plots.



Figure 6.28: Select phase plots for the cancer services model.

Since the location of the EPs are precisely known, the state-space is now sampled around each EP for each of the three planes containing each EP. This is done in conjunction with state-space dominance analysis in the next section in order to understand what is driving the system around each of the points, and to formally determine the stability of EP_2 .

6.4 State-Space Dominance Analysis

The same state-space dominance procedure used in Chapter 5 for the yeast model is now used to determine which feedback loops are dominant in the regions around each EP^{46} . According to Theorem 6.1, it is expected that balancing feedback loops are dominant in regions surrounding stable EPs, and reinforcing loops are potentially dominant in regions surrounding unstable EPs.

6.4.1 Dominance of Stable EP

In the previous section, $EP_3 = [P_S^* = .9015; P_A^* = .8647; S^* = 7795]$ was shown to be stable using the Lyapunov stability theorem. Theorem 6.1 states that each state variable will be dominated by at least one balancing loop in an arbitrarily small region of the EP. Therefore, state-space dominance analysis methods will be used to determine which loops dominate around this EP.

The variable whose behavior is of primary interest for the dominance analysis is P_S . The trajectories, dominant polarity, and necessary and sufficient feedback loops for P_S are evaluated along three orthogonal planes in the state-space, each intersecting at EP_3 . Figure 6.29 shows plots in the P_SP_A -plane where the left plot shows the dominant polarity of P_S (yellow is positive, blue is negative).

 EP_3 happens to lie precisely at the vertex of the four regions in the left graph. As expected, all trajectories coming into the EP eventually enter a region of negative polarity associated with the dominance of balancing feedback loops. The right graph shows the boundaries of the regions in which balancing loop B2 (depletion of non-seekers through word-of-mouth) of the Bass diffusion process governing P_S (Figure 6.1) is necessary (red) and sufficient (blue). This

⁴⁶The Matlab algorithm for the PFD procedure is included in Appendix D. The Matlab script defining the model and pathway decompositions are in Appendix E.



Figure 6.29: State-space dominance analysis of EP_3 in the P_SP_A -plane.

is the only feedback loop that is either necessary or sufficient in the region encompassing the trajectories that are attracted to this EP_3 . There exists other balancing loops that contribute to the dominant polarity, but which are neither sufficient nor necessary. The center bow-tie shaped region bordered by both blue and red represents the region in which the feedback loop B2 is both necessary and sufficient, and thus dominant in the region. Therefore, the lack of people remaining in the non-seeking category is what is chiefly responsible for the slowing of the adoption rate.

Figure 6.30 shows the phase plot, dominant polarity, and dominance regions for EP_3 in the P_AS -plane.

Trajectories travel from the outside to the middle and meet along the diagonal line. EP_3 is located at the vertex of the positive and negative polarity regions in the middle of the graph. Trajectories attracted to the EP eventually enter the region of negative polarity, as expected for a stable EP, and are thus dominated by a balancing feedback loop. The blue and red lines represent the boundaries of the sufficient and necessary regions for balancing loop B2 of the Bass diffusion process (Figure 6.1), just as in the previous graph. No other feedback loops



Figure 6.30: State-space dominance analysis of EP_3 in the P_AS -plane

are necessary or sufficient in this region. B2 is both necessary and sufficient (dominant) in the small wedge in the middle that encompasses the region in which the trajectories meet from both sides.

Lastly, Figure 6.31 shows the phase plot, dominant polarity, and dominance regions for EP_3 in the P_SS -plane.

Trajectories move from the outside edges inward towards the diagonal line in the middle. As in the previous two planes, the EP is located in the middle at the intersection of the regions. It appears that precisely two trajectories in this plane meet towards the EP, one from the right and left, which are both encompassed in the negative polarity region associated with the dominance of balancing feedback loops, as expected. In fact, the same balancing feedback loop (B2) as in the previous planes, also dominates in this plane within the region of negative polarity. The red and blue lines depict the boundaries of the necessary and sufficient regions, respectively. The necessary region is also depicted by the yellow region in the small lower left plot; and the sufficient region, the lower right plot. The intersection of the two yellow regions is where B2 is dominant.



Figure 6.31: State-space dominance analysis of EP_3 in the P_SS -plane

As expected from Theorem 6.1, the state-space region encompassing trajectories attracted to stable EP_3 contains a dominant balancing feedback loop.

6.4.2 Dominance of Unstable EPs

The stability of $EP_2 = [P_S^* = .633; P_A^* = .8647; S^* = 5473.3]$ was undetermined based on the Lyapunov stability theorem, but suspected to be unstable based on the behavior of the chosen Lyapunov function, and since sensitivity analysis did not indicate the existence of a third stable EP. Theorem 6.1 requires that in an arbitrarily small region around a stable EP, each state variable be dominated by at least one balancing loop. The contrapositive of the theorem also provides a sufficient, but not necessary condition for determining instability.

As before, P_S is the state variable whose behavior is of interest. Figure 6.32 shows the trajectories and regions of dominant polarity (yellow - positive; blue - negative), for the three orthogonal planes intersecting at EP_2 .



Figure 6.32: Three orthogonal phase plots centered around EP_2 showing the dominant polarity regions.

In the top two graphs, trajectories come from the outside toward the middle. In the bottom graph, trajectories come from the top and the bottom towards the middle. It is observed that EP_2 is connected to regions of both positive and negative polarity (the regions intersect at EP_2), thus the contrapositive of the theorem cannot be applied. Closer inspection however,

reveals that while all the trajectories coming into the EP in the upper two plots enter a negative polarity region, there appears to be trajectories leaving the EP in the lower phase plot. If this is the case, then EP_2 is unstable, by definition of stability, since trajectories do not stay within an arbitrarily small domain around the EP. It is also observed that the trajectories leaving the EP in the lower plot happen to be contained in a region of positive dominant polarity, which suggests that indeed EP_2 is unstable. Inspection of additional P_SP_A phase plots for values of S slightly less than and greater than the EP value of S = 5473.3 confirms that the trajectories which appear to exit the EP are contained in the dominant polarity region. It is observed that Theorem 6.1 may be useful to prove that EP_3 is unstable with the following additional criteria.

For a stable isolated equilibrium point x = 0, the condition that a balancing loop dominates somewhere inside an arbitrarily small region containing x = 0 can actually be made stronger. Precisely, that a balancing loop must dominate at some point along each trajectory (solution) x(t) within an arbitrarily small region connected to x = 0.

The following corollary to Theorem 6.1 is offered:

Corollary 6.1

Consider the autonomous system $\dot{x} = f(x)$ where $f: D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n . Let x = 0 be an isolated equilibrium point for $\dot{x} = f(x)$. If x = 0is stable, then each state variable x_i is dominated by one or more balancing feedback loops at some point along its trajectory x(t) inside an arbitrarily small state-space region connected to x = 0.

Proof

The proof by contradiction used for Theorem 6.1 applies here so long as $|x(0)| < \delta(\epsilon)$ and assuming, towards contradiction, that there exists a dimension x_i which is not dominated by a balancing loop anywhere along its trajectory for t > 0. This produces the same contradiction that a time t can be found at which point $||x(t)|| > \epsilon$, which contradicts the assumption that x = 0 is stable. \Box

The contrapositive to this corollary is that if trajectories can be found which are connected to an EP and which do not lie within a negative polarity region (i.e. a region dominated by balancing feedback) within an arbitrarily small distance ϵ from the EP, then the EP cannot be stable.

Employing this contrapositive, EP_2 is determined to be unstable since the lower phase plot in Figure 6.32 shows that the single trajectory connected to the EP to the left (along the ridge line) does not intersect a negative polarity (blue) region anywhere to the left of the EP. This finding is robust across the state-space based on sampled phase plots for values of S close to and surrounding the EP.

We now seek to understand which loops are dominant for the trajectories approaching EP_2 along its stable manifold (indicated by the blue regions in the upper two plots), as well as the trajectories leaving EP_2 along its unstable manifold (indicated by the yellow region in the lower plot). Considering the unstable region, Figure 6.33 shows the region in which the reinforcing loops in the model are dominant and produce positive polarity, which encompasses the trajectory exiting EP_2 to the left.

State-space dominance analysis within this yellow region determines that none of the reinforcing loops are individually sufficient, nor are they individually necessary, in the part of the region connected to the EP. This implies that the system is operating at point (0,0) in the dominance framework introduced in Chapter 3, and therefore exhibits behavior which is robust to changes in any individual loop due to redundant mechanisms. Some loops however are more influential than others and contribute significantly to the behavior. One pair of contributing reinforcing loops are associated with R3 in Figure 6.15 in which a low service



Figure 6.33: Region of dominance of the reinforcing loops for EP_2 in the P_SP_A -plane.

capacity lowers the average quality which both increases de-adoption and decreases internal adoption, which lowers the proportion seeking services and thus the number of patients, which causes the service capacity to drop even more. Another pair of contributing reinforcing loops are R1 and R2 in which a low proportion of people receiving services leads to less adoption through word of mouth and greater de-adoption as well, which both cause the proportion seeking services and thus the proportion receiving services to fall further. Two additional mechanisms, not associated with clear feedback loops in the model, but which contribute to the divergent behavior is the effect of low proportion of people who can get services on the proportion who receive services, and its subsequent contributing effect of strengthening the reinforcing loops R1 and R2 described above which act as vicious cycles, pushing the proportion seeking services further downward.

Considering the stable region, Figure 6.34 shows the regions in yellow associated with two different loops being necessary to produce convergent behavior for trajectories coming towards EP_2 in the P_AS -plane.



Figure 6.34: Region of dominance of the balancing loops for EP_2 in P_AS -plane

No loops are individually sufficient in this region. The left plot shows the necessary region for the balancing loop of depletion of internal adoption process in Figure 6.1, and the right plot, the necessary region for the balancing loop of depletion of the de-adoption process.

Finally, Figure 6.35 shows the regions in yellow associated with the necessary and sufficient regions of the influence of the proportion able to get services on the proportion who receives services, on the internal adoption fractional rate, which produces convergent behavior for trajectories coming towards EP_2 in the P_SS -plane. No other causal pathway or loop is individually necessary or sufficient in this space.

The two trajectories coming into EP_2 , from the left and the right, fall within the region in which the causal pathway is both sufficient and necessary (dominant). That is, the slowing approach of P_S towards the EP can be explained by the slowing increase or decrease of the proportion of people able to get services and its effect on the internal adoption fractional rate.



Figure 6.35: Region of dominance of the balancing loops for EP_2 in P_SS -plane

6.5 Discussion of Baseline Results

Returning to the questions at the beginning of the chapter, the results are now discussed. The first two questions ask about the primary influences on reach and quality, and the conditions for which one might increase while the other decreases. The system is found to have a single unstable equilibrium point which acts as a tipping point and pushes the system towards one of two stable equilibrium points. Therefore, two cases are possible, the first being that reach goes to zero while quality goes to one. This occurs if the proportion seeking services is below the tipping point (in which the tipping point depends on the other initial conditions and parameters) such that the system is caught in a vicious cycle (reinforcing feedback loop) in which the low number of patients reduces population outcomes which further reduces the number of people seeking services, and thus the number of patients. The cycle continues until demand goes to zero. Meanwhile, those few remaining who do seek and obtain services enjoy a relatively large service capacity to patient ratio which increases quality for those few patients. Since supply lags demand due to a first order balancing loop, the ratio of capacity to demand remains greater than one as patients go to zero, and quality goes to one. This scenario warns against assessing the performance of a system based on a single metric such as average quality, while ignoring the outcomes for the entire population.

The second case to consider, qualitatively different from the first, is when the proportion of people seeking services is sufficiently large that it is above the tipping point and is pushed towards a stable positive steady-state value, and thus reach (number of patients) also reaches a stable positive steady-state value. In this scenario, the same reinforcing loop as in the previous case acts as a virtuous cycle in which the more people who seek services, the greater the number of patients and population level outcomes, which increases adoption through word-of-mouth and decreases de-adoption, thus leading to even more patients. Quality also reaches a steady-state value determined by the impact of the ratio of service capacity to patients on quality. This is a relationship function that would depend, in the real world, on the efficiency, experience, and general capabilities of the service providers. Whether or not reach or quality increases or decreases in this scenario depends on whether they start above or below their steady-state condition.

One finding is that if the de-adoption fractional rate is greater than adoption fractional rate, it is a sufficient condition for the system to collapse. However if de-adoption is less than adoption, that alone does not prevent the system from collapsing. The point in which the de-adoption fractional rate equals the adoption fractional rate falls to the left of the tipping point condition.

The above findings also reveal a few limitations of the model. In the scenario in which patients decline towards zero there may be other balancing mechanisms which would prevent the demand from going to zero. Additionally, quality may begin to suffer if the ratio of capacity to patients becomes too high causing care provider experience to diminish. Furthermore, the Bass diffusion process permits unlimited adoptions, deadoptions, and re-adoptions. It is possible for the Bass diffusion structure to be modified to suit additional realistic restrictions.

The next set of questions ask about the conditions which produce disparities. According to the model, there are two ways in which different population segments can experience a different health status. From a model perspective, the first is associated with structural differences of the equations. For example, the ability to get services having a different sensitivity to the ratio of capacity to patients for one segment versus another. Or, if two segments have different service providers and the quality for one segment is lower at the same ratio than the quality for the other. Or, if there are structural reasons why the maximum proportion of people who are able to get services is lower for one segment than for another⁴⁷. In this case, two population segments may start with the same initial conditions in statespace, but for one segment, those initial conditions may fall within the region of attraction of the zero equilibrium point (for example, if quality sensitivity is too low), while for the other, it may fall in the region of attraction of the positive equilibrium point. In this case, the two populations would experience a qualitatively different outcome, which is a disparity. An example of this is seen in the sensitivity analysis in which quality sensitivity is reduced from 2 to 1 resulting in P_S going from a positive steady state value to a zero steady-state value. Figure 6.36 shows the reason why: lowering λ_Q to just 1.5 results in no non-zero solutions to P_S^* , since the graphs do not intersect, which means that no matter where the system starts, P_S will go to zero⁴⁸. Even if lowering λ_Q to a value in which there are still two non-zero solutions to P_S^* , it has the effect of increasing the de-adoption fractional rate curve while lowering the internal adoption fraction rate curve, which increases the region of attraction of EP_1 corresponding to $P_S^* = 0$.

The other way population segments can experience disparities is if they start with different initial conditions. Even if the socio-economic factors, risk factors, and service provider quality function and adjustment time was the same for both populations segments (that is, the model parameters and equations are the same), if one segment has a sufficiently lower

 $^{^{47}{\}rm For}$ example, differences due to income levels or the availability of service providers who accept MEDI-CAID.

⁴⁸This is an example of a bifurcation parameter which changes the number of equilibrium points.



Figure 6.36: Graph showing condition in which non-zero solutions do not exist for P_s^* .

proportion of people who have access and who are seeking services, that segment can be trapped in a vicious cycle while the other can be in a different stable region of the statespace, and gravitate towards a positive steady-state. In reality, disparities are likely to be some combination of the two ways described here, but understanding the distinction may be helpful in identifying ways to intervene.

Lastly, it is possible for disparities to continue to exist between segments even as reach and quality improve for both. If both segments are within the region of attraction of the positive EP, and each are starting below their stable steady-state values, then each will experience improvements. However, disparities will exist if the transient responses or steady-state conditions are different. The transient responses are determined primarily by the adjustment times, whereas the steady-state conditions are determined primarily by the sensitivity of ability to get services to the capacity ratio, the sensitivity of quality to the capacity ratio, and the relationship between the de-adoption and adoption curves. That is, how likely one is to adopt or de-adopt based on quality and overall population impact.

6.6 Policy Analysis and Insights

The questions on policy aim to identify effective places to intervene in the system. We also consider the problem of how to scale up services when here is a sudden increase in demand.

6.6.1 Intervention Effectiveness

Assume a situation in which proportion able to get services is at a positive steady-state but proportion seeking services is .4, well below the positive steady-state of .9015 and within the region of attraction of zero. Service capacity is currently equal to the number of patients, and so supply and demand are in equilibrium. We desire to find policies which raise P_S to within 5% of its positive steady-state value in twenty years.

For an advertising or educational intervention, the external adoption fractional rate (FR) would have to be .25/year, meaning each year 25% of the non-seekers become seekers. This is significantly high in order to overcome the high de-adoption FR which exists in this region of state-space, due to low fraction of people receiving services. Now assume the maximum effectiveness of an advertising campaign is only .05/year. In this case, the lowest that initial *proportion seeking services* could be where the intervention is successful is .65 (slightly above the tipping point condition). Further sensitivity analysis indicates that external adoption can be effective in state-space regions near the tipping point (unstable equilibrium), but is relatively weak in state-space regions far from the tipping point. Sensitivity and dominance analysis suggest looking closely at the de-adoption and internal adoption fractional rates. Reducing the maximum de-adoption FR by one third brings the system past the tipping point, and reducing it by one half achieves the desired results.

Now consider if the system is operating near the tipping point. What interventions can bring the system to within 5% of its positive steady-state in ten years? An advertising or educational intervention would require an effectiveness of .08/year. An intervention focused on de-adoption would require reducing the maximum de-adoption FR from .3/year to .24/year. Another potentially effective policy revealed by dominance analysis is to increase the *proportion able to get services*. Here, a successful intervention requires increasing *sensitivity of ability to get services to ratio of capacity to demand* from 2 to 2.5.

Dominance, state-space, and sensitivity analysis all reveal that contrary to what might be expected, adjustment time (AT) of service capacity has very little impact on behavior. It slightly changes the transient response, but does not change the location and stability of the EPs. AT of service capacity may affect the regions of attraction.

6.6.2 Response to External Changes

A sudden increase in the population needing services causes a decrease in the ratio of capacity to patients, which immediately decreases quality and access. If the system is operating well within the region of attraction of the positive EP, it is robust to external disturbances and will adjust to a new stable steady-state. However, if the system is operating near the boundary of the region of attraction, slight disturbances can push the system into the neighboring region of attraction, causing it to collapse. Consider a surge of 20% of people in need of services at year two. As in the previous scenario, *proportion able to get services* is assumed to be at steady-state, and supply equals demand. In one case, shown in Figure 6.37, the initial *proportion seeking services* is .65, slightly above the tipping point and just within the region of attraction of the positive steady-state.

The demand surge is sufficient to push the state into the region of attraction of $P_S * = 0$, which leads to collapse.



Figure 6.37: Testing response to an increase in demand, case 1.

In the second case, initial *proportion seeking services* is .75, providing greater margin to outside disturbances. In this case, the system accommodates the surge and remains within the region of attraction, while both supply and demand adjust to a new stable positive steady-state, as shown in Figure 6.38.



Figure 6.38: Testing response to an increase in demand, case 2.

One implication for the scale-up of services problem is the importance of knowing if a system contains a tipping point, and if so, is the system is operating near or far from it. This indicates how robust the system is to external effects such as changes in demand. Quality is a key variable affecting the tipping point condition, and so if capacity requirements can anticipate future changes in demand, it may be possible to mitigate the effects of an increased demand on quality before it occurs.

6.7 Evaluation of the State-Space Dominance Method

State-space methods are useful for evaluating the origin and destination of trajectories (alpha- and omega-limit sets), which can include any number of equilibria and limit cycles. Methods exist to determine their stability properties and how their quantity, location, and stability change with the parameters of the system (bifurcations). Methods also exist to approximate regions of attraction for asymptotically stable EPs of a system, if they exist.

Dominance methods are useful for evaluating if a system is exhibiting convergent or divergent behavior and which structural elements associated with real-world causes are responsible for producing the convergent or divergent behavior. Dominance methods are typically applied in the time-domain; however, this thesis applies them in state-space. The method was specifically applied in state-space regions encompassing the EPs of the cancer control services model in order to understand which causal mechanisms were dominant and responsible for the behavior. In the process, a formal relationship between dominance and stability appeared to emerge through numerous examples, which was made explicit and proven in Theorem 6.1 and Corollary 6.1, and subsequently used to prove that EP_2 is unstable. This results was also confirmed through sensitivity analysis by starting the simulation at EP_2 and slightly varying the initial conditions. Sensitivity analysis alone, apart from an exhaustive state-space search, would have been unlikely to find the unstable EP. Conducting an exhaustive full-factorial sensitivity analysis over parameter- and state-space can be computationally-prohibitive for large system dynamics models, not to mention time-consuming just to analyze the results. Published sensitivity analysis methods in the SD literature prescribe methods for testing each parameter in isolation, but not in combination [25, 142, 143]. State-space analysis can be used to identify the existence and location of EPs; dominance analysis can be used then to identify the stability of the EPs, which then focuses sensitivity and policy analysis in a more manageable stateand parameter-space.

Conducting dominance analysis in the state-space region around the unstable EP helped identify the forces which cause trajectories to tip towards one steady-state or another. It was found that multiple reinforcing loops were dominant around the unstable EP, which created a tipping point condition. In summary, dominance methods can be applied not just to transient behavior between alpha- and omega-limit sets (as is commonly done), but can also be applied near the limit sets themselves in order to understand the forces governing the source and destinations of trajectories. Additionally, using the contrapositive of Corollary 6.1, one could use dominance methods to locate and determine the stability of EPs in largedimension models.

In summary, each analysis method provides complementary insights and can be used together effectively. Sensitivity analysis, for example, revealed that parameters affecting de-adoption and internal adoption fractional rates d and w were highly sensitive. Stability analysis revealed the specific relationship between d and w and stability. Dominance analysis identified the dominant mechanisms affecting d and w in regions near the unstable EP. Finally, sensitivity analysis was used over a smaller state- and parameter-space, near these regions, to analyze interventions.

Chapter 7

Summary and Conclusions

7.1 Main Findings and Contributions to Systems Science

- 1. Theorem relating stability of dynamic systems to feedback loop dominance. In the field of system dynamics there have been general claims and principles relating feedback loop dominance to behavior, but most are offered without proof or are only applicable to trivially simple systems. It was suspected, however, that a formal relationship should exist between stable modes and balancing loops, and between unstable modes and reinforcing loops, for general nonlinear systems. Theorem 6.1 and Corollary 6.1 establish a formal relationship between stability and loop dominance. This is a significant contribution to the field of system dynamics in that it grounds the theory of dominance using a mathematically rigorous criteria for stability. As a result, Theorem 6.1 and Corollary 6.1 provide a necessary condition for stable equilibrium points and the existence of dominant balancing feedback loops. The contrapositive also provides a sufficient condition for unstable equilibrium points.
- 2. Formal definition and framework for *dominance*. While developing the statespace dominance methods it became evident there was no mathematically rigorous and

formal definition of dominance agreed upon by practitioners in the field. As a result, an exhaustive historical survey of the field was performed through the lens of feedback loop dominance. The systematic review and analysis concluded with a formal and mathematically rigorous framework for defining dominance based on necessary and sufficient conditions. The framework also indicates the robustness or fragility of a system. Shifts in dominance were found to occur when systems exist in fragile states in the dominance framework. The framework captures both structural and behavioral aspects of dominance, which have been used in current dominance methods. It also establishes a mathematical basis and definition for observed phenomena such as *shadow loop dominance* and *multiple loop dominance*.

3. Behavioral-structural procedure for dominance in state-space. The pathway force decomposition procedure was developed to determine dominance in either the time- or state-domain, and was implemented in both Excel and Matlab. One question was whether or not the concept of state-space regions even made sense. In fact, dominant regions were found to exist in state-space and were able to explain the results from previous dominance studies. State-space dominance analysis was performed on well-studied previous examples and directly compared with other methods. It was hypothesized that regions of dominance would directly correspond to regions of attraction of asymptotically stable equilibria. This was proven not to be the case using simple counter-examples such as the logistic model. Dominance regions were found to be typically contained within regions of attraction. Within a region of attraction, there may exist multiple distinct regions of dominance. Another hypothesis was that within a sufficiently small domain containing an asymptotically stable EP, there would exist a single region of dominance responsible for the attraction of the trajectories. However, the logistic model also showed this not to be the case, as different sets of feedback mechanisms were responsible for the attraction on either side of the equilibrium point.

- 4. Formal and mathematical relationships between current dominance methods. A direct comparison was made between each dominance method for several wellstudied models. The pathway force decomposition procedure resulted in consistent results, but furthermore, was able to explain why some methods produced consistent results in some cases, and inconsistent in others. The dominance framework was used to identify the conditions in which different methods produced similar or different results. This also validated the proposed definition of dominance.
- 5. Approach for using dominance, state-space, and sensitivity analysis together.. Analysis of the cancer control services model began with a modest sensitivity analysis of behavior over time, indicating high-level qualitative characteristics such as whether or not the system was stable or unstable, oscillatory or non-oscillatory, containing a single steady-state or multiple steady-states. State-space analysis then identified three EPs, and Lyapunov stability theory was used to confirm the stability of two of the three. Finally, dominance analysis and Corollary 6.1 was used to confirm the stability of the third EP and identify which feedback loops were dominant around each EP, indicating structural sources of behavior. Finally, sensitivity analysis was used to perform policy analyses within a significantly smaller parameter- and state-space.
- 6. Model confidence-building. Using state-space and dominance methods together proved helpful not just for analysis, but for model synthesis and confidence building. For example, in earlier versions of the model, state-space and dominance methods were used to identify and locate regions and conditions in which trajectories were unbounded. These methods then helped identify how to modify the equations to ensure trajectories would be bounded within the logical constraints of the parameters and state values.

7.2 Main Findings and Contributions to Transdisciplinary Science and Public Health

- 1. Framing cancer control services as a dynamic problem. The analysis methods developed in this thesis are designed to investigate the underlying structural causes of dynamic behavior. Framing cancer control services as a dynamic problem, formally defined by a system of dynamic equations, makes the problem accessible to a wide range of analysis tools available in engineering and systems sciences. State-space dominance analysis is a novel approach for analyzing cancer control service systems, and could inform future lines of research. The model and analysis methods are particularly useful for systems which may not exist in an equilibrium state, and for which interactions are determined to be nonlinear. This provides advantages over traditional means such as statistical analysis, in that it directly accounts for how nonlinear interactions play out over time.
- 2. Sources of health disparities. Strategic goal four of the Society for Prevention Research includes using transdisciplinary innovation to study health disparities⁴⁹. Around this topic, the issue of access to care arises frequently. However, analysis of the cancer control services model indicates that the decision of whether or not to seek services could have a greater impact than access, as it belongs to several reinforcing feedback loops. Whether or not these reinforcing loops act as virtuous or vicious cycles depend on how close the system is to a tipping point condition, which is caused by an unstable equilibrium point. The size of the regions of attraction on either side of the tipping point are defined by the system parameters, which may reflect structural differences between two population segments, resulting in qualitatively different system behaviors, and thus, significant health disparities. It is possible for disparities to increase even

⁴⁹Interested readers are directed to the society's website for its strategic plan, goals, and objectives at: http://www.preventionresearch.org/about-spr/mission-statement/

as overall performance increases for both population segments. It is also possible for average quality to increase as the total impact of services decreases due to a decrease in the number of patients.

- 3. Leverage points. Internal adoption mechanisms by word-of-mouth seem to have a greater influence than external mechanisms such as education and advertising. This is likely due to the fact that internal adoption is part of a reinforcing, growth-producing feedback loop. De-adoption (that is, the process by which people decide to no longer seek services), if sufficiently large relative to internal adoption, can cause the system to collapse to a zero equilibrium point. On the other hand, if adoption is sufficiently large relative to de-adoption, this greatly expands the region of attraction of the positive stable equilibrium value, and makes the system more robust.
- 4. Policy resistance. Policy resistance occurs when the system is unresponsive to a policy or intervention. This can occur when a system is operating in a state in which there are no necessary feedback loops (i.e. no critical mechanisms), and multiple redundant feedback loops which collectively dominate the system. This thesis proposed a dominance framework to evaluate the level of robustness with respect to policy resistance and found that, in fact, the cancer control services model exhibits policy resistance when it is operating near but diverging from the tipping point condition. This suggests that no single intervention may be effective without addressing multiple feedback loops. For example, both the adoption and de-adoption process. This is equivalent to addressing both why people make the decision to seek services, and why they make the decision to no longer seek services, since both are part of different dominant reinforcing loops. In business and marketing, this is equivalent to developing strategies which address both the front and the back door. That is, why patients come and why they leave.

7.2.1 Limitations and Potential Lines of Research in Systems Science

Additional models are required to understand how well state-space dominance methods will scale to higher-order systems. Phase plots are useful for two- and three-dimensions at a time, however beyond three dimensions, visualization of trajectories is challenging. Additionally, many problems become significantly more difficult for nonlinear systems when the number of state variables is greater than three, such as detecting limit cycles and estimating regions of attraction. Pragmatically, the utility of the methods in this thesis will depend on the level of automation and integration possible with existing system dynamics software. The methods in this thesis have been implemented in Matlab. Opportunities exist to increase the automation of the methods for higher dimensional spaces. For example, state-space dominance methods could be tailored and automated to scan large regions of state-space in order to locate EPs or even limit cycles. For a particular loop, one can ask over which regions it is dominant or if it is dominant at all. One can ask which loops are dominant around a specific point in state-space.

The concept of robustness is not new in dynamic systems. However, two aspects of robustness were discussed in this thesis and were related to feedback loop dominance. First, robustness was defined using the dominance framework in which systems operating with either multiple sufficient or no necessary structures would exhibit behavior modes robust to changes in the system. This is related to the concept of policy resistance in system dynamics. Second, robustness was described as distance from a tipping point condition, associated with an unstable equilibrium point. This was related to regions of attraction of the system. Future research can investigate how systems change robustness over time and state-space with formal connections to loop dominance, towards the goal of designing robust systems or designing policies effective for robust systems. All models evaluated in this thesis were continuously differentiable, allowing straightforward application of stability theory and dominance analysis. However, models of social systems may not be differentiable (such as those using minimum, maximum, absolute value functions), while others may not be continuous (such as those using if-then-else logical constructs and discontinuous table look-up functions). Opportunities may exist to apply innovations from the field of machine learning (such as automatic differentiation) which impose fewer conditions on systems. In dynamic systems theory, the main issue associated with differentiability is that of existence and uniqueness of solutions, however in practice, existence and uniqueness seems to rarely be an issue for even discontinuous dynamic systems so long as they are constructed using sound principles, such as causality. Using methods such as automatic differentiation may also open up dominance analysis, for the first time, to systems science methods outside of SD such as agent-based and discrete-event simulation. These would be exciting areas for future exploration. Additionally, automatic differentiation would make implementation of PFD easier, in that it would not require the manual derivation of each partial derivative, as it currently does.

Finally, the cancer control services model has a structure similar to other transient growth models such as the market growth model and the world model. The methods would benefit from further evaluation on large models exhibiting oscillatory modes such as supply chain dynamics and the economic long wave model. Also, the addition of perception and measurement delays to the cancer control services model would add oscillatory modes of behavior which could then be evaluated using the same dominance methods.

7.2.2 Limitations and Potential Lines of Research in Public Health

Model analysis revealed that the likelihood of seeking services and ability to obtain services are key influences of health disparities. In addition to being affected by quality and the ratio of supply to demand, as seen in the current model, qualitative group model building sessions have revealed that these factors are also affected by a variety of social determinants of health, which are not explicitly modeled. To what extent is the system governed or constrained by exogenous or endogenous social determinants? If endogenous social determinants can be considered as stocks (that is, they accumulate or are depleted over time), does this effectively add delays on the demand side? For example, community perception of cancer outcomes, distinct from actual outcomes, can influence whether people seek services. Perception is often modeled as a stock (state variable), which is a delay in the system. How does this impact the relationship between supply and demand? Does this qualitatively change the steady-state conditions of the system or just the near-term (transient) response? For example, in dynamic systems, significant delays can produce unstable oscillations in an otherwise stable feedback control system. Additionally, social determinants such as access to transportation, income, and family support may act as latent variables mediating between quality and the decision to seek services and the ability to obtain services.

Similarly, on the supply side, observation or measurement delays may induce instability. How do organizations' measurement of demand, quality, and performance affect their ability to adjust capacity to meet demand? Does this affect the transient response or does it change the stability of the entire system? Also, if the system is supply-constrained, and if two population segments compete for the same limited service capacity, what factors might increase disparities? What kind of policies would help? Under what conditions will the system gravitate towards an equitable equilibrium or one that results in disparities?

Finally, strategic goal two of the Society for Prevention Research includes developing systems science methods to inform the scale-up of evidence-based practices. The sustainable scale-up of health services can be defined as a dynamic problem [67, 14]. Further research leveraging the cancer control services model can be used to develop and test design principles using dynamic systems theory. The model illustrated how some feedback loops dominate the steady-state condition (indicating sustained performance or eventual collapse), while other loops dominate the transient response (how quickly the system rises, falls, or overshoots). Additionally, the location in which a system is operating in state-space affects which loops will come to dominate its trajectory. Figure 7.1 illustrates how the cancer control services model is capable of producing a variety of different scale-up behaviors for different initial conditions and parameter values.



Figure 7.1: Different scale-up behaviors over time for the cancer control services model.

Further research would seek to identify design principles which can help achieve desired scale-up performance.

References

- A.A. AbdelGawad, B.E.A. Abdel-Aleem, Mohamed M Saleh, and Pål I Davidsen. Automated eigenvalue analysis of system dynamics models. page 39, Boston, MA, 2005. System Dynamics Society.
- [2] Kevin A Agatstein. (D-4602-2) Oscillating Systems II: Sustained Oscillation. 1997.
- [3] Hayward R. Jr. Alker and Ann Tickner. (D-2679a) Some issues raised by previous world models. 1977.
- [4] Mark Supplee Ames. A Model of Hiring Practices and Implementation Decisions in a Research Oriented Organization. Master's thesis, Massachusetts Institute of Technology, 1966.
- [5] David F Andersen and Jeppe Sturis. Chaotic structures in generic management models: Pedagogical principles and examples. System Dynamics Review, 4(1-2):218–245, 1988.
- [6] Yaman Barlas. Formal aspects of model validity and validation in system dynamics. System Dynamics Review, 12(3):183–210, 1996.
- [7] Yaman Barlas and Stanley Carpenter. Philosophical roots of model validation: Two paradigms. System Dynamics Review, 6(2):148–166, 1990.
- [8] Frank M Bass. A new product growth for model consumer durables. *Management* science, 15(5):215–227, 1969.
- [9] Philip Knight Bates. A dynamic study of self-induced seasonal cycles. Master's thesis, Massachusetts Institute of Technology, 1960.
- [10] John Bell. (D-1634) Inventory control part II, Response to Sinusoidal inputs. 1972.
- [11] Donald M Berwick. A User's Manual For The IOM's 'Quality Chasm' Report. Health Affairs, 21(3):80–90, 2002.
- [12] William L. Brogan. Modern Control Theory. Prentice Hall, 3rd edition, 1991.
- [13] Robert Y. Cavana and Edwin D. Mares. Integrating critical thinking and systems thinking: from premises to causal loops. System Dynamics Review, 20(3):223–235, 2004.
- [14] David A Chambers, Russell E Glasgow, and Kurt C Stange. The dynamic sustainability framework: addressing the paradox of sustainment amid ongoing change. *Implementation Science*, 8(1):1, 2013.

- [15] Ping Chen. Empirical and theoretical evidence of economic chaos. System Dynamics Review, 4(1-2):81–108, 1988.
- [16] TB CROSSLEY and B Porter. Eigenvalue and eigenvector sensitivities in linear systems theory. International Journal of Control, 10(2):163–170, 1969.
- [17] Pål I Davidsen. The structure-behavior diagram: Understanding the relationship between structure and behavior in complex dynamic systems. pages 127–140, Utrecht, Netherlands, 1992. System Dynamics Society.
- [18] LeRoy E Day. A dynamic model of a weapons logistic system. Master's thesis, Massachusetts Institute of Technology, 1960.
- [19] Dietrich Dörner. The logic of failure: Recognizing and avoiding error in complex situations. Basic Books, 1996.
- [20] J. Duggan and R. Oliva. Methods for Identifying Dominant Structure–Introduction to the Model Analysis Virtual Issue of the Systems Dynamics Review. System Dynamics Review, 2013.
- [21] Terri Duhon. (D-4286) A system dynamics approach to rabbits (S-shaped growth). 1992.
- [22] Robert L Eberlein. Simplifying Dynamic Models by Retaining Selected Behavior Modes. PhD thesis, Massachusetes Institute of Technology, Cambridge, MA, 1984.
- [23] Robert L Eberlein. Simplification and understanding of models. System Dynamics Review, 5(1):51–68, December 1989.
- [24] Willard R Fey. The stability and transient response of industrial organizations. Master's thesis, Massachusetts Institute of Technology, 1961.
- [25] Andrew Ford and Hilary Flynn. Statistical screening of system dynamics models. System Dynamics Review, 21(4):273–303, December 2005.
- [26] David N. Ford. A behavioral approach to feedback loop dominance analysis. System Dynamics Review, 15(1):3–36, March 1999.
- [27] Jay W Forrester. (D-0000) Note to the Faculty Research Seminar. 1956.
- [28] Jay W Forrester. Industrial dynamics: a major breakthrough for decision makers. Harvard Business Review, 36(4):37–66, 1958.
- [29] Jay W Forrester. Industrial Dynamics. Cambridge: MIT Press, 1961.
- [30] Jay W Forrester. (D-4054) Modeling the Dynamic Processes of Corporate Growth (reprinted from IBM symposium). 1964.
- [31] Jay W Forrester. Proceedings of the American Marketing Association 1965 Fall Conference: Modeling of Market and Company Interactions. 1965.

- [32] Jay W Forrester. (D-1123) Planning Under the Dynamic Influences of Complex Social Systems. 1968.
- [33] Jay W Forrester. Industrial dynamics after the first decade. Management Science, 14(7):398–415, 1968.
- [34] Jay W Forrester. Market growth as influenced by capital investment. *Industrial Management Review*, 9(2):83, 1968.
- [35] Jay W Forrester. Principles of Systems. Pegasus Communications, Waltham, MA, 1968.
- [36] Jay W Forrester. Urban Dynamics. MIT Press, Cambridge, MA, 1969.
- [37] Jay W Forrester. (D-3691-2) Nonlinearity in high-order models of social systems. 1985.
- [38] Nathan B Forrester. A dynamic synthesis of basic macroeconomic theory: implications for stabilization policy analysis. PhD thesis, Massachusetts Institute of Technology, 1982.
- [39] Nathan B Forrester. Eigenvalue analysis of dominant feedback loops. In *Proceedings* of the 1st International System Dynamics Society Conference, pages 178–202, 1983.
- [40] N.B. Forrester. A Dynamic Synthesis of Basic Macroeconomic Theory: Implications for Stabilization Policy Analysis. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 1982.
- [41] G. Franklin, J. Powell, and A. Emami-Naeini. Feedback Control of Dynamic Systems. Addison-Wesley, 3rd edition, 1994.
- [42] Katherine L Frohlich and Louise Potvin. Transcending the known in public health practice: the inequality paradox: the population approach and vulnerable populations. *American Journal of Public Health*, 98(2):216–221, 2008.
- [43] William Glass. (D-4243-3) Teaching SD: Looking at Epidemics. 1991.
- [44] Marc Glick and Terri Duhon. (D-4432-2) Generic Structures: S-shaped growth I. 1994.
- [45] Paulo Gonçalves. Eigenvalue and eigenvector analysis of linear dynamic systems. In Proceedings of the 24th International Conference of the System Dynamics Society, 2006.
- [46] Paulo Gonçalves. Behavior modes, pathways and overall trajectories: eigenvector and eigenvalue analysis of dynamic systems. System Dynamics Review, 25(1):35–62, 2009.
- [47] Michael R Goodman. (D-1602) S-shaped growth and simple feedback. 1971.
- [48] Michael R Goodman. Elementary system dynamics structures. Master's thesis, Massachusetts Institute of Technology, 1972.
- [49] Michael R Goodman. Study Notes in System Dynamics. Allen Press, 1974.

- [50] Alan K Graham. (D-2254) Study unit for visiting scholars: the production sector of the national socio-economic model. 1975.
- [51] Alan K Graham. (D-2256-3) Proposal for thesis research: Entrainment of economic sectors in cyclic behavior. 1976.
- [52] Alan K Graham. Principles on the Relationship Between Structure and Behavior of Dynamic Systems. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 1977.
- [53] Alan K Graham. (D-3289) What P.O.S.2 students should know about control theory. 1981.
- [54] Peter Gross and Jeppe Sturis. Hopf bifurcation analysis applied to predator-prey modeling. System Dynamics Review, 8(2):179–186, 1992.
- [55] Burak Güneralp. A principle on structure-behavior relations in system dynamics models. In Proceedings of the 22nd International Conference of the System Dynamics Society, Oxford, UK.
- [56] Burak Güneralp. Exploring structure-behavior relations: Eigenvalues and eigenvectors versus loop polarities. In *Proceedings of the 22nd International Conference of the System Dynamics Society*, Oxford, UK., 2004.
- [57] Burak Güneralp. Towards coherent loop dominance analysis: progress in eigenvalue elasticity analysis. *System Dynamics Review*, 22(3):263–289, September 2006.
- [58] John Hayward. Model Behaviour and the Strengths of Causal Loops: Mathematical Insights and a Practical Method. In Proceedings of the 30th International System Dynamics Conference, 2012.
- [59] John Hayward. Newton's Laws of System Dynamics. In Proceedings of the 33rd International System Dynamics Conference, 2015.
- [60] John Hayward and Graeme P. Boswell. Model behaviour and the concept of loop impact: A practical method. System Dynamics Review, 30(1-2):29–57, January 2014.
- [61] Gary Hirsch, Jack B Homer, Elizabeth Evans, and Ann Zielinski. A system dynamics model for planning cardiovascular disease interventions. *American Journal of Public Health*, 100(4):616–622, 2010.
- [62] Gary B Hirsch. Societal Response to Social Problems. Master's thesis, Massachusetts Institute of Technology, 1971.
- [63] Philip Holmes. Ninety plus thirty years of nonlinear dynamics: Less is more and more is different. International Journal of Bifurcation and Chaos, 15(09):2703–2716, 2005.
- [64] Jack B Homer and Gary B Hirsch. System dynamics modeling for public health: background and opportunities. *American Journal of Public Health*, 96(3):452–458, 2006.

- [65] Jack B Homer, Bobby Milstein, W. Dietz, D. Buchner, and D. Majestic. Obesity population dynamics: exploring historical growth and plausible futures in the US. In Proceedings of the 24th International Conference of the System Dynamics Society, 2006.
- [66] Jack B Homer and Rogelio Oliva. Maps and models in system dynamics: a response to Coyle. System Dynamics Review, 17(4):347–355, 2001.
- [67] Peter S Hovmand, Matt Kreuter, and James W Deering. Designing Public Health Dissemination and Delivery Systems. In *Proceedings of the 30th International Conference* of the System Dynamics Society, St. Gallen, Switzerland, 2012.
- [68] J. Huang, E. Howley, and J. Duggan. An eigenvector approach for analysing linear feedback systems. In *Proceedings of the 27th International System Dynamics Confer*ence, Seoul, South Korea, 2010.
- [69] J. Huang, E. Howley, and J. Duggan. An extension of loop deactivation in the behavioral method. In Proceedings of the 27th International Conference of the System Dynamics Society, Seoul, South Korea, 2010.
- [70] Jinjing Huang, Enda Howley, and Jim Duggan. The Ford method: a sensitivity analysis approach. In Proceedings of the 27th International Conference of the System Dynamics Society, 2009.
- [71] Jinjing Huang, Enda Howley, and Jim Duggan. Observations on the shortest independent loop set algorithm. *System Dynamics Review*, 28(3):276–280, July 2012.
- [72] A. Isidori. Nonlinear Control Systems. Springer Verlag, London, 3rd edition, 1995.
- [73] A. Isidori. Nonlinear Control Systems II. Springer Verlag, London, 1999.
- [74] Chanoch Jacobsen and Richard Bronson. Defining sociological concepts as variables for system dynamics modeling. System Dynamics Review, 3(1):1–7, 1987.
- [75] Andrew P Jones, Jack B Homer, Dara L Murphy, Joyce D K Essien, Bobby Milstein, and Donald A Seville. Understanding Diabetes Population Dynamics Through Simulation Modeling and Experimentation. *American Journal of Public Health*, 96(3):488– 494, March 2006.
- [76] Malcolm M Jones, Alexander L Pugh, and Joseph V Yance. (D-0014) Shoe Model 1. 1958.
- [77] Christian Erik Kampmann. Feedback loop gains and system behavior. In *Proceedings* of the 1996 International System Dynamics Conference, pages 21–25, 1996.
- [78] Christian Erik Kampmann and Rogelio Oliva. Loop eigenvalue elasticity analysis: three case studies. *System Dynamics Review*, 22(2):141–162, June 2006.

- [79] Christian Erik Kampmann and Rogelio Oliva. Analytical methods for structural dominance analysis in system dynamics: An assessment of the current state of affairs. In Proceedings of the 25th International Conference of the System Dynamics Society, Boston, USA, 2007.
- [80] Christian Erik Kampmann and Rogelio Oliva. Structural dominance analysis and theory building in system dynamics. Systems Research and Behavioral Science, 25(4):505– 519, September 2008.
- [81] William O Kermack and Anderson G McKendrick. A contribution to the mathematical theory of epidemics. In Proceedings of the Royal Society of London A: mathematical, physical and engineering sciences, volume 115, pages 700–721. The Royal Society, 1927.
- [82] H.K. Khalil. Nonlinear Systems. Prentice Hall, Upper Saddle River, New Jersey, 3rd edition, 2002.
- [83] William Robert Killingsworth et al. Application of contraction mappings to the control of nonlinear systems. PhD thesis, Massachusetts Institute of Technology, 1972.
- [84] Dong-Hwan Kim. A new approach to finding dominant feedback loops: loop-by-loop simulations for tracking feedback loop gains. System Dynamics, and International Journal of Policy Modeling, 7(2):1–15, 1995.
- [85] David P Kreutzer. (D-3913-2) Introduction to feedback loops and simple structures. 1991.
- [86] Gilbert Levin and Edward Baer Roberts. The dynamics of human service delivery. Ballinger Pub. Co., 1976.
- [87] Hans-Walter Lorenz. Complex dynamics in low-dimensional continuous-time business cycle models: The Šil nikov case. System Dynamics Review, 8(3):233–250, 1992.
- [88] Gilbert W. Low. (D-3042) Using System Dynamics to Simulate the Past. 1979.
- [89] James M Lyneis. (D-2929) Problems and cases in industrial dynamics: solution sets and instructors notes. 1978.
- [90] Leslie A Martin. (D-4476-2) Exploring S-shaped growth. 1996.
- [91] Dennis Meadows and Donella H Meadows. (D-1550) The club of Rome Project on the Predicament of mankind. 1971.
- [92] D.H. Meadows, D.L. Meadows, J. Randers, and W.W. Behrens. The Limits to Growth. Pan Books, London, 1971.
- [93] Donella Meadows. Leverage Points: Places to Intervene in a System. *Donella Meadows Institute*, 1999. The Sustainability Insitute.
- [94] Richard Fowler Miller. Dynamics of product growth in a competitive market. Master's thesis, Massachusetts Institute of Technology, 1963.

- [95] Mohammad Mojtahedzadeh. A Path Taken: Computer Assisted Heuristics for Understanding Dynamic Systems. PhD thesis, Rockefeller College, State University of New York at Albany, Albany, NY, 1997.
- [96] Mohammad Mojtahedzadeh. Do parallel lines meet? How can pathway participation metrics and eigenvalue analysis produce similar results? System Dynamics Review, 24(4):451–478, December 2008.
- [97] Mohammad Mojtahedzadeh. Consistency in explaining model behavior based on its feedback structure. *System Dynamics Review*, 27(4):358–373, October 2011.
- [98] Mohammad Mojtahedzadeh. Constructing Local Stories for Global Insight: Detecting Dominant Structure Forrester's Market Growth Model. In Proceedings of the 33rd International System Dynamics Conference, 2015.
- [99] Mohammad Mojtahedzadeh, David Andersen, and George P. Richardson. Using Digest to implement the pathway participation method for detecting inuential system structure. *System Dynamics Review*, 20(1):1–20, March 2004.
- [100] Mohammad Mojtahedzadeh and George P Richardson. Confusion in the polarity of major loops. In Proceedings of the 13th International Conference of the System Dynamics Society, Tokyo, Japan, 1995.
- [101] Erik Mosekilde and Erik Reimer Larsen. Deterministic chaos in the beer productiondistribution model. System Dynamics Review, 4(1-2):131-147, 1988.
- [102] Ole C Nord and Carl V Swanson. (D-0429) Growth of a New Product. 1962.
- [103] Rogelio Oliva. Model structure analysis through graph theory: partition heuristics and feedback structure decomposition. System Dynamics Review, 20(4):313, 2004.
- [104] Rogelio Oliva and Mohammad Mojtahedzadeh. Keep it simple: A dominance assessment of feedback loops. In Proceedings of the 22nd International Conference of the System Dynamics Society, Oxford, UK, 2004.
- [105] W.J. Palm III. Modeling, Analysis and Control of Dynamic Systems. John Wiley & Sons, Inc., New York, NY, 1983.
- [106] Ruth E. Patterson, Graham A. Colditz, Frank B. Hu, Kathryn H. Schmitz, Rexford S. Ahima, Ross C. Brownson, Kenneth R. Carson, Jorge E. Chavarro, Lewis A. Cho-dosh, Sarah Gehlert, Jeff Gill, Karen Glanz, Debra Haire-Joshu, Karen Louise Herbst, Christine M. Hoehner, Peter S. Hovmand, Melinda L. Irwin, Linda A. Jacobs, Aimee S. James, Lee W. Jones, Jacqueline Kerr, Adam S. Kibel, Irena B. King, Jennifer A. Ligibel, Jeffrey A. Meyerhardt, Loki Natarajan, Marian L. Neuhouser, Jerrold M. Olefsky, Enola K. Proctor, Susan Redline, Cheryl L. Rock, Bernard Rosner, David B. Sarwer, J. Sanford Schwartz, Dorothy D. Sears, Howard D. Sesso, Meir J. Stampfer, S. V. Subramanian, Elsie M. Taveras, Julia Tchou, Beti Thompson, Andrea B. Troxel, Marianne Wessling-Resnick, Kathleen Y. Wolin, and Mark D. Thornquist. The 2011-2016
Transdisciplinary Research on Energetics and Cancer (TREC) initiative: rationale and design. *Cancer Causes & Control: CCC*, 24(4):695–704, April 2013.

- [107] Erik Viggo Pedersen. The analysis of feedback systems. Master's thesis, Massachusetts Institute of Technology, 1967.
- [108] Peyton Z Peebles, Jay Read, and Peter Read. Probability, Random Variables, and Random Signal Principles, volume 3. McGraw-Hill Boston, Mass, USA, 2001.
- [109] Jose Ignacio Perez Arriaga. Selective modal analysis with applications to electric power systems. PhD thesis, Massachusetts Institute of Technology, 1981.
- [110] H Willem Geert Phaff, Jill H Slinger, Burak Güneralp, and C Els van Daalen. Investigating model behavioural analysis: A critical examination of two methods. In Proceedings of the 24th International Conference of the System Dynamics Society, 2006.
- [111] H.G. Phaff. Generalized loop deactivation method: An extension of fords behavioural approach to loop dominance analysis and its applications. In *Proceedings of the 26th International Conference of the System Dynamics Society*, Athens, Greece, 2008.
- [112] Brian Porter and Roger Crossley. Modal control: theory and applications. Taylor & Francis Group, 1972.
- [113] Willard Van Orman Quine. *Methods of logic*. Harvard University Press, 1982.
- [114] Hazhir Rahmandad. Connecting strategy and system dynamics: an example and lessons learned. System Dynamics Review, 31(3):149–172, 2015.
- [115] Hazhir Rahmandad, Rogelio Oliva, Nathaniel D Osgood, and George Richardson. Analytical Methods for Dynamic Modelers. MIT Press, 2015.
- [116] Hazhir Rahmandad, Nelson Repenning, and John D Sterman. Effects of feedback delay on learning. System Dynamics Review, 25(4):309–338, 2009.
- [117] Jørgen Randers. Elements of the System Dynamics Method. Wright Allen Press, 1980.
- [118] Steen Rasmussen, Erik Mosekilde, and John D Sterman. Bifurcations and chaotic behavior in a simple model of the economic long wave. System Dynamics Review, 1(1):92–110, 1985.
- [119] Nelson P. Repenning. Selling system dynamics to (other) social scientists. System Dynamics Review, 19(4):303–327, 2003.
- [120] George P Richardson. Dominant structure. System Dynamics Review, 2(1):68–75, 1986.
- [121] George P. Richardson. Loop polarity, loop dominance, and the concept of dominant polarity (1984). System Dynamics Review, 11(1):67–88, March 1995.

- [122] George P Richardson. Reflections on the foundations of system dynamics. System Dynamics Review, 27(3):219–243, 2011.
- [123] George P Richardson and Alexander I Pugh III. Introduction to System Dynamics Modeling with DYNAMO. Productivity Press Inc., 1981.
- [124] George Peter Richardson. The evolution of the feedback concept in American social science. PhD thesis, Massachusetts Institute of Technology, 1984.
- [125] R Riegelman. Contributory cause: unnecessary and insufficient. Postgraduate Medicine, 66(2):177–179, 1979.
- [126] Dale Runge. (D-2648) Solution to problem set 3 (Principles of Dynamics Systems I). 1977.
- [127] Mohamed M Saleh. The characterization of model behavior and its causal foundation. PhD thesis, 2002.
- [128] Mohamed M Saleh and K.A.H. Bayoumi. A comprehensive eigenvalue analysis of system dynamics models. In Proceedings of the 23rd International Conference of the System Dynamics Society, page 130, Boston, MA., 2005.
- [129] Mohamed M Saleh and Pål I Davidsen. An eigenvalue approach to feedback loop dominance analysis in non-linear dynamic models. In *Proceedings of the 18th International Conference of the System Dynamics Society*, pages 182–183, Bergen, Norway, 2000.
- [130] Mohamed M Saleh and Pål I Davidsen. The origins of behavior patterns. In Proceedings of the 19th International Conference of the Sytem Dyanmics Society, Atlanta, GA, 2001.
- [131] Mohamed M Saleh and Pål I Davidsen. Extending eigenvalue analysis to nonlinear models via incorporating higher order terms of Taylor series expansion. In Proceedings of the 25th International System Dynamics Conference, 2007.
- [132] Mohamed M Saleh, Rogelio Oliva, Christian Erik Kampmann, and Pål I Davidsen. A comprehensive analytical approach for policy analysis of system dynamics models. European Journal of Operational Research, 203(3):673–683, 2010.
- [133] Peter M Senge. (D-3205-7) Principles of Dynamic Systems I Suggested answers. 1980.
- [134] John D Sterman. A behavioral model of the economic long wave. Journal of economic behavior & organization, 6(1):17–53, 1985.
- [135] John D Sterman. Learning in and about complex systems. System Dynamics Review, 10(2-3):291–330, 1994.
- [136] John D Sterman. Business Dynamics: Systems Thinking and Modeling for a Complex World. Irwin McGraw-Hill, New York, 2000.

- [137] Steven H Strogatz. Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering. Perseus Books Publishing, 1994.
- [138] Can Sücüllü and Gönenç Yücel. Behavior analysis and testing software (BATS). In Proceedings of the 32nd International System Dynamics Conference, 2014.
- [139] Carl V Swanson. (D-0730) A study of the stagnation and growth in a product line. 1965.
- [140] Carl V Swanson. (D-0946) Properties of Feedback Systems. 1966.
- [141] Carl V Swanson. (D-0986-1) Notions Useful for the Analysis of Complex Feedback Systems. 1968.
- [142] T. Taylor, D. Ford, and A. Ford. Model analysis using statistical screening: Extensions and example applications. In *Proceedings of the 25th International Conference of the System Dynamics Society*, Boston, MA, 2007.
- [143] Timothy R. B. Taylor, David N. Ford, and Andrew Ford. Improving model understanding using statistical screening. System Dynamics Review, 26(1):73–87, January 2010.
- [144] Ramon L Thomas. Planning for discontinuities. Master's thesis, Massachusetts Institute of Technology, 1980.
- [145] Pierre-François Verhulst. Notice sur la loi que la population suit dans son accroissement. correspondance mathématique et physique publiée par a. Quetelet, 10:113–121, 1838.
- [146] Margarita Vzquez, Manuel Liz, and Javier Aracil. Knowledge and reality: some conceptual issues in system dynamics modeling. *System Dynamics Review*, 12(1):21–37, 1996.
- [147] Rik Wallace, Kip Cooper, and Debbie Hyams. (D-4357-B) The System Dynamics Lexicon. 1993.
- [148] Qifan Wang. (D-3498) Using the Liapunov principles in the analysis of SD models. 1983.
- [149] FH Weymar. (D-0151) Feedback in Single Loop Systems (Class Notes). 1961.
- [150] Jarrod Whitfield Wilcox. The multipiler-accelerator model reformulated. Master's thesis, Massachusetts Institute of Technology, 1967.
- [151] Faustine Williams and Nancy Zoellner. Addressing African-American and White Disparities in Breast Cancer Treatment Delays Community Report. Technical report, Washington University in St. Louis, 2015.

- [152] Q. Zhang and N. Osgood. Local analysis of individual-based viral dynamic models with eigenspace and eigenvalue elasticity analysis. In *Proceedings of the 27th International Conference of the System Dynamics Society*, Albuquerque, NM., 2009.
- [153] Q. Zhang and N. Osgood. Summary function elasticity analysis for an individual-based system dynamics model. In *Proceedings of the Winter Simulation Conference*, pages 405–416, Baltimore, MD, 2010.
- [154] H Zhu. (D-4498) Road Maps Glossary. 1995.

Appendix A

Definitions of *Dominance* From Literature Review

The literature review found the following definitions of *dominance*, listed in chronological order:

Here, a loop dominates the behavior in the sense that if the loop is disconnected or substantially altered, the behavior mode also changes substantially [52].

...In a feedback structure, a loop that is primarily responsible for model behavior over some time interval is known as a dominant loop [123, p. 285].

Links that have large-magnitude eigenvalue elasticities are particularly important. If a small number of elasticities have markedly greater magnitudes than others, then (they) define a dominant subset of model structure [38].

In a first-order system with level x and net rate of change x, a shift in loop dominance is said to occur if and when $\partial \dot{x}/\partial x$ changes sign, that is, when the dominant polarity of the system changes [121].

If we say that at a particular time one feedback loop is stronger or dominant over another we mean that the system is undergoing behavior associated with the dominant type of feedback at that time [43].

The stronger loop is said to have loop dominance [85].

When a positive and negative feedback loop are used together, as shown in Fig 9, the strongest loop is the dominant one. In Fig 9, for one loop to be dominant, it must have a greater effect on the population [21].

We can trace several feedback loop gains simultaneously by simulating model equations, and then we can select a dominant feedback loop which has the largest gain in the specified periods [84].

Loop dominance: A system in which one loop is stronger. In a system with multiple loops, magnitudes and algebraic signs of variables determine what kind of behavior, positive or negative feedback, is dominant at any given time. If the system exhibits exponential growth, then the positive loop is dominant. If asymptotic behavior is evidenced, the negative loop has dominance. S-shape growth is a common behavior of a system in which loop dominance shifts with time [154].

A feedback loop dominates the behavior of a variable during a time interval in a given structure and set of conditions when loop determines the atomic pattern of that variable's behavior [26].

Dominant loops can be seen as a reduced set of closed feedback paths that contribute most to the overall behavior mode of a model [99].

Contributes the most to $\partial \dot{x} / \partial x$ [99].

Mojtahedzadeh then considers each possible pathway and defines the dominant pathway as the one with the largest numerical value and the same sign as PPM_i [80].

By dominant structure we mean particular feedback loops, or possibly external drivers, that are in some sense "important" in shaping the behavior of interest [80].

...the one considered as a dominant loop should exert most significant influence to the behaviour, i.e., when the dominant loop is deactivated, the behaviour diverts most from its original trajectory [69].

...pathways with higher magnitude of frequency (stability) factors are considered dominant in deriving the periodicity (stability) of the cycles. A set of pathways that close the loops are considered the dominant structure [97].

EEA calculates how much each feedback loop influences the eigenvalue, and the one with most influence is considered the dominant loop. This influence is quantified by the loop elasticity e [71].

Dominance is defined as the loop, or minimum combination of loops of like polarity, whose (combined) impact is greater than the sum of all loops of opposite polarity [60].

Appendix B

Summary of Dominance Methods

Method	Dominance Criteria	Strengths	Limitations
Behavioral loop dominance analysis (Ford's behavioral approach; Gener- alised Loop Deac- tivation Method; Extension of deac- tivation method)	Deactivating loop causes change in be- havior mode of specific variable (counterfactual test)	Intuitive relationship between structure and behavior. Identifies when structure determines behavior. Can identify multiple dominant loops.	Limited insight into how mechanisms cause behav- ior. May not always be able to isolate effect of individual loop through deactivation. Time domain only, not ap- plied to state-space.
Loop Eigenvalue Elasticity Analysis (LEEA)	Loop with greatest eigenvalue elasticity with respect to overall system behavior.	Takes into consideration all system parameters. Shown to be appropri- ate for quasi-linear mod- els that exhibit transient or oscillatory behavior. Identifies the relative in- fluence of each loop (and thus, can produce a rank order of loops).	Computationally inten- sive. Non-intuitive relationship between structure and be- havior and interpretation of elasticities. Does not detect whether structure determines be- havior. Existence and uniqueness of independent loop sets. Not applied to specific variables. Relative importance of eigenvalues is subjective. Addressing phantom loops. Difficulties with chaotic systems and individual- based models. Not applied to state-space.
Eigenvector anal- ysis or Dynamic decomposition weights analysis (DDWA)	Loop with greatest in- fluence measure with re- spect to a specific vari- able.	Similar strengths as LEEA. Extends LEEA method to apply to specific variables.	Similar as LEEA, but solves issue of applying to specific variables. Confounds effects of initial conditions and structural elements.
Pathway Participa- tion Metric (PPM)	Pathway or loop with greatest total PPM with respect to a specific vari- able.	Computationally simple. Identifies how influence shifts over time.	Does not detect whether structure determines be- havior (relies on TPPM as proxy for behavior). Always identifies a sin- gle dominant loop using depth-first search. Metric undefined for zero first derivative. Time-domain only, not ap- plied to state-space.
Loop Impact Method	Minimum combination of loops with like po- larity with loop impact greater than opposing loops (with respect to specific variable).	Accounts for initial condi- tions. Handles higher order loops that change polar- ity, hidden (phantom) loops, and loop that self-cancel.	Not a direct link be- tween structure and behav- ior (uses PPM as proxy for behavior). Time-domain only, not ap- plied to state-space.

Table B.1: Summary of strengths and limitations of dominance analysis methods.

Appendix C

Enumerated Findings of Systematic Review

The following list documents the detailed findings from the systematic review, which were used to develop the criteria for a formal definition for dominance.

- 1. Nearly all explanations offered for behavior trace back, in some manner, to the structure of nonlinear dynamic systems.
- 2. The term *dominant* is frequently used to provide explanations of structure-behavior relationships.
- 3. The concepts of *structural dominance* and *feedback loop dominance* seem to have originated entirely within the field of system dynamics.
- 4. The concept of *feedback loop dominance* seems to have followed (and perhaps was motivated) by concepts of dominant modes, poles, and eigenvalues from feedback control engineering.
- 5. Currently, there does not exist one formal and rigorous definition for *dominance* that is agreed upon and widely used among practitioners in the field.

- 6. Despite not having a formal and rigorous definition, there has been considerable progress in developing methods for detecting dominant structure.
- 7. Not having a formal and rigorous definition is a significant limitation for future progress in the field:
 - (a) Many claims pertaining to loop dominance lack sufficient rigor to be proven true or falsified.
 - (b) Unresolved discrepancies exist between different methods of dominance analysis.
 - (c) There is a need for a comprehensive understanding for how different methods of dominance relate to one another.
 - (d) There is a need to establish a formal basis for explaining observed phenomena related to dominance (e.g. shifts in dominance, shadow dominance, etc.).
- 8. Dominance has been used in three distinct ways to describe structure and behavior:
 - (a) To relate behavior to other behaviors (as in *dominant behavior modes*).
 - (b) To relate elements of structure to specific behaviors (definitions use objective standards for dominance)
 - (c) To relate elements of structure to other elements of structure (definitions use relative standards for dominance)
- 9. The notion of *shifts in dominance* originated when studies turned from steady-state oscillatory processes to that of transient growth processes.
 - (a) Shifts in dominance are attributed to nonlinearity in the system.
 - (b) S-shape growth, and specifically the logistic growth equation, became the canonical example of *shifts in dominance*.
- 10. Methods of dominance analysis fall into two categories.

- (a) Exploratory/behavioral methods (i.e. Ford's behavioral method and extensions) define dominance in an objective fashion based on impact to behavior, and detect when dominance occurs, but not why it occurs.
- (b) Formal/structural methods (i.e. LEEA, PPM, DDWA, and Loop Impact method) define dominance in a relative fashion based on normalized metrics, and provide insight into how influence changes over time, but do not determine if structure actually determines behavior.
- 11. The relative utility of formal/structural dominance methods (such as LEEA) (versus traditional analysis methods) remains to be proven over a wide set of realistic models.
- Methods of dominance assume different definitions of behavior, including local patterns, global patterns, local pattern indicators, behavior proxy measures, and eigenvalues.
- 13. Methods of dominance assume different definitions of structure, however the vast majority consider feedback loops as the explanatory element of structure, while a few focus on individual links, variables, or parameters.
- 14. Significant challenges remain in identifying, isolating and analyzing feedback loops as explanatory elements of structure.
 - (a) The identification of independent loop sets.
 - (b) The best way to isolate the effects of a single loop when testing for dominance.
 - (c) Analyzing the role of state space on a feedback loop's influence.
- 15. A new, formal and rigorous definition of dominance requires:
 - (a) Formally defining behavior.
 - (b) Formally defining structural elements, independent from behavior.

- (c) Formally defining the relationship between structural elements and behavior (such as the criteria for when a structural element determines behavior).
- (d) Formally defining the relationships between structural elements (such as when one structure dominates over another, or when dominance shifts from one structure to another).
 - An unambiguous criteria for dominance should determine whether no elements, one element, or multiple elements of structure are dominant.
 - Shifts in dominance as a consequence of a new definition for dominance should be tested against and found consistent with the canonical example of shifts in dominance: the logistic growth process (Verhulst equation).
 - Causes of shifts in dominance have been traditionally attributed to nonlinearities. If a new definition of dominance indicates that a linear system can in fact shift dominance, this should be explained.
- 16. A formal and rigorous definition for dominance should be able to accommodate and explain observed phenomena such as shadow dominance, shared dominance, and multiple dominance.
- A rigorous and formal definition for dominance should consider both behavioral and structural methods.

Appendix D

Matlab Code for Dominance Methods

```
function ssnecessary(x, y, n, f, varargin)
%STATE-SPACE REGIONS OF NECESSARY PATHWAYS
% This function calculates the regions in (x,y) state-space in which
% pathway n, corresponding with the nth anonymous function in the
varargin cell array,
8
   is a necessary pathway for determining the second time-derivative.
8
8
  x - state variabel 1 values (grid array)
8
  y - state variable 2 values (grid array)
\% n - the causal pathway of interest. if n = 0, then plots regions for
8
       all causal pathways
8
  f - fill flag: if true - fills in the regions, otherwise plots
8
       only the region boundaries.
8
  varargin - cell array of one or more anonymous function handles
representing causal
       pathway force contributions
00
    Fnet = varargin{1} (x, y) \cdot 0;
    for i = 1:(nargin-4)
       Fnet = Fnet + varargin{i}(x,y);
                                        % sum of all pathway force
contributions = second derivative
   end
   hold;
    for i = 1: (nargin-4)
        if (i == n) | (n == 0)
            % pathway necessary if removal changes sign of second
           % derivative
           Fnec = (sign(Fnet) ~= sign(Fnet-varargin{i}(x,y)));
            if f
                contourf(x, y, Fnec, 'LevelStep', 1);
            else
                contour(x, y, Fnec, 'LevelStep', 1);
            end
       end
   end
end
```

```
function sssufficient(x, y, n, f, varargin)
STATE-SPACE REGIONS OF SUFFICIENT PATHWAYS
   This function calculates the regions in (x, y) state-space in which
2
2
    pathway n, corresponding with the nth anonymous function in the varargin
   cell array, is a sufficient pathway for determining the second
8
8
   time-derivative of a variable of interest.
8
   x - state variabel 1 values (grid array)
8
   y - state variable 2 values (grid array)
8
8
   n - the causal pathway of interest. if n = 0, then plots regions for
       all causal pathways
8
8
   f - fill flag: if true - fills in the regions, otherwise plots
       only the region boundaries.
8
8
   varargin - cell array of one or more anonymous function handles representing
causal
       pathway force contributions
8
8
    compute the sum of all force contributions which have opposite sign
   of the force contribution of the pathway of interest.
8
    Fopp = zeros(size(x,1), size(x,2), nargin-4);
    for i = 1:size(x,1)
        for j = 1:size(x, 2)
            for k = 1: (nargin-4)
                if (k == n) | (n == 0)
                    for l = 1: (nargin-4)
                        if sign(varargin{l}(x(i,j),y(i,j))) ~=
sign(varargin\{k\}(x(i,j),y(i,j)))
                            Fopp(i,j,k) = Fopp(i,j,k) +
varargin{l}(x(i,j),y(i,j));
                        end
                    end
                end
            end
        end
    end
    hold;
    for i = 1:(nargin-4)
        if (i == n) | (n == 0)
            % pathway is sufficient if its force contribution is greater
            % than the sum of all force contributions of opposite sign
            Fsuf = (abs(varargin{i}(x,y)) > abs(Fopp(:,:,i)));
            if f
                contourf(x, y, Fsuf, 'LevelStep', 1);
            else
                contour(x, y, Fsuf, 'LevelStep', 1);
            end
        end
    end
```

end

```
function sspolaritysign(x, y, f, v, varargin)
STATE-SPACE REGIONS OF POSITIVE AND NEGATIVE POLOARITY
00
   This function draws the regions in (x, y) state-space in which
the polarity
   (sign of second time-derivative multiplied by sign of first
00
time-derivative)
% of a variable of interest is positive and negative.
9
8
  x - state variabel 1 values
  y - state variable 2 values
8
   f - fill flag: if true - fill in the regions, otherwise just
00
shown
00
  the region boundaries.
00
  v - velocity (first time derivative) function of variable of
interest
00
   varargin - cell array of one or more causal pathway force
contributions
8
   (functions) which make-up the total acceleration (second time
00
   derivative).
    Fnet = varargin{1} (x, y) \cdot 0;
    for i = 1:(nargin-4)
        Fnet = Fnet + varargin{i} (x,y);
    end
    PosPolarity = (sign(Fnet).*sign(v(x,y)) > 0);
   hold;
    if f
        contourf(x, y, PosPolarity, 'LevelStep', 1);
         %makes the spacing between contour lines = 1
    else
        contour(x, y, PosPolarity, 'LevelStep', 1);
    end
```

```
end
```

Appendix E

Matlab Script Defining Cancer Model and Pathways

%DESCRIPTION: this script defines the equations and parameters of the % thesis-version of the cancer services supply and demand model for a % single population segment. It also defines the pathway force % contributions for each state variable. clear all; %model parameters N = 10000;%people in need of services (people) a = 0;%advertising or external adoption fractional rate (FR) (1/year) ATa = .5;%adjustment time (AT) of ability to get services (years) Sa = 2; %sensitivity of ability to get services to service ratio (no units) MINw = .01; %minimum internal adoption (word-of-mouth) fractional rate (1/year). %equivalent to # of contacts per person per year multiplied by adoption %fraction or probability. %maximum internal adoption (word-of-mouth) fractional rate (1/year). MAXw = 1;Sw = 7; %sensitivity of internal adoption (word-of-mouth) FR to quality (no units) MINd = .01;%minimum de-adoption fractional rate (1/year) MAXd = .3;%maximum de-adoption fractional rate (1/year) Sd = 7; %sensitivity of de-adoption FR to quality (no units) MIN ATs = .1; %minimum adjustment time of service capacity (years) MAX_ATs = 5; %maximum adjustment time of service capacity (years)
S_ATs = 3; %sensitivity of adjustment time of service capacity to quality (no units) Sq = 2; %sensitivity of quality to service ratio (no units) %auxiliary variable relationship functions g1 = @(x,min,max,s) min+(max-min)./(1+exp(-s.*(x-.5))); g2 = @(x,min,max,s) max-(max-min)./(1+exp(-s.*(x-.5))); g3 = @(x,min,max,s) max-(max-min).*exp(-s.*x); g4 = @(x,min,max,s) min+(max-min).*exp(-s.*x); %derivatives (primes) of relationship functions with respect to x $q_{1p} = 0 (x, min, max, s) (max-min) \cdot s \cdot exp(-s \cdot (x-.5)) \cdot / (1+exp(-s \cdot (x-.5))) \cdot 2;$ g2p = Q(x,min,max,s) -g1p(x,min,max,s);g3p = @(x,min,max,s) (max-min).*s.*exp(-s.*x); g4p = Q(x, min, max, s) - g3p(x, min, max, s);

```
Sauxiliary variables and their derivatives (primes)
D = Q(Ps, Pa) N.*Ps.*Pa;
                                 %demand for services (patients) (people)
Q = Q(Ps, Pa, S) g3(S./D(Ps, Pa), 0, 1, Sq);
                                          %average service quality (no units)
Qp = @(Ps,Pa,S) g3p(S./D(Ps,Pa),0,1,Sq);
w = @(Ps,Pa,S) g1(Q(Ps,Pa,S).*Ps.*Pa,MINw,MAXw,Sw);
                                                     %internal adoption or word-of-mouth
fractional rate (1/year)
wp = @(Ps,Pa,S) g1p(Q(Ps,Pa,S).*Ps.*Pa,MINw,MAXw,Sw);
d = @(Ps,Pa,S) g2(Q(Ps,Pa,S).*Ps.*Pa,MINd,MAXd,Sd);
                                                     %de-adoption fractional rate
(1/vear)
dp = @(Ps,Pa,S) g2p(Q(Ps,Pa,S).*Ps.*Pa,MINd,MAXd,Sd);
Pa goal = @(Ps,Pa,S) g3(S./D(Ps,Pa),0,1,Sa); % goal of proportion able to get services
                %(effect of service ratio on proportion abe to get services) (no units)
Pa goalp = @(Ps, Pa, S) g3p(S./D(Ps, Pa), 0, 1, Sa);
ATS = @(Ps, Pa, S) g1(Q(Ps, Pa, S), MIN ATS, MAX ATS, S ATS);
                                                        %adjustment time of service
capacity (years)
ATsp = @(Ps,Pa,S) g1p(Q(Ps,Pa,S),MIN ATs,MAX ATs,S ATs);
%state variable derivatives (system of ODEs)
%derivative of Ps (proportion seeking services) (no units)
f1 = @(Ps,Pa,S) w(Ps,Pa,S).*Ps.*(1-Ps)+a.*(1-Ps)-d(Ps,Pa,S).*Ps;
%derivative of Pa (proportion able to access services) (no units)
f2 = @(Ps,Pa,S) (Pa goal(Ps,Pa,S)-Pa)./ATa;
%derivative of S (service capacity) (people)
f3 = @(Ps,Pa,S) (D(Ps,Pa)-S)./ATs(Ps,Pa,S);
%Ps (proportion seeking services) (x1) pathway partial derivatives (gains)
% and force contributions
2
  Ps word-of-mouth partials
P311 = @(Ps,Pa,S) Ps.*(1-Ps).*wp(Ps,Pa,S).*Ps.*Pa.*Qp(Ps,Pa,S)./(N.*Ps.*Pa);
F311 = @(Ps,Pa,S) P311(Ps,Pa,S).*f3(Ps,Pa,S);
P111 = @(Ps, Pa, S) -Ps.*(1-Ps).*wp(Ps, Pa, S).*Ps.*Pa.*Qp(Ps, Pa, S).*S./(N.*Pa.*Ps.*Ps);
F111 = @(Ps,Pa,S) P111(Ps,Pa,S).*f1(Ps,Pa,S);
P211 = @(Ps,Pa,S) -Ps.*(1-Ps).*wp(Ps,Pa,S).*Ps.*Pa.*Qp(Ps,Pa,S).*S./(N.*Ps.*Pa.*Pa);
F211 = @(Ps,Pa,S) P211(Ps,Pa,S).*f2(Ps,Pa,S);
P112 = @(Ps,Pa,S) Ps.*(1-Ps).*wp(Ps,Pa,S).*Q(Ps,Pa,S).*Pa;
F112 = @(Ps,Pa,S) P112(Ps,Pa,S).*f1(Ps,Pa,S);
P212 = @(Ps,Pa,S) Ps.*(1-Ps).*wp(Ps,Pa,S).*Q(Ps,Pa,S).*Ps;
F212 = @(Ps,Pa,S) P212(Ps,Pa,S).*f2(Ps,Pa,S);
P113 = @(Ps,Pa,S) w(Ps,Pa,S).*(1-Ps);
F113 = @(Ps,Pa,S) P113(Ps,Pa,S).*f1(Ps,Pa,S);
P114 = @(Ps,Pa,S) -w(Ps,Pa,S).*Ps;
F114 = @(Ps,Pa,S) P114(Ps,Pa,S).*f1(Ps,Pa,S);
  Ps advertising partials
P115 = 0(Ps, Pa, S) -a;
```

```
F115 = @(Ps,Pa,S) P115(Ps,Pa,S).*f1(Ps,Pa,S);
```

```
% Ps de-adoption partials
P312 = @(Ps,Pa,S) -Ps.*dp(Ps,Pa,S).*Ps.*Pa.*Qp(Ps,Pa,S)./(N.*Ps.*Pa);
F312 = @(Ps,Pa,S) P312(Ps,Pa,S).*f3(Ps,Pa,S);
P116 = @(Ps,Pa,S) Ps.*dp(Ps,Pa,S).*Ps.*Pa.*Qp(Ps,Pa,S).*S./(N.*Pa.*Ps.*Ps);
F116 = @(Ps,Pa,S) P116(Ps,Pa,S).*f1(Ps,Pa,S);
P213 = @(Ps, Pa, S) Ps.*dp(Ps, Pa, S).*Ps.*Pa.*Qp(Ps, Pa, S).*S./(N.*Ps.*Pa.*Pa);
F213 = @(Ps,Pa,S) P213(Ps,Pa,S).*f2(Ps,Pa,S);
P117 = @(Ps,Pa,S) -Ps.*dp(Ps,Pa,S).*Q(Ps,Pa,S).*Pa;
F117 = @(Ps,Pa,S) P117(Ps,Pa,S).*f1(Ps,Pa,S);
P214 = @(Ps,Pa,S) -Ps.*dp(Ps,Pa,S).*Q(Ps,Pa,S).*Ps;
F214 = @(Ps,Pa,S) P214(Ps,Pa,S).*f2(Ps,Pa,S);
P118 = @(Ps, Pa, S) - d(Ps, Pa, S);
F118 = @(Ps,Pa,S) P118(Ps,Pa,S).*f1(Ps,Pa,S);
Pa (proportion able to access services) (x2) pathway partial derivatives (gains) P
% and force contributions F
P321 = @(Ps,Pa,S) (1./ATa).*Pa goalp(Ps,Pa,S)./(N.*Ps.*Pa);
F321 = @(Ps,Pa,S) P321(Ps,Pa,S).*f3(Ps,Pa,S);
P121 = @(Ps,Pa,S) (-1./ATa).*Pa goalp(Ps,Pa,S).*S./(N.*Pa.*Ps.*Ps);
F121 = @(Ps,Pa,S) P121(Ps,Pa,S).*f1(Ps,Pa,S);
P221 = @(Ps,Pa,S) (-1./ATa).*Pa goalp(Ps,Pa,S).*S./(N.*Ps.*Pa.*Pa);
F221 = @(Ps,Pa,S) P221(Ps,Pa,S).*f2(Ps,Pa,S);
P222 = @(Ps,Pa,S) -1./ATa;
F222 = @(Ps,Pa,S) P222(Ps,Pa,S).*f2(Ps,Pa,S);
```